Stochastic Differential Equations

Guanyu Chen

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Abstract

This paper is about the basic theory and applications of SDEs. Also, this is a sum up of last term's SDE course. Reference: [6][1][5] [3][4][7][2]

Contents

1 Random Field

1.1 Definitions

Definition 1 (Random Field) *For a set $D \subset \mathbb{R}^d$, a (real-valued) random field $u(x) : x \in D$ is a set of real-valued random variables on a probability space $(\Omega, \mathcal{F}, P)$. We usually speak of realizations of random field, instead of sample paths.*

Definition 2 (second-order random field) *A random field is called second-order random field if $u(x) \in L^2(\Omega)$ for $\forall x \in D$. With its mean and covariance function:*

$$\begin{cases} \mu(x) = \mathbf{E}[u(x)] \\ C(x, y) = Cov(u(x), u(y)) = \mathbf{E}[(u(x) - m(x))(u(y) - m(y))] \end{cases} \tag{1}$$

Definition 3 (Gaussian Random Field) *A second-order random field $u(x) : x \in D$ is called Gaussian random field if*

$$u = u[u(x_1), u(x_2), \cdots, u(x_n)]^T \sim \mathcal{N}(\mu(x), C(x, y)), \ \forall x_i \in D \tag{2}$$

Example 1 ($L^2(D)$-valued random variable) *For $D \subset \mathbb{R}^d$, consider $L^2(D)$-valued R.V. u with $\mu \in L^2(D)$ and $\mathscr{C}$. Then $u(x)$ is a real-valued random field for each $x \in D$, and mean and covariance are well defined.*

Meanwhile, for $\phi, \psi \in L^2(D)$, we have

$$\begin{aligned} \langle \mathscr{C}\phi, \psi \rangle &= Cov\left(\langle u, \phi \rangle_{L^2(D)}, \langle u, \psi \rangle_{L^2(D)}\right) \\ &= E\left[\left(\int_D \phi(x)(u(x) - \mu(x))dx\right)\left(\int_D \psi(y)(u(y) - \mu(y))dy\right)\right] \\ &= \int_D \int_D \phi(x)\psi(y)E[(u(x) - mu(x))(u(y) - mu(y))]dxdy \\ &= \int_D \int_D \phi(x)\psi(y)Cov(u(x), u(y))dxdy \end{aligned} \tag{3}$$

So that

$$(\mathscr{C}\phi)(x) = \int_D Cov(u(x), u(y))\phi(y)dy \tag{4}$$

which is the covariance function of the random field $u(x)$. So, any $L^2(D)$-valued random variable defines a second-order random field, with mean $\mu(x)$ and covariance $C(x, y) = Cov(u(x), u(y))$ which is the kernel of the covariance operator $\mathscr{C}$.

Example 2 (Stationary Random Field) *A second-order random field $u(x) : x \in D$ is called stationary if the mean is constant and covariance function depends only on the difference $x-y$, i.e. $\mu(x) = \mu$, $C(x, y) = C(x-y)$.*

Theorem 1 (Wiener-Khinchin Theorem) *There exists a stationary random field $u(x) : x \in D$ with mean μ and covariance function $c(x)$ that is mean square continuous if and only if the function $c(x) : \mathbb{R}^d \to \mathbb{R}$ is such that*

$$c(x) = \int_{\mathbb{R}^d} e^{iv\cdot x} dF(v) = (2\pi)^{\frac{d}{2}} \hat{f}(x) \tag{5}$$

where $F(v)$ is some measure on $\mathbb{R}^d$ and $\hat{f}(x)$ is the Fourier transform of $f(x)$, f is the density function of F.

Reversely, $f(v) = (2\pi)^{\frac{d}{2}} \check{c}(v)$. If f is non-negative and integrable, then $c(x)$ is a valid covariance function.

Example 3 (Isotropic Random Field) *A stationary random field is called isotropic if its covariance function depends only on the distance between points, i.e.*

$$Cov(x) = c(\|x\|_2) = c^0(r) \tag{6}$$

where c^0 is known as the isotropic covariance function.

1.2 Algorithms

In 2D cases, the covariance matrices of samples of stationary random fields $u(x)$ at uniformly spaced points $x \in D$ are symmetric BTTB matrices.

Definition 4 (Uniformly spaced points) *Let $D = [0, a_1] \times [0, a_2]$, the uniformly spaced points are given by:*

$$x_k = x_{i,j} = (i\Delta x_1, j\Delta x_2)^T,\ i = 0, 1, \cdots, n_1 - 1,\ j = 0, 1, \cdots, n_2 - 1, k = i + jn_1 \tag{7}$$

where $\Delta x_1 = \frac{a_1}{n_1-1}$ and $\Delta x_2 = \frac{a_2}{n_2-1}$.

With $N = n_1n_2$, $u = [u_0, u_1, \cdots, u_{N-1}]^T \sim \mathcal{N}(0, C)$ is the vector of samples of $u(x)$ at the uniformly spaced points. Since $u(x)$ is stationary, C is a $N \times N$ symmetric BTTB matrix with elements:

$$C_{kl} = Cov(u_k, u_l) = c(x_{i+jn_1} - x_{r+sn_1}) \tag{8}$$

where $c(x_k - x_l)$ is the covariance function of $u(x)$.

Theorem 2 *The covariance matrix C is always a symmetric BTTB matrix.*

Since we have the Fourier representation of BCCB matrix and BTTB matrix can by extended to BCCB by even extension, we can use the following algorithm to generate the samples of $u(x)$. So, when the even BCCB extension $\tilde{C} \in \mathbb{R}^{4N\times 4N}$ is non-negative definite, then $N(0, \tilde{C})$ is a valid Gaussian distribution.

Algorithm 1 *Suppose the even BCCB extension $\tilde{C} \in \mathbb{R}^{4N\times 4N}$ is non-negative definite, and the leading principle submatrix $S \in \mathbb{R}^{2N\times 2N}$ is:*

$$S = \begin{pmatrix} \tilde{C}_0 & \tilde{C}_1^T & \cdots & \tilde{C}_{n_2-1}^T \\ \tilde{C}_1 & \tilde{C}_2 & \cdots & \tilde{C}_{n_2-2}^T \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{C}_{n_2-1} & \tilde{C}_{n_2-2} & \cdots & \tilde{C}_0 \end{pmatrix}, \quad \tilde{C}_i = \begin{pmatrix} C_i & B_i \\ B_i & C_i \end{pmatrix} \tag{9}$$

where $C_i, B_i \in \mathbb{R}^{n_1\times n_1}$, $i = 0, 1, \cdots, n_2 - 1$.

Now given $\tilde{u} \sim N(0, \tilde{C})$, let v be the first $2n_1n_2$ elements of $\tilde{u}$, then $v \sim N(0, S)$. Take the first n_1 elements of v per $2n_1$ elements to get $\tilde{v} \sim N(0, C)$.

However, when the even BCCB extension $\tilde{C} \in \mathbb{R}^{4N\times 4N}$ is indefinite, we can avoid this by padding. But sometimes, padding leads to the size of matrix explosion. Approximate circulant embedding may be the only option.

1.3 KL expansion of R.F.

As mentioned before, we have the underlying covariance operator defined by:

$$(\mathscr{C}\phi)(x) = \int_D Cov(u(x), u(y))\phi(y)dy = \int_D c(x - y)\phi(y)dy \tag{10}$$

Hence, for the covariance operator $\mathscr{C}$, we have the eigenfunctions with corresponding eigenvalues $\{v_j, \phi_j\}_{j=1}^{\infty}, v_j \geq v_{j-1}$.

Theorem 3 (L^2 convergence of KL expansion) *Let $D \subset \mathbb{R}^d$, consider a random field $u(x) : x \in D$ and $u \in L^2(\Omega, L^2(D))$, then:*

$$u(x) = \mu(x) + \sum_{j=0}^{\infty} \sqrt{v_j}\phi_j(x)\xi_j \tag{11}$$

where the sum converges in $L^2(\Omega, L^2(D))$,

$$\xi_j = \frac{1}{\sqrt{v_j}} \int_D (u(x) - \mu(x))\phi_j(x)dx \tag{12}$$

The random variables ξ_j have mean zero, unit variance and are pairwise uncorrelated. If u is Gaussian, then ξ_j are i.i.d. Gaussian random variables with zero mean and unit variance.

2 Stationary SPDEs

2.1 Definition

Definition 5 (Stationary SPDE) *Assume given $a, f \in L^2(\Omega, L^2(D))$ are random fields, try to seek $u : \bar{D} \times \Omega \to \mathbb{R}$ in weak sense s.t. $\mathbb{P}$-a.s.:*

$$\begin{cases} -\nabla \cdot (a(x, w)\nabla u(x, w)) = f(x, w), & x \in D \\ u(x, w) = g(x), & x \in \partial D \end{cases} \tag{13}$$

To ensure the existence of solution, we need to impose some conditions on g.

Definition 6 (Weak solution on $D \times \Omega$) *A weak solution to Eq(13) with $g = 0$ is a function $u \in V = L^2(\Omega, H_0^1(D))$ s.t. for any $v \in V$,*

$$a(u, v) = l(v) \tag{14}$$

where

$$\begin{cases} a(u, v) = E\left[\int_D a(x, \cdot)\nabla u(x, \cdot) \cdot \nabla v(x, \cdot) dx\right] \\ l(v) = E\left[\int_D f(x, \cdot)v(x, \cdot) dx\right] \end{cases} \tag{15}$$

If $g \neq 0$, a weak solution to Eq(13) is a function $u \in W = L^2(\Omega, H_g^1(D))$ s.t. for any $v \in V$,

$$a(u, v) = l(v) \tag{16}$$

where $a(\cdot, \cdot) : W \times V \to \mathbb{R}$ and $l : V \to \mathbb{R}$:

$$\begin{cases} a(u, v) = E\left[\int_D a(x, \cdot)\nabla u(x, \cdot) \cdot \nabla v(x, \cdot) dx\right] \\ l(v) = E\left[\int_D f(x, \cdot)v(x, \cdot) dx\right] \end{cases} \tag{17}$$

Theorem 4 (Existence and uniqueness of weak solution) *Note for all $x \in D$*

$$0 < a_{\min} \leq a(x, \cdot) \leq a_{\max} < \infty \tag{18}$$

as a basic assumption.

If $f \in L^2(\Omega, L^2(D)), g = 0$, and Assumption (18) holds, then SPDE 14 has a unique weak solution $u \in V$.

If Assumption (18) holds, $f \in L^2(\Omega, L^2(D))$, and $g \in H^{\frac{1}{2}}(\partial D)$, then SPDE 16 has a unique weak solution $u \in W$.

Assume we have the approximate random fields $\tilde{a}, \tilde{f} : D \times \Omega \to \mathbb{R}$ s.t. (18) holds.

Then as mentioned before, we can expand a, f in terms of (truncated) Karhunen-Loeve expansion as:

$$\begin{cases} a(x, w) = \mu_a(x) + \sum_{i=1}^{N_a} \sqrt{v_i^a}\phi_i^a(x)\xi_i^a(w) \\ f(x, w) = \mu_f(x) + \sum_{i=1}^{N_f} \sqrt{v_i^f}\phi_i^f(x)\xi_i^f(w) \end{cases} \tag{19}$$

where $(v_i^a, \phi_i^a), (v_i^f, \phi_i^f)$ are the eigenpairs of the covariance operators of a, f respectively, and ξ_i^a, ξ_i^f are i.i.d. random variables.

The next question is how to compute:

$$\begin{aligned} a(u, v) &= E\left[\int_D a(x, \cdot)\nabla u(x, \cdot) \cdot \nabla v(x, \cdot) dx\right] \\ &= \int_\Omega \int_D a(x, w)\nabla u(x, w) \cdot \nabla v(x, w) dx dP(w) \end{aligned} \tag{20}$$

Since the truncated KL expansion of $a(x, w)$ depends on a finite number N_a of random variables $\xi_i^a : \Omega \to \Gamma_i$(same as $f(x, w)$), we consider weak form of Eq(13) on $D \times \Gamma$, where $\Gamma = \prod_{i=1}^{N_a} \Gamma_i$.

Definition 7 (finite-dimensional noise) *A function $v \in L^2(\Omega, L^2(D))$ of the form $v(x, \xi(w))$ for $\forall x \in D, w \in \Omega$, where $\xi = [\xi_1, \cdots, \xi_N]^T : \Omega \to \Gamma$, is called a finite-dimensional noise.*

Definition 8 (Weak solution on $D \times \Gamma$) *Let $\tilde{a}(x)$ and $\tilde{f}(x)$ be finite-dimensional noises defined in Eq(19), then the solution to Eq (13) is also finite-dimensional noise. Define*

$$W := L_p^2(\Gamma, H_g^1(D)) = \left\{v : D \times \Gamma \to \mathbb{R} : \int_\Gamma \|v(\xi, \cdot)\|_{H_g^1(D)}^2 d\xi < \infty\right\} \tag{21}$$

A weak solution to Eq(13) on $D \times \Gamma$ is a function $u \in W = L_p^2(\Gamma, H_g^1(D))$ s.t. for any $v \in V = L_p^2(\Gamma, H_0^1(D))$,

$$a(u, v) = l(v) \tag{22}$$

where

$$\begin{cases} a(u, v) = \int_\Gamma p(\xi) \int_D \tilde{a}(x, \xi)\nabla u(x, \xi) \cdot \nabla v(x, \xi) dx d\xi \\ l(v) = \int_\Gamma p(\xi) \int_D \tilde{f}(x, \xi)v(x, \xi) dx d\xi \end{cases} \tag{23}$$

2.2 Stochastic Galerkin Method

Therefore, we have the stochastic Galerkin solution: seek $u_{hk} \in W^{hk} \subset L^2(\Gamma, H_g^1(D))$ s.t. for any $v_{hk} \in V^{hk} \subset L^2(\Gamma, H_0^1(D))$.

By define the inner product:

$$\langle v, w\rangle_p = \int_\Gamma v(\xi)w(\xi)P(\xi)d\xi \tag{24}$$

We can construct a sequence of polynomials $P_i(\xi)$ on Γ. Hence:

$$L_p^2(\Gamma) := \{v : \Gamma \to \mathbb{R} : \|v\|^2_{L_p^2(\Gamma)} = \langle v, v\rangle_p < \infty\} \tag{25}$$

Definition 9 *Note S^k be the set of polynomials of degree k or less on Γ:*

$$\begin{aligned} S^k &= \mathrm{span}\{\prod_{i=1}^M P_i^{\alpha_i}(\xi_i) : \alpha_i \in \mathbb{N}_0, \sum_{i=1}^M \alpha_i \le k\} \\ &= \mathrm{span}\{\psi_1, \psi_2, \cdots, \psi_Q\} \end{aligned} \tag{26}$$

where $P_i(\xi_i)$ is some polynomial. And $Q = \dim S^k = \binom{M+k}{k}$.

We need $S^k \subset L_p^2(\Gamma)$ where $\Gamma \subset \mathbb{R}^M$. If $\{\xi_i\}$ are independent, then the joint density p is:

$$p(\xi) = \prod_{i=1}^M p_i(\xi_i) \tag{27}$$

Recall $V^h = \mathrm{span}\{\phi_i\}_{i=1}^J \subset H_0^1(D)$ is the finite element space, we have tensor product space:

$$V^{hk} := V^h \otimes S^k = \mathrm{span}\{\phi_i\psi_j\}_{i=1,j=1}^{J,Q} \tag{28}$$

Then

$$W^{hk} := V^{hk} \oplus \mathrm{span}\{\phi_{J+1}, \cdots, \phi_{J+J_b}\} \tag{29}$$

where J_b is finite element functions associated with Dirichlet boundary vertices.

Theorem 5 (Stochastic basis functions) *If $\{\xi_i\}$ are independent, suppose that $\{P_i^{\alpha_i}(\xi_i)\}_{\alpha_i=1}^M$ are orthonormal with $\langle\cdot,\cdot\rangle_{p_i}$ on Γ_i. Then the complete orthonormal polynomials $\{\psi_j\}_{j=1}^Q$ are orthonormal with $\langle\cdot,\cdot\rangle_p$ on Γ.*

Then u_{hk} can be written as:

$$u_{hk}(x, \xi) = \sum_{i=1}^J \sum_{j=1}^Q u_{ij}\phi_i(x)\psi_j(\xi) + w_g \tag{30}$$

Theorem 6 (Mean and covariance) *The Galerkin solution can be rewritten as:*

$$\begin{aligned} u_{hk}(x, \xi) &= \sum_{i=1}^J \left(\sum_{j=1}^Q u_{ij}\phi_i(x)\right)\psi_j(\xi) + w_g \\ &= \sum_{j=1}^Q u_j\psi_j(\xi) + w_g \\ &= (u_1(x) + w_g(x))\psi_1(\xi) + \sum_{j=2}^Q u_j(x)\psi_j(\xi) \end{aligned} \tag{31}$$

Then the mean and covariance is

$$\begin{cases} E[u_{hk}] = u_1 + w_g \\ Var(u_{hk}) = \sum_{j=2}^Q u_j^2 \end{cases} \tag{32}$$

2.3 Algorithm

Expand u_{hk} in terms of basis functions $v = \phi_r\psi_s$ for $r = 1, 2, \cdots, J, s = 1, 2, \cdots, Q$, we have the linear system:

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1Q} \\ A_{21} & A_{22} & \cdots & A_{2Q} \\ \vdots & \vdots & \ddots & \vdots \\ A_{Q1} & A_{Q2} & \cdots & A_{QQ} \end{pmatrix}, \mathbf{u} = \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_Q \end{pmatrix}, \mathbf{b} = \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \vdots \\ \mathbf{b}_Q \end{pmatrix} \tag{33}$$

where

$$\mathbf{u}_j = [u_{1j}, u_{2j}, \cdots, u_{Jj}]^T, j = 1, 2, \cdots, Q \tag{34}$$

and each submatrix A_{sj} is a $J \times J$ matrix:

$$A_{sj} = \langle \phi_r, \frac{\partial \psi_s}{\partial x} \rangle_{L^2(D)} \tag{35}$$

3 Semilinear Stochastic PDEs

3.1 Semilinear SPDE

Then we come to the time-dependent SPDE. We study the stochastic semilinear evolution equation:

$$du = [\Delta u + f(u)]dt + G(u)dW(t, x) \tag{36}$$

Definition 10 (Semilinear SPDE) *Simmilar to normal time-dependent PDE, we treat SPDE like this as semilinear SODEs on a Hilbert space, like*

$$du = [-Au + f(u)]dt + G(u)dW(t) \tag{37}$$

where $-A$ is a linear operator that generates a semigroup $S(t) = e^{-tA}$.

Example 4 (Phase-field model)

$$du = [\epsilon\Delta u + u - u^3]dt + \sigma dW(t, x) \tag{38}$$

Example 5 (Fluid Flow)

$$\begin{aligned} u_t &= \epsilon\Delta u - \nabla p - (u \cdot \nabla)u \\ \nabla \cdot u &= 0 \end{aligned} \tag{39}$$

So like we deal with integration of stochastic process like Itos or stratonovich, we need to generalize the Brownian Motion by introducing spatial variable to W(t). Here we define Q-Wiener Process.

3.2 Q wiener process

First, we assume U is a Hilbert space. And $(\Omega, \mathbf{F}, \mathbf{F}_t, \mathbb{R})$ is a filtered probability space.

Definition 11 (Q) *$Q \in \mathcal{L}(U)$ is non-negative definite and symmetric. Further, Q has an orthonormal basis $\{\mathcal{X}_j : j \in \mathcal{N}\}$ of eigenfunctions with corresponding eigenvalues $q_j \geq 0$ such that $\sum_{j\in\mathcal{N}} q_j < \infty$ (i.e., Q is of trace class).*

Definition 12 (Q-Wiener Process) *A U-valued stochastic process $\{W(t) : t \geq 0\}$ is Q-Wiener process if*

- *W(0) = 0 a.s.*
- *W(t) is a continuous function $\mathbb{R}^+ \to U$, for each $\omega \in \Omega$.*
- *W(t) is $\mathcal{F}_t$-adapted and $W(t) - W(s)$ is independent of $\mathcal{F}_s$ for $s \leq t$*
- *$W(t) - W(s) \sim N(0, (t - s)Q)$ for all $0 \leq s \leq t$*

Theorem 7 (Q-Wiener Process) *Assume we have Q defined in 11. Then, $W(t)$ is a Q-Wiener process if and only if*

$$W(t) = \sum_{j=1}^{\infty} \sqrt{q_j}\mathcal{X}_j\beta_j(t) \tag{40}$$

which is converges in $L^2(\Omega, C([0, T], U))$ and $\beta_j(t)$ are iid $\mathcal{F}_t$-Brownian motions and the series converges in $L^2(\Omega, U)$.

Theorem 8 **($H^r_{\text{per}}(0,a)$-valued process)** ...

Theorem 9 **($H^r_0(0,a)$-valued process)** ...

So, in place of $L^2(D)$, we develop the theory on a separable Hilbert space U with norm $\|\cdot\|_U$ and inner product $\langle\cdot,\cdot\rangle_U$ and define the Q-Wiener process $W(t) : t \geq 0$ as a U-valued process.

3.3 Cylindrical Wiener Process

We mention the important case of Q = I, which is not trace class on an infinite-dimensional space U (as $q_j = 1$ for all j) so that the series does not converge in $L^2(\Omega, U)$. To extend the definition of a Q-Wiener process, we introduce the cylindrical Wiener process.

The key point is to introduce a second space U1 such that $U \subset U_1$ and Q = I is a trace class operator when extended to U_1.

Then we can define cylindrical Wiener process:

Definition 13 (Cylindrical Wiener Process) *Let U be a separable Hilbert space. The cylindrical Wiener process (also called space-time white noise) is the U-valued stochastic process W(t) defined by*

$$W(t) = \sum_{j=1}^{\infty} \mathcal{X}_j \beta_j(t)$$

where $\{\mathcal{X}_j\}$ is any orthonormal basis of U and $\beta_j(t)$ are iid $\mathcal{F}_t$-Brownian motions.

Theorem 10 *If for the second Hilbert space U_1, and the inclusion map $\mathcal{I} : U \to U_1$ is Hilbert-Schmidt. Then, the cylindrical Wiener process is a Q-Wiener process well-defined on U_1(Converges in $L^2(U, U_1)$).*

3.4 Ito integral solution

Here we consider the Ito integral $\int_0^t B(s)dW(s)$ for a Q-Wiener process $W(s)$. Since dW_t takes value in Hilbert space U, and we treat SPDE in Hilbert space H, the integral will also take value in Hilbert space H.

Hence, $B(s)$ should be $\mathcal{L}^2_0(U_0, H)$-valued process, where $U_0 \subset U$ known as Cameron-Martin space. So, $B(s)$ is an operator from U_0 to H. Then, we consider the set of operator B.

Definition 14 **(L^2_0 space)** *Let $U_0 := \{Q^{\frac{1}{2}}u : u \in U\}$, the set of linear operators $B : U_0 \to H$ is noted as L^2_0 s.t.*

$$\|B\|_{L^2_0} := \left(\sum_{j=1}^{\infty} \|BQ^{\frac{1}{2}}\mathcal{X}_j\|^2\right)^{\frac{1}{2}} = \|BQ^{\frac{1}{2}}\|_{\text{HS}(U_0,H)} < \infty \tag{41}$$

Remark 1 *If G is invertible, L^2_0 is the space of Hilbert-Schmidt operators* $\text{HS}(U_0, H)$.

Definition 15 *The stochastic integral can be defined by*

$$\int_0^t B(s)dW(s) := \sum_{j=1}^{\infty} \int_0^t B(s)\sqrt{q_j}\mathcal{X}_j d\beta_j(s) \tag{42}$$

So, we can have the truncated form:

$$\int_0^t B(s)dW^J(s) = \sum_{j=1}^{J} \int_0^t B(s)\sqrt{q_j}\mathcal{X}_j d\beta_j(s) \tag{43}$$

3.5 Semilinear SPDE

Consider the semilinear SPDE:

$$du = [-Au + f(u)]dt + G(u)dW(t) \tag{44}$$

given the initial condition $u_0 \in H$ and $A : \mathcal{D} \subset H \to H$ is a linear operator, $f : H \to H$ and $G : H \to L^2_0$.

Example 6 *Consider the stochastic heat equation:*

$$du = \Delta u dt + \sigma dW(t, x), u(0, x) = u_0(x) \in L^2(D) \tag{45}$$

where D is a bounded domain in $\mathbb{R}^d$ and σ is a constant. Also, homogeneous Dirichlet boundary condition is imposed on D. Hence,

$$H = U = L^2(D), f(u) = 0, G(u) = \sigma I \tag{46}$$

We see that $A = -\Delta$ with domain $\mathcal{D}(A) = H^2(D) \cap H^1_0(D)$.

In the deterministic setting of PDEs, there are a number of different concepts of solution. Here is the same for SPDEs. We can also define strong solution, weak solution and mild solution.

Definition 16 (strong solution) *A predictable H -valued process $\{u(t) : t \in [0, T]\}$ is called a strong solution if*

$$u(t) = u_0 + \int_0^t [-Au(s) + f(u(s))]ds + \int_0^t G(u(s))dW(s), \quad \forall t \in [0, T] \tag{47}$$

Definition 17 (weak solution) *A predictable H -valued process $\{u(t) : t \in [0, T]\}$ is called a weak solution if*

$$\langle u(t), v\rangle = \langle u_0, v\rangle + \int_0^t [-\langle u(s), Av\rangle + \langle f(u(s)), v\rangle]ds + \int_0^t \langle G(u(s))dW(s), v\rangle, \quad \forall t \in [0, T], v \in \mathcal{D}(A) \tag{48}$$

where

$$\int_0^t \langle G(u(s))dW(s), v\rangle := \sum_{j=1}^{\infty} \int_0^t \left\langle G(u(s))\sqrt{q_j}\mathcal{X}_j, v\right\rangle d\beta_j(s).$$

Definition 18 (mild solution) *A predictable H -valued process $\{u(t) : t \in [0, T]\}$ is called a mild solution if for $t \in [0, T]$*

$$u(t) = \mathrm{e}^{-tA}u_0 + \int_0^t \mathrm{e}^{-(t-s)A} f(u(s))ds + \int_0^t \mathrm{e}^{-(t-s)A}G(u(s))dW(s),$$

where e^{-tA} is the semigroup generated by $-A$. The right hand side is also called stochastic convolution.

Example 7 (stochastic heat equation in one dimension) *Consider the weak solution of 1D heat SPDE with $D = (0, \pi)$, so that $-A$ has eigenfunctions $\phi_j(x) = \sqrt{2/\pi}\sin(jx)$ and eigenvalues $\lambda_j = j^2$ for $j \in \mathbb{N}$. Suppose that $W(t)$ is a Q -Wiener process and the eigenfunctions $\mathcal{X}_j$ of Q are the same as the eigenfunctions ϕ_j of A. A weak solution satisfies: $\forall v \in \mathcal{D}(A)$,*

$$\begin{aligned} \langle u(t), v\rangle_{L^2(0,\pi)} &= \langle u_0, v\rangle_{L^2(0,\pi)} + \int_0^t \langle -u(s), Av\rangle_{L^2(0,\pi)}ds \\ &+ \sum_{j=1}^{\infty} \int_0^t \sigma\sqrt{q_j} \left\langle \phi_j, v\right\rangle_{L^2(0,\pi)} d\beta_j(s) \end{aligned} \tag{49}$$

Assume $u(t) = \sum_{j=1}^{\infty} \hat{u}_j(t)\phi_j$ for $\hat{u}_j(t) := \langle u(t), \phi_j\rangle_{L^2(0,\pi)}$. Take $v = \phi_j$, we have

$$\hat{u}_j(t) = \hat{u}_j(0) + \int_0^t (-\lambda_j)\, \hat{u}_j(s)ds + \int_0^t \sigma\sqrt{q}_j d\beta_j(s). \tag{50}$$

Hence, $\hat{u}_j(t)$ satisfies the SODE

$$d\hat{u}_j = -\lambda_j\hat{u}_j dt + \sigma\sqrt{q_j}d\beta_j(t) \tag{51}$$

Therefore, each coefficient $\hat{u}_j(t)$ is an Ornstein-Uhlenbeck (OU) process (see Examples 8.1 and 8.21), which is a Gaussian process with variance

$$\operatorname{Var}(\hat{u}_j(t)) = \frac{\sigma^2 q_j}{2\lambda_j}\left(1 - \mathrm{e}^{-2\lambda_j t}\right) \tag{52}$$

For initial data $u_0 = 0$, we obtain, by the Parseval identity (1.43),

$$\|u(t)\|^2_{L^2(\Omega, L^2(0,\pi))} = \mathbb{E}\left[\sum_{j=1}^{\infty} |\hat{u}_j(t)|^2\right] = \sum_{j=1}^{\infty} \frac{\sigma^2 q_j}{2\lambda_j}\left(1 - \mathrm{e}^{-2\lambda_j t}\right). \tag{53}$$

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