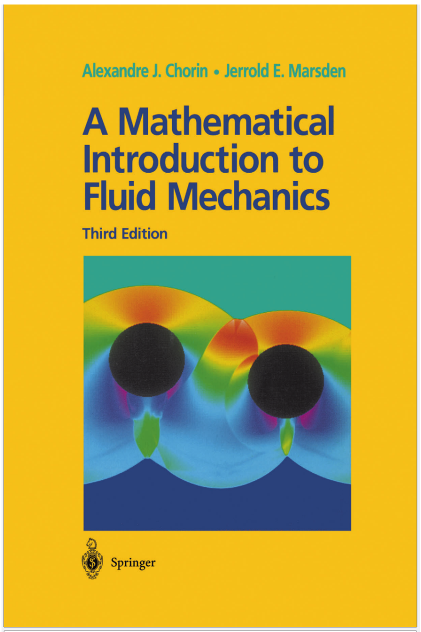
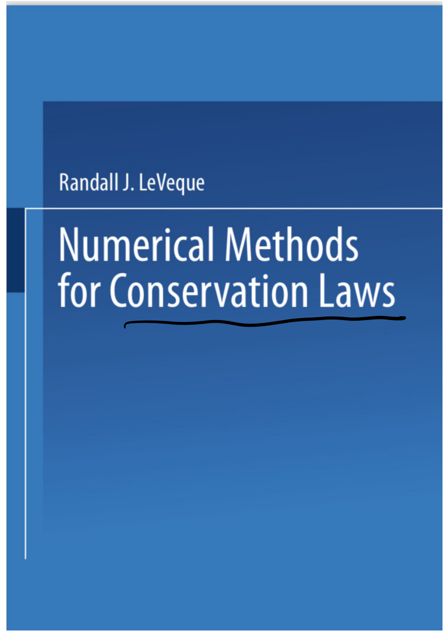
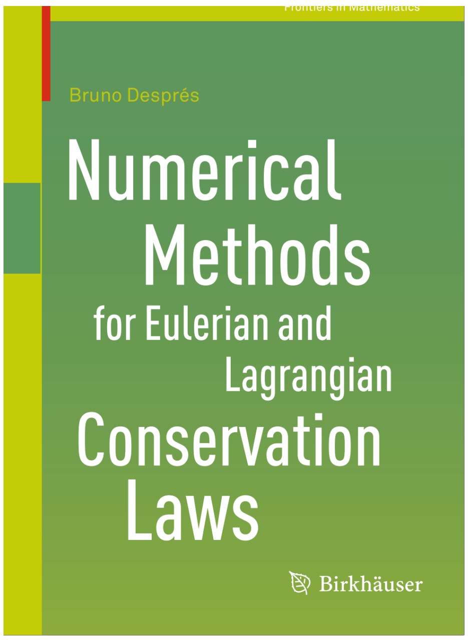
# Conservation Law: Theory and Numerical Method.

Reference:




Discontinuous Galerkin Methods: General Approach and Stability

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Part 2: Numerical

1. FD Scheme. { LTE, Stability, Lax Theorem, CFL Condition }
2. Discontinuous Solution (Linear Eq + Discontinuous Initial)
3. Conservation Form for nonlinear equation.
4. Numerical Flux.

(*) 5 Brief Introduction to DG.

1. Derive. (e.g. Euler's equation for Ideal fluid system)

Preliminaries:

① $\int_{\Omega} \nabla\cdot F \, dx = \int_{\partial\Omega} F\cdot n \, dS$ (Gauss-Green Formula).

② $\frac{d}{dt}\int_{\Omega(t)} f(x,t)\,dx = \int_{\Omega(t)} \frac{\partial f}{\partial t}\,dx + \int_{\partial\Omega(t)} f\,v\cdot n\,dS$ (Reynold's Transport Theorem)

$v$: 区域 $\Omega(t)$ 的速度.

1.1. Mass Conservation Law.


$\Omega(t) = \phi_0^t(\Omega_0)$

$$\left(\begin{aligned} \frac{d\phi_0^t(p)}{dt} &= u(\phi_0^t(p), t) \\ \phi_0^0(p) &= p \end{aligned}\right)$$

$m_t \equiv m_0$.

$$m(t) = \int_{\Omega(t)} \rho(x,t)\,dx \equiv C \;\Rightarrow\; \frac{d}{dt}\int_{\Omega(t)} \rho(x,t)\,dx = 0.$$

$$\overset{\text{Reynold.}}{\Longrightarrow} \int_{\Omega(t)} \frac{\partial \rho}{\partial t}\,dx + \int_{\partial\Omega(t)} \rho u\cdot n\,dS = 0.$$

$$\overset{\text{Div.}}{\Longrightarrow} \int_{\Omega(t)} \left[\frac{\partial \rho}{\partial t} + \nabla\cdot(u\rho)\right] dx = 0. \quad ①$$

$\forall\, \Omega_0, \forall t$. ① Satisfies.

$$\Rightarrow \frac{\partial \rho}{\partial t} + \nabla\cdot(u\rho) = 0.$$

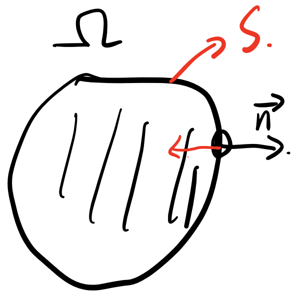
($\rho$: 密度 density. $\vec{u}$: 速度 velocity).

1.2 Momentum Conservation Law (动量守恒).

$$Ft = \Delta(mv) \Rightarrow F = \frac{d(\text{momentum})}{dt}\ ?$$

$I_?$

$mv$




$p\vec{n}$ (压强)

$-p\vec{n}$

$$F = -\int_{\partial\Omega(t)} p\vec{n}\, dS.$$

$$\frac{d}{dt}\int_{\Omega(t)} \rho v\, dx = -\int_{\partial\Omega(t)} p\vec{n}\, dS$$

Reynold $\Rightarrow$
$$\int_{\Omega(t)} \frac{\partial(\rho v)}{\partial t} dx + \int_{\partial\Omega(t)} (\rho \underset{\substack{\downarrow \\ \text{张量积}}}{v \otimes v}) \cdot n\, dS + \int_{\partial\Omega(t)} p n\, dS = 0.$$

Divergence $\Rightarrow$
$$\int_{\Omega(t)} \left[\frac{\partial(\rho v)}{\partial t} + \nabla\cdot(\rho v\otimes v + p)\right] dx = 0$$

$$\Rightarrow \frac{\partial(\rho v)}{\partial t} + \nabla\cdot(\rho v \otimes v + p) = 0.$$

Energy Conservation: $(E)_t + \nabla\cdot(v(E+p)) = 0.$

Rem: $E = E_{kinetic} + E_{internal}$.

Euler's equation:
$$\frac{\partial}{\partial t}\begin{bmatrix}\rho \\ \rho v \\ E\end{bmatrix} + \nabla_x \cdot \begin{bmatrix}\rho v \\ \rho v\otimes v + p \\ v(E+p)\end{bmatrix} = \vec{0}$$

$\Delta$. discontinuous Solution.

2. General Form

$\Omega \subseteq \mathbb{R}^m$ $\quad F: \mathbb{R}^d \to \mathbb{R}^d$. $(C^1)$-map.

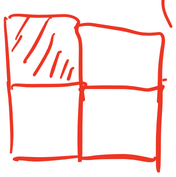
$u: \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^d$.

$$\frac{\partial u}{\partial t} + \nabla_x \cdot (F(u)) = 0.$$

(For Euler eq: $u = \begin{bmatrix} \rho \\ \rho v \\ E \end{bmatrix}$, $F(u) = \begin{bmatrix} \rho v \\ \rho v \otimes v + p \\ v(E+p) \end{bmatrix}$).

## 2-1 Integration Form

$$\Rightarrow \underbrace{\frac{d}{dt}\int_\Omega u(x,t)\,dx}_{\text{(积分平均)}} = -\int_\Omega \nabla_x \cdot (F(u))\,dx = \underbrace{-\int_{\partial\Omega} F(u)\cdot n\,dS}_{\text{(Flux)}}.$$



(Eulerian Scheme).

## 2.2. Characteristic Line.

$$u_t + a u_x = 0 \Rightarrow \frac{d\,u(x(t),t)}{dt} = 0.$$

2.2.1: (m = d = 1)

$$u_t + f'(u)\,u_x = 0$$

$\Delta$. $x'(t) = f'(u(x(t),t))$

$$\Rightarrow \frac{d\,u(x(t),t)}{dt} = u_t + u_x \cdot x'(t) = f'(u)u_x + u_t = 0.$$

$$\Rightarrow u(x(t),t) \equiv u(x(0),0)$$

2.2.2 m > 1, d = 1

$$u_t + \nabla\cdot(f(u)) = u_t + f'(u)\,\nabla\cdot u = 0.$$

$$x'(t) = f'(u(x(t),t)) \quad \begin{bmatrix} \vdots \\ \vdots \end{bmatrix}.$$

General Case: $(m>1, d>1)$

$$u_t + \nabla\cdot(F(u)) = u_t + J_F \nabla\cdot u = 0$$

$$x'(t) = \lambda_p(u(x(t),t)) \qquad \begin{bmatrix} \vdots \end{bmatrix}.$$

$\lambda_p$: p-th eigenvalue of $J_F$.

(Lagrangian Scheme).

# 3. One-Dimensional Conservation Laws $(m=d=1)$

## 3.1 $f(u) = au$, $a\in\mathbb{R}$ constant.

$$\Rightarrow \begin{cases} u_t + a u_x = 0. \\ u(x,0) = \varphi(x) \end{cases} \Rightarrow x(t;x_0) = x_0 + at.$$

$$\Rightarrow u(x,t) = u(x-at, 0) = \varphi(x-at).$$

$$(\varphi\in C^k(\mathbb{R}) \Rightarrow u\in C^k(\mathbb{R}^2))$$

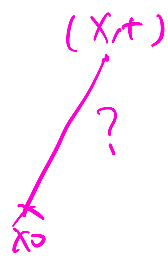
## 3.2. Burgers' equation.

$$f(u) = \frac{1}{2}u^2. \Rightarrow \text{Equation: } \begin{cases} u_t + u u_x = 0. \\ u(x,0) = \varphi(x) \end{cases}$$

$\phi(t;x_0)$: 初始点在$x_0$的特征线

$(\phi(0;x_0) = x_0)$.

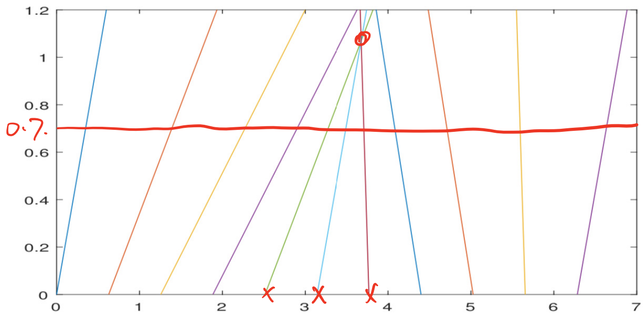
$$\begin{cases} \dfrac{d\phi}{dt} = u(\phi(t;x_0), t). \\ \phi(0;x_0) = x_0 \end{cases}$$




$$u(\phi(t;x_0),t) = u(\phi(0;x_0),0) = \varphi(x_0)$$

$$\Rightarrow \phi(t;x_0) = x_0 + \varphi(x_0)\cdot t.$$

$$\begin{cases} u(x,t) = u(x_0,0) = \varphi(x_0). \\ x = x_0 + \varphi(x_0)t = x_0 + ut. \end{cases} \Rightarrow u(x,t) = \varphi(x-ut). \ (\text{隐函数}).$$




不适定!

$T > T_b$. $\begin{cases} 1^\circ. \text{无古典解.} \\ 2^\circ. \text{弱解不唯一.} \end{cases}$

$g(x) = \frac{1}{2} + \sin x$.

$E(\varepsilon)$: $\begin{cases} u_t + u u_x + \varepsilon u_{xx} = 0 \\ u(x,0) = \varphi(x) \end{cases}$ ⇒ 解适定.

$u_\varepsilon \xrightarrow[\text{weak limit}]{\varepsilon \to 0} \tilde{u}$ → Entropy Solution (熵解)

# 4. Weak Form

## 4.1 Weak Solution

$$u_t + \nabla_x \cdot (f(u)) = 0$$

$$\phi \in C_c^\infty(\mathbb{R}^d \times (0, +\infty))$$

$$\int_0^\infty \int_{\mathbb{R}^d} [\phi u_t + \phi \nabla_x \cdot (f(u))] \, dx dt = 0$$

$$\Rightarrow \int_0^\infty \int_{\mathbb{R}^d} (f(u) \nabla \cdot \phi + u \phi_t) \, dx dt = -\int_{\mathbb{R}^d} \phi(x,0) u(x,0) \, dx$$

(weak Form).

## 4.2 Entropy Condition.

If the $C^1$ function $\psi, \eta$. satisfies:

$$\psi' = \eta' f'$$

Then $(\psi, \eta)$ is the entropy pair. $\eta$: entropy function.

$\psi$: entropy flux

Def (Entropy Condition).

If $\forall$ convex entropy functions $\eta$. Corresponding flux $\psi$.

$$(\eta(u))_t + \nabla\cdot(\psi(u)) \le 0.$$

is satisfied in the weak sense.

Then $u$ satisfies the Entropy Condition!

$$\left( \int_{x_1}^{x_2} \int_{t_1}^{t_2} \left( (\eta(u))_t + \nabla\cdot(\psi(u)) \right) dt\, dx \le 0 \right).$$

e.g.

$$\begin{cases} u_t + u u_x = 0. \\ u(x,0) = \begin{cases} u_L, & x \le 0 \\ u_R, & x > 0. \end{cases} \end{cases}$$

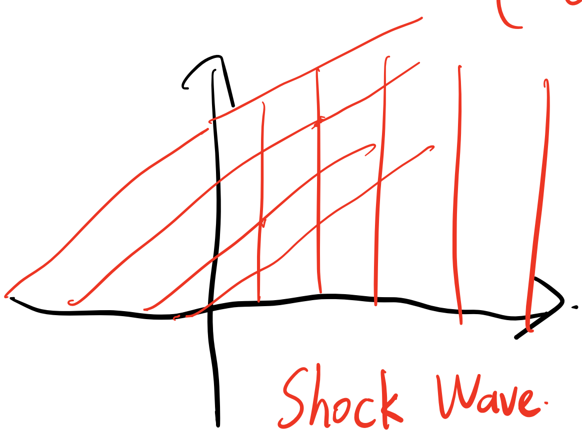
$$\eta(u) = u^2, \quad \psi(u) = \frac{2}{3}u^3.$$

Entropy Condition: $(x_2-x_1)(u_L^2-u_R^2)+\frac{2}{3}\Delta t(u_R^3-u_L^3)$

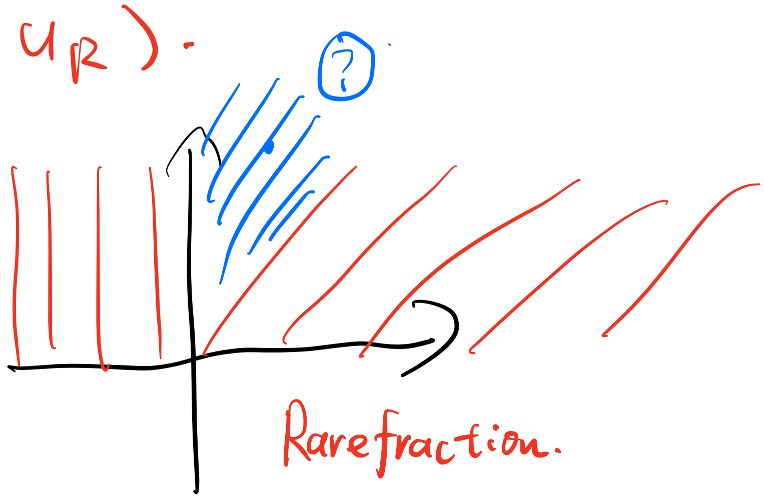
$+ O(\Delta t^2) \le 0.$

The entropy shock wave exists $\Leftrightarrow (u_L-u_R)^3 > 0.$

$(u_L > u_R).$



Shock Wave.



Rarefaction.

# 5. Shock wave & Rarefraction.

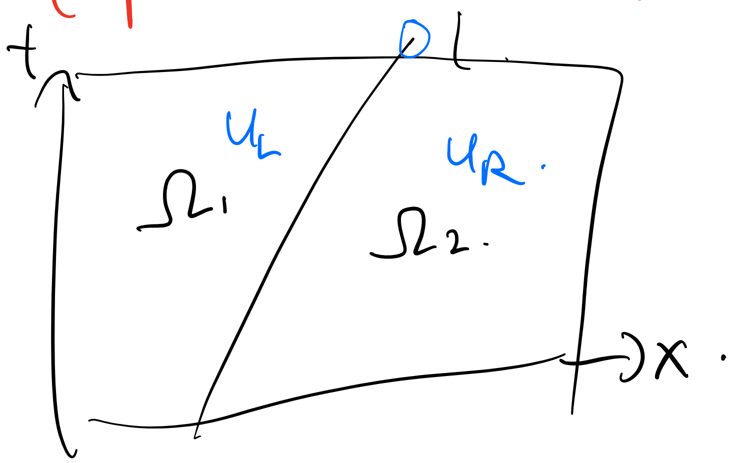
e.g. $\begin{cases} u_t + u u_x = 0. \\ u(x,0) = \begin{cases} u_L, & x \le 0 \\ u_R, & x > 0. \end{cases} \end{cases}$

## 5.1. $u_L > u_R$: Weak Solution.

$$u(x,t) = \begin{cases} u_L, & x < st. \\ u_R, & x > st. \end{cases}$$

$s = \frac{u_L + u_R}{2}$ is the shock speed.

(speed at which the discontinuity travels).



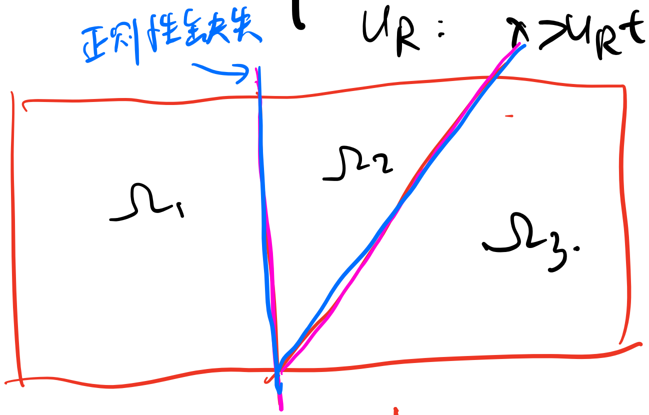

$u \in C(\Omega_1) \cap C(\Omega_2)$. $u$ is discontinuous on $l$.

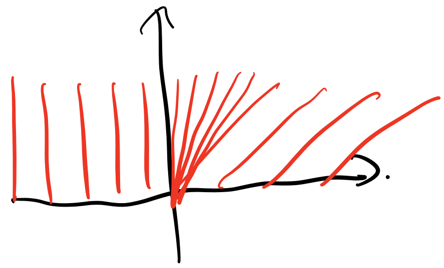
$u(x,t)$: Shock wave. $s = \frac{f(u_L) - f(u_R)}{u_L - u_R} = \frac{[f]}{[u]}$.

## 5.2. $u_L < u_R$: Infinitely Many weak solutions

rarefraction wave:

$$u(x,t)=\begin{cases} u_L: & x<u_Lt. \\ \frac{x}{t}: & u_Lt\le x\le u_Rt. \\ u_R: & x>u_Rt. \end{cases}$$



- Stable to pertubations.
- Entropy Solution.

# Part 2: Numerical.

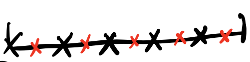
空间: FD (finite difference)

时间: FE (Forward Euler).

## 1. Linear Conservation Law + Smooth IV + Periodic BC

$$\begin{cases} u_t + Au_x = 0, & x\in[0,1]. \\ u(x,0)=u_0(x) \\ u(0,t)=u(1,t). \end{cases}$$

Step 1: Mesh.



$x_0=0$ $\qquad$ $1=x_N$.

$x_i = ih \ (h=1/N)$ $\qquad$ $x_{j+\frac{1}{2}} := (j+\frac{1}{2})h$ )

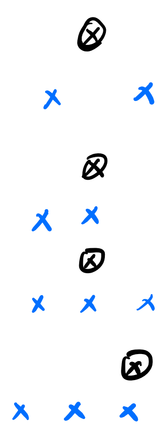
$k\in\mathbb{R}^+$, $t_n = nk$.

Step 2: FD Scheme. $U_j^n \approx u(x_j,t_n)$.

Periodic BC: $U_{i+N}^n = U_i^n$.

- Lax-Friedriches. $U_j^{n+1} = \frac{1}{2}(U_{j-1}^n + U_{j+1}^n) - \frac{k}{2h}A(U_{j+1}^n - U_{j-1}^n)$
- Upwind $(A>0)$: $U_j^{n+1} = U_j^n - \frac{kA}{h}(U_j^n - U_{j-1}^n)$
- Lax-Wendroff: (自行搜索).
- Beam-Warming:

$$U^{n+1} = H_k U^n.$$



$N\times N$ matrix.

step3: LTE (Local Truncation Error).

$$L_k(x,t) := \frac{1}{k}\Big[\underset{\substack{\uparrow\\ \text{exact!}}}{u(x,t+k)} - \underset{\substack{\uparrow\\ \text{numerical}}}{\underline{H_k(u(\cdot,t);x)}}\Big]$$

(u(x,t) exact!)

Def: The method is of order p. if $\forall$ init-data. compact support.

$\exists\ C_L$. s.t.

$$\|L_k(\cdot,t)\| \le C_L k^p.$$

Consistent: $\lim_{k\to 0} \|L_k(\cdot,t)\| = 0$.

e.g. (L-F).

$$L_k(x,t) = \frac{1}{k}\Big[u(x,t+k) - \frac{1}{2}\big(u(x-h,t)+u(x+h,t)\big)\Big] + \frac{1}{2h}A\big[u(x+h,t)-u(x-h,t)\big].$$

$$= u_t + A u_x + \frac{1}{2}\Big(k u_{tt} - \frac{h^2}{k}u_{xx}\Big) + O(h^2).$$

(前提: $\frac{h}{k} = O(1)$ ) $\Rightarrow$. $L_k = O(k)$

Step4: Stability.

LTE: $u(x,t+k) = H_k(u(\cdot,t);x) + k\,L_k(x,t)$

Global Error:

$$E_k(x_j,nk) = u(x_j,nk) - U_j^n.$$

Since $H_k$ is linear, we have:

$$E_k(\cdot, t+k) = H_k E_k(\cdot, t) - k L_k(\cdot, t).$$

$$\Rightarrow \quad E_k(\cdot, t_n) = H_k^n \underbrace{E_k(\cdot, 0)}_{\text{Initial.}} - k \sum_{i=1}^{n} H_k^{n-i} \underbrace{L_k(\cdot, t_{i-1})}_{\text{LTE.}}$$

$$\| H_k^n \| \le C_s. \quad \forall\, nk \le T,\ k < k_0.$$

△. (Lax Equivalence Theorem).

For a consistent linear method. stability $\Leftrightarrow$ convergence

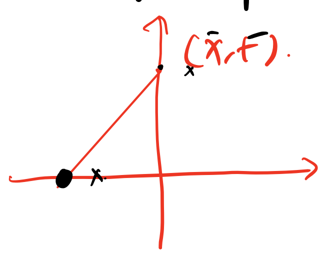
For L-F method. stable $\Leftrightarrow$. $\left| \frac{Ak}{h} \right| \le 1.$

$$Cr := \frac{Ak}{h} \ : \text{ Courant number}$$

△ CFL Condition (necessary condition).

$$\begin{cases} u_t + A u_x = 0. \ (A>0) \\ u(x,0) = u_0(x) \end{cases}$$

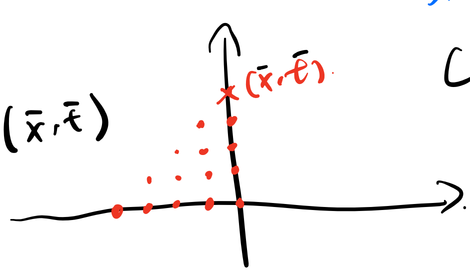
① Domain of dependence. (依赖区域)




$D(\bar{x}, \bar{t})$.

② Numerical D. O. D.

(Upwind Scheme).



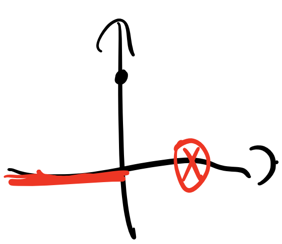

$D_h^k(\bar{x}, \bar{t})$

CFL Condition:

Numerical Stable

$$\Rightarrow D(\bar{x}, \bar{t}) \le \underline{D_0(\bar{x}, \bar{t})}.$$

$$\lim_{h \to 0} D_h^k(\bar{x}, \bar{t}) := D_0(\bar{x}, \bar{t})$$



$\Rightarrow$ Unstable!

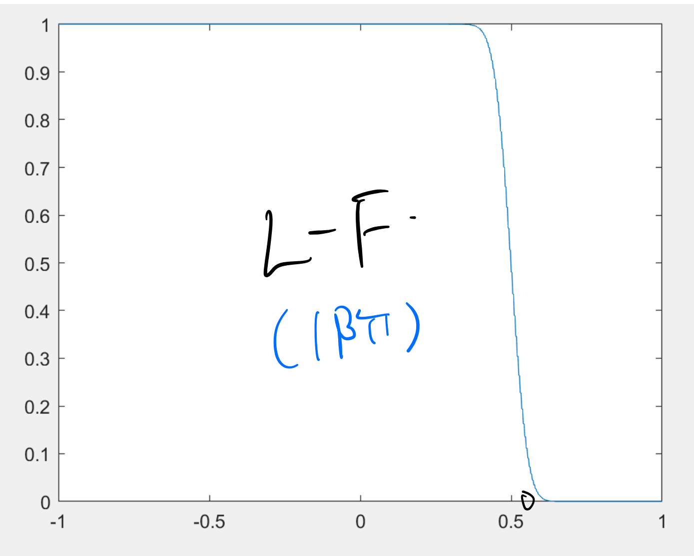
# 2. Discontinuous Solution.

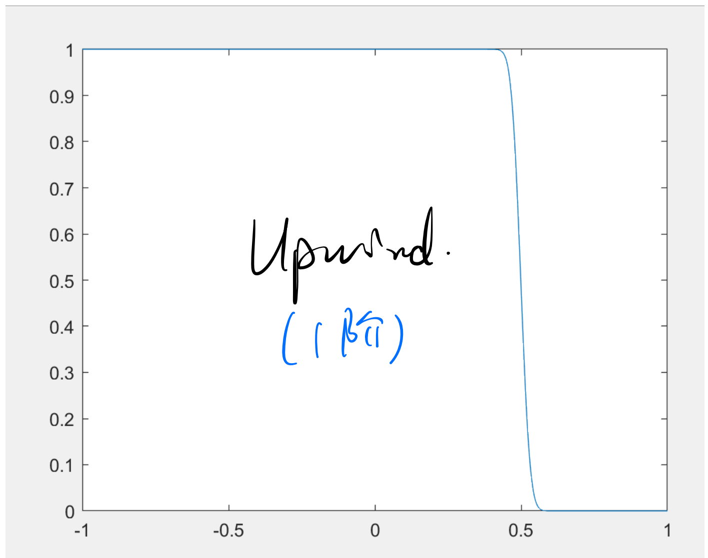
$$\begin{cases} u_t + a u_x = 0 \\ u_0(x) = \begin{cases} 1, & x<0 \\ 0, & x>0 \end{cases} \end{cases} \Rightarrow \text{Exact: } u_0(x,t).$$

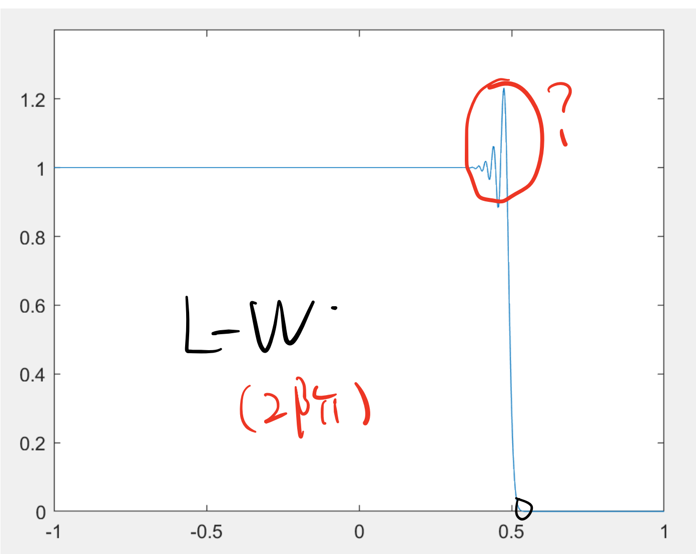
$(LTE(\sigma) \to \infty)$.

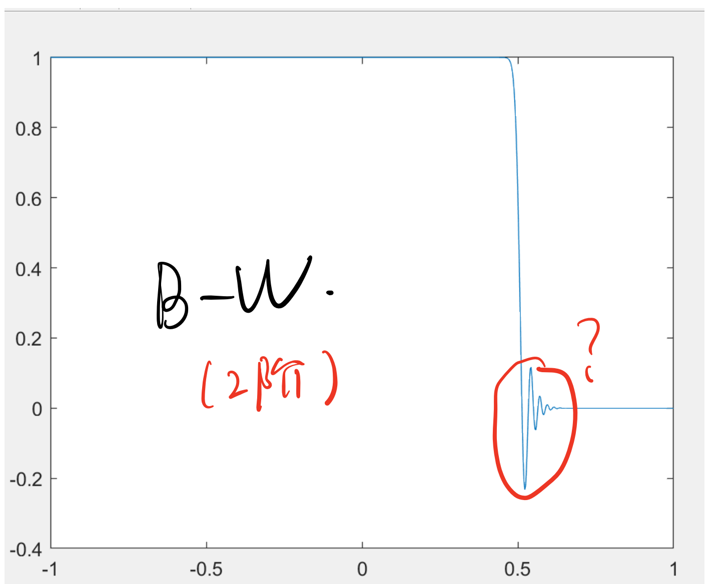
$\Omega = [-1,1]$. Dirichlet Boundary value.

$a=1$, $\frac{k}{h}=0.5$, $h=0.0025$.




## 2-1 Modified Equation.

PDE: $u_t + au_x = 0$ $\xrightarrow{FD}$ Numerical Scheme

?

(LF: $U_j^{n+1} = \frac{1}{2}(U_{j-1}^n + U_{j+1}^n) - \frac{k}{2h}A(U_{j+1}^n - U_{j-1}^n)$)

LTE: $L_k = O(k)$

PDE?

① LTE:

$$L_k(x,t) = u_t + Au_x + \frac{1}{2}\left(ku_{tt} - \frac{h^2}{k}u_{xx}\right) + O(h^2)$$

$\Rightarrow$ PDE: $u_t + Au_x + \frac{1}{2}\left(ku_{tt} - \frac{h^2}{k}u_{xx}\right) = 0$. = 阶逼近.

$u_t + Au_x = 0$.

$\Downarrow$ $u_{tt} = A^2 u_{xx} + O(k)$.

$$u_t + Au_x = \frac{h^2}{2k}\left(1 - \frac{k^2}{h^2}a^2\right)u_{xx}.$$

(Related to $h, k$).

(限定: $\frac{ak}{h} < 1$). $\Rightarrow u_t + Au_x = Du_{xx}$ $(D>0)$.

Lax-Wendroff: $u_t + Au_x = \frac{h^2}{6}a\left(\frac{a^2k^2}{h^2} - 1\right)u_{xxx}$.

$$D := \frac{ah^2}{6}\left(\frac{a^2k^2}{h^2} - 1\right) < 0.$$

$$u_0(x,0) = \begin{cases} 1, & x<0 \\ 0, & x>0. \end{cases}$$

$u_t + Au_x = Du_{xx}$.

2-2. Fourier Expansion & Wave Solution.

$u(x,t;\xi) = e^{i(\xi x - c(\xi)t)}$. $\xi$ 很大 $\Rightarrow$ high Oscillation.

$u(x,t) = \int_{\mathbb{R}} \hat{u}(\xi,0)\, u(x,t;\xi)\, d\xi$. (存在高频波!)

$u_0(x,0)=\varphi(x)\in C^{\infty}$. Everything is OK!

$$\hat{u}(\xi,0)\sim e^{-|\xi|}.$$

$$u_0(x,0) \Rightarrow \hat{u}(\xi,0) \xrightarrow{|\xi|\to\infty} O\left(\frac{1}{|\xi|}\right).$$

L-F: $u_t+Au_x=Du_{xx}$.

$$u(x,t)=e^{i(\xi x-c(\xi)t)}$$

$$\Rightarrow c(\xi)=A\xi-iD\xi^2.$$

$$\Rightarrow u(x,t)=e^{i\xi(x-At)-D\xi^2}.$$

$$|\xi|\to\infty \Rightarrow |u|\sim O(e^{-D|\xi|^2})$$

(不会出现明显高频波)

L-W: $u_t+Au_x+Du_{xxx}=0$ $(D>0)$

$$c(\xi)=A\xi-D\xi^3.$$

$$\Rightarrow u(x,t;\xi)=e^{i\xi(x-At-D\xi^2t)}.$$

$$|u|\equiv O(1) \to$$ 高频波不会消散.

① phase velocity: 波峰移速.

$$c_p(\xi):=\frac{c(\xi)}{\xi}=A-D\xi^2<A.$$

② group velocity: the velocity of the envelope of wave

$C_g(\xi):=c'(\xi)=A-3D\xi^2<A.$

高频波：移速慢于波速！$\longrightarrow$ 振荡部分出现在波前.