Ref: «Stochastic Differential Equations An Introduction with Application». by OKsendal, «Nonlinear Expectation, Nonlinear Evaluations and Risk Measures» by Shige Peng and «Backward Stochastic Differential Equation in Finance» by N.El Karoui, S.Peng, M.C. Quenez. «Backward Stochastic Differential Equations» by Jianfeng Zhang

1. Martingale representation thm.

$(\Omega, \mathcal{F}, \mathbb{P})$: a probability space. $(B_t)_{t\geq 0}$ a standard d-dim B.M.

$\mathcal{F}_t = \sigma\{B_s : 0 \leq s \leq t \ \vee N\}$ where N is the collection of $\mathbb{P}$-null sets in Ω.

$L^2(\Omega, \mathcal{F}_T, \mathbb{P}) := \{$ r.v. $X \subset \mathcal{F}_T$ and $\int_\Omega |X|^2 d\mathbb{P} < +\infty \}$

$\mathcal{L}^2(0,T; \Omega, \mathcal{F}_t, \mathbb{P}) := \{$ an $\mathcal{F}_t$-adapted process $\phi_t(\omega)$ s.t. $\mathbb{E}[\int_0^T |\phi_t|^2 dt] < +\infty \}$.

Recall that we have prove if $v \in \mathcal{L}^2(0,T)$, the Itô integral.

$$X_t = X_0 + \int_0^t v(s,\omega) dB_s \qquad t \geq 0$$

is always a martingale w.r.t. $\mathcal{F}_t$, now we'll give the converse result: Any $\mathcal{F}_t$-adapted martingale in $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ can be representation as an Itô integral.

Lemma 1. Fix $T > 0$, the set of r.v.s

$$\{\phi(B_{t_1}, \cdots B_{t_n}) : t_i \in [0,T], \phi \in C_0^\infty(\mathbb{R}^n)\}$$

(thus cylindrical function) is dense in $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$.

Lemma 2. The linear span of r.v.s of the type

$$\exp\{\int_0^T h(t) dB_t - \tfrac{1}{2}\int_0^T h^2(t) dt\}, \quad h \in L^2(0,T) \text{ (deterministic)}$$

simple function

is dense in $L^2(\Omega, \mathcal{F}_T; \mathbb{P})$.

Thm 3. Let $F \in L^2(\Omega, \mathcal{F}_T; \mathbb{P})$ Then there exists a unique stochastic process $f(t,\omega) \in \mathcal{L}^2(0,T)$ s.t.

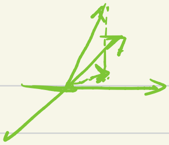
$$F(\omega) = \mathbb{E}[F] + \int_0^T f(t,\omega) dB_t.$$

Rmk: $f(t,\omega)$ can be express in terms of the Malliavin derivation of F.

Thm 4. Suppose M_t is an $\mathcal{F}_t$-adapted martingale and $M_t \in L^2(\Omega, \mathbb{P})$. $\forall t \geq 0$. there exist a unique stochastic process $g(s,\omega) \in \mathcal{L}^2(0,t)$ s.t.

$$M_t(\omega) = \mathbb{E}[M_0] + \int_0^t g(s,\omega) dB_s \quad \text{a.s.} \quad \forall t \geq 0$$

2. Existence, uniqueness of BSDEs



$$\begin{cases} dX_t = \mu(t,X_t)dt + \sigma(t,X_t)dW_t \\ X_0 = \xi_0 \end{cases}$$

$\Downarrow$?

$$\begin{cases} dX_t = \mu(t,X_t)dt + \sigma(t,X_t)dW_t \\ X_T = \xi_1 \end{cases} \qquad (*)$$

$\sigma(t,X_t) \to X_t$.

Consider the BSDE

$$\begin{cases} -dY_t = f(t,Y_t,Z_t)dt - Z_t^* dW_t \\ Y_T = \xi \in \mathcal{F}_T \end{cases}$$

$f \to g(t,Y_t,Z_t)$

or equivalently

$$Y_t = \xi + \int_t^T f(s,Y_s,Z_s)ds - \int_t^T Z_s^* dW_s$$

where: · W_t is an n-dim B.M.

· The terminal value is an $\mathcal{F}_T$-measurable r.v $\xi: \Omega \mapsto \mathbb{R}^d$

· The generator $f: \Omega \times \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^{n\times d} \mapsto \mathbb{R}^d$ and $\mathcal{P}\otimes\mathcal{B}^d\otimes\mathcal{B}^{n\times d}$-measurable.

We need to find a pair of $\mathcal{F}_t$-adapted process $(Y_t, Z_t) \in \mathcal{L}^2(0,T;\mathbb{R}^m\times\mathbb{R}^{m\times d})$ satisfying BSDE $(*)$

Qn: Why Z_t is needed as a part of solution?

If we consider $\begin{cases} dY_t = f'(t,Y_t)dt \\ Y_T = \xi \end{cases}$ where f' is a deterministic function

Y_0 is a map of Y_T which is $\mathcal{F}_T$-measurable not $\mathcal{F}_0$.

Hence we need Z_t as a part of solution to fit measurability.

Lemma 5. For a fixed $\xi \in L^2(\Omega, \mathcal{F}_T; \mathbb{R})$, and $g_0(\cdot)$ satisfying

$$\mathbb{E}\Big[\int_0^T g_0(t)dt\Big]^2 < \infty$$

there exist a unique pair of process $(y_t, z_t) \in \mathcal{L}^2(0,T;\mathbb{R}\times\mathbb{R}^n)$ s.t.

$$y_t = \xi + \int_t^T g_0(s)ds - \int_t^T z_s dB_s$$

In particular we have the following prior estimate.

$$|y_0|^2 + \mathbb{E}\Big[\int_0^T\big(\tfrac{\beta}{2}|y_s|^2 + |z_s|^2\big)e^{\beta s}ds\Big] \le \mathbb{E}|\xi|^2 e^{\beta T} + \frac{2}{\beta}\mathbb{E}\Big[\int_0^T g_0(s)^2 e^{\beta s}ds\Big], \ \forall \beta \in \mathbb{R}.$$

Proof: Uniqueness is a simple corollary of prior estimate.

Existence: define $M_t = \mathbb{E}\big[\xi + \int_0^T g_0(s)ds \,\big|\, \mathcal{F}_t\big]$, $(\mathbb{E}[M_t|\mathcal{F}_s] = M_s)$

M_t is a square integrable $\mathcal{F}_t$-martingale., by martingale representation thm.

$\exists!\ z_t \in \mathcal{L}^2(0,T)$ s.t. $\quad M_t = M_0 + \int_0^t z_s dB_s$

thus $\quad M_t = M_T - \int_t^T z_s dB_s = \xi + \int_0^T g_0(s)ds - \int_t^T z_s dB_s$

let $y_t := M_t - \int_0^t g_0(s)ds$, $\quad y_t = \xi + \int_t^T g_0(s)ds - \int_t^T z_s dB_s$.

With some prior estimates, we assume

① For each $(y,z) \in \mathbb{R}^m \times \mathbb{R}^{m\times d}$, $g(\cdot, y, z) \subseteq \mathcal{F}_t$ and

$$\int_0^T |g(\cdot, 0, 0)|\, ds \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$$

② For each $y, y' \in \mathbb{R}^m$ and $z, z' \in \mathbb{R}^{m\times d}$

$$|g(t,y,z) - g(t,y',z')| \le c\,(|y-y'| + |z-z'|)$$

Thm 6. For any terminal condition $\xi \in L^2(\mathcal{F}_T)$, BSDE has a unique solution $(Y, Z) \in \mathcal{L}^2(0,T; \mathbb{R}^m \times \mathbb{R}^{m\times d})$.

Proof: Prior estimate + fixed points thm or Picard iteration.

$$Y_t^{k+1} = \xi + \int_t^T f(s, Y_s^k, Z_s^k)\, ds - \int_t^T (Z_s^{k+1})^* dW_s$$

Rmk: Since Y_t is $\mathcal{F}_t$-adapted, and $\mathcal{F}_0$ is a trivial σ-field. Y_0 is a constant!

3. Formulation of linear BSDE

We next consider the linear BSDE with suitable condition, and provide a representation formula for its solution. ($d=m=1$ for simplity)

$$Y_t = \xi + \int_t^T [\alpha_s Y_s + Z_s \beta_s + f_s^\circ]\, ds - \int_t^T Z_s\, dB_s$$

we define $\Gamma_t := \exp\{\int_0^t \beta_s dB_s + \int_0^t [\alpha_s - \frac{1}{2}|\beta_s|^2]\, ds\}$

thus $d\Gamma_t = \beta_t \Gamma_t\, dB_t + \alpha_s \Gamma_t\, dt$. (adjoint process)

apply Itô formula to $\Gamma_t Y_t$. we have

$$d(\Gamma_t Y_t) = \Gamma_t\, dY_t + Y_t\, d\Gamma_t + d\langle Y_t, \Gamma_t\rangle$$

where $dY_t = -(\alpha_t Y_t + Z_t \beta_t + f_s^\circ)\, dt + Z_t\, dB_t$

$$d(\Gamma_t Y_t) = -\Gamma_t f_t^\circ\, dt + \Gamma_t [Y_t \beta_t^T + Z_t]\, dB_t$$

denote $\hat{Y}_t = \Gamma_t Y_t$; $\hat{Z}_t := \Gamma_t [Y_t \beta_t^T + Z_t]$; $\hat{\xi} := \Gamma_T \xi$; $\hat{f}_t^\circ = \Gamma_t f_t^\circ$

$$\hat{Y}_t = \hat{\xi} + \int_t^T \hat{f}_s^\circ\, ds - \int_t^T \hat{Z}_s\, dB_s$$

according to Lemma 5.

$$\hat{Y}_t = \mathbb{E}[\hat{\xi} + \int_t^T \hat{f}_s^\circ\, ds \,|\, \mathcal{F}_t]$$

$$Y_t = \Gamma_t^{-1} \mathbb{E}[\hat{\xi} + \int_t^T \hat{f}_s^\circ\, ds \,|\, \mathcal{F}_t].$$

$Z_t = ?$ $\longleftrightarrow$ Martingale re thm

4. Comparison Theorem

For $\quad Y_t^1 = \xi^1 + \int_t^T f^1(s, Y_s^1, Z_s^1)\, ds - \int_t^T (Z_s^1)^* dW_s$

and $\quad Y_t^2 = \xi^2 + \int_t^T f^2(s, Y_s^1, Z_s^1)\, ds - \int_t^T (Z_s^2)^* dW_s$

We suppose ① $\xi^1 \geq \xi^2$ $\mathbb{P}$ a.s.

② $f^1(t, y, z) \geq f^2(t, y, z)$ $d\mathbb{P} \otimes dt$ a.a.

Then we have a.e. $Y_t^1 \geq Y_t^2$ $(0 \leq t \leq T)$

Rmk: The comparison for SDE needs some more conditions on σ. (see e.g. GTM 113 Prop 2.18).

5. Application

We consider asset pricing:

riskless asset: $\quad dP_t^0 = P_t^0 r_t\, dt \qquad r_t$: short rate.

n risky securities: $\quad dP_t^i = P_t^i [\, b_t^i dt + \sum_{j=1}^n \sigma_t^{i,j} dW_t^j \,]$ $\quad b_t^i$: appreciation rate

Assume (The market is dynamically complete)

- r_t, b_t are predictable bounded and nonnegative.
- σ, σ^{-1} (volatility matrix) are predictable. bdd. $\quad \sigma = (\sigma^{i,j})$
- $\exists$ bdd θ (risk premium) s.t. $\quad b_t - r_t \mathbf{1} = \sigma_t \theta_t \quad d\mathbb{P} \otimes dt$ a.s

π_t^i: the wealth invested in the i-th stock.

$\pi^0 := V - \sum_{i=1}^n \pi_t^i \qquad V_t = V_0 + \int_0^t \sum_{i=0}^n \pi_t^i \frac{dP_t^i}{P_t^i} \qquad \pi^* = (\pi^1, \cdots \pi^n)$

thus $\qquad dV_t = dV_0 + \sum_{i=1}^n \pi_t^i \cdot \frac{dP_t^i}{P_t^i}$

$$dV_t = \sum_{i=1}^n \pi_t^i [\, b_t^i dt + \sum_{j=1}^n \sigma_t^{i,j} dW_t^j \,] + \pi_t^0 \cdot r_t\, dt$$
$$= r_t V_t\, dt + \pi_t^* (b_t - r_t \mathbf{1})\, dt + \pi_t^* \sigma_t\, dW_t$$
$$= r_t V_t\, dt + \pi_t^* \sigma_t [\, dW_t + \theta_t dt \,]$$

(V, π) is a trading strategy.

A fair price of positive contingent claim is to find (X, π) s.t.

$$\begin{cases} dX_t = r_t X_t\, dt + \pi_t^* \sigma_t \theta_t\, dt + \pi_t^* \sigma_t\, dW_t \\ X_T = \xi \in \mathcal{F}_T \text{ and } \xi \geq 0 \end{cases}$$

where ξ is the future price, the X_0 is a fair price. at $t = 0$