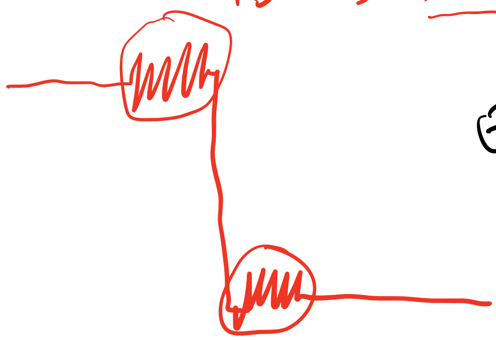
Discontinuous Galerkin (DG) for Hyperbolic / Conservation / (A-D)

1D → <u>Analysis</u> (L^2-stable., TVD → deliminate)

total variation

error estimate $\|u_h - u\|_{L^2} \sim O(h^k)$



⊕ limiter (限制器).

2D → Algorithm.

一. Introduction.

DG → FEM

$Lu = f \xrightarrow{\text{convert}}$ weak form.

$$\begin{cases} -\Delta u = f & \text{in } \Omega. \\ u|_{\partial\Omega} = u_0 & \text{on } \partial\Omega \end{cases}$$

Set $\varphi \in C_c^\infty(\Omega)$.

$$\langle -\Delta u, \varphi \rangle = \langle f, \varphi \rangle. \xrightarrow{D.T.} \langle \nabla u, \nabla\varphi \rangle = \langle f, \varphi \rangle \quad (*)$$

$$\forall \varphi \in C_c^\infty(\Omega)$$

$u \in H^2(\Omega)$. $u \in V_h^k(\Omega)$.

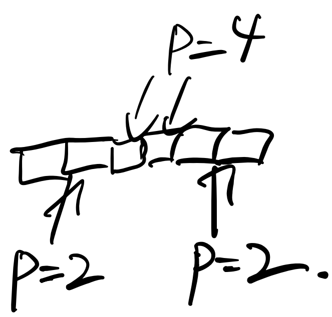
分片 k次多项式. $u \in C^{k-1}(\Omega)$.

$\forall \varphi \in V_h^k(\Omega)$.

u: Distontinuous. 分片k次多项式. 无任何连续性条件!

advantage:

- High parallel efficiency.
- High flexibility.
- p-adaptivity. (自适应).



- Local data structure.

Disadvantage:

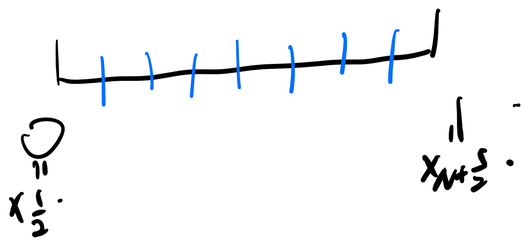
- DoF.
- Stiffness matrix: condition number.
- Complex error analysis.

二. DG Method for Conservation Law.

2.1: Scheme

- 1D. time-dependent conservation.

$$\begin{cases} u_t + (f(u))_x = 0, & x \in \Omega := [0,1]. \\ u(x,0) = u_0(x) & u: \text{periodic BC}. \end{cases}$$



$h = 1/N$. $\quad x_{j+\frac{1}{2}} := jh$. $\quad I_j := [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$.

$$V_h^k := \{ u \in L^2(\Omega) : u \in P^k(I_j),\ 1 \le j \le N \}.$$

for $v \in V_h^k$.

$$\int_{I_j} u_t v\,dx + \int_{I_j} (f(u))_x v\,dx = 0.$$

分部. $\Longrightarrow$

$$\int_{I_j} u_t v\,dx - \int_{I_j} f(u) v_x\,dx + (f(u)v)\Big|_{x^-_{j+\frac{1}{2}}} - (f(u)v)\Big|_{x^+_{j-\frac{1}{2}}} = 0$$

Seek $u \in L^2(V_h^k; [0,T])$ satisfies the weak form.

$v\big|_{x^-_{j+\frac{1}{2}}} := v(x^-_{j+\frac{1}{2}})$. $\quad v\big|_{x^+_{j-\frac{1}{2}}} := v(x^+_{j-\frac{1}{2}})$.

Δ $\underbrace{\left(f(u)\right)}\big|_{x^-_{j+\frac{1}{2}}}$, $\underbrace{\left(f(u)\right)}\big|_{x^+_{j-\frac{1}{2}}}$? $\longrightarrow$ approximation

$\downarrow \qquad\qquad \downarrow$

$f(u(x^-_{j+\frac{1}{2}}))$ $\quad f(u(x^+_{j-\frac{1}{2}}))$ X

Discontinuous.

Δ Numerical Flux

$\hat{f}(u^-, u^+)\big|_{x_{j+\frac{1}{2}}}$.

① Consistency (一致性): $\hat{f}(u,u) = f(u)$.

② Lipschitz Continuous: $\hat{f}(u^-, u^+)$ at least Lip-cont. rel. to. u^-, u^+.

③ Monotonicity: $\hat{f}(u^-, u^+)$. 关于 $u^- \nearrow$. $u^+ \searrow$.

- L-F flux.

$$\hat{f}^{LF}(u^-, u^+) = \frac{1}{2}\left(f(u^-) + f(u^+) - \alpha(u^+ - u^-)\right),$$

$$\alpha = \max_u |f'(u)|.$$

Numerical Scheme. (Spatial Discretization).

$$\int_{I_j} u_t v \, dx - \int_{I_j} f(u) v_x \, dx + \underbrace{\hat{f}_{j+\frac{1}{2}}} v\big|_{x^-_{j+\frac{1}{2}}} - \hat{f}_{j-\frac{1}{2}} v\big|_{x^+_{j-\frac{1}{2}}} = 0.$$

Numerical Flux.

Time Discretization. 1阶: FE.
3 order RK.

$$u_t = \mathcal{L}(u,t):$$

$$\begin{cases} u^{(1)} = u^n + \Delta t \mathcal{L}(u^n, t^n) \\ u^{(2)} = \frac{3}{4}u^n + \frac{1}{4}u^{(1)} + \frac{1}{4}\Delta t \mathcal{L}(u^{(1)}, t^n + \Delta t) \\ u^{n+1} = \frac{1}{3}u^n + \frac{2}{3}u^{(2)} + \frac{2}{3}\Delta t \mathcal{L}(u^{(2)}, t^n + \frac{1}{2}\Delta t) \end{cases}$$

2.2 L^2-Stability.

- Entropy weak solution.

Entropy inequality: $(U(u))_t + (F(u))_x \leq 0$.

$\forall$ Convex function $U(u)$

and Entropy Flux: $F(u) = \int U'(u) f'(u) du$.

$(\underline{U, F})$: Entropy-pair.

- cell entropy inequality.

$$\frac{d}{dt}\int_{I_j} U(u_h)dx + \hat{F}_{j+\frac{1}{2}} - \hat{F}_{j-\frac{1}{2}} \leq 0.$$

$U(u) := \frac{1}{2}u^2$. $\hat{F}$: 待定.

Proof: $\int_{I_j} u_t v dx - \int_{I_j} f(u) v_x dx + \hat{f}_{j+\frac{1}{2}} v(x^-_{j+\frac{1}{2}}) - \hat{f}_{j-\frac{1}{2}} v(x^+_{j-\frac{1}{2}}) = 0$

$$B_j(u_h; v_h) := \int_{I_j} u_t v dx - \int_{I_j} f(u) v_x dx + \hat{f}_{j+\frac{1}{2}} v(x^-_{j+\frac{1}{2}}) - \hat{f}_{j-\frac{1}{2}} v(x^+_{j-\frac{1}{2}}) = 0.$$

$V_h := u_h \Rightarrow$

$$\frac{d}{dt}\int_{I_j} U(u_h)\,dx - \int_{I_j} f(u_h)\,du_h + \hat{f}_{j+\frac{1}{2}}\, u_h(x^-_{j+\frac{1}{2}}) - \hat{f}_{j-\frac{1}{2}}\, u_h(x^+_{j-\frac{1}{2}}) = 0$$

如何定义 $\hat{F}$？

$$\tilde{F} := \int f(u)\,du \Rightarrow \int_{I_j} f(u_h)\,du_h = \tilde{F}(u_h(x^-_{j+\frac{1}{2}})) - \tilde{F}(u_h(x^+_{j-\frac{1}{2}}))$$

$$\hat{F}_{j+\frac{1}{2}} := -\tilde{F}(u_h(x^-_{j+\frac{1}{2}})) + \hat{f}_{j+\frac{1}{2}}\, u_h(x^-_{j+\frac{1}{2}})$$

$$\hat{F}_{j-\frac{1}{2}} = -\tilde{F}(u_h(x^-_{j-\frac{1}{2}})) + \hat{f}_{j-\frac{1}{2}}\, u_h(x^-_{j-\frac{1}{2}})$$

$$\Rightarrow \frac{d}{dt}\int_{I_j} U(u_h)\,dx + \hat{F}_{j+\frac{1}{2}} - \hat{F}_{j-\frac{1}{2}} + \underbrace{\Theta_{j-\frac{1}{2}}}_{\text{余项}} = 0$$

$$\Theta_{j-\frac{1}{2}} = -\tilde{F}(u_h(x^-_{j-\frac{1}{2}})) + \hat{f}_{j-\frac{1}{2}}\, u_h(x^-_{j-\frac{1}{2}}) + \tilde{F}(u_h(x^+_{j-\frac{1}{2}})) - \hat{f}_{j-\frac{1}{2}}\, u_h(x^+_{j-\frac{1}{2}})$$

$$= \hat{f}_{j-\frac{1}{2}}(u_h^- - u_h^+) + \underbrace{f(\xi)}_{\text{MVT}}(u_h^+ - u_h^-)$$

$$= \left(\hat{f}(\xi,\xi) - \hat{f}(u^-,u^+)\right)(u^+ - u^-) \geq 0$$

WLOG: $u^+ \geq u^- \Rightarrow u^+ \geq \xi \geq u^-$

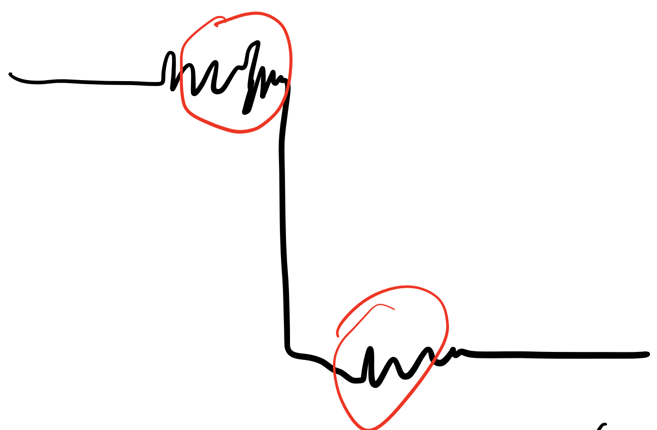
$\hat{f}(\xi,\xi) \geq \hat{f}(u^-,\xi) \geq \hat{f}(u^-,u^+)$.

$\Rightarrow$ Q.E.D.

Cor: $\sum_j \left(\frac{d}{dt} \int_{I_j} U(u_h)dx + \hat{F}_{j+\frac{1}{2}} - \hat{F}_{j-\frac{1}{2}} \right) \leq 0$.

$\Rightarrow \frac{d}{dt} \int_0^1 U(u_h)dx \leq 0$.

$\Rightarrow \frac{d}{dt} \|u_h\|_{L^2} \leq 0 \Rightarrow L^2$-Stable!



2-3. Limiter / total variation.

FE + DG.

$\Rightarrow \int_{I_j} \frac{u_h^{n+1} - u_h^n}{\Delta t} v_h dx - \int_{I_j} f(u_h^n)(v_h)_x dx + \hat{f}^n_{j+\frac{1}{2}} v_h^- - \hat{f}^n_{j-\frac{1}{2}} v_h^+$.

→ u_h^{n+1}: post_process (后处理)

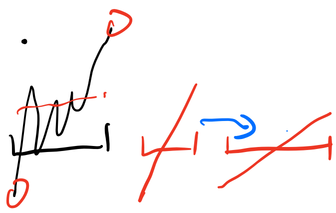
- ① shouldn't change cell average.

$$\bar{u}_h^{n+1,pre} = \bar{u}_h^{n+1}$$

- ② ✗ affect accuracy in $\underbrace{\text{smooth regions}}$.

↓ $u_h^n = u_h^{n,pre}$.

$$\bar{u}_i := \frac{1}{\Delta x_i}\int_{I_i} u_h\,dx.$$



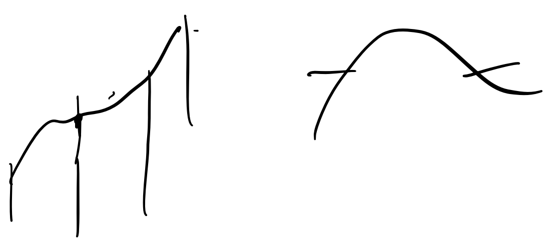
I_i.

$$\left.\begin{aligned}\tilde{u}_i &:= u_h(x^-_{i+\frac{1}{2}}) - \bar{u}_i\\ \tilde{\tilde{u}}_i &:= \bar{u}_i - u_h(x^+_{i-\frac{1}{2}})\end{aligned}\right\} \to \text{不能太大} \to \text{limiter (斜率限制)}.$$

$$\tilde{u}_i^{(mod)} := m(\tilde{u}_i, \underbrace{\Delta_+\bar{u}_i}_{\text{右}}, \underbrace{\Delta_-\bar{u}_i}_{\text{左}}). \qquad \tilde{\tilde{u}}_i^{(mod)} := m(\tilde{\tilde{u}}_i, \Delta_+\bar{u}_i, \Delta_-\bar{u}_i).$$

$$\Delta_+\bar{u}_i = \bar{u}_{i+1} - \bar{u}_i, \quad \Delta_-\bar{u}_i := \bar{u}_i - \bar{u}_{i-1}$$

$$m(a_1,\dots,a_l) := \begin{cases} s\min\{|a_1|,\dots,|a_l|\}: & s = \operatorname{sign}(a_1) = \cdots = \operatorname{sign}(a_l).\\ 0; & \text{otherwise.}\end{cases}$$



$$u_h^{(mod)}(x^-_{i+\frac{1}{2}}) := \bar{u}_i + \tilde{u}_i^{(mod)}$$

$$u_h^{(mod)}(x^+_{i-\frac{1}{2}}) = \bar{u}_i - \tilde{\tilde{u}}_i^{(mod)}$$

△ TVM. (total variation in means). diminishing.

$$\mathrm{TVM}(u_h) := \sum_i |\Delta_+\bar{u}_i|.$$

TVDM: $\quad TVM(u_h^{n+1}) \le TVM(u_h^n)$.

Lem (Harten).

$$u_i^{n+1} = u_i^n + C_{i+\frac{1}{2}} \Delta_+ u_i^n - D_{i-\frac{1}{2}} \Delta_- u_i^n \qquad \text{Periodic BC.}$$

$$C_{i+\frac{1}{2}} \ge 0; \quad D_{i+\frac{1}{2}} \ge 0, \quad C_{i+\frac{1}{2}} + D_{i+\frac{1}{2}} \le 1.$$

$$\Rightarrow TV(u^{n+1}) \le TV(u^n).$$

$$TV(u) := \sum_i |\Delta_+ u_i|.$$

pf:
$$\begin{cases} u_i^{n+1} = u_i^n + C_{i+\frac{1}{2}} \Delta_+ u_i^n - D_{i-\frac{1}{2}} \underbrace{\Delta_- u_i^n}_{\Delta_+ u_{i-1}^n} \\ u_{i+1}^{n+1} = u_{i+1}^n + C_{i+\frac{3}{2}} \Delta_+ u_{i+1}^n - D_{i+\frac{1}{2}} \underbrace{\Delta_- u_{i+1}^n}_{\Delta_+ u_i^n} \end{cases}$$

$$\Rightarrow \Delta_+ u_i^{n+1} = (1 - C_{i+\frac{1}{2}} - D_{i+\frac{1}{2}}) \Delta_+ u_i^n + C_{i+\frac{3}{2}} \Delta_+ u_{i+1}^n + D_{i-\frac{1}{2}} \Delta_+ u_{i-1}^n.$$

$$\Rightarrow \sum_i |\Delta_+ u_i^{n+1}| \le \sum_i (1 - C_{i+\frac{1}{2}} - D_{i+\frac{1}{2}}) |\Delta_+ u_i^n| + \sum_i C_{i+\frac{1}{2}} |\Delta_+ u_i^n| + \sum_i D_{i+\frac{1}{2}} |\Delta_+ u_i^n|$$

$$\le \sum_i |\Delta_+ u_i^n| \Rightarrow \text{TVDM}$$

TVDM:

proof: $V_n = 1$.

$$\Rightarrow \int_{I_i} \frac{u_h^{n+1,pre} - u_h^n}{\Delta t} dx + \hat{f}^n_{i+\frac{1}{2}} - \hat{f}^n_{i-\frac{1}{2}} = 0.$$

$$\lambda_i := \frac{\Delta t}{\Delta x_i}. \qquad \hat{f}^n_{i+\frac{1}{2}} := \hat{f}(\underline{u}^-, \underline{u}^+)$$

$$\Rightarrow \bar{u}_i^{n+1,pre} = \bar{u}_i - \lambda_i \Big(\hat{f}(\bar{u}_i + \tilde{u}_i, \bar{u}_{i+1} - \tilde{\tilde{u}}_{i+1}) - \hat{f}(\bar{u}_{i-1} + \tilde{u}_{i-1}, \bar{u}_i - \tilde{\tilde{u}}_i) \Big)$$

$$u_i^{n+1} = u_i^n + C_{i+\frac{1}{2}} \Delta_+ u_i^n - D_{i-\frac{1}{2}} \Delta_- u_i^n$$

$$C_{i+\frac{1}{2}} = -\lambda_i \frac{\hat{f}(\bar{u}_i + \tilde{u}_i, \bar{u}_{i+1} - \tilde{\tilde{u}}_{i+1}) - \hat{f}(\bar{u}_i + \tilde{u}_i, \bar{u}_i - \tilde{\tilde{u}}_i)}{\Delta_+ u_i^n}.$$

$$D_{i-\frac{1}{2}} = -\lambda_i \frac{\hat{f}(\bar{u}_{i-1} + \tilde{u}_{i-1}, \bar{u}_i - \tilde{\tilde{u}}_i) - \hat{f}(\bar{u}_i, \tilde{u}_i, \bar{u}_i - \tilde{\tilde{u}}_i)}{\Delta_- \bar{u}_i}$$

$$C_{i+\frac{1}{2}} := -\lambda_i \frac{\hat{f}(\bar{u}_i + \tilde{u}_i, \bar{u}_{i+1} - \tilde{\tilde{u}}_{i+1}) - \hat{f}(\bar{u}_i + \tilde{u}_i, \bar{u}_i - \tilde{\tilde{u}}_i)}{\bar{u}_{i+1} - \tilde{\tilde{u}}_{i+1} - \bar{u}_i + \tilde{\tilde{u}}_i} \cdot \frac{\Delta_+ \bar{u}_i + \tilde{\tilde{u}}_i - \tilde{\tilde{u}}_{i+1}}{\Delta_+ \bar{u}_i}$$

$\because = -\lambda_i A \cdot B$.

$\Rightarrow -L_2 \le A \le 0$.

关于第2参数的Lip.

由于Limiter的表达式:

$$\begin{cases} |\tilde{u}_i| \le |\Delta_+ u_i| \\ |\tilde{\tilde{u}}_{i+1}| \le |\Delta_- u_{i+1}| = |\Delta_+ u_i|. \\ \mathrm{Sign}(\tilde{u}_i) = \mathrm{Sign}(\tilde{\tilde{u}}_{i+1}) = \mathrm{Sign}(\Delta_+ u_i). \end{cases}$$

$\Rightarrow 0 \le B \le 2$

$\therefore \begin{cases} 0 \le C_{i+\frac{1}{2}} \le 2L_2\lambda_i. \\ 0 \le D_{i-\frac{1}{2}} \le 2L_1\lambda_i. \end{cases}$ $\Rightarrow$ choose Δt s.t. $\lambda_i \le \dfrac{1}{2(L_1+L_2)}$.

$\Rightarrow$ TVDM #

Δ doesn't affect accuracy in smooth.

$$\begin{cases} \tilde{u}_i = \frac{1}{2} u_x(x_i)\Delta x_i + O(h^2) \\ \tilde{\tilde{u}}_i = \frac{1}{2} u_x(x_i)\Delta x_i + O(h^2) \end{cases}$$

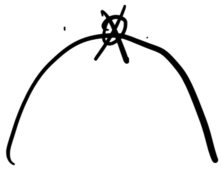
$$\begin{cases} \Delta_+ \bar{u}_i = \frac{1}{2} u_x(x_i)(\Delta x_i + \Delta x_{i+1}) + O(h^2) \\ \Delta_- \bar{u}_i = \frac{1}{2} u_x(x_i)(\Delta x_i + \Delta x_{i-1}) + O(h^2) \end{cases}$$

① smooth + monotone.

$\Rightarrow \tilde{u}_i, \tilde{\tilde{u}}_i \le \Delta_+, \Delta_-$

$\Rightarrow$ unmodified.

② TVB corrected limiter.



$$\tilde{m}(a_1,\dots,a_l) = \begin{cases} a_1: & |a_1| \le \underline{Mh^2}, \\ m(a_1,\dots,a_l) & \text{otherwise.} \end{cases}$$

2-4 Error Estimates.

If: Exact Solution: smooth. $\longrightarrow$ L^2 error estimates.

$$\begin{cases} u_t+u_x=0. \\ \hat{f}(u^-,u^+) := u^- \quad \text{(up wind)} \end{cases}$$

Thm: $\|u-u_h\|_{L^2} \le \underset{\substack{\downarrow \\ C(u).}}{C} h^{k+1}$

Pf: DG Scheme: $B_i(u_h; v_h)=0$. $\forall v_h \in V_h$, $\forall i$.

Exact Solution: $B_i(u; v_h)=0$. $\forall i$.

$B_i \rightarrow$ bilinear.

$\Rightarrow$ $\underline{B_i(u-u_h; v_h)=0}$ (Projection)

Def: $P: C^\infty(\Omega) \to V_h$.

$$\begin{cases} \int_{I_i} (Pw-w)\, v_h \, dx = 0. \quad \forall v_h \in P^{k-1}(I_i). \\ Pw(x^-_{i+\frac{1}{2}}) := w(x_{i+\frac{1}{2}}). \end{cases}$$

$$\Rightarrow \|Pw - w\|_{L^2} \leq C(w) h^{k+1}.$$

Target: $u - u_h$.

$e_h := Pu - u_h$, $\quad \varepsilon_h := \underline{u - Pu}$.

$$\|\varepsilon_h\| \leq C(w) h^{k+1}$$

$$B_i(u - u_h; v_h) = 0 \Rightarrow B_i(e_h; v_h) = -B_i(\varepsilon_h; e_h)$$

$$v_h := e_h. \Rightarrow B_i(e_h; e_h) = -B_i(\varepsilon_h; e_h).$$

① $B_i(e_h; e_h) = \frac{1}{2} \frac{d}{dt} \int_{I_i} e_h^2 dx + \hat{F}_{i+\frac{1}{2}} - \hat{F}_{i-\frac{1}{2}} + \Theta_{i-\frac{1}{2}}$.

$\Theta_{i-\frac{1}{2}} \geq 0$.

② $B_i(\varepsilon_h; e_h) = \int_{I_i} (\varepsilon_h)_t e_h dx - \int_{I_i} \varepsilon_h (e_h)_x dx$

$\quad + \varepsilon_h(x_{i+\frac{1}{2}}^-) e_h(x_{i+\frac{1}{2}}^-) - \varepsilon_h(x_{i-\frac{1}{2}}^-) e_h(x_{i-\frac{1}{2}}^+)$.

$\varepsilon_h(x_{i+\frac{1}{2}}^-) = 0$.

$$\Rightarrow -B_i(\varepsilon_h; e_h) = -\int_{I_i} (\varepsilon_h)_t e_h dx$$

$$\leq \frac{1}{2} \left(\int_{I_i} ((\varepsilon_h)_t)^2 dx + \int_{I_i} e_h^2 dx \right)$$

$$\Rightarrow \frac{d}{dt}\int_0^1 e_h^2\,dx \leq \int_0^1 e_h^2\,dx + Ch^{2k+2}.$$

Gronwall

$$\Rightarrow \quad \|e_h\| \leq Ch^{k+1}.$$

$$\Rightarrow \|u-u_h\| \leq Ch^{k+1} \qquad \#.$$