

Modeling Neural Random Field through Operator-based Diffusion

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Abstract

1 Introduction

Some references: [8, 2, 3, 6, 1, 5]

Definition 1. Random field $\mathcal{M}(x, \omega)$, where $x \in D$ and $\omega \in \Omega$, is defined as:

$$\begin{aligned} \mathcal{M}(x, \cdot) & \text{ is a random variable defined on the probability space } (\Omega, \mathcal{F}, P), \\ \mathcal{M}(\cdot, \omega) & \text{ is a deterministic function of } x. \end{aligned} \quad (1)$$

Classical methods to simulate random field are based on polynomial chaos expansion B and Karhunen-Loeve expansion C, See [4]. Random field can be regarded as a Hilbert space $(L^2(\Omega, H))$ -valued random variable.

Definition 2 ($L^p(\Omega, H)$ space). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and H is a Hilbert space with norm $\|\cdot\|$. Then $\mathcal{L}^p(\Omega, H)$ with $1 \leq p < \infty$ is the space of H -valued \mathcal{F} -measurable random variables $X : \Omega \rightarrow H$ with $\mathbf{E}[\|X\|^p] < \infty$ and a Banach space with norm:

$$\|X\|_{\mathcal{L}^p(\Omega, H)} := \left(\int_{\Omega} \|X(\omega)\|^p dP(\omega) \right)^{\frac{1}{p}} = \mathbf{E}[\|X\|^p]^{\frac{1}{p}} \quad (2)$$

Then we can define the inner product:

$$\langle X, Y \rangle_{\mathcal{L}^2(\Omega, H)} := \int_{\Omega} \langle X(\omega), Y(\omega) \rangle_H dP(\omega) \quad (3)$$

Definition 3 (uncorrelated, covariance operator). Let H be a Hilbert space. A linear operator $\mathcal{C} : H \rightarrow H$ is the covariance of H -valued random variables X and Y if

$$\langle \mathcal{C}_{XY} \phi, \psi \rangle_H = \text{Cov}(\langle X, \phi \rangle_H, \langle Y, \psi \rangle_H), \forall \phi, \psi \in H \quad (4)$$

Example 1 ($H = \mathbb{R}^d$). In finite dimensional case $H = \mathbb{R}^d$, the covariance matrix coincides with the covariance operator.

$$\begin{aligned} \text{Cov}(\langle X, \phi \rangle, \langle Y, \psi \rangle) &= \text{Cov}(\phi^T X, \psi^T Y) \\ &= \mathbf{E}[\phi^T (X - \mu_X)(Y - \mu_Y)^T \psi] = \phi^T \mathbf{E}[(X - \mu_X)(Y - \mu_Y)^T] \psi \\ &= \phi^T \text{Cov}(X, Y) \psi = \langle C_{XY} \phi, \psi \rangle \end{aligned} \quad (5)$$

Example 2 ($H = L^2(D)$). When $X = Y$ noted as $u(x, \omega) \in H = L^2(D)$, the covariance operator: The covariance operator \mathcal{C} can be defined as:

$$\langle \mathcal{C}_u \phi, \psi \rangle_{L^2(D)} = \int_D \int_D \text{Cov}(u(x), u(y)) \phi(x) \psi(y) dx dy \quad (6)$$

So that

$$(\mathcal{C}_u \phi)(x) = \int_D \text{Cov}(u(x), u(y)) \phi(y) dy \quad (7)$$

That is any $L^2(D)$ -valued random variable $u(x)$ can defines a R.F. with $\mu(x)$ and $C(x, y)$ equal to the kernel of \mathcal{C} .

Definition 4 (H-valued Gaussian random variable). *Let H be a Hilbert space. An H -valued random variable X is Gaussian if $\langle X, \phi \rangle$ is a real-valued Gaussian random variable for all $\phi \in H$.*

2 For Stationary RF on \mathbb{R}^d

About SPDE approaches for random field simulation:

Theorem 1 (Wiener-Khinchin Theorem). *There exists a stationary random field $u(x) : x \in D$ with mean μ and covariance function $c(x)$ that is mean square continuous if and only if the function $c(x) : \mathbb{R}^d \rightarrow \mathbb{R}$ is such that*

$$c(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ik \cdot x} dF(k) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ik \cdot x} S(k) dk = (\mathcal{F}^{-1} S)(x) \quad (8)$$

where $F(k)$ is some measure on \mathbb{R}^d called spectral distribution and $\hat{S}(x)$ is the Fourier transform of $S(k)$, called spectral density. Reversely, $S(k) = (\mathcal{F}c)(k) = \hat{c}(k)$. If $S(k)$ is non-negative and integrable, then $c(x)$ is a valid covariance function.

Theorem 2 (Spectral Density of Random Field). *Assume $u(x)$ has zero mean, then*

$$S_u(k) = \frac{1}{(2\pi)^d} \mathbb{E}[|\hat{u}(k)|^2] \quad (9)$$

By defining the pseudo-differential operators, the class of SPDEs is defined by:

$$\mathcal{L}_g u = W, \mathcal{L}_g = \mathcal{F}^{-1} g \mathcal{F} \quad (10)$$

where $g : \mathbb{R}^d \rightarrow \mathbb{C}$ must be a sufficiently regular and Hermitian-symmetric function, that is it must satisfy: $g(k) = \overline{g(-k)}$, $\bar{\cdot}$ denotes the complex conjugate. So if we have $\mathcal{L}_g u = W$, then:

$$u = \mathcal{L}_g^{-1} W \quad (11)$$

Theorem 3. *The spectral density of $\mathcal{L}_g u$ and of u are related by:*

$$S_{\mathcal{L}_g u}(k) = |g(k)|^2 S_u(k) \quad (12)$$

Generally, if

$$\mathcal{L}_g u = w \quad (13)$$

where w is a GeRF source term, then $S_w(k) = |g(k)|^2 S_u(k)$. Therefore, when $w = W$, $S_u = \frac{1}{(2\pi)^d |g(k)|^2} \mathbb{E}[|\hat{W}(k)|^2] = \frac{1}{|g(k)|^2}$. Then,

$$u(x) = \mathcal{L}_g^{-1} w(x) = \mathcal{L} \sqrt{\frac{S_u}{S_w}} w(x) \quad (14)$$

Then consider the existence of the function.

Theorem 4. *Let $w(x)$ be a real stationary GeRF over \mathbb{R}^d , and let $g : \mathbb{R}^d \rightarrow \mathbb{C}$ be a symbol function. Then for (13), there exists a unique stationary solution $u(x)$ if and only if: there exists $N \in \mathbb{N}$ s.t.*

$$\int_{\mathbb{R}^d} \frac{dS_w(k)}{|g(k)|^2 (1 + \|k\|^2)^N} < \infty \quad (15)$$

and

$$S_u(k) = |g(k)|^{-2} S_w(k) \quad (16)$$

Moreover, $S_u(k)$ is unique if and only if $|g| > 0$.

Hence the key is the symbol function $g(k)$. The following theorem shows that solutions of SPDEs with White Noise source term is the starting point of more general solutions, when the source term can be any stationary GeRF.

Theorem 5. *Let $w(x)$ be a real stationary GeRF over \mathbb{R}^d with covariance distribution $C_w(x)$. Let g be a symbol function over \mathbb{R}^d such that $\frac{1}{g}$ is smooth with polynomially bounded derivatives of all orders. Then, there exists a unique stationary solution to (13) and its covariance distribution is given by*

$$C_u(x) = C_u^W * C_w(x) \quad (17)$$

where C_u^W is the covariance function of the solution to the SPDE with White Noise source term.

Proof. The proof is straightforward by using Weiner-Khinchin theorem. \square

For any precision operator which is a polynomial in the Laplacian, $Q = p(-\Delta)$, such as the Matern operator with $\nu \in \mathbb{N}$, this results in a polynomial $F(Q) = p(\|k\|^2)$.

2.1 Matern Field

The important relationship that we will make use of is that a Gaussian field $u(x)$ with the Matern covariance is a solution to the linear fractional stochastic partial differential equation (SPDE):

$$\mathcal{L}^{\alpha/2} u(x) = (\kappa^2 - \Delta)^{\alpha/2} u(x) = W(x), \quad x \in D \in \mathbb{R}^d, \alpha = \nu + d/2, \kappa > 0, \nu > 0, \quad (18)$$

where $\nu = \alpha - d/2$, $\rho = \frac{\sqrt{2\nu}}{\kappa}$ is the range parameter, Δ is the Laplacian operator, $W(x)$ is a spatial Gaussian white noise with unit variance. We will name any solution to Equ (18) a Matern field in the following.

Theorem 6 (Spectral Solution of Matern Field). *The solution of u solved by Equ (18) is given by:*

$$u(x) = \mathcal{F}^{-1} \left[\frac{\hat{W}(k)}{(\kappa^2 + \|k\|^2)^{\alpha/2}} \right] (x) \quad (19)$$

where \mathcal{F} is defined in (A). And the covariance function of u is given by:

$$c(x) = \frac{\sigma^2}{2^{\nu-1} \Gamma(\nu)} (\kappa \|x\|)^{\nu} K_{\nu}(\kappa \|x\|) \quad (20)$$

where $\nu = \alpha - d/2$, $\rho = \frac{\sqrt{2\nu}}{\kappa}$, $\sigma^2 = \frac{\Gamma(\nu)}{(4\pi)^{d/2} \kappa^{2\nu} \Gamma(\alpha)}$

Weiner-Khinchin theorem + Spectral Theorem.

2.2 Generalized Matern Field

Consider the following SPDE:

$$(\kappa^2 + (-\Delta)^{\gamma})^{\alpha/2} u(x) = \mathcal{F}^{-1} \left((\kappa^2 + \|k\|^{2\gamma})^{\alpha/2} \mathcal{F} u \right) (x) = W(x), \quad x \in D \in \mathbb{R}^d \quad (21)$$

Hence the solution is:

$$u(x) = \mathcal{F}^{-1} \left[\frac{\hat{W}(k)}{(\kappa^2 + \|k\|^{2\gamma})^{\alpha/2}} \right] (x) \quad (22)$$

Therefore the spectral density is:

$$S_u(k) = \frac{1}{(\kappa^2 + \|k\|^{2\gamma})^{\alpha}} \quad (23)$$

So when $\gamma = 1$, it becomes the Matern Field. Since the spectral density $S_u(k) \in L^2(\mathbb{R}^d)$ if and only if $\alpha\gamma > \frac{d}{2}$.

Generally, we can define the pseudo-differential operator through symbol function $g(k)$.

3 Spatial-Temporal General Random Field on $\mathbb{R}^d \times (0, T)$

3.1 Stein Model

Proposed in [7], we define the spatial-temporal white noise $\mathcal{W}(x, t)$ as Gaussian noise that is white in time but correlated in space.

$$\left(b(s^2 - \frac{d}{dt^2})^\beta + a(\kappa^2 - \Delta)^\alpha\right)^{\nu/2} u(x, t) = W, \quad (x, t) \in D \times (0, T) \quad (24)$$

Consider when $D = \mathbb{R}^d$, where the space-time spectral density of the stationary solution is given by:

$$S_u(k_s, k_t) = \frac{1}{(b(s^2 + k_t^2)^\beta + a(\kappa^2 + k_s^2)^\alpha)^\nu} \quad (25)$$

We note the spatio-temporal symbol function as:

$$g(k_s, k_t) : (k_s, k_t) \rightarrow (b(s^2 + k_t^2)^\beta + a(\kappa^2 + k_s^2)^\alpha)^{\nu/2} \quad (26)$$

When α, β, ν are positive and $\frac{d}{\alpha\nu} + \frac{1}{\beta\nu} = 2$, [7] shows that the spectral density is finite and the corresponding random field is mean square continuous.

Theorem 7. When $\kappa, s, a, b > 0$ and α, β, ν are not null, $g(k_s, k_t)$ satisfies Thm 5, then for any stationary GeRF X , the SPDE:

$$\left(b(s^2 - \frac{d}{dt^2})^\beta + a(\kappa^2 - \Delta)^\alpha\right)^{\nu/2} U(x, t) = X(x, t) \quad (27)$$

has a unique stationary solution $U(x, t)$ with covariance function:

$$C_U(x, y, t, s) = C_U^W * C^X(x - y, t - s) \quad (28)$$

3.2 Evolution Equations Model

Here we consider the following model:

$$\frac{\partial^\beta u}{\partial t^\beta} + \mathcal{L}_g u = w(x, t) \quad (29)$$

where \mathcal{L}_g is a pseudo-differential operator with symbol $g(k)$ and $w(x, t)$ is a stationary spatio-temporal GeRF.

$$g(k_s, k_t) = (ik_t)^\beta + g(k_s) \quad (30)$$

3.3 Advection-Diffusion SPDE

This is poeposed in [6]. The equation is given by:

$$\left[\frac{\partial}{\partial t} - \nabla \cdot (\Sigma \nabla) + \mu \nabla + C\right] u(x, t) = w(x, t) \quad (31)$$

where Σ is the diffusion matrix, μ is the advection velocity, C is the drift coefficient. Here we set the diffusion matrix as:

$$\Sigma = \frac{1}{\rho^2} \begin{pmatrix} \cos \theta & \sin \theta \\ -\gamma \sin \theta & \gamma \cos \theta \end{pmatrix}^T \begin{pmatrix} \cos \theta & \sin \theta \\ -\gamma \sin \theta & \gamma \cos \theta \end{pmatrix} \quad (32)$$

where ρ is the correlation length and γ is the anisotropy ratio, $\theta \in [0, \pi/2)$. With $\gamma = 1$, it becomes isotropic. Similarly the spectral density is given by:

$$\begin{aligned} S_u(k_s, k_t) &= \frac{S_w(k_s, k_t)}{|i(k_t + \mu k_s) + (C + k_s^T \Sigma k_s)|^2} \\ &= \frac{S_w(k_s, k_t)}{(k_t + \mu k_s)^2 + (C + k_s^T \Sigma k_s)^2} \\ &= \frac{S_w(k_s, k_t)}{|g_u|^2} \end{aligned} \quad (33)$$

By Wiener-Khinchin theorem, the covariance function is given by:

$$C_u(x, t) = \frac{1}{(2\pi)^d} \int S_w \frac{e^{-i\mu k_s t - (k_s^T \Sigma k_s + C)|t|}}{2(k_s^T \Sigma k_s + C)} e^{ik_s x} dk_s \quad (34)$$

Specifically, when $\mu = 0, \Sigma = 0$, the covariance function is given by:

$$C_u(x, t) = \frac{e^{-C|t|}}{2C} C_w(x, t) \quad (35)$$

However $\mu(x)$ may not be constant.

3.4 Generic class of non-stationary models

Similar to ADSPDE, we consider:

$$\frac{\partial u}{\partial t} + [-\nabla \cdot (\Sigma(x, t) \nabla) + \mu(x, t) \cdot \nabla + \kappa^2(x, t)]^{\alpha/2} u(x, t) = w(x, t) \quad (36)$$

where μ, Σ, κ are functions of x, t , and $w(x, t)$ is a GeRF driven by Equ (18).

4 Experiments

We must pay particular attention to the change in normalization factors when moving from the continuous to the discrete setting. Specifically:

- Regarding the simulation of white noise: We use $N(0, 1/h)$ to approximate the characteristic variance $\delta(x - y)$ of continuous white noise. However, when calculating the spectral density (54), we need to carefully incorporate appropriate normalization factors:

$$S_u(k) = \frac{1}{N^{\frac{1}{h}}} \frac{\mathbb{E} [|DFT(W)(k)|^2]}{(\kappa^2 + \|k\|^2)^\alpha} \quad (37)$$

- Fourier transform definitions: Special attention must be given to the definition of the Fourier transform. Different normalization conventions will result in distinct normalization factors and, consequently, different results.

First we test the Matern Covariance function:

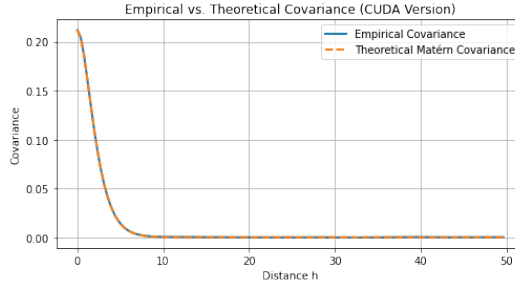


Figure 1: 1D stationary covariance function

4.1 1D random fields

Here are some examples of 1D random fields computed by classical methods in Fig 2.

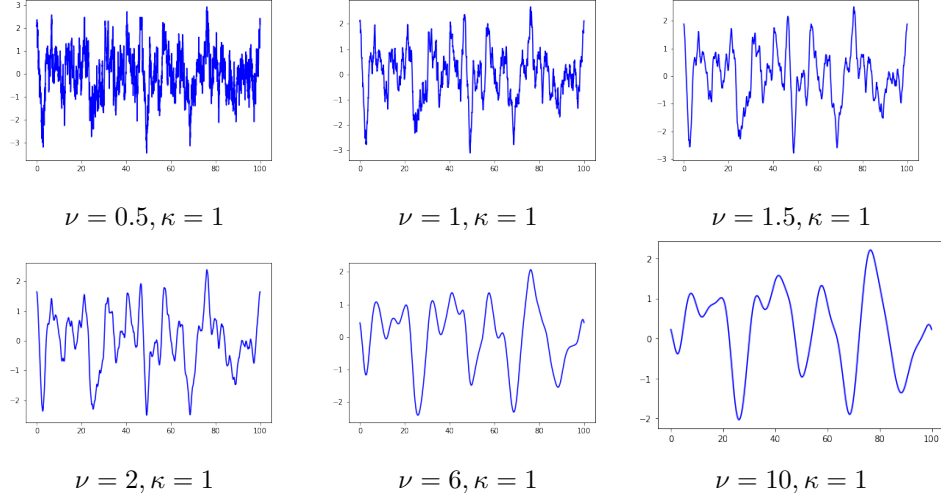


Figure 2: 1D Random Field (Normalized)

Use MLP to learn the spectral density of the random field, and then generate the random field by inverse Fourier transform.

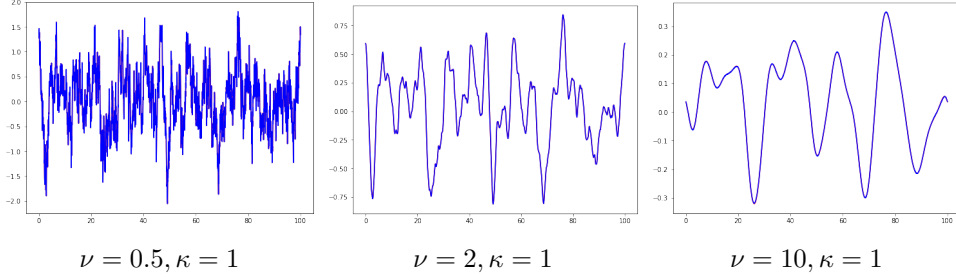


Figure 3: 1D Neural Random Field

Assume discrete samples of the stochastic simulator $\mathcal{M}(x, \omega)$ are given as:

$$\mathcal{T}_r := \{(x_i^r, \mathcal{M}(x_i^r, \omega^r)) : i = 1, 2, \dots, N^r\}, \quad r = 1, 2, \dots, R. \quad (38)$$

in the form of discrete evaluations of the stochastic simulator on R trajectories, where for every r , $\{x_i^r\}_{i=1}^{N^r}$ is an i.i.d. sample from the input distribution μ , and $\mathcal{M}(x_i^r, \omega^r)$ is the output of the stochastic simulator. Here we assume for notational simplicity that $N^r = N$ for all r .

That is we have R samples $u(x)$. By the definition of spectral density, we can have the estimation of $S_u(k)$:

$$S_u(k) = \frac{1}{(2\pi)^d} \frac{1}{R} \sum_{r=1}^R \left(\|\mathcal{F}u_r(k)\|^2 \right) \quad (39)$$

For the above example, the solution can be rewritten as:

$$u(x) = \mathcal{F}^{-1} \left[\hat{W}(k) (S_u(k))^{1/2} \right] (x) \quad (40)$$

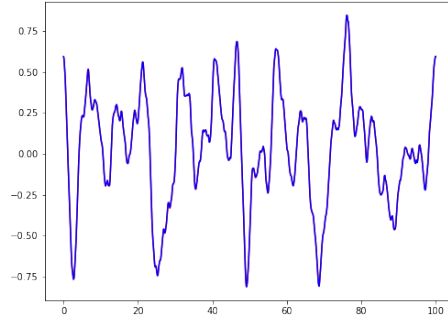


Figure 4: 1D Data-Driven Random Field

4.2 2D random fields

2D is similar. Neural 2D random field is verified.

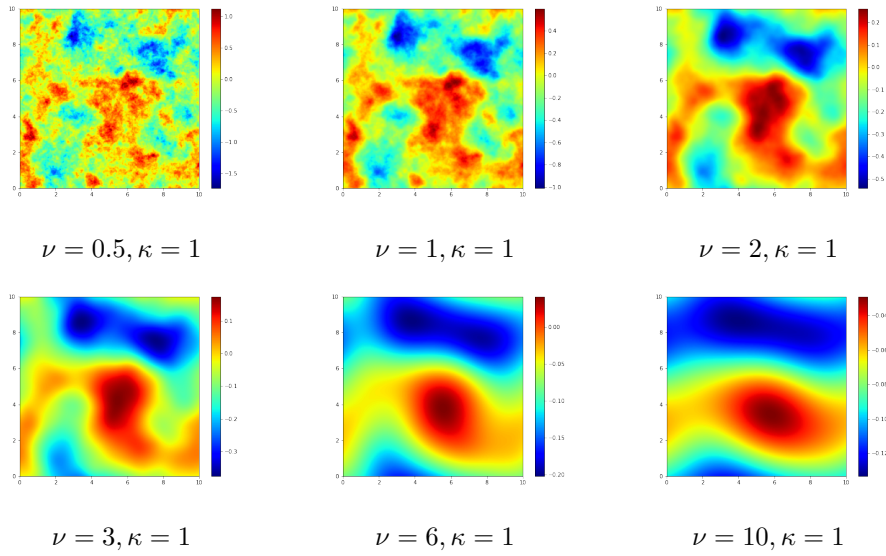


Figure 5: 2D Random Field

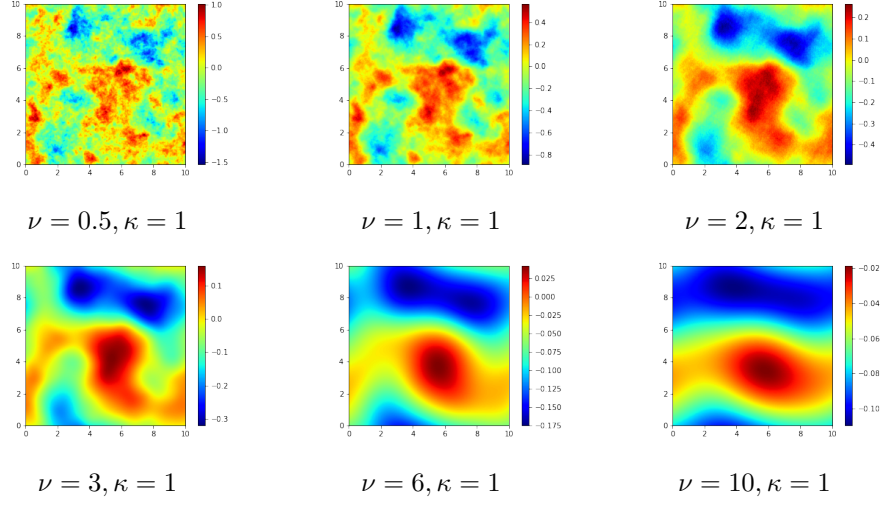


Figure 6: 2D Neural Random Field

5 Conclusion

Q1: For general SPDE, given the equation, how to simulate the random field? Or inversely, given the random field data, how to infer the equation?

Q2: What is the relationship between the equation and the covariance function? For $\forall \mathcal{M}(x, \omega)$, can we find a differential operator \mathcal{L} and a noise \mathcal{W} s.t. $\mathcal{L}\mathcal{M} = \mathcal{W}$?

Q3: Generally speaking, given the covariance function of $\mathcal{M}(x, \omega)$, how can we simulate the random field with DNN?

Consider:

- Generative Model: GAN, VAE, Flow-based model
- Neural Operator: DeepONet, FNO, etc.

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A Fourier Transform

Definition 5. Define the Fourier transform of $u(x)$ as:

$$\begin{cases} \mathcal{F}u(k) = \int_{\mathbb{R}^d} u(x)e^{-ikx} dx, \\ \mathcal{F}^{-1}u(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} u(k)e^{ikx} dk. \end{cases} \quad (41)$$

B Polynomial Chaos Expansion

A spectral expansion in $L_\mu(D)$ is called chasos expansion. By defining the inner product in $L_\mu(D)$ as:

$$\langle f, g \rangle_\mu = \int_D f(x)g(x)d\mu(x), \quad (42)$$

the space of $L_\mu(D)$ is a Hilbert space.

Theorem 8. Let $\{\phi_i(x)\}_{i=1}^\infty$ be an orthonormal basis of $L_\mu(D)$, i.e.

$$\begin{aligned} 1. & \int_D \phi_i(x)\phi_j(x)d\mu(x) = \delta_{ij}, \\ 2. & \phi_i(x) \text{ is dense in } L_\mu(D). \end{aligned} \quad (43)$$

Then the chasos expansion of $\mathcal{M}(x, \omega) \in L_\mu(D)$ is given by:

$$\mathcal{M}(x, \omega) = \sum_{i=1}^{\infty} c_i \phi_i(x), \quad (44)$$

where $c_i(x) = \langle \mathcal{M}, \phi_i \rangle_\mu$ are the coefficients of the expansion.

C Karhunen-Loeve Expansion

Let $\mu(x)$ be the mean function and $C(x, y)$ be the covariance function of $\mathcal{M}(x, \omega)$. Assume that D is bounded, $C(x, y)$ is continuous and $\mathcal{M}(x, \cdot)$ has finite variables for all $x \in D$.

Theorem 9. The Karhunen-Loeve expansion of $\mathcal{M}(x, \omega)$ is given by:

$$\mathcal{M}(x, \omega) = \mu(x) + \sum_{i=1}^{\infty} \lambda_i \phi_i(x) \xi_i(\omega), \quad (45)$$

where λ_i and $\phi_i(x)$ are the eigenvalues and eigenfunctions of the covariance operator \mathcal{C} :

$$(\mathcal{C}\phi_i)(x) = \int_D \text{Cov}(\mathcal{M}(x, \omega), \mathcal{M}(y, \omega)) \phi_i(y) dy = \int_D C(x, y) \phi_i(y) dy = \lambda_i \phi_i(x). \quad (46)$$

where $C(x, y), x, y \in D$ is the covariance function of $\mathcal{M}(x, \omega)$. The KL-random variables $\xi_i(\omega)$ are the result of the projection of $\mathcal{M}(x, \omega)$ onto the eigenfunctions $\phi_i(x)$:

$$\xi_i(\omega) = \frac{1}{\sqrt{\lambda_i}} \int_D (\mathcal{M}(x, \omega) - \mu(x)) \phi_i(x) dx. \quad (47)$$

Note that both $\{\phi_i(x)\}_{i=1}^\infty$ and $\{\xi_i(\omega)\}_{i=1}^\infty$ are orthonormal bases, one capturing the “spatial” variation of $\mathcal{M}(x, \omega)$ over D (in terms of x), the other capturing the stochastic variation of $\mathcal{M}(x, \omega)$ (in terms of ω).

Theorem 10 (Mercer’s theorem). The covariance function $C(x, y)$ of $\mathcal{M}(x, \omega)$ can be expressed as:

$$C(x, y) = \sum_{i=1}^{\infty} \lambda_i \phi_i(x) \phi_i(y). \quad (48)$$

It follows that the average variance of the random field over the domain D is equal to $\sum_{i=1}^{\infty} \lambda_i$.

In most cases, the integral eigenvalue problem in Equ (9) is difficult: analytically solved and high-dimensional.

D Some proofs

Proof of Thm 2.

$$\begin{aligned}
S_u(k) &= (\mathcal{F}c)(k) = \int_{\mathbb{R}^d} e^{-ikh} c(h) dh \\
&= \int_{\mathbb{R}^d} e^{-ik(x+h-x)} \mathbb{E}[u(x+h)u(x)] dh \\
&= \mathbb{E} \left[\int_{\mathbb{R}^d} e^{-ik(x+h)} u(x+h) e^{ikx} u(x) dx \right] \\
&= \frac{1}{(2\pi)^d} \mathbb{E} \left[|(\mathcal{F}u)(k)|^2 \right]
\end{aligned} \tag{49}$$

□

Proof of Thm 6. Do Fourier transform on Equ (18):

$$\left\{ \mathcal{F}(\kappa^2 - \Delta)^{\alpha/2} u \right\}(k) = (\kappa^2 + \|k\|^2)^{\alpha/2} (\mathcal{F}u)(k) \tag{50}$$

then we have

$$(\mathcal{F}u)(k) = \hat{u}(k) = \frac{\hat{W}(k)}{(\kappa^2 + \|k\|^2)^{\alpha/2}} \tag{51}$$

Therefore, u can be written as:

$$u(x) = \mathcal{F}^{-1} \left[\frac{\hat{W}(k)}{(\kappa^2 + \|k\|^2)^{\alpha/2}} \right] \tag{52}$$

Then the stationary covariance function of u is given by:

$$c(x) = Cov(u(x), u(0)) \tag{53}$$

By the definition of spectral density Equ (9) and Equ (51) we have:

$$S_u(k) = \frac{1}{(2\pi)^d} \frac{\mathbb{E} \left[|\hat{W}(k)|^2 \right]}{(\kappa^2 + \|k\|^2)^\alpha} = \frac{1}{(\kappa^2 + \|k\|^2)^\alpha} \tag{54}$$

Then we have the variance of u:

$$c(0) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} S_u(k) dk = \frac{\Gamma(\nu)}{(4\pi)^{d/2} \kappa^{2\nu} \Gamma(\alpha)} := \sigma^2 \tag{55}$$

By Wiener-Khinchin theorem, we have:

$$\begin{aligned}
c(x) &= (\mathcal{F}^{-1} S_u)(x) = \mathcal{F}^{-1} \left[\frac{1}{(\kappa^2 + \|k\|^2)^\alpha} \right] \\
&= \frac{\|x\|^\nu K_\nu(\kappa\|x\|)}{(4\pi)^{d/2} 2^{\nu-1} \kappa^\nu \Gamma(\alpha)} \\
&= \frac{\sigma^2}{2^{\nu-1} \Gamma(\nu)} (\kappa\|x\|)^\nu K_\nu(\kappa\|x\|)
\end{aligned} \tag{56}$$

□

Remark 1. To make $c(0) = 1$, we can multiple a constant factor σ_1 to $S_u(k)$:

$$\sigma_1^2 = \frac{\Gamma(\alpha) \kappa^{2\nu} (4\pi)^{d/2}}{\Gamma(\nu)} \tag{57}$$

Then the corresponding function of u is:

$$u(x) = \mathcal{F}^{-1} \left[\frac{\sigma_1 \hat{W}(k)}{(\kappa^2 + \|k\|^2)^{\alpha/2}} \right] \tag{58}$$