About Stochastic Partial Differential Equations

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Abstract

This report focuses on numerical methods for stochastic partial differential equations (SPDEs), covering various aspects such as the generation of (stationary) stochastic processes and random fields, numerical methods for stochastic ordinary differential equations (SODEs), SPDEs with random data and semilinear SPDEs. To facilitate solving SPDEs, we also implemented the finite element method (FEM) for standard partial differential equations (PDEs).

Contents

1 Priori

1.1 Hilbert space-valued random variable

Definition 1 ($L^p(\Omega, H)$ space). *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and H is a Hilbert space with norm $\|\cdot\|$. Then $\mathcal{L}^p(\Omega, H)$ with $1 \leq p < \infty$ is the space of H-valued $\mathcal{F}$-measurable random vaiables $X : \Omega \to H$ with $\mathbf{E}[\|X\|^p] < \infty$ and a Banach space with norm:*

$$\|X\|_{\mathcal{L}^p(\Omega,H)} := \left(\int_\Omega \|X(\omega)\|^p dP(\omega)\right)^{\frac{1}{p}} = \mathbf{E}[\|X\|^p]^{\frac{1}{p}} \tag{1}$$

Then we can define the inner product:

$$\langle X, Y\rangle_{\mathcal{L}^2(\Omega,H)} := \int_\Omega \langle X(\omega), Y(\omega)\rangle dP(\omega) \tag{2}$$

Definition 2 (uncorrelated, covariance operator). *Let H be a Hilbert space. A linear operator $\mathcal{C} : H \to H$ is the covariance of H-valued random variables X and Y if*

$$\langle \mathcal{C}\phi, \psi\rangle = \operatorname{Cov}\left(\langle X, \phi\rangle, \langle Y, \psi\rangle\right), \forall \phi, \psi \in H \tag{3}$$

specially, we show that in finite dimensional case, the covariance matrix conincides with the covariance operator. when $H = \mathbb{R}^d$,

$$\begin{aligned} &\operatorname{Cov}\left(\langle X, \phi\rangle, \langle Y, \psi\rangle\right) = \operatorname{Cov}\left(\phi^T X, \psi^T Y\right) \\ =&\mathbf{E}\left[\phi^T (X - \mu_X)(Y - \mu)^T \psi\right] = \phi^T \mathbf{E}\left[(X - \mu_X)(Y - \mu)^T\right] \psi \\ =&\langle C\phi, \psi\rangle \end{aligned} \tag{4}$$

Definition 3 (H-valued Gaussian random variable). *Let H be a Hilbert space. An H-valued random variable X is Gaussian if $\langle X, \phi\rangle$ is a real-valued Gaussian random variable for all $\phi \in H$.*

1.2 Hilbert-Schmidt operator

Definition 4 (Hilbert-Schmidt operator). *Let U, H be two separable Hilbert spaces with norms $\|\cdot\|, \|\cdot\|_U$ respectively. For an orthonormal basis $\{\phi_j\}$ of U, define the Hilbert-Schmidt norm:*

$$\|L\|_{\mathrm{HS}(U,H)} := \left(\sum_{j=1}^\infty \|L\phi_j\|_H^2\right)^{\frac{1}{2}} \tag{5}$$

where $\mathrm{HS}(U, H) := \{L \in \mathcal{L}(U, H) : \|L\|_{\mathrm{HS}(U,H)} < \infty\}$ is a Banach space with Hilbert-Schmidt norm. And $L \in \mathrm{HS}(U, H)$ is called Hilbert-Schmidt operator.

Definition 5 (Integral operator with kernel G). *For a domain D and a kernel $G \in L^2(D \times D)$, define the integral operator L by*

$$(Lu)(x) = \int_D G(x, y)u(y)dy, x \in D, u \in L^2(D) \tag{6}$$

Furthermore, L is a Hilbert-Schmidt operator.

1.3 Operator theory

Theorem 1 (Sobolev embedding theorem). *1.Let $W^{r,p}(\mathbf{R}^n)$. Here k is a non-negative integer and $1 \leq p < \infty$. If $k > \ell, p < n$ and $1 \leq p < q < \infty$ are two real numbers such that $\frac{1}{p} - \frac{r}{n} = \frac{1}{q} - \frac{\ell}{n}$, then*

$$W^{r,p}(\mathbf{R}^n) \subseteq W^{\ell,q}(\mathbf{R}^n) \tag{7}$$

Specially, if $\ell = 0$, then $\frac{1}{p} - \frac{r}{n} = \frac{1}{q}$, then $W^{r,p}(\mathbf{R}^n) \subseteq L^q(\mathbf{R}^n)$.

2.If $n < pr$ and $\frac{1}{p} - \frac{r}{n} = -\frac{s+\alpha}{n}$, then $W^{r,p}(\mathbf{R}^n) \subseteq C^{s,\alpha}(\mathbf{R}^n)$.

Definition 6 (domain of operator). *For a linear operator $A : \mathcal{D}(A) \subset H \to H$, the domain of A is defined as $\mathcal{D}(A)$*

Theorem 2 (Dirichlet Boundary Condition)**.** *Consider the Dirichlet problem for Possion equation: for $f \in L^2(0,1)$, find $u \in H^2(0,1)$ s.t.*

$$\begin{aligned} u_{xx} &= f, \quad x \in (0,1) \\ u(0) &= u(1) = 0 \end{aligned} \tag{8}$$

We also assume $u \in H_0^1(0,1)$. By Sobolev embedding theorem, $u \in H_0^1(0,1) \subset C([0,1])$. Then, Laplacian with Dirichlet conditions can be defined as:

$$Au := -u_{xx}, u \in \mathcal{D}(A) = H^2(0,1) \cap H_0^1(0,1) \tag{9}$$

Definition 7 (Periodic Boundary Condition)**.** *...*

Definition 8. *If A is a linear operator from $\mathcal{D}(A) \subset H$ to Hilbert space H, with an orthonormal basis of eigenfunctions $\{\phi_j\}$ and corresponding increasing eigenvalues $\{\lambda_j\}$, then A^α is defined as:*

$$A^\alpha u = \sum_{j=1}^{\infty} \lambda_j^\alpha \langle u, \phi_j \rangle \phi_j \tag{10}$$

and the domain $\mathcal{D}(A^\alpha)$ is the set of all $u \in H$ such that $A^\alpha u \in H$.

2 Semilinear Stochastic PDEs

2.1 Semilinear SPDE

Then we come to the time-dependent SPDE. We study the stochastic semilinear evolution equation:

$$du = [\Delta u + f(u)]dt + G(u)dW(t,x) \tag{11}$$

Definition 9 (Semilinear SPDE)**.** *Simmilar to normal time-dependent PDE, we treat SPDE like this as semi-linear SODEs on a Hilbert space, like*

$$du = [-Au + f(u)]dt + G(u)dW(t) \tag{12}$$

where $-A$ is a linear operator that generates a semigroup $S(t) = e^{-tA}$.

Example 1 (Phase-field model)**.**

$$du = [\epsilon \Delta u + u - u^3]dt + \sigma dW(t,x) \tag{13}$$

Example 2 (Fluid Flow)**.**

$$\begin{aligned} u_t &= \epsilon \Delta u - \nabla p - (u \cdot \nabla) u \\ \nabla \cdot u &= 0 \end{aligned} \tag{14}$$

So like we deal with integration of stochastic process like Itos or stratonovich, we need to generalize the Brownian Motion by introducing spatial variable to W(t). Here we define Q-Wiener Process.

2.2 Q wiener process

First, we assume U is a Hilbert space. And $(\Omega, \mathbf{F}, \mathbf{F}_t, \mathbb{R})$ is a filtered probability space.

Definition 10 (Q)**.** *$Q \in \mathcal{L}(U)$ is non-negative definite and symmetric. Further, Q has an orthonormal basis $\{\mathcal{X}_j : j \in \mathcal{N}\}$ of eigenfunctions with corresponding eigenvalues $q_j \geq 0$ such that $\sum_{j \in \mathcal{N}} q_j < \infty$ (i.e., Q is of trace class).*

Definition 11 (Q-Wiener Process)**.** *A U-valued stochastic process $\{W(t) : t \geq 0\}$ is Q-Wiener process if*

- *W(0) = 0 a.s.*
- *W(t) is a continuous function $\mathbb{R}^+ \to U$, for each $\omega \in \Omega$.*
- *W(t) is $\mathcal{F}_t$-adapted and $W(t) - W(s)$ is independent of $\mathcal{F}_s$ for $s \leq t$*
- *$W(t) - W(s) \sim N(0, (t-s)Q)$ for all $0 \leq s \leq t$*

Theorem 3 (Q-Wiener Process)**.** *Assume we have Q defined in 10. Then, $W(t)$ is a Q-Wiener process if and only if*

$$W(t) = \sum_{j=1}^{\infty} \sqrt{q_j}\mathcal{X}_j\beta_j(t) \tag{15}$$

which is converges in $L^2(\Omega, C([0,T],U))$ and $\beta_j(t)$ are iid $\mathcal{F}_t$-Brownian motions and the series converges in $L^2(\Omega, U)$.

Theorem 4 ($H^r_{\text{per}}(0,a)$-valued process)**.** *...*

Theorem 5 ($H^r_0(0,a)$-valued process)**.** *...*

So, in place of $L^2(D)$, we develop the theory on a separable Hilbert space U with norm $\|\cdot\|_U$ and inner product $\langle\cdot,\cdot\rangle_U$ and define the Q-Wiener process $W(t) : t \geq 0$ as a U-valued process.

2.3 Cylindrical Wiener Process

We mention the important case of Q = I, which is not trace class on an infinite-dimensional space U (as $q_j = 1$ for all j) so that the series does not converge in $L^2(\Omega, U)$. To extend the definition of a Q-Wiener process, we introduce the cylindrical Wiener process.

The key point is to introduce a second space U1 such that $U \subset U_1$ and Q = I is a trace class operator when extended to U_1.

Then we can define cylindrical Wiener process:

Definition 12 (Cylindrical Wiener Process)**.** *Let U be a separable Hilbert space. The cylindrical Wiener process (also called space-time white noise) is the U-valued stochastic process W(t) defined by*

$$W(t) = \sum_{j=1}^{\infty} \mathcal{X}_j\beta_j(t)$$

where $\{\mathcal{X}_j\}$ is any orthonormal basis of U and $\beta_j(t)$ are iid $\mathcal{F}_t$-Brownian motions.

Theorem 6. *If for the second Hilbert space U_1, and the inclusion map $\mathcal{I} : U \to U_1$ is Hilbert-Schmidt. Then, the cylindrical Wiener process is a Q-Wiener process well-defined on U_1(Converges in $L^2(U, U_1)$).*

2.4 Ito integral solution

Here we consider the Ito integral $\int_0^t B(s)dW(s)$ for a Q-Wiener process $W(s)$. Since dW_t takes value in Hilbert space U, and we treat SPDE in Hilbert space H, the integral will also take value in Hilbert space H.

Hence, $B(s)$ should be $\mathcal{L}^2_0(U_0, H)$-valued process, where $U_0 \subset U$ known as Cameron-Martin space. So, $B(s)$ is an operator from U_0 to H. Then, we consider the set of operator B.

Definition 13 (L^2_0 space)**.** *Let $U_0 := \{Q^{\frac{1}{2}}u : u \in U\}$, the set of linear operators $B : U_0 \to H$ is noted as L^2_0 s.t.*

$$\|B\|_{L^2_0} := \left(\sum_{j=1}^{\infty} \|BQ^{\frac{1}{2}}\mathcal{X}_j\|^2\right)^{\frac{1}{2}} = \|BQ^{\frac{1}{2}}\|_{\text{HS}(U_0,H)} < \infty \tag{16}$$

Remark 1. *If G is invertible, L^2_0 is the space of Hilbert-Schmidt operators* $\text{HS}(U_0, H)$*.*

Definition 14. *The stochastic integral can be defined by*

$$\int_0^t B(s)dW(s) := \sum_{j=1}^{\infty} \int_0^t B(s)\sqrt{q_j}\mathcal{X}_j d\beta_j(s) \tag{17}$$

So, we can have the truncated form:

$$\int_0^t B(s)dW^J(s) = \sum_{j=1}^{J} \int_0^t B(s)\sqrt{q_j}\mathcal{X}_j d\beta_j(s) \tag{18}$$

2.5 Semilinear SPDE

Consider the semilinear SPDE:

$$du = [-Au + f(u)]dt + G(u)dW(t) \tag{19}$$

given the initial condition $u_0 \in H$ and $A : \mathcal{D} \subset H \to H$ is a linear operator, $f : H \to H$ and $G : H \to L_0^2$.

Example 3. *Consider the stochastic heat equation:*

$$du = \Delta u dt + \sigma dW(t,x), u(0,x) = u_0(x) \in L^2(D) \tag{20}$$

where D is a bounded domain in $\mathbb{R}^d$ and σ is a constant. Also, homogeneous Dirichlet boundary condition is imposed on D. Hence,

$$H = U = L^2(D), f(u) = 0, G(u) = \sigma I \tag{21}$$

We see that $A = -\Delta$ with domain $\mathcal{D}(A) = H^2(D) \cap H_0^1(D)$.

In the deterministic setting of PDEs, there are a number of different concepts of solution. Here is the same for SPDEs. We can also define strong solution, weak solution and mild solution.

Definition 15 (strong solution)**.** *A predictable H -valued process $\{u(t) : t \in [0,T]\}$ is called a strong solution if*

$$u(t) = u_0 + \int_0^t [-Au(s) + f(u(s))]ds + \int_0^t G(u(s))dW(s), \quad \forall t \in [0,T] \tag{22}$$

Definition 16 (weak solution)**.** *A predictable H -valued process $\{u(t) : t \in [0,T]\}$ is called a weak solution if*

$$\langle u(t), v\rangle = \langle u_0, v\rangle + \int_0^t [-\langle u(s), Av\rangle + \langle f(u(s)), v\rangle]ds + \int_0^t \langle G(u(s))dW(s), v\rangle, \quad \forall t \in [0,T], v \in \mathcal{D}(A) \tag{23}$$

where

$$\int_0^t \langle G(u(s))dW(s), v\rangle := \sum_{j=1}^{\infty} \int_0^t \left\langle G(u(s))\sqrt{q_j}\mathcal{X}_j, v\right\rangle d\beta_j(s).$$

Definition 17 (mild solution)**.** *A predictable H -valued process $\{u(t) : t \in [0,T]\}$ is called a mild solution if for $t \in [0,T]$*

$$u(t) = \mathrm{e}^{-tA}u_0 + \int_0^t \mathrm{e}^{-(t-s)A} f(u(s))ds + \int_0^t \mathrm{e}^{-(t-s)A} G(u(s))dW(s),$$

where e^{-tA} is the semigroup generated by $-A$. The right hand side is also called stochastic convolution.

Example 4 (stochastic heat equation in one dimension)**.** *Consider the weak solution of 1D heat SPDE with $D = (0,\pi)$, so that $-A$ has eigenfunctions $\phi_j(x) = \sqrt{2/\pi}\sin(jx)$ and eigenvalues $\lambda_j = j^2$ for $j \in \mathbb{N}$. Suppose that $W(t)$ is a Q -Wiener process and the eigenfunctions $\mathcal{X}_j$ of Q are the same as the eigenfunctions ϕ_j of A. A weak solution satisfies: $\forall v \in \mathcal{D}(A)$,*

$$\begin{aligned} \langle u(t), v\rangle_{L^2(0,\pi)} &= \langle u_0, v\rangle_{L^2(0,\pi)} + \int_0^t \langle -u(s), Av\rangle_{L^2(0,\pi)} ds \\ &+ \sum_{j=1}^{\infty} \int_0^t \sigma\sqrt{q_j}\, \langle \phi_j, v\rangle_{L^2(0,\pi)}\, d\beta_j(s) \end{aligned} \tag{24}$$

Assume $u(t) = \sum_{j=1}^{\infty} \hat{u}_j(t)\phi_j$ for $\hat{u}_j(t) := \langle u(t), \phi_j\rangle_{L^2(0,\pi)}$. Take $v = \phi_j$, we have

$$\hat{u}_j(t) = \hat{u}_j(0) + \int_0^t (-\lambda_j)\, \hat{u}_j(s)ds + \int_0^t \sigma\sqrt{q}_j d\beta_j(s). \tag{25}$$

Hence, $\hat{u}_j(t)$ satisfies the SODE

$$d\hat{u}_j = -\lambda_j \hat{u}_j dt + \sigma\sqrt{q_j} d\beta_j(t) \tag{26}$$

Therefore, each coefficient $\hat{u}_j(t)$ is an Ornstein-Uhlenbeck (OU) process (see Examples 8.1 and 8.21), which is a Gaussian process with variance

$$\operatorname{Var}(\hat{u}_j(t)) = \frac{\sigma^2 q_j}{2\lambda_j}\left(1 - \mathrm{e}^{-2\lambda_j t}\right) \tag{27}$$

For initial data $u_0 = 0$, we obtain, by the Parseval identity (1.43),

$$\|u(t)\|^2_{L^2(\Omega, L^2(0,\pi))} = \mathbb{E}\left[\sum_{j=1}^{\infty} |\hat{u}_j(t)|^2\right] = \sum_{j=1}^{\infty} \frac{\sigma^2 q_j}{2\lambda_j}\left(1 - \mathrm{e}^{-2\lambda_j t}\right). \tag{28}$$