Cogn Comput (2016) 8:543–554
DOI 10.1007/s12559-016-9397-5

Three-Way Decisions and Cognitive Computing

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Received: 2 February 2016 / Accepted: 2 March 2016 / Published online: 15 March 2016

Abstract A trisecting-and-acting model explains three-way decisions (3WD) in terms of two basic tasks. One task is to divide a universal set into three pair-wise disjoint regions called a trisection or a tri-partition of the universal set. The other task is to act upon objects in one or more regions by developing appropriate strategies. 3WD are a class of effective ways and heuristics commonly used in human problem solving and information processing. We argue that 3WD are built on solid cognitive foundations and offer cognitive advantages and benefits. We demonstrate the flexibility and general applicability of 3WD by using examples from across many fields and disciplines.

Keywords Cognitive computing · Information processing · Problem solving · Three-way decisions · Tri-partitions · Trisecting-and-acting model

Introduction

Cognitive computing may be viewed as either a field of study of brain-inspired mechanisms and systems [18, 19, 42] or the third era of computing characterized by human–machine interaction and symbiosis [32, 55], namely the cognitive era after the tabulating era and the programming era [22]. Understanding the mind is a fundamental issue in cognitive computing. In explaining how the mind works, Minsky [41] suggests that we have many ways to think and many ways to represent things. We can easily switch among different ways, extend the range of our ways, and create new ways if old ways do not work. In this context, a specific topic of cognitive computing may be the investigation and understanding of a wide spectrum of human ways of problem solving, decision making, computing, and information processing. The goal of the present paper is to explore a theory of three-way decisions (3WD) [69] that model a particular class of human ways of problem solving and information processing.

Basic ideas of three-way decisions are to divide a universal set into three pair-wise disjoint regions, or more generally a whole into three distinctive parts, and to act upon each region or part by developing an appropriate strategy [72]. The term "three-way decisions" embraces all aspects of a decision-making process, including tasks such as data and evidence collection and analysis for supporting decision making, reasoning, and computing in order to arrive at a particular decision, and justification and explanation of a decision. The unique feature of three-way decisions is a type of three-way approaches (i.e., the division of a whole into three parts) to problem solving and information processing. We may replace "decisions" in "three-way decisions" by other words to have specific interpretations such as three-way computing, three-way processing, three-way classification [72], three-way analysis [48], three-way clustering [76], three-way recommendation [78], and many others.

Principles and methods of three-way decisions are commonly used in everyday life. For example, temporally we have past–present–future; spatially we have left–middle–right, top–middle–bottom, or front–middle—back; volumetrically we have small–medium–large; judgmentally we have for–neutral–against, or acceptance–non-commitment–rejection [14]. Human cognition and problem solving rely on such a three-way division of a whole into three parts. Three-way decisions have been widely applied

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in many fields and disciplines. We consider several examples as an illustration. In medical decision making, based on observed symptoms and tests, a doctor makes a decision of treatment, further test, or non-treatment [43]. In social judgement theory, one has the notions of acceptance, non-commitment, and rejection [52]. In sequential hypothesis testing, one accepts a hypothesis, rejects the hypothesis, or continues testing [59]. In a peer review process, an editor may accept a paper, reject the paper, or carry out a further review [61].

At the same time, it is surprising to find that there still does not exist a unified formal description of three-way decisions. For this reason, Yao [69] introduced a theory of three-way decisions based on his earlier work on three-way decisions with rough sets [66–68]. Since its introduction, there has been a fast growing interest, resulting in several edited books [20, 34, 75] and extensive research that extends and applies three-way decisions (for example, see a sample of recently published papers [3, 17, 27–31, 35, 46, 51, 64, 76, 78–80]). Explorations of three-way decisions have been made in relation to several other theories, including, for example, rough sets [44], interval sets [65], three-way approximations of fuzzy sets [11, 77], shadowed sets [45], orthopairs (i.e., a pair of disjoint sets) [4, 5], and squares of oppositions [6, 12, 71]. The theory of three-way decisions embraces ideas from these theories and introduces its own notions, concepts, methods, and tools.

A theory of three-way decisions aims at modeling a class of human ways of problem solving and information processing, with intended applications in designing and implementing intelligent systems. It may be framed as a multidisciplinary, interdisciplinary, and transdisciplinary study. We can draw results from across many disciplines and integrate them for the purpose of three-way decisions. The main objective of this paper is to examine cognitive computing perspectives on three-way decisions. After introducing a trisecting-and-acting model of three-way decisions, we discuss its cognitive basis, advantages, and benefits. We provide a brief discussion on modes of three-way decisions that realize such cognitive benefits.

Three-way decisions rely on human heuristics and, in some situations, may have undesirable cognitive biases and errors [2, 21]. We must carefully consider the conditions when it is advantageous to apply three-way decisions. When developing a theory or a model, we typically prefer these with a greater generality. The generality of three-way decisions may be, at the same time, a potential weakness. The model of three-way decisions, as outlined in this paper, may be too general to provide some common tool to all the situations where the model can be applied. As future research, we plan to develop specific models that aim at solving particular classes of problems.

A Trisecting-and-Acting Model of Three-Way Decisions

We provide a basic formulation of a trisecting-and-acting model of three-way decisions [72] and use several examples to explain fundamental concepts of the model.

Basic Formulation

Three-way decisions are an effective way of problem solving that aims at making fast, low cost, and/or high-benefit decisions with some tolerance of errors. Three-way decisions are based on a heuristic that divides a whole into three parts and acts on some or all of the three parts, which turns complexity into simplicity in many situations. From a wide range of studies related to three-way decisions, we extract the two basic ingredients of trisecting and acting to build a simple, and yet sufficiently general and flexible, trisecting-and-acting model [72]. By making a series of three-way decisions, we arrive at a more general model of sequential three-way decisions [70].

In the trisecting-and-acting model, we formally interpret the whole and the parts in a set theoretical setting. We assume that a whole is formed with a universal set of individual elements. A subset of elements is a part of the whole; disjoint subsets represent distinctive parts. With respect to trisecting, we divide a universal set of elements into three pair-wise disjoint parts, also called regions [72]. The set of three regions is called a trisection or a tri-partition of the universal set. With respect to acting, we design and develop most effective strategies for acting upon the three parts.

Figure 1 depicts the basic ideas of the trisecting-and-acting model. Suppose U denotes a universal set. The triplet $(\mathrm{Region\,I}, \mathrm{Region\,II}, \mathrm{Region\,III})$ is called a tri-partition of U and satisfies the following two properties:

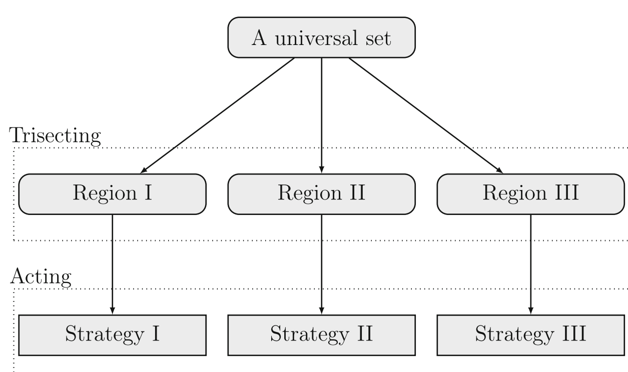

Fig. 1 Trisecting-and-acting model

$$\begin{aligned}
&(\mathrm{i})(\mathrm{Region\,I}) \cap (\mathrm{Region\,II}) = \emptyset,\\
&\quad(\mathrm{Region\,I}) \cap (\mathrm{Region\,III}) = \emptyset,\\
&\quad(\mathrm{Region\,II}) \cap (\mathrm{Region\,III}) = \emptyset,\\
&(\mathrm{ii})(\mathrm{Region\,I}) \cup (\mathrm{Region\,II}) \cup (\mathrm{Region\,III}) = U.
\end{aligned}$$

We do not require that all regions are non-empty sets; some of the three regions may be the empty set. In the strict sense, a tri-partition is not a standard partition of a set that requires each block to be a non-empty set. We use the term "tri-partition" in a loose sense of "partition." Once a tri-partition is constructed, we develop three different strategies, i.e., Strategy I, Strategy II, and Strategy III, to act on the three regions. Some of the three strategies may be the same.

The identification and explicit investigation of different strategies for different regions are distinguishing features of three-way decisions. In formulating the basic trisecting-and-acting model, we treat tri-partitions and strategies as primitive notions. We do not attempt to interpret or formulate them by using other notions; otherwise, we can only have a specific model applicable to special situations. We are not concerned with explanations of the three regions nor procedures for producing the three regions. Similarly, we are not concerned with a detailed description of strategies nor methods for deriving the strategies. Although the basic model is simple, it captures the essential ideas of three-way decisions.

Even with the simple model, we can still have several guidelines on meaningful uses of the model. (1) The separation of trisecting and acting is mainly for the purpose of a conceptual understanding and explanation. In many situations, the two tasks in fact weave together as one and cannot be easily separated. A meaningful trisection of U depends on the strategies and actions used for acting on the three regions. Effective strategies of acting depend on an appropriate trisection of U. The two are just different sides of the same coin; we cannot have one without a consideration of the other. It is desirable that we both separate and integrate the tasks of trisecting and acting. (2) There exist many tri-partitions of the universal set on one hand and many strategies on the other. We must design measures to quantify the quality of both tri-partitions and strategies. Once such quantitative measures are introduced, we may formulate a problem of three-way decisions as an optimization problem with two related components, one for trisecting and the other for acting. (3) The basic model can be used to construct, recursively, a more general model of sequential three-way decisions in which one uses three-way decisions by treating a region obtained from the previous stage as the universal set. (4) Three-way decisions are only advantageous in certain situations. It may be important to identify the conditions under which three-way decisions are better than, for example, two-way decisions, m-way decisions ($m > 3$), and other methods [68]. (5) Three-way decisions implement a commonly used heuristic. When solving a particular problem, it is important first to evaluate the applicability of three-way decisions, rather than a blind application. (6) From the basic model, we can derive more concrete models by using specific interpretations of trisecting and acting.

Examples of Three-Way Decisions

We use three examples to further explain fundamental concepts of three-way decisions.

Triage This is a classical example of three-way decisions that was used to prioritize treatment of the wounded during war. The basic version of triage divides a set of the wounded into three categories [23]:

1. Those who are likely to live, regardless of what care they receive;
2. Those who are likely to die no matter what is done for them;
3. Those for whom immediate care will make a difference.

The division of the wounded into three categories is relatively easy. The tri-partition leads to strategies of treating "the wounded according to the seriousness of their injuries and urgency for need of medical care, regardless of their rank or nationality" [23]. Modern refined triage uses fundamentally the same ideas, but is enriched by better evaluation procedures, more categories, and associated strategies. This example clearly shows the importance and value of developing winning strategies in three-way decisions.

Text Analysis Some of the tasks of text analysis are to identify a set of significant words and to determine their significance values. Following the pioneer work of Luhn [36], one can divide a set of words into three regions based on their frequencies. In Fig. 2 (adapted from van Rijsbergen [58]), the dotted inverted U curve represents the

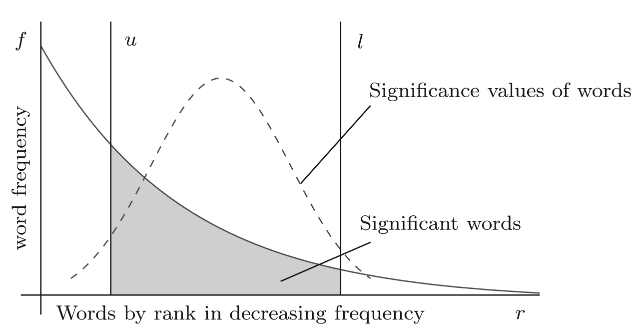

Fig. 2 Three-way classification of words

significance values of words. Given a pair of a high threshold u and a low threshold l, we can divide words into three regions. High-frequency words ranked before u and low-frequency words ranked after l are nonsignificant words, and medium-frequency words ranked between u and l are significant words. Once the three regions are constructed, one can design strategies to further process words in each region. Salton and McGill [50] discuss two strategies called phrase transformation and thesaurus transformation. The phrase transformation combines two or more high-frequency words into a medium-frequency phrase. The thesaurus transformation puts a group of low-frequency words with a similar meaning into a thesaurus class. The frequency of the class is the sum of the frequencies of all words in the class, which makes the class significant. Through the two transformations, one turns nonsignificant words into significant phrases and classes. This example shows that with three-way decisions we may change disadvantages into advantages in some situations.

Management of Students Many schools, colleges, and universities typically monitor student progress based on a three-way classification of students. For example, according to GPA (Grade Point Average), one can divide students into the groups of high-, medium-, and low-GPA students, respectively. A different action is taken for each group of students. A high-GPA student may be awarded a scholarship and a low-GPA student may receive a warning letter. Usually, nothing is done for medium-GPA students. It is surprising that we seldom question the rationales and impacts of these strategies, particularly, the strategy of doing nothing for medium-GPA students. We may easily convince ourselves that the actions taken for the two groups of the low- and high-GPA students are of a limited impact. They are kinds of actions expected by students. We may also easily imagine the positive impact on the medium-GPA students when we identify their strengths and offer them assistance on how to improve. The recognition of their strengths may encourage students to do better and prevent them from falling into the low-GPA group. Let us imagine a specific example that a medium-GPA student performs very well in a particular course. An unexpected letter of congratulation sent to the student may trigger a chain reaction. The encouragement might make the student do better in other courses. In summary, doing something for medium-GPA students may be a high-impact strategy for turning them into high-GPA students. This example also calls for a closer examination of commonly used strategies in three-way decisions.

An Evaluation-Based Model

Based on the three examples, one can formulate an evaluation-based model that produces three regions by using an evaluation function and a pair of thresholds on the values of the evaluation function [69]. The model partially addresses the issue of trisecting a universal set into three regions.

Let $(\mathbf{L}, \succeq)$ denote a totally ordered set, namely $\succeq$ is reflexive, antisymmetric, transitive, and any two values in $\mathbf{L}$ can be compared under $\succeq$. For two values $a, b \in \mathbf{L}$, if $a \succeq b$, we also write $b \preceq a$. Suppose $v: U \longrightarrow \mathbf{L}$ is an evaluation function that maps an object in U to a value in $\mathbf{L}$. For an object $x \in U$, $v(x)$ is its evaluation status value (ESV). The ESV of an object may be interpreted as its quality or desirability. Based on their ESVs, one can arrange objects in U in a decreasing order: objects with higher values to the left and objects with lower values to the right. By using a pair of thresholds (α, β) from $\mathbf{L}$ with $\alpha \succ \beta$ (i.e., $\alpha \succeq \beta$ and $\neg(\beta \succeq \alpha)$), one can divide U into three regions:

$$\begin{aligned} \text{Region I}(v) &= \{x \in U \mid v(x) \succeq \alpha\}, \\ \text{Region II}(v) &= \{x \in U \mid \alpha \succ v(x) \succ \beta\}, \\ \text{Region III}(v) &= \{x \in U \mid \beta \succeq v(x)\}. \end{aligned} \tag{1}$$

To be consistent with the decreasing ordering of objects from left to right according to their evaluation status values, Yao and Yu [74] call the three regions the left, middle, and right regions, respectively, or simply, L region, M region, and R region. Under the condition $\alpha \succ \beta$, the three regions are pair-wise disjoint and their union is the universe U.

The evaluation-based model covers a large class of problems to which three-way decisions are applicable. In some situations, the relation $\succeq$ may be intuitively interpreted as the "better than or the same as" relation. That is, if $v(x) \succeq v(y)$, we say that "x is better than or the same as y." Thus, Region I, Region II, and Region III consist of the subsets of more desirable, neutral, and less desirable objects, respectively. A tri-partition seems to be more meaningful than a bipartition (i.e., a pair of disjoint subsets of U, representing desirable and undesirable subsets of objects, respectively). Classifying an object in the middle as either desirable or undesirable might be difficult. With the tri-partition, it seems natural to act differently on the three sets of objects. The case of management of students is an example under this interpretation.

The case of text analysis shows that there are situations in which the middle region is preferred. Gladwell [15] discusses two examples that an inverted U curve shows the advantages of the middle. In one example, the horizontal axis represents the wealth of a family in an increasing order, the vertical axis represents the easiness of parenting, and an inverted U curve shows their relationship. Before reaching a threshold, an increase in wealth makes parenting easier; after another threshold, a further increase in wealth makes parenting harder. In another example, the class size (i.e., number of students in a class) has a similar relationship with the academic performance of students as depicted by an inverted U curve. In these two examples, as illustrated by Fig. 2, we can use three-way classification to provide simple and elegant explanations of the advantages of wealth and class size. The families and classes in the middle region have advantages over the other two regions. There are many more examples of the preferred middle. When interpreting many medical tests, middle-ranged values indicate normality and low- and high-ranged values indicate anomalies.

Research Issues of Three-Way Decisions

The previous examples show that three-way decisions enable us to focus properly the efforts on some of the three regions and, sometimes, to turn disadvantages into advantages. With respect to the two components of the trisecting-and-acting framework, we recall from an earlier paper [72] two lists of future research topics.

For trisecting, we have the following topics:

- Methods for trisecting a universe: In addition to evaluation-based methods, we may investigate other approaches. One possibility is to construct a trisection through a pair of bisections. A second possibility is to consider a ranking of objects, which is a fundamental notion in theories of decision making. The three regions correspond to the top, middle, and bottom segments of the ranking. A third possibility is to seek for a statistical interpretation of a trisection [73].
- Methods for constructing evaluation functions: Probabilistic three-way classifications use probability as an evaluation function [67]. In general, we may consider other types of evaluation functions, including, for example, possibility functions, fuzzy membership functions, Bayesian confirmation measures, similarity measures, and subsethood measures. It is necessary to interpret an evaluation function in operable notions. Another possibility is the fusion of a set of evaluation functions, as in the cases of multi-agent systems and group decision making.
- Methods for determining the pair of thresholds: For an evaluation-based model, we need to investigate ways to compute and to interpret a pair of thresholds. An optimization framework can be designed to achieve such a goal. That is, a pair of thresholds should induce a trisection that optimizes a given objective function. By designing different objective functions for different applications, we gain a great flexibility. For example, we may use information-theoretical measures, cost-based measures, complexity-based measures, and granularity-based measures in designing and interpreting an objective function.

For acting, we have the following topics:

- Descriptive rules for three regions: To fully understand three regions and to design effective strategies and actions for processing them, as a prerequisite, we must be able to describe and to represent the three regions. Descriptive rules summarize the main features of the three regions, with each rule characterizing a portion of a specific region.
- Predictive rules for three regions: We can also construct predictive rules from the three regions to make decisions for new instances. While descriptive rules may contain redundant conditions in order to obtain a more complete description, predictive rules should be simple and more general without redundant conditions.
- Actionable rules for transferring objects between regions: In some situations, it is desirable to transfer objects from one region to another, as demonstrated by the two examples of text analysis and management of students. We can learn and use actionable rules in order to make such a move possible.
- Effective strategies and actions for processing three regions: For different regions, we must design the most suitable and effective strategies for processing. It may be sufficient to focus on one of the three regions. It may also happen that we must consider two or three regions simultaneously.
- Comparative studies: We can perform comparative studies by considering a pair of regions. In so doing, we can identify the differences between, and similarities of, the two regions. Descriptive rules may play a role in such comparative studies. The results of comparative study may also lead to actionable rules for transferring objects among regions.

There are mutual dependencies between trisecting and acting. A meaningful trisection depends on the strategies and actions for processing three regions; effective strategies and actions depend on an appropriate trisection. We need to search for a most optimal and matching combination of trisecting and acting. Interactions of strategies for different regions may play an important role in three-way decisions. In designing strategies for processing one region, we may have to consider the strategies for processing other regions.

In some situations, we first take consideration only of the fact that a universal set is to be divided into three regions; we develop strategies without detailed information about the three regions. In fact, the actual division may only be possible after we have a clear formulation of strategies for processing the three regions. That is, we form a most suitable trisection of the universe based on a given set of strategies.

Cognitive Basis and Advantages of Three-Way Decisions

By weaving ideas from theories of computation and evolution, Pinker [47] provides a beautiful explanation of how the mind works. We may state two important implications from Pinker's model. The computational theory of the mind may unlock the mechanisms used to achieve many brain functions. From an evolutionary point of view, human ways of problem solving are optimal or close to optimal. If we want to accept three-way decisions as a class of human ways of problem solving, we must answer questions regarding its mechanisms, cognitive basis, and advantages. This section examines cognitive aspects of three-way decisions, and the next section examines the mechanism aspects.

The Magical Number Three

Our ability to abstract is one of the keys to effective problem solving. Through abstraction, we transform a complex problem into a simpler one by considering only the essentials and omitting less relevant details. Organization and categorization, as a means of abstraction, are essential to mental life [47]. By grouping a set of objects into a category, we can talk about the group at a more abstract level as a whole without the influence or distraction of minute differences among individual objects. In three-way decisions, we use three parts or groups as a basis of problem solving. The cognitive basis for using three groups can be explained in terms of the magical number three in many cognition related activities.

A profound finding from a study by Miller [40] is that we have a limited capacity for processing information due to limitations of short-term working memory. To accommodate such limitations and to decrease cognitive overload, a technique of chunking (i.e., clustering and categorization) is used to transform information into a manageable number of units or chunks. Although there is a general agreement on the limited capacity, there does not exist an agreement on the exact number of chunks. Miller's study shows that we can normally process about five to nine chunks. Later studies suggest that the actual number of chunks is smaller and is from three to five [9].

In an empirical study that asks 26 individuals to identify possible causes for several phenomena, Kováč [24] reports that "the average number of causes for any of the events or phenomena given by any of the respondents that I asked was only three, within the range of 2.00 ± 0.98 and 3.15 ± 1.41." Kováč offers a first possible explanation based on the limited capacity. Davidson [10] offers two more explanations, bringing the number of possible explanations to three. A second possible explanation is that "the propensity to think in threes corresponds to that well-established aesthetic principle, the 'rule of three', which suggests that groups of three are simply more innately pleasing." A third possible explanation is that the respondents are too busy and do not spend much time and efforts on speculating on causation; they just quickly provide their responses. Thus, "to reply with just one or two possibilities might have suggested a paucity of imagination, while three at least gives the impression that a reasonable effort has been applied" [10]. According to Davidson, three is "a number that is at once more aesthetically pleasing, more intellectually satisfying and probably just as simplistic." For more discussions on the magical number three from different perspectives, one may read some earlier papers [13, 26, 56].

From the perspective of mathematical logic, Warfield [60] argues that the number of parts or chunks is exactly three. In addition to the number of parts, Warfield considers the number of interactions among different parts. With a trisection of a universal set, we can consider the numbers of a) single regions, b) combinations of two regions, and c) the combination of all three regions. The total number is $3 + 3 + 1 = 7$, which is exactly seven. If we use four parts, we would have a total of $4 + 6 + 4 + 1 = 15$, which is far more beyond the capacity as suggested by seven.

On the one hand, the number three characterizes the limited capacity of human information processing; on the other hand, it also invokes an esthetic pleasure in our search for simplicity and beauty. Both aspects are related to human perception and cognition. It is therefore not surprising to find that the uses of three parts or groups, as an appropriate number of units or chunks in human problem solving, have appeared across different disciplines. In addition to examples discussed earlier, we give several new examples. In studies of early childhood development, the National Scientific Council on the Developing Child distinguishes three kinds of response to stress: "positive, tolerable and toxic" [57]. According to such a three-way classification, one can design different strategies for dealing with different types of response to stress. In marketing, Smith [54] points out that versioning strategies often follow a qualitatively three-part "good-better-best" progression, with increasing number of features and increasing pricing. Shu and Carlson [53] suggest that the optimal number of positive claims is three in order to produce the most positive impression of a product or a service. In a much wider context, Clayton [7] points out that lists of three things (e.g., three words, three phrases, or three sentences) are powerful speech patterns commonly used by great speakers. As an interesting note, he remarks, “The emotional power of lists of three is so great that even when lists of four are used, we typically remember only three.” Within a cognitive perspective, McCutchen [39] investigates the ways that limitations of working memory, as characterized by the magical number three, plus or minus two, affect the writing process. By examining decision-making processes in law, Clermont [8] finds that the number three forms psychological bases for standards of decisions. As an example of a three-way classification, “a continuum of standards divides in practice into the three distinct standards of preponderance of the evidence, clear and convincing evidence, and proof beyond a reasonable doubt” [8]. In composing visual images, the rule of thirds suggests to divide a medium into thirds both horizontally and vertically. The resulting invisible nine rectangles and four intersections allow effective positioning of the primary element in a design [33].

Cognitive Advantages

By using a tri-partition with three regions, three-way decisions are built on a firm cognitive basis as demonstrated by the magical number three. At the same time, three-way decisions have the desirable advantages with respect to cognitive load, cognitive simplicity, cognitive fluency, and cognitive flexibility.

Two-Way Decisions and Three-Way Decisions Concepts are the basic units of thought that underlie human intelligence and communication. Following the ideas from Port-Royal Logic of Arnauld and Nicole [1], one may understand a concept jointly as a pair of an intension (or comprehension) and an extension (or denotation). The intension of a concept consists of all properties or attributes (more generally, some formulas of a language) that are valid for all those objects to which the concept applies. The extension of a concept is the set of objects that are instances of the concept. In many situations, precisely determining the extension of a concept, that is, divide a universal set into two disjoint regions, may not be cognitively easy. Nevertheless, it may be relatively easy to determine typical instances and typical non-instances of a concept. For other objects, it might be difficult to determine their belongingness to a concept without extra cognitive efforts. Consider an example of medical diagnosis. For patients who obviously suffer or obviously do not suffer from a particular disease, a medical doctor can easily make a decision. For patients whose situations are not very clear, the doctor needs considerably more efforts in diagnosing. By introducing a third region, three-way decisions extend two-way decisions. Three-way decisions reduce cognitive load and lead to cognitive simplicity in the sense that a doctor can make quick and right decisions for some patients and focus more efforts on some other patients.

Alternatively, three-way decisions can be interpreted in terms of two-way decisions. Our limited ability to make a “yes” decision for some objects and a “no” decision for some other objects leads, in fact, to two models of two-way decisions. In general, the two models are constructed with respect to two evaluation functions [69]. Let $A \subseteq U$ denote a subset of objects with a “yes” decision. The set complement $A^c = U - A$ is the set of objects without a “yes” decision. The pair (A, A^c) is a two-way classification of U. Similarly, we can form another two-way classification $(\overline{A}, \overline{A}^c)$, where $\overline{A}$ and $\overline{A}^c$ are the subsets of objects with and without a “no” decision, respectively. A three-way classification of U may be constructed based on the two two-way classifications as $(A \cap \overline{A}^c, (A \cap \overline{A}) \cup (A^c \cap \overline{A}^c), A^c \cap \overline{A})$. The first and the third regions consist of objects with a consistent decision with respect to the two models of two-way decisions. The second region is the union of two subsets: $A \cap \overline{A}$ consists of objects with contradictory decisions (i.e., both a “yes” and a “no” decisions) and $A^c \cap \overline{A}^c$ consists of objects with neither a “yes” nor a “no” decision.

Three-Way Decisions Based on Two Polarities and the Middle A tri-partition used in three-way decisions provides a natural way to interpret, understand, and represent many problems in terms of two polarities and the middle. Two of the three regions represent the two opposite polarities, and the third one represents the middle. In this way, we have a beautiful symmetric description: The two polarities are symmetric with reference to the middle. There are several reasons that call for three-way decisions based on two polarities and the middle.

As stated earlier, classifying objects based on only two polarities may be inappropriate under some circumstances. When facing uncertain or incomplete information, we may not have strong evidence to support classifying an object into either of the two opposite regions. It might be more meaningful to classify the objects into the third middle region. Following the explanation for the case of medical three-way decision making, objects in the middle region require further investigations. This usually involves collecting new evidence or carrying out further testing.

The two polarities and the middle may represent a simple discrete and qualitative approximation of a continuous whole. We commonly divide time, space, volume, and judgements based on two polarities and the middle, as exemplified by past–present–future, left–middle–right, small–medium–large, and for–neutral–against, respectively. Another compelling example is the uses of three degrees or forms of comparison of adjectives and adverbs in English, namely positive, comparative, and superlative degrees. Intuitively, these three forms of adjectives and adverbs provide the necessary vocabularies to describe and to explain the three regions in three-way decisions. For example, versioning strategies in pricing interpret and explain the three classes as the good, better, and best versions of a product or a service [54].

The introduction of the middle to the two polarities has cognitive advantages. By looking at a problem through two polarities and the middle, we have a high-level, qualitative description. The description immediately offers two effective procedures for trisecting. When determining the three regions, we can either move from the two opposite poles to the middle or move from the middle to the two opposite poles. Decision making for objects close to the two poles is relatively easy. Moving toward the two poles leads to low cognitive efforts and high cognitive fluency. Consider an example of assigning fuzzy membership values, in which 0 and 1 represent the two opposite poles and 0.5 represents the middle. As pointed out by Pedrycz [45], studies have shown that people are usually far more confident when assigning values close 1 or 0 to objects and are more hesitate when assigning values around 0.5.

Two-Way Decisions and m-Way Decisions With the aid of the evaluation-based model, we can explain m-way decisions, where m is an odd number greater than three, through a sequence of three-way decisions. Without loss of generality, we assume $\mathbf{1}$ and $\mathbf{0}$ are the maximum and the minimum values of $\mathbf{L}$, respectively. Given a pair of thresholds (α_1, β_1) with $\mathbf{1} \succeq \alpha_1 \succ \beta_1 \succeq \mathbf{0}$, according to Eq. (1), we divide U in the following three regions:

$$\begin{aligned} \text{Region I}(v) &= \{x \in U \mid v(x) \succeq \alpha_1\}, \\ \text{Region II}(v) &= \{x \in U \mid \alpha_1 \succ v(x) \succ \beta_1\}, \\ \text{Region III}(v) &= \{x \in U \mid \beta_1 \succeq v(x)\}. \end{aligned} \tag{2}$$

With respect to the interpretation based on two polarities and the middle, we are more certain and confident about the classification of objects in the Region I and Region II. For objects in the middle Region II, we can further divide them into three regions according to another pair of thresholds (α_2, β_2) with $\mathbf{1} \succeq \alpha_1 \succ \alpha_2 \succ \beta_2 \succ \beta_1 \succeq \mathbf{0}$ as follows:

$$\begin{aligned} \text{Region II.I}(v) &= \{x \in \text{Region II}(v) \mid v(x) \succeq \alpha_2\}, \\ \text{Region II.II}(v) &= \{x \in \text{Region II}(v) \mid \alpha_2 \succ v(x) \succ \beta_2\}, \\ \text{Region II.III}(v) &= \{x \in \text{Region II}(v) \mid \beta_2 \succeq v(x)\}. \end{aligned} \tag{3}$$

By combining the results of the two three-way decisions, we obtained a five-way decision characterized by the quintuple (Region I, Region II.I, Region II.II, Region II.III, Region III). We can further divide the region Region II.II to derive seven-way decisions and, continually, other m-way decisions.

The construction of m-way decisions based on three-way decisions provides us a cognitive flexibility in problem solving. Once we find that three-way decisions do not give us enough number of regions, we can further divide the middle regions to obtain more regions. Instead of constructing m regions directly, the sequential construction based on three-way decisions is cognitively simple and less demanding. One only needs to consider three regions in each step.

Modes of Three-Way Decisions

The effectiveness of three-way decisions relies on developing appropriate strategies to act on objects in each of the three regions. One possible way to evaluate strategies is based on the relative values or degrees of importance we associate with the three regions. More specifically, we consider three possible structures of a tri-partition, from unordered three regions without any preferences, non-linearly ordered three regions, to a linearly ordered three regions. They give rise to three modes of three-way decisions with respect to the task of acting in the trisecting-and-acting model.

Unordered Three Regions Without Any Preferences

In some situations, we do not have a clear preference among different regions. This may happen for several reasons. The three regions may have the same value and, consequently, they all have the same rank, which is equivalent to unordered three regions. It may also be the case that a comparison of any two regions is a meaningless operation. In other words, a trisection is only a classification of the universal set. We treat all three regions the same without comparing them.

For unordered three regions, strategies for different regions may be constructed independent of each other. We aim at an overall optimization by making full use of each region. An example of using unordered three regions is a method of multilevel analysis. A central idea of multilevel analysis is to investigate the same problem at several levels, with each level captures a particular aspect of the problem. Although levels are ordered with respect to some criteria, they are typically unordered with respect to their degrees of importance or values. It may be difficult to say which levels are more important than other levels. A common practice in multilevel analysis is the use of three levels. Marr [38] suggests that one can explain an information-processing system at three levels, namely a computational theory level, a representation and algorithm level, and a hardware implementation level. For an understanding of international politics, Rourke [49] combines an individual-level analysis, a state-level analysis, and a system-level analysis. In studies of complex systems and complex networks, one may adopt a tri-level consisting of microscopic, mesoscopic, and macroscopic levels of description and explanation [25, 62]. In economics, there is also tri-level analysis characterized by microeconomic, mesoeconomic, and macroeconomic levels [63]. The power of multilevel analysis comes from the integration of results from individual analyses.

Three-way decisions with unordered three regions can draw ideas from multilevel analysis. We focus on both an independent analysis of each region and a seamless integration of results from analyzing the three regions.

Non-Linearly Ordered Three Regions

A nonlinear order requires that at least one pair of regions is not comparable. Let $\succ_p$ denote a preference relation used to rank three regions. The preferences can be determined by pair-wise comparison of regions according to some criteria. Categorically speaking, we consider three possible forms of non-linearly ordered three regions generated by preferences:

(a) Only two out of the three regions are comparable, e.g., Region I $\succ_p$ Region II,
(b) One region is preferred to other two regions, e.g., Region II $\succ_p$ Region I, Region II $\succ_p$ Region III,
(c) Two regions are preferred to the third region, e.g., Region I $\succ_p$ Region II, Region III $\succ_p$ Region II.

With respect to the three forms, we examine different strategies for action.

Case (a) In this case, we have only two regions worthy consideration and one of them is preferred or has a higher priority than the other; there is no need to consider the third region. In other words, it is neither necessary nor meaningful to compare the two regions with the third region. Strategies are needed to act on objects in the two regions according to their priorities. Consider again the example of medical diagnosis. With three-way decisions, a doctor divides a set of patients into three groups, namely a group of patients who obviously have a disease, the group of patients who may have the disease, and a group of patients who obviously do not have the disease. With respect to the needs for medical treatment, the first group has a higher priority than the second group and it is not necessary to consider the third group. Accordingly, a doctor may form strategies of immediately treating patients in the first group, requesting further tests for patients in the second group, and doing nothing for the third group.

The structure of Case (a) suggests strategies that focus on two regions according to their priorities. The elimination of the third region prevents unnecessary efforts on processing some objects, which may lead to efficiency in problem solving.

Case (b) The structure in Case (b) is perhaps one of the most used structures for three-way decisions. We can intuitively explain this structure based on two polarities and the middle. Without loss of generality, we may assume that Region I and Region III correspond to the two poles, representing two extremes, and Region II corresponds to the middle. The structure of Case (b) may be understood as the preference of the middle, the principle of the middle way, or the principle of extreme aversion. In other words, people tend to avoid extremes and are more likely to choose an intermediate option. For examples of the preference of the middle, we recall the earlier mentioned examples of text analysis, family wealthy and parenting, and class size and student performance. In these examples, an inverted U curve vividly and pictorially describes the preference of the middle. The strategies of versioning and pricing capitalize on the customers' preference of the middle by making most money off the middle versions of a product or a service in a three-way, namely good–better–best, classification of versions [54].

Many more examples of the preference of the middle can also be found in everyday life. For example, Marinoff [37] discusses the value and the power of the middle way as a mode of moderate thought and behavior by drawing inspiration from three of the world's greatest thinkers: Aristotle, Buddha, and Confucius. Hartshorne [16] examines a philosophy of the middle way, namely wisdom as moderation.

Three-way decisions with the structure of Case (b) suggest strategies that focus more efforts on the middle region. Actions on objects in the middle region will bring out the most value from a tri-partition. In many situations, it is advantageous to take the middle way and to avoid the extremes.

Case (c) The structure of Case (c) is the reverse of Case (b). In this case, two regions are preferred to the third region. For an example of this structure, consider again a medical test with a numerical result. Either a too high value or a too low value may indicate anomalies that require attention, further investigation, or treatment; a value in the middle range indicates normality that does not require further investigation. Three-way decisions with the structure of Case (c) lead to strategies that focus on two regions. By treating the two regions as abnormal instances, three-way decisions can be applied to anomaly detection and prevention.

Another interpretation of the structure of Case (c) is that it represents the degree of easiness to determine and process elements in the two regions corresponding to the two opposite poles. Under this interpretation, we can form strategies for fast processing of two regions and slow processing of the third region. These strategies are in line with the proposal that there are two systems used by humans for fast and slow thinking, respectively [21].

Linearly Ordered Three Regions

With linearly ordered three regions, we can rank the three regions one after another. Without loss of generality, we consider the following linear order:

$$\text{Region I} \succ_p \text{Region II} \succ_p \text{Region III}. \tag{4}$$

It indicates that Region I is the most preferred region, Region III is the least preferred region, and Region II is in the middle. In an evaluation-based model, a linear order may be obtained directly from the ordering according to the ESVs of objects.

Examples of linearly ordered three regions are three-way classifications of students according to their GPA and three-way classifications of customers based on their opinions being positive, neutral, or negative. Linearly ordered three regions require strategies that prevent objects from moving to less preferred regions and make objects move to more preferred regions. In the case of three-way classification of students, effective strategies encourage students to move from low-GPA regions to high-GPA regions. In the case of customer relation management, strategies are designed to retain customers with positive opinions, to entice customers with neutral opinions to move to the positive region with some incentives, and to control potential damage made by customers with negative opinions.

Linearly ordered three regions appear in many real world applications. Three-way decisions with a linear order search for strategies that help objects to move from less preferred regions to more preferred regions. The simplicity of linearly ordered three regions is useful in realizing the value of three-way decisions.

Conclusion

In this paper, we examined the interplay of three-way decisions and cognitive computing. Our basic assumption is that three-way decisions are a special class of human ways to problem solving and information processing. As such, it is imperative to investigate the cognitive basis and advantages of three-way decisions. The arguments presented in the paper represent results from our preliminary investigation, supplemented by an extensive list of references from different disciplines. As a multidisciplinary study, future research on three-way decisions must draw inspirations and insights into many different fields.

We identified trisecting and acting as the two basic tasks of three-way decisions, which gives rise to a trisecting-and-acting model of three-way decisions. Within this model, we presented an evaluation-based approach to trisecting and a preference-based approach to acting. They may be considered as examples of possible solutions. It is necessary to search for other methods of trisecting and acting.

The discussions of this paper mainly focused on the advantages of three-way decisions. As an ending note to the conclusion, we need to point out blind spots of three-way decisions. Recall that three-way decisions are based on human heuristics. They may suffer from human heuristic biases and cognitive errors [2, 21]. There is no guarantee that three-way decisions will always work. It is important to investigate the conditions under which three-way decisions will work .

Acknowledgments This work was supported in part by a Discovery Grant from NSERC, Canada. The author thanks Professor Amir Hussain for his encouragement during the preparation of this paper. The author thanks Ling Wei, Jingyu Chen, Shunling Wang, Jianjun Qi, Mengjun Hu and Ruisi Ren for their constructive comments.

Funding This study was funded by NSERC, Canada.

Compliance with Ethical Standards

Conflicts of interest Yiyu Yao declares no conflict of interest.

Informed Consent All procedures followed were in accordance with the ethical standards of the responsible committee on human experimentation (institutional and national) and with the Helsinki Declaration of 1975, as revised in 2008 (5). Additional informed consent was obtained from all patients for which identifying information is included in this article.

Human and Animal Rights This article does not contain any studies with human participants or animals performed by any of the authors.

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