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SHORT COMMUNICATION

COMPARISON OF THE PROBABILISTIC APPROXIMATE CLASSIFICATION AND THE FUZZY SET MODEL

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It is shown that the generalized notion (probabilistic approximate classification) of rough sets can be conveniently described by the concept of fuzzy sets. A discussion of the proper choice of the definition for the membership function of the intersection (union) of fuzzy sets is also presented. However, from the point of view of the probabilistic approximation space, it is argued that there does not exist a universal definition for the fuzzy intersection (union) operation.

Keywords: Rought set, Fuzzy set, Approximate classification, Probabilistic approximation space, Fuzzy set operations.

1. Introduction

The theory of rough sets was proposed by Pawlak [5, 7] based on the concept of approximate (rough) classification. Recently, we introduced the notion of probabilistic approximate classification [10] which provides a framework to generalize Pawlak's original idea. We showed in particular that the basic principles of inductive learning can be precisely formulated and better understood with this new approach. A question which naturally arises is whether the concept of probabilistic approximate classification is related to that in the fuzzy set theory introduced by Zadeh [12]. One of the main objectives in this communication is to clarify this important issue.

Since the introduction of fuzzy set theory [12], there has been ample evidence to support the need to model some real world problems using the concept of fuzzy sets. However, the issue of searching for self-consistent definitions [1, 3, 4, 8, 9, 11, 14] to compute the membership function of the intersection (union) of fuzzy sets remains somewhat controversial. One of the main reasons for this confusion is, perhaps, due to the failure of recognizing that these set operators should be defined 'external' to the theory of fuzzy sets itself. We demonstrate in this paper that the 'correct' choice of fuzzy set operators depends solely on the nature of the physical problem one attempts to model. It seems unreasonable to expect that there exists a universal way of defining the fuzzy set operators. Any attempt to justify the choice of these operators without taking into account the physical meaning of the membership grades may inevitably lead to inconsistent results.

In subsequent discussions, we will first provide some background information on the theory of rough sets as originally suggested by Pawlak [5]. We then show how the concept of rough sets can be generalized from the vantage point of the probability theory. Finally, we will demonstrate that, if we choose some appropriate definitions for the union and intersection of fuzzy subsets (other than the usual Max and Min operators), the approach of probabilistic approximate classification to deal with 'imprecise concepts' is indeed closely related to the fuzzy set model. However, it must be emphasized that the manner in which one interprets the membership grades becomes crucial in the understanding of the relationship between these two apparently different approaches.

2. Fuzzy sets

Zadeh [12] introduced the concept of fuzzy sets as an extension of the ordinary set theory.

It is well known that given a universe E, the union and intersection of fuzzy sets X, Y may be defined as follows:

$$\begin{aligned} k_{X\cup Y}(e) &= \mathrm{Max}(k_X(e), k_Y(e)), \\ k_{X\cap Y}(e) &= \mathrm{Min}(k_X(e), k_Y(e)), \end{aligned} \tag{1}$$

for every $e \in E$, where $k(e)$ is the membership function. The complement $E - Y$ of a fuzzy set Y is defined by the membership function

$$k_{E-Y}(e) = 1 - k_Y(e), \quad e \in Y. \tag{2}$$

Bellman and Zadeh [2] suggested the following alternative definitions for the fuzzy intersection and union:

$$\begin{aligned} k_{X\cap Y}(e) &= k_X(e) \cdot k_Y(e), \\ k_{X\cup Y}(e) &= k_X(e) + k_Y(e) - k_X(e) \cdot k_Y(e), \end{aligned} \tag{3}$$

for every $e \in E$. Both of the definitions (1) and (3) reduce to the ordinary set intersection and union when the membership grade is either 0 or 1.

3. Approximate classification of a set

Let E denote a set of objects and let V be a set of 'properties'. We assume that each object $e \in E$ can be uniquely described by a subset of properties Des(e) of V, where Des(e) is referred to as the description of e. Thus, the description function $\mathrm{Des}: \mathrm{E} \rightarrow 2^V$ induces an equivalence relation θ_D on E such that for $e_1, e_2 \in E$,

$$(e_1, e_2) \in \theta_D \quad \text{iff} \quad \mathrm{Des}(e_1) = \mathrm{Des}(e_2).$$

The pair $A = \langle E, \theta_D \rangle$ is called an 'approximation space'. Let $\mathrm{Des}(D_i)$ denote the description of an equivalence class (elementary set) D_i of the relation θ_D. It follows that objects in any elementary set D_1 of θ_D are indistinguishable from one another with respect to the description $\mathrm{Des}(D_i)$. That is, any two objects, $e_1, e_2 \in E$, have the same description, $\mathrm{Des}(e_1) = \mathrm{Des}(e_2) = \mathrm{Des}(D_i)$, if and only if they both belong to the same elementary set D_i of θ_D (i.e., $(e_1, e_2) \in \theta_D$).

Let $Y \subseteq E$ be an arbitrary subset of objects. We want to classify the set Y based on the set of descriptions $\{\mathrm{Des}(D_i)\}$ of the elementary sets of θ_D. In order to measure how well this set of descriptions can actually determine the membership of objects in Y, Pawlak [6] introduced the following notions of approximate classification:

(i) The A-lower approximation of Y in $A = \langle E, \theta_D \rangle$ is defined as

$$\underline{A}(Y) = \{e \in E \mid e \in D_i \text{ and } D_i \subseteq Y\}.$$

$\underline{A}(Y)$ is the union of all those elementary sets each of which is contained by Y.

(ii) The A-upper approximation of Y in $A = \langle E, \theta_D \rangle$ is defined as

$$\bar{A}(Y) = \{e \in E \mid e \in D_i \text{ and } D_i \cap Y \neq \emptyset\}.$$

$\bar{A}(Y)$ is the union of all those elementary sets each of which has a non-empty intersection with Y.

(iii) The set $\bar{A}(Y) - \underline{A}(Y)$ is called the A-doubtful region of Y in $A = \langle E, \theta_D \rangle$.

(iv) The set $E - \bar{A}(Y)$ is referred to as the negative region of Y in A.

Note that a set $Y \subseteq E$ may or may not have a clearly defined boundary based on the descriptions of the elementary sets, and this leads to the concept of 'rough sets'. Any set characterized by its lower and upper approximations in an approximation space is called a rough set. We say that sets Y and Y' are 'roughly equal' (denoted by $Y \approx Y'$) in the approximation space $A = \langle E, \theta_D \rangle$ if $\underline{A}(Y) = \underline{A}(Y')$ and $\bar{A}(Y) = \bar{A}(Y')$.

Recall that objects in an approximation space $A = \langle E, \theta_D \rangle$ are indistinguishable if they belong to the same elementary set. It is clear that we are unable to say with certainty whether or not any object in the doubtful region, $\bar{A}(Y) - \underline{A}(Y)$, is a member of the set Y based solely on the descriptions of the elementary sets of θ_D. Therefore, the membership grade for any object that exists in the doubtful region is *not* defined within the framework of the theory of rough sets. For example, the choice for $k_Y(e) = \frac{1}{2}$, $e \in \bar{A}(Y) - \underline{A}(Y)$, is therefore totally arbitrary [7].

The second problem associated with rough sets is that the set operations for the 'rough' union and intersection are not *explicitly* defined as in the theory of fuzzy sets. It is also not clear from the algebraic properties [7] of the rough sets how these set operations should be specified in a consistent manner.

4. Probabilistic approximate classification

Given a set of objects E, let P be a probability measure defined on the σ-algebra F of subsets of objects (elementary events) in the probability space

(E, F, P). The triple $A_p = \langle E, \theta_D, P \rangle$ is called the probabilistic (stochastic) approximation space [10], where θ_D is an equivalence relation induced by the description function $\mathrm{Des}: E \to 2^V$ as defined in Section 3.

Let $Y \subseteq E$ be a subset of objects representing an 'expert' concept. Our objective is to classify the set Y based on the set of descriptions $\{\mathrm{Des}(D_i)\}$ of the elementary sets of θ_D. In order to specify the membership function for the objects in Y, we first introduce the notions of *probabilistic* approximate classification as follows:

(i) The A_p-lower approximation of Y in $A_p = \langle E, \theta_D, P \rangle$ is defined as

$$\underline{A}_p(Y) = \{e \in E \mid e \in D_i \text{ and } P(Y \mid D_i) = 1\}.$$

$\underline{A}(Y)$ is in fact the union of all those elementary sets with conditional probabilities $P(Y \mid D_i) = 1$. We also refer to $\underline{A}_p(Y)$ as the positive region $\mathrm{POS}_A(Y)$ of the concept Y.

(ii) The A_p-upper approximation of Y in A_p is defined as

$$\bar{A}_p(Y) = \{e \in E \mid e \in D_i \text{ and } 0 < P(Y \mid D_i) \leq 1\}.$$

$\bar{A}_p(Y)$ is the union of all those elementary sets with conditional probabilities $0 < P(Y \mid D_i) \leq 1$.

(iii) The set $\mathrm{BND}_A(Y) = \bar{A}_p(Y) - \underline{A}_p(Y)$ is called the doubtful region or the boundary region of Y. That is, $\mathrm{BND}_A(Y)$ consists of all those elements in E with conditional probabilities $0 < P(Y \mid D_i) < 1$.

(iv) The set $\mathrm{NEG}_A(Y) = \{e \in E \mid e \in D_i \text{ and } P(Y \mid D_i) = 0\} = E - \bar{A}_p(Y)$ is the negative region of Y in A_p.

We can now define the membership function of a fuzzy subset Y of E in the probabilistic approximation space $A_p = \langle E, \theta_D, P \rangle$ as follows:

$$k_Y(e) = P(Y \mid D_i) \quad \text{iff} \quad e \in D_i \text{ (i.e. } \mathrm{Des}(e) = \mathrm{Des}(D_i)). \tag{4}$$

It is clear from the above definition that $k_Y(e)$ is the probability that an object $e \in E$ is a member of the subset Y if $\mathrm{Des}(e)$ matches the description $\mathrm{Des}(D_i)$ of the elementary set of D_i.

Given any two fuzzy subsets X, Y in $A_p = \langle E, \theta_D, P \rangle$, the intersection $X \cap Y$ is also a fuzzy subset, and its membership function is, therefore, defined by

$$k_{X \cap Y}(e) = P(X \cap Y \mid D_i), \quad e \in D_i. \tag{5}$$

Similarly, the membership function of the union, $X \cup Y$ can be written as

$$\begin{aligned} k_{X \cup Y}(e) &= P(X \cup Y \mid D_i) \\ &= P(X \mid D_i) + P(Y \mid D_i) - P(X \cap Y \mid D_i), \quad e \in D_i. \end{aligned} \tag{6}$$

From the above discussions, we have clearly shown that the notion of probabilistic approximate classification of a subset of objects $Y \subseteq E$ in a given probabilistic approximation space $A_p = \langle E, \theta_D, P \rangle$ is, in fact, equivalent to defining a fuzzy subset Y characterized by the membership function $k_Y(e) = P(Y \mid D_i)$, $e \in D_i$.

Furthermore, if we assume that subsets X, Y of E are probabilistically independent in every D_i (obviously, not necessarily true in general), then our definitions for the fuzzy intersection and union immediately reduce to the special case suggested by Bellman and Zadeh [2], namely, for every $e \in D_i$,

$$k_{X\cap Y}(e) = P(X \cap Y \mid D_i) = P(X \mid D_i) \cdot P(Y \mid D_i) = k_X(e) \cdot k_Y(e) \quad (7)$$

and

$$k_{X\cup Y}(e) = P(X \cup Y \mid D_i) = P(X \mid D_i) + P(Y \mid D_i) - P(X \cap Y \mid D_i) = k_X(e) + k_Y(e) - k_X(e) \cdot k_Y(e). \quad (8)$$

Thus, our analysis on fuzzy sets based on the notion of probabilistic approximate classification clearly demonstrates that applying the *product* operator to compute the membership function of intersecting fuzzy sets is valid only if the assumption of probabilistic independence holds in the approximation space under consideration.

5. Conclusion

Since the introduction of the theory of fuzzy sets by Zadeh, there have been many disagreements among researchers on the interpretation of the membership function. Zadeh himself strongly disagrees with those who argue that the fuzzy set concept is merely a disguised form of subjective probability. It was emphasized by Zadeh [13] that the fuzzy set theory is, in essence, aimed at the development of a body of concepts and techniques for dealing with sources of uncertainty or imprecision, which are non-statistical in nature. There have also been considerable discussions of the choice of appropriate definitions for the fuzzy intersection and union operations.

In this article, we have discussed these issues from a different point of view which is based on the notion of probabilistic approximate classification. The results of our analysis clearly indicate that when using fuzzy set concepts to model real world problems, a purely mathematical justification of the fuzzy set operations is not sufficient [9, 11]. In fact, there does not exist a 'universal' definition for the membership function of the intersection (union) of fuzzy sets. For example, in our model the membership function is defined by the conditional probabilities in a probability space. Obviously, in this case neither the minimum operator nor the product operator is suitable for defining the membership function of intersecting fuzzy sets. This is not to say that fuzzy set concepts are not applicable to deal with problems which are statistical in nature. However, in order to correctly define the membership function for fuzzy set operations, it is absolutely essential to understand the physical meaning of the membership grades in a particular application.

Pawlak [7] showed that the concept of a rough set is different from that of a fuzzy set. However, we have demonstrated that the generalized notion of rough sets based on the probabilistic approximation space can indeed be conveniently described by the concept of fuzzy sets if proper fuzzy set operations are employed as we have presented.

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