Rough Set Approximations in an Incomplete Information Table

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Abstract. We present a new method for constructing and interpreting rough set approximations in an incomplete information table in four steps. Step 1: we introduce the notion of conjunctively definable concepts in a complete table. Step 2: we suggest a slightly different version of Pawlak rough set approximations in a complete table by using the family of conjunctively definable concepts. Step 3: we adapt a possible-world semantics that interprets an incomplete table as a family of complete tables. Correspondingly to conjunctively definable concepts in a complete table, we introduce the notion of conjunctively definable interval concepts in an incomplete table. Step 4: we study rough set approximations in an incomplete table by using the family of conjunctively definable interval concepts. Our method focuses on a conceptual understanding of rough set approximations for the purpose of rule induction. It avoids difficulties with existing approaches with respect to semantical interpretations.

1 Introduction

Analyzing an incomplete information table for rule induction is an important topic in rough set theory. Following Pawlak's formulations of rough set approximations using equivalence relations in a complete table [18,19], the majority of commonly used approaches is to construct a similarity or tolerance relation on a set of objects and to define generalized rough set approximations by using similarity classes. A fundamental difficulty with this type of approaches is that a partially defined similarity relation does not truthfully and fully reflect the available partial knowledge given in an incomplete table. For this reason, many authors have proposed and studied different definitions of similarity relations [3,7,9–12,17,21,22]. However, those solutions are not entirely satisfactory. It is necessary to study the family of all possible similarity relations in an incomplete table [13].

Yao [24] argued that there are two sides of rough set theory. The conceptual formulation focuses on the meanings of various concepts and notions of rough set theory. The computational formulation focuses on methods for constructing

Y.Y. Yao—This work is partially supported by a Discovery Grant from NSERC, Canada.

L. Polkowski et al. (Eds.): IJCRS 2017, Part II, LNAI 10314, pp. 200–215, 2017.
DOI: 10.1007/978-3-319-60840-2_14

these concepts and notions. Pawlak's formulation based on an equivalence relation in a complete table is an example of computational formulations. A conceptual formulation of rough sets uses a description language and explains rough set approximations in terms of the definability of sets under the description language [3,4,15,24]. To obtain a semantically sound and superior interpretation of rough set approximations in an incomplete table, we adopt the notion of a possible-world semantics of an incomplete table, that is, we truthfully represent an incomplete table by using a family of all its possible complete tables. The possible-world semantics of an incomplete table was used in interpreting incomplete databases by Lipski [14]. Several authors, for example, Li and Yao [13], Sakai et al. [16,20] and Hu and Yao [8], have adopted this semantics to study rough sets. By representing an incomplete table as a family of complete tables, we have an advantage of simply using any existing approaches to analyze an incomplete table, without the need to introduce new approaches. With the possible-world semantics of an incomplete table, we have shown earlier that, correspondingly to definable sets, one has the notion of definable interval sets [8]. Continuing with our study, in this paper we investigate rough set approximations by using definable interval concepts.

There are two formulations of rough sets. One uses a pair of lower and upper approximations. The other uses three pair-wise disjoint positive, boundary and negative regions. The latter has led to the introduction of three-way decisions with rough sets [25,26]. The positive and negative regions can be used to learn acceptance and rejection (i.e., rule-out) rules, respectively. However, we cannot learn such acceptance or rejection rules from the boundary region. Therefore, it is sufficient and meaningful to investigate, in this paper, only the positive and negative regions.

For simplicity, we only consider conjunctive rules in which the left-hand-side of a rule contains only logic conjunctions. We only study conjunctively definable concepts in a complete table and conjunctively definable interval concepts in an incomplete table. This enables us to arrive at the main results of this paper: rough set approximations are families of conjunctively definable concepts in a complete table and are families of conjunctively definable interval concepts in an incomplete table.

2 Conjunctively Definable Concepts and Rule Induction

An important task in rough set theory is to construct decision rules to classify objects. The left-hand-side of a conjunctive decision rule is a conjunction of conditions. For such a purpose, we introduced the notion of structured rough set approximations in a complete information table by using conjunctively definable concepts [28]. Compared with Pawlak rough set approximations [18,19], the structured approximations not only give the same definable part of a given set but also reveal their internal structure in terms of conjunctively definable concepts. This facilities the learning process of decision rules.

2.1 Conjunctively Definable Concepts in a Complete Information Table

An information table provides the context for concept analysis with rough sets [23]. According to whether the information is complete or not, there are two types of information tables, namely, complete and incomplete information tables. Formally, a complete information table T is represented by a tuple:

$$T = (OB, AT, \{V_a \mid a \in AT\}, \{I_a : OB \rightarrow V_a \mid a \in AT\}), \tag{1}$$

where OB is a finite nonempty set of objects as rows, AT is a finite nonempty set of attributes as columns, V_a is the domain of an attribute $a \in AT$ and I_a is an information function mapping each object to a unique value in the domain of the attribute a.

A description language is commonly used to describe the objects in an information table. In this paper, we consider a description language DL_0 that contains only logic conjunctions and is a sublanguage of the commonly used one in rough set analysis [3,4,14,15,18,24]:

(1) Atomic formulas : $\forall\, a \in AT, v \in V_a, (a = v) \in \mathrm{DL}_0$;

(2) If $p, q \in \mathrm{DL}_0$, and p and q do not share any attribute, then $p \wedge q \in \mathrm{DL}_0$.

By demanding that p and q do not share any attribute, we actually consider a subset of conjunctive formulas in which each attribute appears at most once.

Given a formula in DL_0, an object satisfies the formula if it takes values on the corresponding attributes as specified by the formula. Formally, for an object $x \in OB$, an attribute $a \in AT$, a value $v \in V_a$ and two formulas $p, q \in \mathrm{DL}_0$, the satisfiability $\models$ is defined as:

$$\begin{aligned} &(1)\;\; x \models (a = v) \text{ iff } I_a(x) = v; \\ &(2)\;\; x \models p \wedge q \text{ iff } x \models p \text{ and } x \models q. \end{aligned} \tag{2}$$

The set of objects satisfying a formula describes the semantics or meaning of the formula.

Definition 1. *Given a formula $p \in \mathrm{DL}_0$, the following set of objects:*

$$m(p) = \{x \in OB \mid x \models p\}, \tag{3}$$

is called the meaning set of p.

Finding the meaning set of a formula is an easy task. However, given an arbitrary set of objects, there might not be a formula in the description language whose meaning set contains exactly these given objects. In other words, such a set cannot be described or defined with respect to the description language. In this sense, we may divide all sets of objects into two categories by their definability, that is, definable and undefinable sets. According to the school of Port-Royal Logic [1,2], a concept is represented by a pair of its intension and extension, where the intension describes the properties of the concept and the extension is the set of instances of the concept. Based on the ideas from formal concept analysis, Yao [23] represented a conjunctively definable concept as a pair of a conjunctive formula and the corresponding conjunctively definable set, which makes the meaning of the set explicit for the purpose of rule induction. D'eer et al. [3,4] adopted the conjunctively definable concepts and presented a semantically sound approach to Pawlak rough set and covering-based rough set models, which focuses on the conceptual understanding of those models.

Definition 2. *A pair of a formula and a set of objects (p, X) is a conjunctively definable concept if $p \in \mathrm{DL}_0$ and $X = m(p)$. The set $m(p)$ is called a conjunctively definable set.*

The family of all conjunctively definable concepts is denoted by $\mathrm{CDEF}(OB) = \{(p, m(p)) \mid p \in \mathrm{DL}_0\}$. It should be noted that a conjunctively definable set may be defined by more than one formula.

2.2 Approximating a Set by Structured Positive and Negative Regions

Suppose that a subset of objects $X \subseteq OB$ consists of instances of a concept, that is, X is the extension of the concept. A fundamental issue of rough set theory is to describe the concept or its extension X by using definable concepts or sets. With respect to the family of conjunctively definable concepts $\mathrm{CDEF}(OB)$, we use a pair of positive and negative regions to approximate X. Instead of using the standard definition, we adopt the definition of structured approximations [28].

Definition 3. *Given a set of objects $X \subseteq OB$, the following families of conjunctively definable concepts:*

$$\begin{aligned} \mathrm{SPOS}(X) &= \{(p, m(p)) \in \mathrm{CDEF}(OB) \mid m(p) \subseteq X, m(p) \neq \emptyset\}, \\ \mathrm{SNEG}(X) &= \{(p, m(p)) \in \mathrm{CDEF}(OB) \mid m(p) \subseteq X^c, m(p) \neq \emptyset\}, \end{aligned} \tag{4}$$

are called the structured positive and negative regions of X, respectively.

Figure 1 demonstrates the relationships between a conjunctively definable set $m(p)$ in the structured positive and negative regions and the set of objects X, respectively. To construct the structured positive region, we collect all conjunctively definable concepts whose extensions are included in the given set of objects. In this way, we explicitly indicate the composition of the family of conjunctively definable concepts used to approximate the given set. Similarly, to construct the structured negative region, we collect all conjunctively definable concepts whose extensions are included in the complement set of the given set of objects. There might be two conjunctively definable concepts with the same conjunctively definable set but different formulas. We include all the possible formulas for a conjunctively definable set in defining a region. There might be redundant conjunctively definable concepts in each of the two regions.

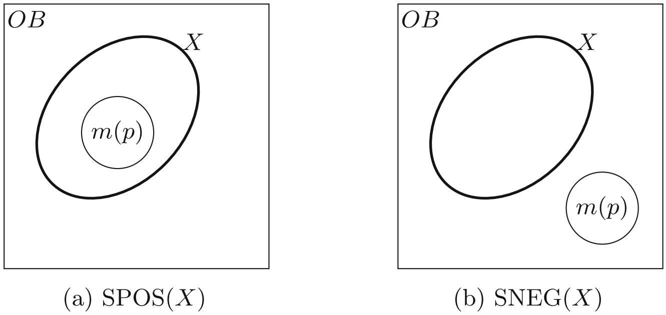


Fig. 1. Relationships between X and $m(p)$ in the two regions

The structured positive and negative regions cover the same sets of objects as defined by the lower approximation of X and X^c, respectively, in Pawlak's framework [18,19], that is:

$$\begin{aligned}\underline{apr}(X) &= \bigcup\{m(p) \mid (p, m(p)) \in \mathrm{SPOS}(X)\},\\ \underline{apr}(X^c) &= \bigcup\{m(p) \mid (p, m(p)) \in \mathrm{SNEG}(X)\}.\end{aligned} \tag{5}$$

They also cover the same sets of objects in the standard unstructured positive and negative regions of X, that is, $\mathrm{POS}(X)$ and $\mathrm{NEG}(X)$, respectively [25]:

$$\begin{aligned}\mathrm{POS}(X) &= \bigcup\{m(p) \mid (p, m(p)) \in \mathrm{SPOS}(X)\},\\ \mathrm{NEG}(X) &= \bigcup\{m(p) \mid (p, m(p)) \in \mathrm{SNEG}(X)\}.\end{aligned} \tag{6}$$

The two sets of objects $\mathrm{POS}(X)$ and $\mathrm{NEG}(X)$ may not contain all objects in OB. The set

$$\mathrm{BND}(X) = (\mathrm{POS}(X) \cup \mathrm{NEG}(X))^c \tag{7}$$

is the boundary region of X. We define the structured boundary region as follows:

$$\mathrm{SBND}(X) = \{(p, m(p)) \in \mathrm{CDEF}(U) \mid m(p) \subseteq \mathrm{BND}(X), m(p) \neq \emptyset\}. \tag{8}$$

If we want to define the structured boundary region of X by using X directly, we need to consider conjunctively definable concepts with specific properties. The details are omitted in this paper.

2.3 Acceptance and Rejection Rules

According to the three regions, we can construct a three-way decision model for rough set theory [25]. From the positive region, we build rules of acceptance for accepting an object to be an instance of the concept represented by X by examining the descriptions of the object. In the same way, we can build rules of rejection for rejecting an object to be an instance of the concept represented by X. For the boundary region, we cannot make such a definite decision. For this reason, in this paper, we are not interested in building rules from the boundary region.

Given a set of objects $X \subseteq OB$, we can build two sets of acceptance and rejection rules from the structured positive and negative regions, respectively. The explicit representation of the intension of a conjunctively definable concept in the structured regions makes this task much simpler. From a conjunctively definable concept $(p, m(p))$ in the structured positive region $\mathrm{SPOS}(X)$, one may immediately get an acceptance rule by taking the formula p as the left-hand-side, that is:

$$\text{If an object } x \text{ satisfies } p, \text{ then accept } x \in X, \text{ denoted by } p \rightarrow X. \tag{9}$$

Similarly, from a conjunctively definable concept $(q, m(q))$ in the structured negative region $\mathrm{SNEG}(X)$, one may immediately get a rejection rule by taking the formula q as the left-hand-side, that is:

$$\text{If an object } x \text{ satisfies } q, \text{ then reject } x \in X, \text{ denoted by } q \rightarrow \neg X. \tag{10}$$

Since there might be redundancy in the two regions, redundant rules may exist in the derived sets of acceptance and rejection rules. Removing redundant rules, that is, finding a rule redundant, is a future research topic.

3 Approximations in an Incomplete Information Table

The notion of definable interval sets was presented to investigate the definability in an incomplete information table [8]. By considering the conjunctive formulas, we present the notion of conjunctively definable interval concepts and use it to define two types of structured positive and negative regions to approximate a set of objects in an incomplete information table. Our approach focuses on a conceptual understanding of the approximations with incomplete information.

3.1 Conjunctively Definable Interval Concepts

An interval set is typically defined by a lower bound and an upper bound [27]. The interval set contains the family of all sets between the two bounds.

Definition 4. *Suppose OB is a universe of objects. An interval set is defined by:*

$$\mathcal{A} = [A_l, A_u] = \{A \subseteq OB \mid A_l \subseteq A \subseteq A_u\}. \tag{11}$$

A_l and A_u are the lower and upper bounds, respectively, and they satisfy the condition $A_l \subseteq A_u$.

The value of an object on an attribute is unique. However, due to incomplete information, we may not know this unique value. Instead, a set of values is known to be possible. Let 2^{V_a} denote the power set of a set of values V_a, that is, the family of all subsets of V_a. An incomplete information table is represented by the following tuple:

$$\widetilde{T} = (OB, AT, \{V_a \mid a \in AT\}, \{\widetilde{I}_a : OB \to 2^{V_a} - \{\emptyset\} \mid a \in AT\}), \tag{12}$$

where OB is a finite nonempty set of objects as rows, AT is a finite nonempty set of attributes as columns, V_a is the domain of an attribute a and $\widetilde{I}_a$ is the information function mapping one object to a nonempty subset of values in the domain of a. We assume that all attributes are applicable to all objects, and demand a nonempty subset of values for every object on every attribute.

Lipski [14] presented a possible-world semantics that interprets an incomplete table as a collection of complete tables. It provides a method to study an incomplete table through a family of complete tables.

Definition 5. *For an incomplete table $\widetilde{T}=(OB, AT, \{V_a \mid a \in AT\}, \{\widetilde{I}_a : OB \to 2^{V_a} - \{\emptyset\} \mid a \in AT\})$, a complete table $T = (OB, AT, \{V_a \mid a \in AT\}, \{I_a : OB \to V_a \mid a \in AT\})$ is called a completion of $\widetilde{T}$ if and only if it satisfies the following condition:*

$$\forall x \in OB, a \in AT, I_a(x) \in \widetilde{I}_a(x). \tag{13}$$

That is, a completion takes exactly one value from the incomplete table for every object on every attribute. The family of all completions of $\widetilde{T}$ is denoted by $\mathrm{COMP}(\widetilde{T}) = \{T \mid T \text{ is a completion of } \widetilde{T}\}$. Since in the incomplete table $\widetilde{T}$, $\widetilde{I}_a(x)$ represents all possibilities of the actual value of x on a, the family $\mathrm{COMP}(\widetilde{T})$ is the collection of all possibilities of the actual table. In other words, once the information becomes complete, we will get a completion in $\mathrm{COMP}(\widetilde{T})$.

For a formula $p \in \mathrm{DL}_0$, we can get a meaning set $m(p|T)$ in each completion $T \in \mathrm{COMP}(\widetilde{T})$. By collecting all the meaning sets of p in the family $\mathrm{COMP}(\widetilde{T})$, we get a family of sets that interprets p in the incomplete table $\widetilde{T}$.

Definition 6. *For a formula $p \in \mathrm{DL}_0$ in an incomplete table $\widetilde{T}$, its meaning set is defined as:*

$$\widetilde{m}(p) = \{m(p|T) \mid T \in \mathrm{COMP}(\widetilde{T})\}, \tag{14}$$

where $m(p|T)$ is the meaning set of p in a completion T.

Our definition of the meaning set of a formula is related to the formulation proposed by Grzymala-Busse et al. [5,6]. In particular, they called an atomic formula an attribute-value pair, and a conjunctive formula a complex. However, they defined the meaning set as a set of objects; we define it as a family of sets of objects.

The meaning set of a formula in an incomplete table is actually an interval set, which is formally stated in the following theorem whose proof is given in Appendix A.

Theorem 1. *For a formula $p \in \mathrm{DL}_0$ in an incomplete table $\widetilde{T}$, its meaning set $\widetilde{m}(p)$ is an interval set with $\cap\widetilde{m}(p)$ as the lower bound and $\cup\widetilde{m}(p)$ as the upper bound:*

$$\widetilde{m}(p) = [\cap\widetilde{m}(p), \cup\widetilde{m}(p)]. \tag{15}$$

By Definition 6, the interval set $\widetilde{m}(p)$ is actually the family of all possibilities of the actual meaning set of p. In this sense, the sets $\cap\widetilde{m}(p)$ and $\cup\widetilde{m}(p)$ are the lower and upper bounds of the actual meaning set of p. Thus, we denote the sets $\cap\widetilde{m}(p)$ and $\cup\widetilde{m}(p)$ as $m_*(p)$ and $m^*(p)$, respectively. Accordingly, Eq. (15) can be written as $\widetilde{m}(p) = [m_*(p), m^*(p)]$. The two bounds can be interpreted in terms of the family $\mathrm{COMP}(\widetilde{T})$ as follows:

$$\begin{aligned} m_*(p) &= \bigcap_{T \in \mathrm{COMP}(\widetilde{T})} m(p|T) = \{x \in OB \mid \forall T \in \mathrm{COMP}(\widetilde{T}), x \in m(p|T)\}, \\ m^*(p) &= \bigcup_{T \in \mathrm{COMP}(\widetilde{T})} m(p|T) = \{x \in OB \mid \exists T \in \mathrm{COMP}(\widetilde{T}), x \in m(p|T)\}. \end{aligned} \tag{16}$$

By Theorem 1, the meaning set of a conjunctive formula in DL_0 in an incomplete table is an interval set. Such an interval set is considered to be conjunctively definable. By explicitly giving the conjunctive formulas, we define a conjunctively definable interval concept as a pair of a formula and its meaning set in an incomplete table.

Definition 7. *A pair of a formula and an interval set $(p, \mathcal{A})$ is a conjunctively definable interval concept if $p \in \mathrm{DL}_0$ and $\mathcal{A} = \widetilde{m}(p)$. The interval set $\widetilde{m}(p)$ is called a conjunctively definable interval set.*

The family of all conjunctively definable interval concepts is denoted as $\mathrm{CDEFI}(OB) = \{(p, \widetilde{m}(p)) \mid p \in \mathrm{DL}_0\} = \{(p, [m_*(p), m^*(p)]) \mid p \in \mathrm{DL}_0\}$.

3.2 Two Types of Structured Positive and Negative Regions in an Incomplete Table

By using the family $\mathrm{CDEFI}(OB)$ instead of the family $\mathrm{CDEF}(OB)$, we generalize the structured positive and negative regions in a complete table into two types of structured positive and negative regions in an incomplete table.

Given a set of objects X in a complete table, its structured positive and negative regions are defined by considering the set-theoretic inclusion relationships between a conjunctively definable set, that is, the meaning set of a conjunctive formula, and X and X^c, respectively. With respect to an incomplete table, the meaning set of a formula becomes a conjunctively definable interval set. The family $\mathrm{CDEFI}(OB)$ is consequently used to define the structured positive and negative regions. We consider the component-wise inclusion relationships between a conjunctively definable interval set and X and X^c, that is, the set-theoretic inclusion between a set in the interval set and X and X^c. This leads to two types of structured positive and negative regions.

Definition 8. *For a set of objects X in an incomplete table $\widetilde{T}$, we define two types of structured positive and negative regions of X as follows:*

$$\begin{aligned}
(1)\quad \mathrm{SPOS}_*(X) &= \{(p,\widetilde{m}(p)) \in \mathrm{CDEFI}(OB) \mid \widetilde{m}(p) \neq [\emptyset,\emptyset], \forall S \in \widetilde{m}(p), S \subseteq X\},\\
\mathrm{SNEG}_*(X) &= \{(p,\widetilde{m}(p)) \in \mathrm{CDEFI}(OB) \mid \widetilde{m}(p) \neq [\emptyset,\emptyset], \forall S \in \widetilde{m}(p), S \subseteq X^c\};\\
(2)\quad \mathrm{SPOS}^*(X) &= \{(p,\widetilde{m}(p)) \in \mathrm{CDEFI}(OB) \mid \exists S \in \widetilde{m}(p), S \neq \emptyset, S \subseteq X\},\\
\mathrm{SNEG}^*(X) &= \{(p,\widetilde{m}(p)) \in \mathrm{CDEFI}(OB) \mid \exists S \in \widetilde{m}(p), S \neq \emptyset, S \subseteq X^c\}. \qquad (17)
\end{aligned}$$

It should be noted that the intersection of the two regions $\mathrm{SPOS}^*(X)$ and $\mathrm{SNEG}^*(X)$ may not be empty since we use the existence of the set S to define these two regions. Suppose $(p, [\emptyset, m^*(p)])$ is a conjunctively definable interval concept where $m^*(p) \cap X = S_1 \neq \emptyset$ and $m^*(p) \cap X^c = S_2 \neq \emptyset$. Since $S_1, S_2 \subseteq m^*(p)$, we have $S_1, S_2 \in [\emptyset, m^*(p)]$. By Definition 8 and the fact that $S_1 \subseteq X$ and $S_2 \subseteq X^c$, the concept $(p, [\emptyset, m^*(p)])$ will be included in both $\mathrm{SPOS}^*(X)$ and $\mathrm{SNEG}^*(X)$.

By Definition 6, the meaning set $\widetilde{m}(p)$ is actually the collection of the meaning sets of p in all the completions. Thus, we can re-write Definition 8 as given in the following theorem.

Theorem 2. *For a set of objects X in an incomplete table $\widetilde{T}$, its two types of structured positive and negative regions can be equivalently expressed as:*

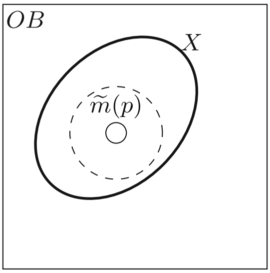
$$\begin{aligned}
(1)\quad \mathrm{SPOS}_*(X) = \{&(p,\widetilde{m}(p)) \in \mathrm{CDEFI}(OB) \mid \widetilde{m}(p) \neq [\emptyset,\emptyset], \forall T \in \mathrm{COMP}(\widetilde{T}),\\
&m(p|T) \subseteq X\},\\
\mathrm{SNEG}_*(X) = \{&(p,\widetilde{m}(p)) \in \mathrm{CDEFI}(OB) \mid \widetilde{m}(p) \neq [\emptyset,\emptyset], \forall T \in \mathrm{COMP}(\widetilde{T}),\\
&m(p|T) \subseteq X^c\};\\
(2)\quad \mathrm{SPOS}^*(X) = \{&(p,\widetilde{m}(p)) \in \mathrm{CDEFI}(OB) \mid \exists T \in \mathrm{COMP}(\widetilde{T}), m(p|T) \neq \emptyset,\\
&m(p|T) \subseteq X\},\\
\mathrm{SNEG}^*(X) = \{&(p,\widetilde{m}(p)) \in \mathrm{CDEFI}(OB) \mid \exists T \in \mathrm{COMP}(\widetilde{T}), m(p|T) \neq \emptyset,\\
&m(p|T) \subseteq X^c\}. \qquad (18)
\end{aligned}$$

By the fact that $m_*(p) \subseteq m^*(p)$, the two types of structured regions could be computed in terms of the two bounds of the interval sets, which is given in the following theorem.

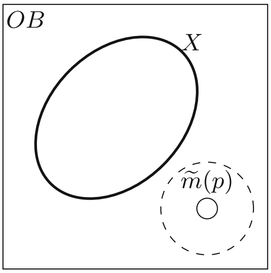
Theorem 3. *For a set of objects X in an incomplete table $\widetilde{T}$, its two types of structured positive and negative regions can be computed as:*

$$\begin{aligned}
(1)\quad \mathrm{SPOS}_*(X) = \{&(p,[m_*(p),m^*(p)]) \in \mathrm{CDEFI}(OB) \mid m^*(p) \neq \emptyset, m^*(p) \subseteq X\},\\
\mathrm{SNEG}_*(X) = \{&(p,[m_*(p),m^*(p)]) \in \mathrm{CDEFI}(OB) \mid m^*(p) \neq \emptyset, m^*(p) \subseteq X^c\};\\
(2)\quad \mathrm{SPOS}^*(X) = \{&(p,[m_*(p),m^*(p)]) \in \mathrm{CDEFI}(OB) \mid (m_*(p) \neq \emptyset \wedge m_*(p) \subseteq X)\\
&\vee(m_*(p) = \emptyset \wedge m^*(p) \cap X \neq \emptyset)\},\\
\mathrm{SNEG}^*(X) = \{&(p,[m_*(p),m^*(p)]) \in \mathrm{CDEFI}(OB) \mid (m_*(p) \neq \emptyset \wedge m_*(p) \subseteq X^c)\\
&\vee(m_*(p) = \emptyset \wedge m^*(p) \cap X^c \neq \emptyset)\}. \qquad (19)
\end{aligned}$$

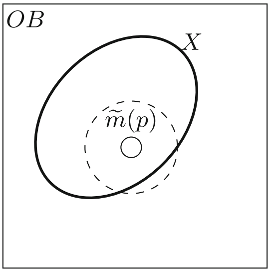
By Theorem 3, we call $\mathrm{SPOS}_*(X)$ and $\mathrm{SNEG}_*(X)$ the upper-bound structured positive and negative regions, respectively; and $\mathrm{SPOS}^*(X)$ and $\mathrm{SNEG}^*(X)$ the lower-bound structured positive and negative regions, respectively. The relationships between the set of objects X and a conjunctively definable interval set $\widetilde{m}(p)$ in the four regions can be depicted by Fig. 2. We use two concentric circles to represent $\widetilde{m}(p)$, one with solid line to represent the lower bound and the other with dashed line to represent the upper bound. There are other possibilities of the relationships in Fig. 2. We only focus on the upper bound for $\mathrm{SPOS}_*(X)$ and $\mathrm{SNEG}_*(X)$, and the lower bound for $\mathrm{SPOS}^*(X)$ and $\mathrm{SNEG}^*(X)$.



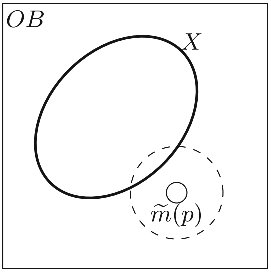
(a) $\mathrm{SPOS}_*(X)$



(b) $\mathrm{SNEG}_*(X)$



(c) $\mathrm{SPOS}^*(X)$



(d) $\mathrm{SNEG}^*(X)$

Fig. 2. Relationships between X and $\widetilde{m}(p)$ in the four regions

We have discussed three equivalent forms for the four regions. Definition 8 provides a direct generalization of the structured positive and negative regions in a complete table. Theorem 2 clarifies the semantical meanings of the four regions in terms of the family of completions. Theorem 3 offers a computational formalization by using the bounds of the interval sets. Theorem 2 could be viewed as a conceptual model of the regions and Theorem 3 as a computational model [24].

One may easily verify the properties of the four regions stated in the following theorem by using any of the three forms.

Theorem 4. *For two sets of objects $X, Y \subseteq OB$, the following properties are satisfied:*

$$\begin{aligned}
&(1)\ \ \mathrm{SPOS}_*(X) \cap \mathrm{SNEG}_*(X) = \emptyset;\\
&(2)\ \ \mathrm{SPOS}_*(X) \subseteq \mathrm{SPOS}^*(X),\\
&\qquad \mathrm{SNEG}_*(X) \subseteq \mathrm{SNEG}^*(X);\\
&(3)\ \ X \subseteq Y \Longrightarrow \mathrm{SPOS}_*(X) \subseteq \mathrm{SPOS}_*(Y),\\
&\qquad X \subseteq Y \Longrightarrow \mathrm{SNEG}_*(X) \supseteq \mathrm{SNEG}_*(Y),\\
&\qquad X \subseteq Y \Longrightarrow \mathrm{SPOS}^*(X) \subseteq \mathrm{SPOS}^*(Y),\\
&\qquad X \subseteq Y \Longrightarrow \mathrm{SNEG}^*(X) \supseteq \mathrm{SNEG}^*(Y);\\
&(4)\ \ \mathrm{SPOS}_*(X \cap Y) = \mathrm{SPOS}_*(X) \cap \mathrm{SPOS}_*(Y),\\
&\qquad \mathrm{SNEG}_*(X \cap Y) \supseteq \mathrm{SNEG}_*(X) \cap \mathrm{SNEG}_*(Y),\\
&\qquad \mathrm{SPOS}^*(X \cap Y) = \mathrm{SPOS}^*(X) \cap \mathrm{SPOS}^*(Y),\\
&\qquad \mathrm{SNEG}^*(X \cap Y) \supseteq \mathrm{SNEG}^*(X) \cap \mathrm{SNEG}^*(Y);\\
&(5)\ \ \mathrm{SPOS}_*(X \cup Y) \supseteq \mathrm{SPOS}_*(X) \cup \mathrm{SPOS}_*(Y),\\
&\qquad \mathrm{SNEG}_*(X \cup Y) = \mathrm{SNEG}_*(X) \cap \mathrm{SNEG}_*(Y),\\
&\qquad \mathrm{SPOS}^*(X \cup Y) \supseteq \mathrm{SPOS}^*(X) \cup \mathrm{SPOS}^*(Y),\\
&\qquad \mathrm{SNEG}^*(X \cup Y) = \mathrm{SNEG}^*(X) \cap \mathrm{SNEG}^*(Y);\\
&(6)\ \ \mathrm{SPOS}_*(X \cap Y) \subseteq \mathrm{SPOS}_*(X \cup Y),\\
&\qquad \mathrm{SNEG}_*(X \cap Y) \supseteq \mathrm{SNEG}_*(X \cup Y),\\
&\qquad \mathrm{SPOS}^*(X \cap Y) \subseteq \mathrm{SPOS}^*(X \cup Y),\\
&\qquad \mathrm{SNEG}^*(X \cap Y) \supseteq \mathrm{SNEG}^*(X \cup Y).
\end{aligned} \tag{20}$$

The properties in Theorem 4 correspond to the properties of positive and negative regions of rough sets in a complete table.

3.3 An Example

We give an example to illustrate the ideas of constructing the two types of structured positive and negative regions in an incomplete table. Suppose we have an incomplete table given in Table 1.

Table 1. An incomplete table $\widetilde{T}$

	a	b	c
o_1	{1}	{3}	{6}
o_2	{1}	{4}	{6}
o_3	{1, 2}	{5}	{6}
o_4	{1}	{4, 5}	{6}
o_5	{2}	{5}	{7}
o_6	{2}	{5}	{7}

The table $\widetilde{T}$ could be equivalently represented by the family of its completions given in Table 2.

Table 2. The family of completions COMP($\widetilde{T}$)

	a	b	c
o_1	{1}	{3}	{6}
o_2	{1}	{4}	{6}
o_3	**{1}**	{5}	{6}
o_4	{1}	**{4}**	{6}
o_5	{2}	{5}	{7}
o_6	{2}	{5}	{7}

(a) A completion T_1

	a	b	c
o_1	{1}	{3}	{6}
o_2	{1}	{4}	{6}
o_3	**{1}**	{5}	{6}
o_4	{1}	**{5}**	{6}
o_5	{2}	{5}	{7}
o_6	{2}	{5}	{7}

(b) A completion T_2

	a	b	c
o_1	{1}	{3}	{6}
o_2	{1}	{4}	{6}
o_3	**{2}**	{5}	{6}
o_4	{1}	**{4}**	{6}
o_5	{2}	{5}	{7}
o_6	{2}	{5}	{7}

(c) A completion T_3

	a	b	c
o_1	{1}	{3}	{6}
o_2	{1}	{4}	{6}
o_3	**{2}**	{5}	{6}
o_4	{1}	**{5}**	{6}
o_5	{2}	{5}	{7}
o_6	{2}	{5}	{7}

(d) A completion T_4

The formulas in DL_0 and the family CDEFI(OB) are given by Tables 3 and 4, respectively. We take $p_1 = (a = 1)$ as an example. The meaning sets of p_1 in the four completions are:

$$\begin{aligned} m(p_1|T_1) &= \{o_1, o_2, o_3, o_4\}, \\ m(p_1|T_2) &= \{o_1, o_2, o_3, o_4\}, \\ m(p_1|T_3) &= \{o_1, o_2, o_4\}, \\ m(p_1|T_4) &= \{o_1, o_2, o_4\}. \end{aligned} \tag{21}$$

By Definition 6, the meaning set of p_1 in $\widetilde{T}$ is:

$$\begin{aligned} \widetilde{m}(p_1) &= \{m(p_1|T), m(p_2|T), m(p_3|T), m(p_4|T)\} \\ &= \{\{o_1, o_2, o_4\}, \{o_1, o_2, o_3, o_4\}\}. \end{aligned} \tag{22}$$

By Theorem 1, $\widetilde{m}(p_1)$ is an interval set:

$$\widetilde{m}(p_1) = [\{o_1, o_2, o_4\}, \{o_1, o_2, o_3, o_4\}], \tag{23}$$

which is a conjunctively definable interval set. The corresponding conjunctively definable interval concept is:

$$\begin{aligned} \mathcal{C}_1 &= (p_1, [\{o_1, o_2, o_4\}, \{o_1, o_2, o_3, o_4\}]) \\ &= (a = 1, [\{o_1, o_2, o_4\}, \{o_1, o_2, o_3, o_4\}]). \end{aligned} \tag{24}$$

Table 3. The formulas in DL_0 in the example

Label	Formula	Label	Formula
p_1	$a = 1$	p_{19}	$(b = 3) \wedge (c = 7)$
p_2	$a = 2$	p_{20}	$(b = 4) \wedge (c = 6)$
p_3	$b = 3$	p_{21}	$(b = 4) \wedge (c = 7)$
p_4	$b = 4$	p_{22}	$(b = 5) \wedge (c = 6)$
p_5	$b = 5$	p_{23}	$(b = 5) \wedge (c = 7)$
p_6	$c = 6$	p_{24}	$(a = 1) \wedge (b = 3) \wedge (c = 6)$
p_7	$c = 7$	p_{25}	$(a = 1) \wedge (b = 3) \wedge (c = 7)$
p_8	$(a = 1) \wedge (b = 3)$	p_{26}	$(a = 1) \wedge (b = 4) \wedge (c = 6)$
p_9	$(a = 1) \wedge (b = 4)$	p_{27}	$(a = 1) \wedge (b = 4) \wedge (c = 7)$
p_{10}	$(a = 1) \wedge (b = 5)$	p_{28}	$(a = 1) \wedge (b = 5) \wedge (c = 6)$
p_{11}	$(a = 2) \wedge (b = 3)$	p_{29}	$(a = 1) \wedge (b = 5) \wedge (c = 7)$
p_{12}	$(a = 2) \wedge (b = 4)$	p_{30}	$(a = 2) \wedge (b = 3) \wedge (c = 6)$
p_{13}	$(a = 2) \wedge (b = 5)$	p_{31}	$(a = 2) \wedge (b = 3) \wedge (c = 7)$
p_{14}	$(a = 1) \wedge (c = 6)$	p_{32}	$(a = 2) \wedge (b = 4) \wedge (c = 6)$
p_{15}	$(a = 1) \wedge (c = 7)$	p_{33}	$(a = 2) \wedge (b = 4) \wedge (c = 7)$
p_{16}	$(a = 2) \wedge (c = 6)$	p_{34}	$(a = 2) \wedge (b = 5) \wedge (c = 6)$
p_{17}	$(a = 2) \wedge (c = 7)$	p_{35}	$(a = 2) \wedge (b = 5) \wedge (c = 7)$
p_{18}	$(b = 3) \wedge (c = 6)$		

Given a set of objects $X = \{o_1, o_3, o_5\}$ and its complement set $X^c = \{o_2, o_4, o_6\}$, the four regions of X are as follows:

$$\begin{aligned} &(1)\ \ \mathrm{SPOS}_*(X) = \{\mathcal{C}_3, \mathcal{C}_8, \mathcal{C}_{16}, \mathcal{C}_{18}, \mathcal{C}_{24}, \mathcal{C}_{34}\}, \\ &\ \ \mathrm{SNEG}_*(X) = \{\mathcal{C}_4, \mathcal{C}_9, \mathcal{C}_{20}, \mathcal{C}_{26}\}; \\ &(2)\ \ \mathrm{SPOS}^*(X) = \{\mathcal{C}_3, \mathcal{C}_8, \mathcal{C}_{10}, \mathcal{C}_{16}, \mathcal{C}_{18}, \mathcal{C}_{22}, \mathcal{C}_{24}, \mathcal{C}_{28}, \mathcal{C}_{34}\}, \\ &\ \ \mathrm{SNEG}^*(X) = \{\mathcal{C}_4, \mathcal{C}_9, \mathcal{C}_{10}, \mathcal{C}_{20}, \mathcal{C}_{26}, \mathcal{C}_{28}\}. \end{aligned} \tag{25}$$

There is redundancy in these four regions. For example, the two concepts $\mathcal{C}_4$ and $\mathcal{C}_{26}$ have the same extension but different intensions. Accordingly, the regions including both $\mathcal{C}_4$ and $\mathcal{C}_{26}$, that is, $\mathrm{SNEG}_*(X)$ and $\mathrm{SNEG}^*(X)$, have redundancy in them.

Table 4. The family CDEFI(OB) in the example

Label	Intension	Extension	Label	Intension	Extension
$\mathcal{C}_1$	p_1	$[\{o_1, o_2, o_4\}, \{o_1, o_2, o_3, o_4\}]$	$\mathcal{C}_{19}$	p_{19}	$[\emptyset, \emptyset]$
$\mathcal{C}_2$	p_2	$[\{o_5, o_6\}, \{o_3, o_5, o_6\}]$	$\mathcal{C}_{20}$	p_{20}	$[\{o_2\}, \{o_2, o_4\}]$
$\mathcal{C}_3$	p_3	$[\{o_1\}, \{o_1\}]$	$\mathcal{C}_{21}$	p_{21}	$[\emptyset, \emptyset]$
$\mathcal{C}_4$	p_4	$[\{o_2\}, \{o_2, o_4\}]$	$\mathcal{C}_{22}$	p_{22}	$[\{o_3\}, \{o_3, o_4\}]$
$\mathcal{C}_5$	p_5	$[\{o_3, o_5, o_6\}, \{o_3, o_4, o_5, o_6\}]$	$\mathcal{C}_{23}$	p_{23}	$[\{o_5, o_6\}, \{o_5, o_6\}]$
$\mathcal{C}_6$	p_6	$[\{o_1, o_2, o_3, o_4\}, \{o_1, o_2, o_3, o_4\}]$	$\mathcal{C}_{24}$	p_{24}	$[\{o_1\}, \{o_1\}]$
$\mathcal{C}_7$	p_7	$[\{o_5, o_6\}, \{o_5, o_6\}]$	$\mathcal{C}_{25}$	p_{25}	$[\emptyset, \emptyset]$
$\mathcal{C}_8$	p_8	$[\{o_1\}, \{o_1\}]$	$\mathcal{C}_{26}$	p_{26}	$[\{o_2\}, \{o_2, o_4\}]$
$\mathcal{C}_9$	p_9	$[\{o_2\}, \{o_2, o_4\}]$	$\mathcal{C}_{27}$	p_{27}	$[\emptyset, \emptyset]$
$\mathcal{C}_{10}$	p_{10}	$[\emptyset, \{o_3, o_4\}]$	$\mathcal{C}_{28}$	p_{28}	$[\emptyset, \{o_3, o_4\}]$
$\mathcal{C}_{11}$	p_{11}	$[\emptyset, \emptyset]$	$\mathcal{C}_{29}$	p_{29}	$[\emptyset, \emptyset]$
$\mathcal{C}_{12}$	p_{12}	$[\emptyset, \emptyset]$	$\mathcal{C}_{30}$	p_{30}	$[\emptyset, \emptyset]$
$\mathcal{C}_{13}$	p_{13}	$[\{o_5, o_6\}, \{o_3, o_5, o_6\}]$	$\mathcal{C}_{31}$	p_{31}	$[\emptyset, \emptyset]$
$\mathcal{C}_{14}$	p_{14}	$[\{o_1, o_2, o_4\}, \{o_1, o_2, o_3, o_4\}]$	$\mathcal{C}_{32}$	p_{32}	$[\emptyset, \emptyset]$
$\mathcal{C}_{15}$	p_{15}	$[\emptyset, \emptyset]$	$\mathcal{C}_{33}$	p_{33}	$[\emptyset, \emptyset]$
$\mathcal{C}_{16}$	p_{16}	$[\emptyset, \{o_3\}]$	$\mathcal{C}_{34}$	p_{34}	$[\emptyset, \{o_3\}]$
$\mathcal{C}_{17}$	p_{17}	$[\{o_5, o_6\}, \{o_5, o_6\}]$	$\mathcal{C}_{35}$	p_{35}	$[\{o_5, o_6\}, \{o_5, o_6\}]$
$\mathcal{C}_{18}$	p_{18}	$[\{o_1\}, \{o_1\}]$			

4 Conclusions

We have proposed a new semantically sound framework to study rough set approximations in an incomplete table. In a complete table, we use the family of conjunctively definable concepts to define a pair of structured positive and negative regions in order to approximate a set of objects. These two structured regions correspond to Pawlak rough set approximations and the standard positive and negative regions. Following the same argument, in an incomplete table we introduced the notion of conjunctively definable interval concepts. By using the family of conjunctively definable interval concepts, we construct two types of structured positive and negative regions. These regions are semantically meaningful in the sense that the possible-world semantics fully and truthfully reflects all partial information of an incomplete table. By adopting possible-world semantics of an incomplete table, we transform the study of an incomplete table into a study of a family of complete tables. By using concepts instead of sets to construct the regions, we explicitly include the intensions, that is, the formulas, which makes the rule induction much simpler. As future work, we will investigate rule induction in an incomplete table based on the structured regions introduced in this paper.

A Appendix: Proof of Theorem 1

We prove Theorem 1 by verifying $\widetilde{m}(p) \subseteq [\cap\widetilde{m}(p), \cup\widetilde{m}(p)]$ and $[\cap\widetilde{m}(p), \cup\widetilde{m}(p)] \subseteq \widetilde{m}(p)$.

(1) $\widetilde{m}(p) \subseteq [\cap\widetilde{m}(p), \cup\widetilde{m}(p)]$.
For any set $S \in \widetilde{m}(p)$, it is evident that $\cap\widetilde{m}(p) \subseteq S \subseteq \cup\widetilde{m}(p)$, which means $S \in [\cap\widetilde{m}(p), \cup\widetilde{m}(p)]$. Thus, $\widetilde{m}(p) \subseteq [\cap\widetilde{m}(p), \cup\widetilde{m}(p)]$.

(2) $[\cap\widetilde{m}(p), \cup\widetilde{m}(p)] \subseteq \widetilde{m}(p)$.
By Definition 6, for any set $S \in [\cap\widetilde{m}(p), \cup\widetilde{m}(p)]$, we prove that $S \in \widetilde{m}(p)$ by constructing a completion $T \in \mathrm{COMP}(\widetilde{T})$ in which $S = m(p|T)$. Suppose $p = (a_1 = v_1) \wedge (a_2 = v_2) \wedge \cdots \wedge (a_m = v_m)$ and $A_p = \{a_1, a_2 \ldots, a_m\}$. The completion T is constructed as given in Table 5. One can easily verify that in Table 5, the objects in S satisfy p and the objects in $OB - S$ do not satisfy p. That is, $S = m(p|T)$. Thus, $[\cap\widetilde{m}(p), \cup\widetilde{m}(p)] \subseteq \widetilde{m}(p)$.

Table 5. A completion T in which $S = m(p|T)$

Objects	Attributes	
	A_p	$AT - A_p$
S	$\forall a_i \in A_p, I_{a_i}(x) = v_i$	$\forall a \in AT - A_p, I_a(x) \in \widetilde{I}_a(x)$
$OB - S$	$\exists a_i \in A_p, I_{a_i}(x) \in \widetilde{I}_{a_i}(x) - \{v_i\}$	

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