Useful Inequalities $\{x^2 \geqslant 0\}$ v0.32 · August 2, 2020

Cauchy-Schwarz $\left(\sum_{i=1}^n x_i y_i\right)^2 \le \left(\sum_{i=1}^n x_i^2\right)\left(\sum_{i=1}^n y_i^2\right)$

Minkowski $\left(\sum_{i=1}^n |x_i+y_i|^p\right)^{\frac{1}{p}} \le \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |y_i|^p\right)^{\frac{1}{p}}$ for $p \ge 1$.

Hölder $\sum_{i=1}^n |x_i y_i| \le \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}\left(\sum_{i=1}^n |y_i|^q\right)^{1/q}$ for $p,q>1,\ \frac{1}{p}+\frac{1}{q}=1$.

Bernoulli

$(1+x)^r \ge 1+rx$ for $x \ge -1,\ r \in \mathbb{R}\setminus(0,1)$. Reverse for $r \in [0,1]$.

$(1+x)^r \le 1+(2^r-1)x$ for $x\in[0,1],\ r\in\mathbb{R}\setminus(0,1)$.

$(1+x)^n \le \frac{1}{1-nx}$ for $x\in[-1,0],\ n\in\mathbb{N}$.

$(1+x)^r \le 1+\frac{rx}{1-(r-1)x}$ for $x\in[-1,\frac{1}{r-1}),\ r>1$.

$(1+nx)^{n+1} \ge (1+(n+1)x)^n$ for $x\in\mathbb{R},\ n\in\mathbb{N}$.

$x^y > \frac{x}{x+y}$ for $x>0,\ y\in(0,1)$.

$(a+b)^n \le a^n + nb(a+b)^{n-1}$ for $a,b\ge 0,\ n\in\mathbb{N}$.

exponential

$e^x \ge \left(1+\frac{x}{n}\right)^n \ge 1+x$, $\quad \left(1+\frac{x}{n}\right)^n \ge e^x\left(1-\frac{x^2}{n}\right)$ for $n\ge 1,\ |x|\le n$.

$e^x \le 1+x+x^2$ for $x<1.79$; $\quad xe^x \ge x+x^2+\frac{x^3}{2}$ for $x\in\mathbb{R}$.

$e^x \ge x^e$ for $x\ge 0$; $\quad \frac{x^n}{n!}+1 \le e^x \le \left(1+\frac{x}{n}\right)^{n+x/2}$ for $x,n>0$.

$a^x \le 1+(a-1)x$; $\quad a^{-x} \le 1-\frac{(a-1)}{a}x$ for $x\in[0,1],\ a\ge 1$.

$\frac{1}{2-x} < x^x < x^2-x+1$, for $x\in(0,1]$; $\quad e^x+e^{-x} \le 2e^{x^2/2}$, for $x\in\mathbb{R}$.

$x^{1/r}(x-1) \le rx(x^{1/r}-1)$ for $x,r\ge 1$.

$x^y+y^x > 1$; $\quad e^x > \left(1+\frac{x}{y}\right)^y > e^{\frac{xy}{x+y}}$ for $x,y>0$.

$2-y-e^{-x-y} \le 1+x \le y+e^{x-y}$; $\quad e^x \le x+e^{x^2}$ for $x,y\in\mathbb{R}$.

$\left(1+\frac{x}{p}\right)^p \ge \left(1+\frac{x}{q}\right)^q$ for $(i)\ x>0,\ p>q>0$,

$(ii)\ -p<-q<x<0$, $(iii)\ -q>-p>x>0$. Reverse for:

$(iv)\ q<0<p,\ -q>x>0$, $(v)\ q<0<p,\ -p<x<0$.

logarithm

$\frac{x}{1+x} \le \ln(1+x) \le x$ for $x>-1$; $\quad \ln(x) \le n(x^{\frac{1}{n}}-1)$ for $x,n>0$.

$\frac{2}{2+x} \le \frac{1}{\sqrt{1+x+x^2/12}} \le \frac{\ln(1+x)}{x} \le \frac{1}{\sqrt{x+1}} \le \frac{2+x}{2+2x}$ for $x>-1$.

$\ln(n+1) < \ln(n)+\frac{1}{n} \le \sum_{i=1}^n \frac{1}{i} \le \ln(n)+1$

$|\ln(x)| \le \frac{1}{2}\left|x-\frac{1}{x}\right|$ for $x>0$; $\quad \ln(1+x) \ge \frac{x}{2}$ for $x\in[0,\sim 2.51]$.

trigonometric $x-\frac{x^3}{2} \le x\cos x \le \frac{x\cos x}{1-x^2/3} \le x\sqrt[3]{\cos x} \le x-x^3/6 \le x\cos\frac{x}{\sqrt{3}} \le \sin x$,

hyperbolic $x\cos x \le \frac{x^3}{\sinh^2 x} \le x\cos^2(x/2) \le \sin x \le (x\cos x+2x)/3 \le \frac{x^2}{\sinh x}$,

$\max\left\{\frac{2}{\pi}, \frac{\pi^2-x^2}{\pi^2+x^2}\right\} \le \frac{\sin x}{x} \le \cos\frac{x}{2} \le 1 \le 1+\frac{x^2}{3} \le \frac{\tan x}{x}$ for $x\in\left[0,\frac{\pi}{2}\right]$.

square root

$2\sqrt{x+1}-2\sqrt{x} < \frac{1}{\sqrt{x}} < \sqrt{x+1}-\sqrt{x-1} < 2\sqrt{x}-2\sqrt{x-1}$ for $x\ge 1$.

$1-\frac{x}{2}-\frac{x^2}{2} \le \sqrt{1-x} \le 1-\frac{x}{2}$ for $x\le 1$.

binomial

$\max\left\{\frac{n^k}{k^k}, \frac{(n-k+1)^k}{k!}\right\} \le \binom{n}{k} \le \frac{n^k}{k!} \le \left(\frac{en}{k}\right)^k$; $\quad \binom{n}{k} \le \frac{n^n}{k^k(n-k)^{n-k}} \le \frac{2^n}{\sqrt{n/2}}$.

$\frac{n^k}{4k!} \le \binom{n}{k}$ for $\sqrt{n}\ge k\ge 0$; $\quad \frac{4^n}{\sqrt{\pi n}}\left(1-\frac{1}{8n}\right) \le \binom{2n}{n} \le \frac{4^n}{\sqrt{\pi n}}\left(1-\frac{1}{9n}\right)$.

$\binom{n_1}{k_1}\binom{n_2}{k_2} \le \binom{n_1+n_2}{k_1+k_2}$ for $n_1\ge k_1\ge 0,\ n_2\ge k_2\ge 0$.

$\frac{\sqrt{\pi}}{2}G \le \binom{n}{\alpha n} \le G$ for $G=\frac{2^{nH(\alpha)}}{\sqrt{2\pi n\alpha(1-\alpha)}}$, $H(x)=-\log_2(x^x(1-x)^{1-x})$.

$\sum_{i=0}^d \binom{n}{i} \le \min\left\{n^d+1, \left(\frac{en}{d}\right)^d, 2^n\right\}$ for $n\ge d\ge 1$.

$\sum_{i=0}^{\alpha n}\binom{n}{i} \le \min\left\{\frac{1-\alpha}{1-2\alpha}\binom{n}{\alpha n},\ 2^n\cdot\exp\left(-2n\left(\frac{1}{2}-\alpha\right)^2\right)\right\}$ for $\alpha\in(0,\frac{1}{2})$.

Stirling $e\left(\frac{n}{e}\right)^n \le \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{1/(12n+1)} \le n! \le \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{1/12n} \le en\left(\frac{n}{e}\right)^n$

means $\min x_i \le \frac{n}{\sum x_i^{-1}} \le \left(\prod x_i\right)^{1/n} \le \frac{1}{n}\sum x_i \le \sqrt{\frac{1}{n}\sum {x_i}^2} \le \frac{\sum x_i^2}{\sum x_i} \le \max x_i$

power means $M_p \le M_q$ for $p\le q$, where $M_p=\left(\sum_i w_i|x_i|^p\right)^{1/p}$, $w_i\ge 0$, $\sum_i w_i=1$.
In the limit $M_0=\prod_i |x_i|^{w_i}$, $M_{-\infty}=\min_i\{x_i\}$, $M_\infty=\max_i\{x_i\}$.

Lehmer $\frac{\sum_i w_i|x_i|^p}{\sum_i w_i|x_i|^{p-1}} \le \frac{\sum_i w_i|x_i|^q}{\sum_i w_i|x_i|^{q-1}}$ for $p\le q,\ w_i\ge 0$.

log mean $\sqrt{xy} \le \left(\frac{\sqrt{x}+\sqrt{y}}{2}\right)(xy)^{\frac{1}{4}} \le \frac{x-y}{\ln(x)-\ln(y)} \le \left(\frac{\sqrt{x}+\sqrt{y}}{2}\right)^2 \le \frac{x+y}{2}$ for $x,y>0$.

Heinz $\sqrt{xy} \le \frac{x^{1-\alpha}y^\alpha+x^\alpha y^{1-\alpha}}{2} \le \frac{x+y}{2}$ for $x,y>0,\ \alpha\in[0,1]$.

Maclaurin-Newton ${S_k}^2 \ge S_{k-1}S_{k+1}$ and $(S_k)^{1/k} \ge (S_{k+1})^{1/(k+1)}$ for $1\le k<n$,
$S_k=\frac{1}{\binom{n}{k}}\sum_{1\le i_1<\cdots<i_k\le n} a_{i_1}a_{i_2}\cdots a_{i_k}$, and $a_i\ge 0$.

Jensen $\varphi\left(\sum_i p_i x_i\right) \le \sum_i p_i\varphi(x_i)$ where $p_i\ge 0$, $\sum p_i=1$, and φ convex.
Alternatively: $\varphi(\mathrm{E}[X]) \le \mathrm{E}[\varphi(X)]$. For concave φ the reverse holds.

Chebyshev $\sum_{i=1}^n f(a_i)g(b_i)p_i \ge \left(\sum_{i=1}^n f(a_i)p_i\right)\left(\sum_{i=1}^n g(b_i)p_i\right) \ge \sum_{i=1}^n f(a_i)g(b_{n-i+1})p_i$
for $a_1\le\cdots\le a_n,\ b_1\le\cdots\le b_n$ and f,g nondecreasing, $p_i\ge 0$, $\sum p_i=1$.
Alternatively: $\mathrm{E}\big[f(X)g(X)\big] \ge \mathrm{E}\big[f(X)\big]\mathrm{E}\big[g(X)\big]$.

rearrangement $\sum_{i=1}^n a_ib_i \ge \sum_{i=1}^n a_ib_{\pi(i)} \ge \sum_{i=1}^n a_ib_{n-i+1}$ for $a_1\le\cdots\le a_n$,
$b_1\le\cdots\le b_n$ and π a permutation of $[n]$. More generally:
$\sum_{i=1}^n f_i(b_i) \ge \sum_{i=1}^n f_i(b_{\pi(i)}) \ge \sum_{i=1}^n f_i(b_{n-i+1})$
with $\big(f_{i+1}(x)-f_i(x)\big)$ nondecreasing for all $1\le i<n$.
Dually: $\prod_{i=1}^n(a_i+b_i) \le \prod_{i=1}^n(a_i+b_{\pi(i)}) \le \prod_{i=1}^n(a_i+b_{n-i+1})$ for $a_i,b_i\ge 0$.

Weierstrass $\prod_i (1-x_i)^{w_i} \ge 1 - \sum_i w_i x_i$ where $x_i \le 1$, and either $w_i \ge 1$ (for all i) or $w_i \le 0$ (for all i).
If $w_i \in [0,1]$, $\sum w_i \le 1$ and $x_i \le 1$, the reverse holds.

Kantorovich $\left(\sum_i x_i{}^2\right)\left(\sum_i y_i{}^2\right) \le \left(\frac{A}{G}\right)^2 \left(\sum_i x_i y_i\right)^2$ for $x_i, y_i > 0$,
$0 < m \le \frac{x_i}{y_i} \le M < \infty$, $A = (m+M)/2$, $G = \sqrt{mM}$.

Nesbitt $\sum_i^n \frac{a_i}{S-a_i} \ge \frac{n}{n-1}$ for $a_i \ge 0$, $S = \sum_{i=1}^n a_i$.

sum & integral $\int_{L-1}^{U} f(x)\,dx \le \sum_{i=L}^{U} f(i) \le \int_{L}^{U+1} f(x)\,dx$ for f nondecreasing.

Cauchy $f'(a) \le \frac{f(b)-f(a)}{b-a} \le f'(b)$ where $a < b$, and f convex.

Hermite $\varphi\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b \varphi(x)\,dx \le \frac{\varphi(a)+\varphi(b)}{2}$ for φ convex.

Gibbs $\sum_i a_i \log \frac{a_i}{b_i} \ge a \log \frac{a}{b}$ for $a_i, b_i \ge 0$, or more generally:
$\sum_i a_i \varphi\left(\frac{b_i}{a_i}\right) \le a\,\varphi\left(\frac{b}{a}\right)$ for φ concave, and $a = \sum a_i$, $b = \sum b_i$.

Chong $\sum_{i=1}^{n} \frac{a_i}{a_{\pi(i)}} \ge n$ and $\prod_{i=1}^{n} a_i{}^{a_i} \ge \prod_{i=1}^{n} a_i{}^{a_{\pi(i)}}$ for $a_i > 0$.

Schur $x^t(x-y)^k(x-z)^k + y^t(y-z)^k(y-x)^k + z^t(z-x)^k(z-y)^k \ge 0$
where $x, y, z, t, k \ge 0$.

Young $\left(\frac{1}{px^p} + \frac{1}{qy^q}\right)^{-1} \le xy \le \frac{x^p}{p} + \frac{y^q}{q}$ for $x, y, p, q > 0$, $\frac{1}{p} + \frac{1}{q} = 1$.

Shapiro $\sum_{i=1}^{n} \frac{x_i}{x_{i+1}+x_{i+2}} \ge \frac{n}{2}$ where $x_i > 0$, $(x_{n+1}, x_{n+2}) := (x_1, x_2)$,
and $n \le 12$ if even, $n \le 23$ if odd.

Hadamard $(\det A)^2 \le \prod_{i=1}^{n} \sum_{j=1}^{n} A_{ij}^2$ where A is an $n \times n$ matrix.

Schur $\sum_{i=1}^n \lambda_i^2 \le \sum_{i,j=1}^n A_{ij}^2$ and $\sum_{i=1}^k d_i \le \sum_{i=1}^k \lambda_i$ for $1 \le k \le n$.
A is an $n \times n$ matrix. For the second inequality A is symmetric.
$\lambda_1 \ge \cdots \ge \lambda_n$ the eigenvalues, $d_1 \ge \cdots \ge d_n$ the diagonal elements.

Ky Fan $\dfrac{\prod_{i=1}^n x_i^{a_i}}{\prod_{i=1}^n (1-x_i)^{a_i}} \le \dfrac{\sum_{i=1}^n a_i x_i}{\sum_{i=1}^n a_i(1-x_i)}$ for $x_i \in [0, \frac{1}{2}]$, $a_i \in [0,1]$, $\sum a_i = 1$.

Aczél $\left(a_1 b_1 - \sum_{i=2}^n a_i b_i\right)^2 \ge \left(a_1^2 - \sum_{i=2}^n a_i^2\right)\left(b_1^2 - \sum_{i=2}^n b_i^2\right)$
given that $a_1^2 > \sum_{i=2}^n a_i^2$ or $b_1^2 > \sum_{i=2}^n b_i^2$.

Mahler $\prod_{i=1}^{n} (x_i + y_i)^{1/n} \ge \prod_{i=1}^{n} x_i^{1/n} + \prod_{i=1}^{n} y_i^{1/n}$ where $x_i, y_i > 0$.

Abel $b_1 \cdot \min_k \sum_{i=1}^{k} a_i \le \sum_{i=1}^{n} a_i b_i \le b_1 \cdot \max_k \sum_{i=1}^{k} a_i$ for $b_1 \ge \cdots \ge b_n \ge 0$.

Milne $\left(\sum_{i=1}^n (a_i + b_i)\right)\left(\sum_{i=1}^n \frac{a_i b_i}{a_i + b_i}\right) \le \left(\sum_{i=1}^n a_i\right)\left(\sum_{i=1}^n b_i\right)$

Carleman $\sum_{k=1}^n \left(\prod_{i=1}^k |a_i|\right)^{1/k} \le e \sum_{k=1}^n |a_k|$

sum & product $\left|\prod_{i=1}^n a_i - \prod_{i=1}^n b_i\right| \le \sum_{i=1}^n |a_i - b_i|$ for $|a_i|, |b_i| \le 1$.
$\prod_{i=1}^n (t + a_i) \ge (t+1)^n$ where $\prod_{i=1}^n a_i \ge 1$, $a_i > 0$, $t > 0$.

Radon $\sum_i \frac{x_i^p}{a_i^{p-1}} \ge \frac{(\sum_i x_i)^p}{(\sum_i a_i)^{p-1}}$ for $x_i, a_i \ge 0$, $p \ge 1$ (rev. if $p \in [0,1]$).

Karamata $\sum_{i=1}^n \varphi(a_i) \ge \sum_{i=1}^n \varphi(b_i)$ for $a_1 \ge a_2 \ge \cdots \ge a_n$, $b_1 \ge \cdots \ge b_n$,
and $\{a_i\} \succeq \{b_i\}$ (majorization), i.e. $\sum_{i=1}^t a_i \ge \sum_{i=1}^t b_i$ for all $1 \le t \le n$,
with $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$, and φ convex (for concave φ the reverse holds).

Muirhead $\sum_\pi x_{\pi(1)}^{a_1} \cdots x_{\pi(n)}^{a_n} \ge \sum_\pi x_{\pi(1)}^{b_1} \cdots x_{\pi(n)}^{b_n}$, sums over permut. π of $[n]$,
where $a_1 \ge \cdots \ge a_n$, $b_1 \ge \cdots \ge b_n$, $\{a_k\} \succeq \{b_k\}$, $x_i \ge 0$.

Hilbert $\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{m+n} \le \pi \left(\sum_{m=1}^\infty a_m^2\right)^{\frac{1}{2}} \left(\sum_{n=1}^\infty b_n^2\right)^{\frac{1}{2}}$ for $a_m, b_n \in \mathbb{R}$.
With $\max\{m,n\}$ instead of $m+n$, we have 4 instead of π.

Hardy $\sum_{n=1}^\infty \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^p \le \left(\frac{p}{p-1}\right)^p \sum_{n=1}^\infty a_n^p$ for $a_n \ge 0$, $p > 1$.

Mathieu $\frac{1}{c^2 + 1/2} < \sum_{n=1}^\infty \frac{2n}{(n^2+c^2)^2} < \frac{1}{c^2}$ for $c \ne 0$.

Kraft $\sum 2^{-c(i)} \le 1$ for $c(i)$ depth of leaf i of binary tree, sum over all leaves.

LYM $\sum_{X \in \mathcal{A}} \binom{n}{|X|}^{-1} \le 1$, $\mathcal{A} \subset 2^{[n]}$, no set in $\mathcal{A}$ is subset of another set in $\mathcal{A}$.

FKG $\Pr[x \in \mathcal{A} \cap \mathcal{B}] \ge \Pr[x \in \mathcal{A}] \cdot \Pr[x \in \mathcal{B}]$, for $\mathcal{A}, \mathcal{B}$ monotone set systems.

Sauer-Shelah $|\mathcal{A}| \le |\mathrm{str}(\mathcal{A})| \le \sum_{i=0}^{\mathrm{vc}(\mathcal{A})} \binom{n}{i}$ for $\mathcal{A} \subseteq 2^{[n]}$, and
$\mathrm{str}(\mathcal{A}) = \{X \subseteq [n] : X \text{ shattered by } \mathcal{A}\}$, $\mathrm{vc}(\mathcal{A}) = \max\{|X| : X \in \mathrm{str}(\mathcal{A})\}$.

Khintchine $\sqrt{\sum_i a_i^2} \ge \mathrm{E}\left[\left|\sum_i a_i r_i\right|\right] \ge \frac{1}{\sqrt{2}}\sqrt{\sum_i a_i^2}$ where $a_i \in \mathbb{R}$, and
$r_i \in \{\pm 1\}$ random variables (r.v.) i.i.d. w.pr. $\frac{1}{2}$.

Bonferroni $\Pr\left[\bigvee_{i=1}^{n} A_i\right] \le \sum_{j=1}^{k} (-1)^{j-1} S_j$ for $1 \le k \le n$, k odd (rev. for k even),
$S_k = \sum_{1 \le i_1 < \cdots < i_k \le n} \Pr\left[A_{i_1} \wedge \cdots \wedge A_{i_k}\right]$ where A_i are events.

Bhatia-Davis $\mathrm{Var}[X] \le \left(M - \mathrm{E}[X]\right)\left(\mathrm{E}[X] - m\right)$ where $X \in [m, M]$.

Samuelson $\mu - \sigma\sqrt{n-1} \le x_i \le \mu + \sigma\sqrt{n-1}$ for $i = 1, \ldots, n$,
where $\mu = \sum x_i/n$, $\sigma^2 = \sum (x_i - \mu)^2/n$.

Markov

$\Pr\big[|X| \ge a\big] \le \mathrm{E}\big[|X|\big]/a$ where X is a r.v., $a > 0$.

$\Pr\big[X \le c\big] \le (1-\mathrm{E}[X])/(1-c)$ for $X \in [0,1]$ and $c \in \big[0, \mathrm{E}[X]\big]$.

$\Pr\big[X \in S\big] \le \mathrm{E}[f(X)]/s$ for $f \ge 0$, and $f(x) \ge s > 0$ for all $x \in S$.

Chebyshev

$\Pr\big[\big|X - \mathrm{E}[X]\big| \ge t\big] \le \mathrm{Var}[X]/t^2$ where $t > 0$.

$\Pr\big[X - \mathrm{E}[X] \ge t\big] \le \mathrm{Var}[X]/(\mathrm{Var}[X] + t^2)$ where $t > 0$.

2^{nd} *moment*

$\Pr\big[X > 0\big] \ge (\mathrm{E}[X])^2/(\mathrm{E}[X^2])$ where $\mathrm{E}[X] \ge 0$.

$\Pr\big[X = 0\big] \le \mathrm{Var}[X]/(\mathrm{E}[X^2])$ where $\mathrm{E}[X^2] \ne 0$.

k^{th} *moment*

$$\Pr\big[|X-\mu| \ge t\big] \le \frac{\mathrm{E}\left[(X-\mu)^k\right]}{t^k} \quad \text{and}$$

$$\Pr\big[|X-\mu| \ge t\big] \le C_k \left(\frac{nk}{et^2}\right)^{k/2} \quad \text{for } X_i \in [0,1] \text{ } k\text{-wise indep. r.v.,}$$

$X = \sum X_i$, $i = 1, \ldots, n$, $\mu = \mathrm{E}[X]$, $C_k = 2\sqrt{\pi k}e^{1/6k}$, k even.

2^{nd} *and* 4^{th}

$$\mathrm{E}\big[|X|\big] \ge \frac{\left(\mathrm{E}\left[X^2\right]\right)^{3/2}}{\left(\mathrm{E}\left[X^4\right]\right)^{1/2}} \quad \text{where } 0 < \mathrm{E}\left[X^4\right] < \infty.$$

$$\Pr\Big[X \ge \frac{\sigma}{2\sqrt{t}}\Big] > 0 \quad \text{where } \mathrm{E}[X] = 0,\ \mathrm{E}[X^2] = \sigma^2,\ 0 < \mathrm{E}[X^4] \le t\sigma^4.$$

Chernoff

$\Pr[X \ge t] \le F(a)/a^t$ for X r.v., $\Pr[X = k] = p_k$,

$F(z) = \sum_k p_k z^k$ probability gen. func., and $a \ge 1$.

$$\Pr\big[X \ge (1+\delta)\mu\big] \le \left(\frac{e^\delta}{(1+\delta)^{(1+\delta)}}\right)^\mu \le \exp\left(\frac{-\mu\delta^2}{3}\right)$$

for X_i i.r.v. from [0,1], $X = \sum X_i$, $\mu = \mathrm{E}[X]$, $\delta \ge 0$ resp. $\delta \in [0,1)$.

$$\Pr\big[X \le (1-\delta)\mu\big] \le \left(\frac{e^{-\delta}}{(1-\delta)^{(1-\delta)}}\right)^\mu \le \exp\left(\frac{-\mu\delta^2}{2}\right) \text{ for } \delta \in [0,1).$$

Further from the mean: $\Pr\big[X \ge R\big] \le 2^{-R}$ for $R \ge 2e\mu$ ($\approx 5.44\mu$).

$$\Pr\big[X \ge t\big] \le \frac{\binom{n}{k}p^k}{\binom{t}{k}} \quad \text{for } X_i \in \{0,1\} \text{ } k\text{-wise i.r.v., } \mathrm{E}[X_i] = p,\ X = \textstyle\sum X_i.$$

$$\Pr\big[X \ge (1+\delta)\mu\big] \le \tbinom{n}{\hat{k}}p^{\hat{k}} \Big/ \tbinom{(1+\delta)\mu}{\hat{k}} \quad \text{for } X_i \in [0,1] \text{ } k\text{-wise i.r.v.,}$$

$k \ge \hat{k} = \lceil \mu\delta/(1-p) \rceil$, $\mathrm{E}[X_i] = p_i$, $X = \sum X_i$, $\mu = \mathrm{E}[X]$, $p = \frac{\mu}{n}$, $\delta > 0$.

Hoeffding

$$\Pr\big[\big|X - \mathrm{E}[X]\big| \ge \delta\big] \le 2\exp\left(\frac{-2\delta^2}{\sum_{i=1}^n (b_i - a_i)^2}\right) \quad \text{for } X_i \text{ i.r.v.,}$$

$X_i \in [a_i, b_i]$ (w. prob. 1), $X = \sum X_i$, $\delta \ge 0$.

A related lemma, assuming $\mathrm{E}[X] = 0$, $X \in [a,b]$ (w. prob. 1) and $\lambda \in \mathbb{R}$:

$$\mathrm{E}\big[e^{\lambda X}\big] \le \exp\left(\frac{\lambda^2(b-a)^2}{8}\right)$$

Kolmogorov

$$\Pr\big[\max_k |S_k| \ge \varepsilon\big] \le \frac{1}{\varepsilon^2}\mathrm{Var}[S_n] = \frac{1}{\varepsilon^2}\sum_i \mathrm{Var}[X_i]$$

where $X_1, \ldots, X_n$ are i.r.v., $\mathrm{E}[X_i] = 0$,

$\mathrm{Var}[X_i] < \infty$ for all i, $S_k = \sum_{i=1}^k X_i$ and $\varepsilon > 0$.

Paley-Zygmund

$$\Pr\big[X \ge \mu\,\mathrm{E}[X]\,\big] \ge 1 - \frac{\mathrm{Var}[X]}{(1-\mu)^2\,(\mathrm{E}[X])^2 + \mathrm{Var}[X]} \quad \text{for } X \ge 0,$$

$\mathrm{Var}[X] < \infty$, and $\mu \in (0,1)$.

Vysochanskij-Petunin-Gauss

$\Pr\big[\big|X - \mathrm{E}[X]\big| \ge \lambda\sigma\big] \le \frac{4}{9\lambda^2}$ if $\lambda \ge \sqrt{\frac{8}{3}}$,

$\Pr\big[\big|X - m\big| \ge \varepsilon\big] \le \frac{4\tau^2}{9\varepsilon^2}$ if $\varepsilon \ge \frac{2\tau}{\sqrt{3}}$,

$\Pr\big[\big|X - m\big| \ge \varepsilon\big] \le 1 - \frac{\varepsilon}{\sqrt{3}\tau}$ if $\varepsilon \le \frac{2\tau}{\sqrt{3}}$.

Where X is a unimodal r.v. with mode m,

$\sigma^2 = \mathrm{Var}[X] < \infty$, $\tau^2 = \mathrm{Var}[X] + (\mathrm{E}[X] - m)^2 = \mathrm{E}[(X-m)^2]$.

Etemadi

$$\Pr\big[\max_{1 \le k \le n} |S_k| \ge 3\alpha\big] \le 3 \max_{1 \le k \le n} \big(\Pr\big[\,|S_k| \ge \alpha\big]\big)$$

where X_i are i.r.v., $S_k = \sum_{i=1}^k X_i$, $\alpha \ge 0$.

Doob

$\Pr\big[\max_{1 \le k \le n} |X_k| \ge \varepsilon\big] \le \mathrm{E}\big[|X_n|\big]/\varepsilon$ for martingale (X_k) and $\varepsilon > 0$.

Bennett

$$\Pr\Big[\sum_{i=1}^n X_i \ge \varepsilon\,\Big] \le \exp\left(-\frac{n\sigma^2}{M^2}\,\theta\Big(\frac{M\varepsilon}{n\sigma^2}\Big)\right) \quad \text{where } X_i \text{ i.r.v.,}$$

$\mathrm{E}[X_i] = 0$, $\sigma^2 = \frac{1}{n}\sum \mathrm{Var}[X_i]$, $|X_i| \le M$ (w. prob. 1), $\varepsilon \ge 0$,

$\theta(u) = (1+u)\log(1+u) - u$.

Bernstein

$$\Pr\Big[\sum_{i=1}^n X_i \ge \varepsilon\,\Big] \le \exp\left(\frac{-\varepsilon^2}{2(n\sigma^2 + M\varepsilon/3)}\right) \quad \text{for } X_i \text{ i.r.v.,}$$

$\mathrm{E}[X_i] = 0$, $|X_i| < M$ (w. prob. 1) for all i, $\sigma^2 = \frac{1}{n}\sum \mathrm{Var}[X_i]$, $\varepsilon \ge 0$.

Azuma

$$\Pr\big[\big|X_n - X_0\big| \ge \delta\big] \le 2\exp\left(\frac{-\delta^2}{2\sum_{i=1}^n {c_i}^2}\right) \quad \text{for martingale } (X_k) \text{ s.t.}$$

$\big|X_i - X_{i-1}\big| < c_i$ (w. prob. 1), for $i = 1, \ldots, n$, $\delta \ge 0$.

Efron-Stein

$$\mathrm{Var}[Z] \le \frac{1}{2}\,\mathrm{E}\left[\sum_{i=1}^n \left(Z - Z^{(i)}\right)^2\right] \quad \text{for } X_i, {X_i}' \in \mathcal{X} \text{ i.r.v.,}$$

$f : \mathcal{X}^n \to \mathbb{R}$, $Z = f(X_1, \ldots, X_n)$, $Z^{(i)} = f(X_1, \ldots, {X_i}', \ldots, X_n)$.

McDiarmid

$$\Pr\big[\big|Z - \mathrm{E}[Z]\big| \ge \delta\big] \le 2\,\exp\left(\frac{-2\delta^2}{\sum_{i=1}^n c_i^2}\right) \quad \text{for } X_i, {X_i}' \in \mathcal{X} \text{ i.r.v.,}$$

Z, $Z^{(i)}$ as before, s.t. $\big|Z - Z^{(i)}\big| \le c_i$ for all i, and $\delta \ge 0$.

Janson

$$M \le \Pr\big[\textstyle\bigwedge \overline{B}_i\big] \le M\exp\left(\frac{\Delta}{2 - 2\varepsilon}\right) \quad \text{where } \Pr[B_i] \le \varepsilon \text{ for all } i,$$

$M = \prod(1 - \Pr[B_i])$, $\Delta = \sum_{i \ne j, B_i \sim B_j} \Pr[B_i \wedge B_j]$.

Lovász

$$\Pr\big[\textstyle\bigwedge \overline{B}_i\big] \ge \prod(1 - x_i) > 0 \quad \text{where } \Pr[B_i] \le x_i \cdot \prod_{(i,j) \in D}(1 - x_j),$$

for $x_i \in [0,1)$ for all $i = 1, \ldots, n$ and D the dependency graph.

If each B_i mutually indep. of all other events, except at most d,

$\Pr[B_i] \le p$ for all $i = 1, \ldots, n$, then if $ep(d+1) \le 1$ then $\Pr\big[\bigwedge \overline{B}_i\big] > 0$.
