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Exploring the role of entanglement in sets of behaviors from prepare-and-measure scenarios

Explorando o papel do emaranhamento em conjuntos de comportamentos de cenários de prepara-e-mede

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Tese apresentada ao Instituto de Matemática, Estatística e Computação Científica da Universidade Estadual de Campinas como parte dos requisitos exigidos para a obtenção do título de Doutor em Matemática Aplicada.

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"The fundamental problem of communication is that of reproducing at one point either exactly or approximately a message selected at another point."

Claude Shannon

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Resumo

O estudo da comunicação quântica é o ramo da ciência que lida com a transmissão de informações usando sistemas quânticos. Esse ramo usa propriedades exclusivas dos sistemas quânticos, como superposição e emaranhamento, para realizar tarefas como codificação densa e distribuição quântica de chaves. Cenários de prepara-e-mede são abstrações úteis que fornecem uma base comum para esses protocolos. Nesses cenários, o procedimento envolve preparar o sistema em um determinado estado, transferi-lo de um local para outro e o medir de maneira a observar suas propriedades. Nos últimos anos, o uso do emaranhamento como recurso em cenários de prepara-e-mede começou a ser sistematicamente investigado, mas muitas questões cruciais ainda permanecem sem resposta. Neste trabalho, pretendemos explorar tais cenários e fornecer respostas a algumas das questões seminais nesta área. Especificamente, mostramos que em cenários onde o emaranhamento é um recurso livre, as mensagens quânticas são equivalentes às mensagens clássicas com o dobro da capacidade. Além disso, provamos que é sempre vantajoso para as partes compartilhar estados emaranhados de dimensão maior que a mensagem transmitida. Além disso, mostramos que estados unsteerable não fornecem nenhuma vantagem em tarefas clássicas de comunicação. Isso prova que nem todos os estados emaranhados são recursos úteis nesses cenários e estabelece um vínculo interessante entre a steering quântico e a não-classicidade em cenários de prepara-e-mede. No geral, este trabalho fornece novos insights sobre o potencial de usar o emaranhamento como um recurso em tarefas de comunicação e destaca a importância de mais pesquisas nesta área.

Palavras-chave: Informação quântica. Comunicação quântica. Cenários de prepara-e-mede. Assistência por emaranhamento.

Abstract

Quantum communication is the branch of science that deals with the transmission of information using quantum mechanical systems. It uses the unique properties of quantum systems, such as superposition and entanglement, to accomplish tasks like dense coding and quantum key distribution. Prepare-and-measure scenarios are useful abstractions that provide a common basis for these protocols. The procedure involves preparing the system in a particular state, transferring it from one location to another, and then measuring it to observe its properties. In recent years, the use of entanglement as a resource in prepare-and-measure scenarios has started to be systematically investigated, but many crucial questions still remain unanswered. In this work, we aim to explore such scenarios and provide answers to some of the seminal questions in this field. Specifically, we show that in scenarios where entanglement is a free resource, quantum messages are equivalent to classical messages with twice the capacity. Additionally, we prove that it is always advantageous for the parties to share entangled states of dimension greater than the transmitted message. Furthermore, we show that unsteerable states do not provide any advantage in classical communication tasks. This proves that not all entangled states are useful resources in these scenarios, and establishes an interesting link between quantum steering and nonclassicality in prepare-and-measure scenarios. Overall, this work provides new insights into the potential of using entanglement as a resource in communication tasks and highlights the importance of further research in this field.

Keywords: Quantum Information. Quantum communication. Prepare-and-measure scenarios. Entanglement assistance.

List of abbreviations and acronyms

PM	Prepare-and-measure
EA-PM	Entanglement-assisted prepare-and-measure
RAC	Random access codes
CCP	Communication complexity problems
POVM	Positive operator-valued measure
CPTP	Completely positive trace-preserving
SDP	Semidefinite program

List of symbols

$\mathbb{R}, \mathbb{C}$	Real and complex numbers
$\log_2$	Binary logarithm, i.e. logarithm to the basis 2
A, B, AB	Physical systems and joint systems
$\mathcal{H}, \mathcal{H}_A$	Hilbert space
$\mathcal{H}_{AB}$	Hilbert spaces corresponding to a joint quantum system AB
ρ	Quantum state (density matrix)
$\|\psi\rangle, \|a\rangle$	Pure states
ρ_{AB}	Quantum state of a joint system AB
$\|\psi\rangle\langle\psi\|, \|a\rangle\langle a\|$	Projection operator
$\langle\psi\|\psi'\rangle$	Inner product
Tr	Trace
Tr_A	Partial trace over system A
$\mathcal{L}(\mathcal{H})$	Linear operators acting on $\mathcal{H}$
$\mathbb{1}_A$	Identity matrix over a Hilbert space $\mathcal{H}_A$
$[d]$	Is the set of the first d non-negative integers, *i.e.*, $[d] := \{0, \cdots, d-1\}$

List of papers

The content of this thesis is based on results developed in the following paper:

- *Interplays between classical and quantum entanglement-assisted communication scenarios*

  Carlos Vieira, Carlos de Gois, Lucas Pollyceno and Rafael Rabelo

  arxiv: 2205.05171

The author has also authored the following research papers:

- *No-broadcasting theorem for non-signaling boxes and assemblages*

  Carlos Vieira, Adrian Solymos, Cristhiano Duarte and Zoltán Zimboras

  arxiv: 2211.14351

- *Bell Non-Locality in Many-Body Quantum Systems with Exponential Decay of Correlations.*

  Carlos Vieira, Cristhiano Duarte, Raphael Campos Drumond and Marcelo Terra Cunha

  Brazilian Journal of Physics 51 (6), 1603-1616

Contents

Introduction

It was realized during the end of the last century that quantum information processing offered enormous advantages over traditional classical methods. With the use of quantum systems, it was shown to be possible to implement qualitatively superior cryptographic protocols, like the BB84 and E91 distribution protocol [1,2], and highly efficient algorithms, like Shor's factoring algorithm [3]. Combined with Holevo's theorem [4] and Schumacher's compression scheme [5,6], these codes form the basis for what we know now as quantum information theory.

Considering quantum information's advantages for computation and cryptography, it is natural to wonder whether quantum information can also be used to improve communication. At the same time, Holevo's theorem states that the transmission of any classical message, in terms of quantum bits, costs the same as transmission in terms of classical bits [4]. This result casts doubt on the existence of an advantage of using quantum communication. Nevertheless, several outstanding communication protocols have been found where quantum systems can be advantageous over their classical siblings. Among these, dense coding and random access codes (RAC) stand out [7–10]. Behind the scenes, something is connecting both protocols. They are instances of a more general scenario, named prepare-and-measure (PM) [11].

A prepare-and-measure scenario involves one agent (Alice) preparing a physical system, based on an input x, and sending it to a second agent (Bob). Following that, based on an input y, Bob measures the incoming system and produces output b. Operationally, this prepare-and-measure scenario is entirely characterized by the conditional probabilities $\mathrm{p(b|x,\ y)}$. These distributions, in turn, describe Alice's and Bob's correlations and are limited by the amount of communication stated between Alice and Bob.

Being one of the simplest correlation scenarios that presume communication, they are indispensable ingredients in quantum information processing protocols such as semi-device independent dimension certification [11–15]. They are also crucial in the analysis of quantum communication networks [16,17]. Additionally, they provide means for self-testing states [18,19], distributing quantum keys [20,21], generating randomness [22–24], and playing an essential part in the discussion of the informational principles of quantum theory [25,26].

Besides the mentioned cases of RAC and dense coding, PM quantum behaviors lead to advantages over classically achievable behaviors in several other communication tasks. This feature, as in many other correlation scenarios, explains the significant attention the PM scenario has been receiving. A cornerstone of quantum information to determine *how* and *which* quantum systems outperform their classical counterparts. The possible behaviors in a PM scenario naturally depend on the available resources. For instance, one may discuss nonclassicality by comparing quantum against classical communication aided by shared randomness. This is arguably the most studied instance of PM scenarios, but even so, there are still many unresolved questions. A paradigmatic example is the connection between measurement incompatibility and nonclassicality. It has been previously shown that incompatible measurements are insufficient for the emergence of nonclassical behaviors in the PM scenario [27]. In a particular case of random access codes, it was proven necessary [28].

More recently, a generalization to *entanglement-assisted* prepare-and-measure (EA-PM) scenarios began to be systematically investigated [29–36]. Similarly to standard PM scenarios, we may consider either quantum or classical communication. However, for EA-PM scenarios, the devices may include a pre-established correlation between them through a shared quantum state ρ_{AB}. In this case, the quantum entanglement enables higher performance in a handful of paradigmatic communication tasks, such as entanglement-assisted random access codes [20] and dense coding [32]. Under the hood, these advantages mean that the sets of behaviors can be significantly distinct, depending on the allowed resources. It is thus of paramount importance to understand *how* and *why* they differ. This is exactly the purpose of this present work.

This thesis delves deeper into the findings outlined in [37] by conducting a more detailed and complete analysis to gain a more comprehensive understanding. The thesis is structured into two chapters, the first focuses on the study of prepare-and-measure scenarios where the parties involved have classical correlations. In this chapter, we introduce the concepts of classical and quantum behavior sets (**Sec. 1.1**). We review the random access codes protocol, thus presenting an example of a protocol with quantum advantage (**Sec. 1.2**). Following that, we discuss the Holevo's bound (**Sec. 1.3**) and the Frenkel and Weiner theorem (**Sec. 1.4**) to understand how delicate the existence of quantum advantages over classical communication is. We end this chapter by presenting the first key result from [37], which establishes that measurement incompatibility is necessary for nonclassicality in PM scenarios (**Sec. 1.5**).

The second chapter is devoted to the entanglement-assisted prepare-and-measure scenarios *i.e.*, prepare-and-measure scenarios where the parts are quantumly correlated. We begin by introducing the definitions of behaviors obtained with classical and quantum communication on the entanglement assistance paradigm (**Sec. 2.1**). We proceed with the presentation of the dense coding protocol and its adequacy in a prepare-and-measure scenario (**Sec. 2.2**). In the next section, we discuss the optimality of this protocol (**Sec. 2.3**). In the light of the previous discussion, we present a geometric approach to the relationships between the sets of quantum and classical behavior in EA-PM

(**Sec. 2.4**). The following four sections present the other results obtained in [37]. The first result is a comparison of the sets of entanglement-assisted behaviors with classical against quantum communication. In this regard, we derive a chain of inclusions between the sets of behaviors and show that, in the limit of arbitrary-dimensional entanglement, the correlations implied by quantum and classical communication are identical (**Sec. 2.5**). Furthermore, we introduce a family of PM inequalities for each dimension of the message d, which are always violated if the parties share a pair of maximally entangled qubits (**Sec. 2.6**). Moreover, we extend the result of [33] showing that for any dimension of the communicated message, a high-dimensional entangled state leads to better performance (**Sec. 2.7**). We also show that unsteerable states do not provide an advantage in classical communication tasks (**Sec. 2.8**), therefore solving two of the open questions proposed in [34]. We conclude the thesis with a summary of our results and an in-depth discussion of possible future works. To facilitate the reading, we have also written some mathematical appendices where the reader can find the proofs of some theorems.

1 Prepare-and-measure scenarios

The prepare-and-measure (PM) scenario is one of the simplest and most fundamental correlation scenarios. From a physical point of view, a preparation apparatus produces a physical system and then sends it, over a communication channel, to a measurement device that reads out information from the received state. Protocols based on quantum communication may provide an advantage over classical protocols. In addition to quantum communication, a resource that leads to different behaviors is the pre-established correlation between the preparation and measurement devices. Generally, there are two classical possibilities: complete independence and shared randomness. In this chapter, we will explore these two possibilities in both classical and quantum cases and discuss the difference among these by exploring RAC formalism. We go on to review Holevo's bound and Frenkel and Weiner's theorem. Finally, we prove that measurement incompatibility is necessary for quantum advantage. This is the first original result of this thesis and is presented in [37].

1.1 The different sets of behaviors

A prepare-and-measure scenario (PM scenario) is composed of two agents, usually called Alice and Bob. Alice corresponds to the "preparation" part and Bob to the "measurement" part. An experiment, or protocol, in this scenario, works as follows: Alice randomly chooses an input x from an alphabet containing n_X distinct symbols, *i.e.*, $x \in [n_X] := \{0, \cdots, n_X - 1\}$. Similarly, Bob also randomly chooses an input $y \in [n_Y] := \{0, \cdots, n_Y - 1\}$. To this point, Alice and Bob do not know what input has been chosen by each other. The next step is for Alice to send[1] a message to Bob, being the latter conditionally dependent on the input choose x, and possibly described by a classical or a quantum system. Bob is then asked to provide an output $b \in [n_B] := \{0, \cdots, n_B - 1\}$, which, in turn, depends on its choosen input y. In **Fig 1**, we illustrate this process.

[1] In this thesis, we will reserve the term "send" for the physical act of sending an object between agents. On the other hand, we will use the term "transmit" as the most abstract notion of information traffic. For example, when we say that Alice *sends* a qudit to Bob, she is actually sending a physical system over a communication channel. Conversely, when we say that Alice *transmits* one bit of information to Bob, we are not necessarily saying that a classical channel was used to transmit that bit. However, some physical system was sent from Alice to Bob, and Bob interacting with this system gets one bit of information.

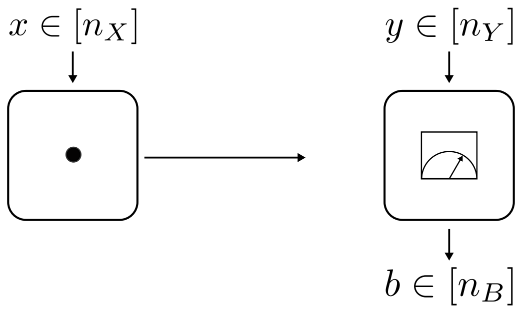


Figure 1 – Operational representation of a PM scenario. The left box performs a preparation, conditioned on input $x \in [n_X]$, and sends the prepared system to the right box, which is a measurement device with measurement choices $y \in [n_Y]$ and output $b \in [n_B]$.

In this text, we will consider the message sent from Alice to Bob as a *latent* variable, *i.e.*, an unobservable variable that is not directly observed or measured. This way, the observable variables in the experiment will be x, y, and b. By repeating this process several times and collecting the data, Alice and Bob can estimate the conditional probability that Bob will output b given that Alice and Bob's inputs were x and y, respectively. We denote this probability by $p(b|x, y)$ and the vector $\boldsymbol{p}$ whose elements correspond to conditional probabilities for every combination of x, y and b, we call *behavior* [11, 38]. These probability distributions are limited by the amount of communication carried by the physical systems from Alice to Bob. This thesis focuses on analyzing properties and relationships between the different sets of behaviors that can be obtained from different resources used by the agents.

Before proceeding to these different cases, it is essential to mention that the scenario becomes trivial if Alice is allowed to send unlimited information. If Alice can send her input to Bob, he will know both variables x, y. Hence, with a local random source, Bob can simulate *any* behavior $\boldsymbol{p}$. Therefore, we must require that the message be constrained so it cannot encode all of Alice's input information. The most common approach, which will be the only one used in this thesis, is by limiting the dimension of the exchanged message, *i.e.,* the alphabet size for classical messages and the dimension of the Hilbert space for quantum messages [9, 11, 18]. Another exciting proposal is limiting the mutual information between Alice's input and the message transmitted to Bob [39, 40]. Recently, an intermediate idea between these two concepts was developed in [41].

1.1.1 Classical behaviors

In this subsection, we will describe the set of behaviors that can be obtained when Alice is sending classical information to Bob. Such behaviors will generally be called *classical behaviors*. Inspired by Information Theory [42], when we say that Alice sends classical information of dimension d to Bob, we consider that she sends one *d*it to Bob. A dit represents a logical state with one of d possible values. These values are commonly represented as a natural number in $[d] = \{0, \cdots, d-1\}$ [43]. The amount of information is limited by the dimension of the message, *i.e.*, the size d. As discussed earlier, the limitation on the message dimension is always required. If $d \geqslant n_X$, Alice can send her input to Bob, and they can simulate any behavior $\boldsymbol{p}$. We will not be interested in this situation since any extra resource is incapable of generating any difference in the set of behaviors. Therefore, we will always consider that $d < n_X$.

Let us rank the classical behaviors into three groups, depending on what kind of resources Alice and Bob own. The first set is called the set of *d-deterministic behaviors*, in which the agents' strategy is given by: (i) Alice using a fixed *encoding function* that transforms her input in a dit; (ii) this dit is sent to Bob; (iii) Bob chooses one of a set of *decoding functions* to apply in the message and returns this result as his output.

Definition 1 (d-deterministic behaviors [44])**.** *A behavior $\boldsymbol{p} = \{p(b|x,y)\}_{x,y,b}$ is a d-deterministic behavior if there exists an encoding function $\mathcal{E} : [n_X] \to [d]$ and a tuple $(\mathcal{D}_0, \cdots, \mathcal{D}_{n_Y-1})$ of decoder functions, with $\mathcal{D}_y : [d] \to [n_B]$ for all $y \in [n_Y]$, such that*

$$p(b|x,y) = \delta(b, (\mathcal{D}_y \circ \mathcal{E})(x)). \tag{1.1}$$

We denote the set of all d-deterministic behaviors by $D_d(n_X, n_Y, n_B)$.

It follows that $D_d(n_X, n_Y, n_B)$ always has a finite number of elements. In fact, both the sets of functions $\mathcal{E} : [n_X] \to [d]$ and the set of functions $\mathcal{D} : [d] \to [n_B]$ are finite. Thus, the number of possible combinations of these encoding and decoding functions resulting in eq. (1.1), is also finite.

Remark 1. *From now on, we will sometimes omit the labels (n_X, n_Y, n_B) on the different types of sets of behaviors to be presented. Moreover, it should be clear that whenever we are comparing two different sets, the comparison is made concerning the same set of labels (n_X, n_Y, n_B). For instance, when we say that $D_d \subseteq D_{d'}$, for $d \leqslant d'$, what we mean is that $D_d(n_X, n_Y, n_B) \subseteq D_{d'}(n_X, n_Y, n_B)$.*

Remark 2. *An alternative way to define d-classical determinist behaviors is given by the probability distributions that can be decomposed as:*

$$p(b|x,y) = \sum_{a=0}^{d-1} p_D(a|x) p_D(b|y,a), \tag{1.2}$$

where: (i) the variable "a" denotes the classical message sent by Alice; and (ii) where $p_D(a|x)$ and $p_D(b|y,a)$ are deterministic, i.e., for every x there is $a_0 \in \{0, \cdots, d-1\}$ such that $p_D(a_0|x) = 1$ and $p_D(a|x) = 0$ for every $a \in \{0, \cdots, d-1\}$ with $a \neq a_0$. A similar property is also valid for $p_D(b|y,a)$. We call $p_D(a|x)$ by encoding probability distribution and $p_D(b|y,a)$ by decoding probability distribution. In ***Appendix A.1****, we show that these two definitions are equivalent.*

The second type of behavior is the *d-classical behavior*. In this circumstance, Alice and Bob can use non-deterministic encoding and decoding probability distribution.

Definition 2 (d-classical behavior)**.** *A behavior* $\boldsymbol{p} = \{p(b|x,y)\}_{x,y,b}$ *is a d-classical behavior if there exists (i) for each* $x \in [n_X]$*, an encoding probability distribution* $\{p(a|x)\}_{a\in[d]}$ *for Alice; and (ii) for each* $y \in [n_Y]$ *and* $a \in [d]$*, a decoding probability distribution* $\{p(b|y,a)\}_{b\in[n_B]}$ *for Bob, such that:*

$$p(b|x,y) = \sum_{a=0}^{d-1} p(a|x)p(b|y,a), \tag{1.3}$$

where the variable "a" denotes the classical message sent by Alice. We denote the set of all d-classical behaviors by $C_d(n_X, n_Y, n_B)$*.*

Comparing eq. (1.2) and eq. (1.3) it follows that $D_d(n_X, n_Y, n_B)$ is a subset of $C_d(n_X, n_Y, n_B)$. However, unlike D_d, the set C_d is not finite. An easy way to see this is taking $p(a|x) = \delta(a,0)$ and *any* $p(b|y,a)$. Thus, $p(b|x,y) = p(b|0,y)$, and since the family of all probability distributions $p(b|0,y)$ is not finite, it follows that C_d has an infinite number of elements. Therefore, $D_d \subsetneq C_d$.

Remark 3. *Every time we write* $X \subseteq Y$*, we mean that* X *is a subset of* Y *that might be equal to* Y*. On the other hand, when we write* $X \subsetneq Y$*, then* X *is a proper (or strict) subset of* Y*, that is, every element of* X *is in* Y*, but* Y *contains at least one element that does not belong to* X*. Finally,* $X \nsubseteq Y$ *means that* X *is not a subset of* Y*, i.e., there is at least one element of* X *that does not belong to* Y*.*

Before proceeding with the third and last type of classical behavior, let us introduce some general definitions. The first definition is of a *convex set*, followed by the definition of a *convex hull* of a set.

Definition 3 (Convex set)**.** *A set* $X \subseteq \mathbb{R}^n$ *is convex if* $x, y \in X$ *and* $\alpha \in [0,1]$ *implies in* $\alpha x + (1-\alpha)y$ *belongs to* X*.*

Definition 4 (Convex hull)**.** *The convex hull of a set* $X \subseteq \mathbb{R}^n$ *is the set of all convex combinations of points in* X*. We denote this set by* $\bar{X}$*.*

It follows that the convex hull of a set is a convex set.

Returning to the PM scenarios, we will introduce the set of *SR-d-classical behaviors*. In this case, we assume that the agents can be classically correlated. This correlation is generated through a random variable source Λ that draws a classical input λ (which we assume to be independent of Alice's and Bob's inputs) and distributes this result to both parties [45] before the experiment starts. In this circumstance, we say that Alice and Bob have a source of shared (or public) randomness available.

Definition 5 (SR-d-classical behavior)**.** *A behavior $\boldsymbol{p} = \{p(b|x,y)\}_{x,y,b}$ is a d-classical behavior with shared randomness (SR-d-classical) if there are:*

1. *a probability distribution $\{\pi(\lambda)\}_{\lambda\in\Lambda}$;*

2. *for each $x \in [n_X], \lambda \in \Lambda$, an encoding probability distribution $\{p(a|x,\lambda)\}_{a\in[d]}$ for Alice;*

3. *for each $y \in [n_Y]$, $a \in [d]$ and $\lambda \in \Lambda$, a decoding probability distribution $\{p(b|y,a,\lambda)\}_{b\in[n_B]}$ for Bob,*

such that:

$$p(b|x,y) = \sum_{\lambda}\sum_{a=0}^{d-1} \pi(\lambda)p(a|x,\lambda)p(b|y,a,\lambda). \tag{1.4}$$

We denote the set of all SR-d-classical behaviors by $\bar{C}_d(n_X, n_Y, n_B)$.

In **Fig 2** we illustrate this process.

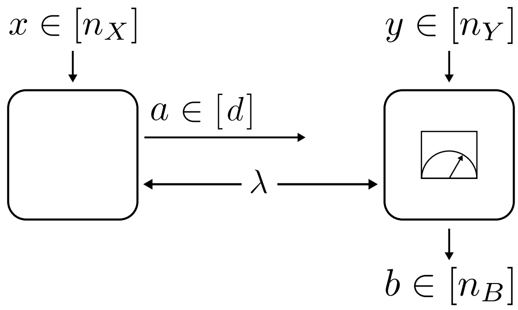


Figure 2 – Operational representations of a PM scenario with classical communication. Alice and Bob share a random variable $\lambda \in \Lambda$. Alice (on the left-hand side) owns a preparing device that sends a classical dit a to Bob based on the shared variable λ and her input $x \in [n_X]$. Bob holds a measuring device which, based on its input $y \in [n_Y]$ and the shared variable λ, does some post-processing on the dit a received from Alice to provide an output $b \in [n_B]$.

By fixing a λ, we have that $p(b|x,y,\lambda) = \sum_{a=0}^{d-1} p(a|x,\lambda)p(b|y,a,\lambda)$ is a d-classical behavior (see **Def. 2**). Then, the behavior

$$p(b|x,y) = \sum_{\lambda}\pi(\lambda)p(b|x,y,\lambda) = \sum_{\lambda}\sum_{a=0}^{d-1}\pi(\lambda)p(a|x,\lambda)p(b|y,a,\lambda),$$

is a convex combination of the d-classical behaviors. In this way, the set $\bar{C}_d$ is the convex hull of the C_d (see **Def. 4**). So, $\bar{C}_d$ is a convex set and $C_d \subseteq \bar{C}_d$. Hence, the addition of shared randomness in the experiment results in the convexification of the set of behaviors. The same will occur for the other sets of behaviors to be introduced in this text.

1.1.2 Quantum behaviors

A generalization of the scheme described in the last section is when Alice sends a quantum system to Bob. A qu*d*it is the quantum mechanical analog of a classical *d*it. It is a quantum unit of information that can be executed in appropriate d-level quantum systems [6].

Before introducing the concept of *quantum behaviors* in PM scenarios, let us briefly introduce some basic definitions of quantum theory. As already discussed, the state of a classical dit is represented by one value out of d possibilities. In contrast, the general state of a qudit can be a *superposition* of vectors in an d-dimensional orthonormal basis. This basis is usually denoted as $\{|0\rangle, \cdots, |d-1\rangle\}$ and is called *computational basis* [46, 47]. Generally, a (pure) quantum state of a qudit is a vector in $\mathbb{C}^d$ and is usually described by a linear superposition of the basis states:

$$|\psi\rangle = \sum_{i=0}^{d-1} \alpha_i |i\rangle, \tag{1.5}$$

where α_i are complex numbers satisfying a normalization constraint, $|\alpha_0|^2 + \cdots + |\alpha_{d-1}|^2 = 1$. Thus, the complete characterization of a (pure) qudit requires $d-1$ complex numbers[2].

The analog of a classical random variable is given by a probability distribution over superpositions of pure quantum states, called *mixed state* [48, 49]. Given a set of pure states $\{|\psi_i\rangle\}$ and a probability distribution $\{p_i\}$, the associated mixed state is characterized by the matrix $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, where $|\psi_i\rangle\langle\psi_i|$ is the projection operator on the vector space generated by the vector $|\psi_i\rangle\langle\psi_i|$. Mixed states are also called *density matrix*. In the particular case of qudits, these matrices are elements of $\mathcal{L}(\mathbb{C}^d)$, where $\mathcal{L}(\mathbb{C}^d)$ represents the set of operators acting on the vector space $\mathbb{C}^d$.

An equivalent way of defining the set of density matrices of a qudit can be stated as all $\rho \in \mathcal{L}(\mathbb{C}^d)$ satisfying [50]:

1. *Hermiticity:* ρ is hermitian, *i.e.*, $\rho = \rho^\dagger$;
2. *Positivity:* ρ is positive semi-definite, *i.e.*, $\langle\psi| \rho |\psi\rangle \geqslant 0 \ \forall \ |\psi\rangle \in \mathbb{C}^d$;
3. *Normalization:* ρ has unit trace, *i.e.*, $\mathrm{Tr}(\rho) = 1$.

The *only* way to extract information from a quantum system is through the process of *quantum measurements* [51]. In a quantum measurement, a test is performed on a quantum system, where one of a fixed number of responses is obtained. The predictions of these measurements are, in general, probabilistic [52].

[2] this dimension reduction can be understood as a result of normalization and elimination of the global phase.

A quantum measurement is described by a set of *measurement operators* defined in the Hilbert space of the system. Each operator is associated with a possible measurement result, and its mathematical nature varies according to the measurement class considered. The most general class of measurements are the *positive-operator-valued measures* (POVMs) [53–55].

A POVM[3] is a set of positive operators $\{M_b\}$ whose sum results in the identity operator, that is,

1. $M_b \geqslant 0 \; \forall \; b$;
2. $\sum_b M_b = \mathbb{1}$.

The POVM element M_b (called effect) is associated with the measurement result b. Provided the system's state is ρ, the probability of such result to be obtained is given by Born's rule [56]:

$$p(b) = \mathrm{Tr}(\rho M_b).$$

To summarise, information encoded in a quantum system is represented by density matrices, and the way to extract information from this system is through POVM measurements. In this way, the quantum implementation of the PM scenario can be described as follows: (i) for each received input $x \in [n_X]$, Alice's device will prepare a quantum state ρ_x and send it to Bob; (ii) based on his input $y \in [n_Y]$, Bob will perform a POVM measurement $\{M_{b|y}\}$ on the received system. The results of these measurements are probabilistic and given by Born's rule, $\mathrm{Tr}\big(\rho_x M_{b|y}\big)$.

As in the case of classical communication, a limitation on the amount of information is required for the experiment to be interesting. Here we will again limit the dimension of the message. We will assume that Alice is sending qudits to Bob, *i.e.*, $\rho_x \in \mathcal{L}(\mathbb{C}^d)$. Moreover, we will assume that the dimension d is smaller than the number of different inputs available to Alice receives, *i.e.*, $d < n_X$. More concretely, we define the set of quantum behaviors as follows:

Definition 6 (d-quantum behaviors [11, 38])**.** *A behavior $\{p(b|x,y)\}_{b,x,y}$ is a d-quantum behavior if there exists a set of quantum states $\{\rho_x\}_{x\in[n_X]} \subset \mathcal{L}(\mathbb{C}^d)$ and for each $y \in [n_Y]$, a n_B-outcome POVM $\{M_{b|y}\}$ on $\mathcal{L}(\mathbb{C}^d)$, such that:*

$$p(b|x,y) = \mathrm{Tr}\big(\rho_x M_{b|y}\big). \tag{1.6}$$

We denote the set of all d-quantum behaviors by $Q_d(n_X, n_Y, n_B)$.

[3] In the situation where more than one POVM measurement is performed in the same experiment, we will label the POVM measurements by $\{M_{b|y}\}$, where the index y represents which POVM measurement is being performed and the index b represent the measurement result.

We now consider the situation where Alice and Bob have *shared randomness* available to them. In this case, we can assume that a classical random variable λ is drawn, and the obtained result is reported to Alice and Bob. Alice can use λ to decide which quantum systems to send to Bob. Likewise, Bob can also rely on λ to choose the measurement to implement (see **Fig. 3 for an illustration**). Concretely, the set of possible behaviors to be obtained by Alice and Bob with the addition of shared randomness is given by:

Definition 7 (SR-d-quantum behaviors [11,38])**.** *A behavior $\{p(b|x,y)\}_{b,x,y}$ is a d-quantum behavior with shared randomness (SR-d-quantum) if there exists*

1. *a probability distribution $\{\pi(\lambda)\}_{\lambda\in\Lambda}$;*
2. *for each $x \in [n_X]$ and $\lambda \in \Lambda$, a quantum states $\rho_x^\lambda \in \mathcal{L}(\mathbb{C}^d)$;*
3. *for each $y \in [n_Y]$ and $\lambda \in \Lambda$, a n_B-outcome POVM $\{M_{b|y}^\lambda\}$ on $\mathcal{L}(\mathbb{C}^d)$;*

such that:

$$p(b|x,y) = \sum_\lambda \pi(\lambda)\operatorname{Tr}\big(\rho_x^\lambda M_{b|y}^\lambda\big). \tag{1.7}$$

We denote the set of all SR-d-quantum behaviors by $\bar{Q}_d(n_X, n_Y, n_B)$.

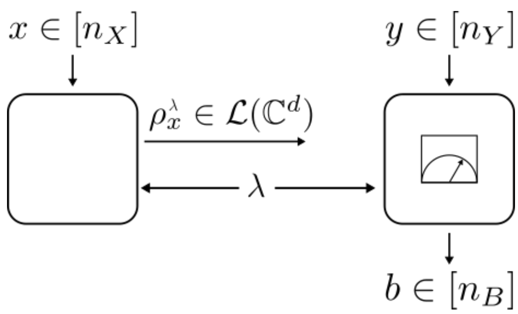


Figure 3 – Operational representations of a PM scenario with quantum communication. Classical pre-established correlations between Alice (left side) and Bob (right side) are given by a random variable $\lambda \in \Lambda$. Alice devices, based on her input $x \in [n_X]$ and on the shared variable λ, prepares a quantum dit ρ_x^λ. This qudit is sent to Bob, who performs a quantum measurement $\{M_{b|y,\lambda}\}$ based on his input $y \in [n_Y]$ and the shared variable λ. The result of this measurement, $b \in [n_B]$, is provided by Bob as his output.

Specifying λ, $p(b|x,y,\lambda) = \operatorname{Tr}\big(\rho_x^\lambda M_{b|y}^\lambda\big)$ is a d-quantum behavior (see **Def. 6**). As a result,

$$p(b|x,y) = \sum_\lambda \pi(\lambda)p(b|x,y,\lambda) = \sum_\lambda \operatorname{Tr}\big(\rho_x^\lambda M_{b|y}^\lambda\big),$$

is a convex combination of the d-quantum behaviors. Thus, the set $\bar{Q}_d$ is the convex hull of the Q_d (see **Def. 4**). So, Q_d is a subset of $\bar{Q}_d$, which in turn is a convex set. It is straightforward to verify that every d-classical behavior is also a d-quantum behavior, *i.e.*, $C_d \subseteq Q_d$. To simulate a classical behavior $p(b|x,y) = \sum_{a=0}^{d-1} p(a|x)p(b|y,a)$, it is enough that Alice and Bob use states and measurements diagonal on the same basis [38, 40]. Indeed, let $\{|a\rangle\}_{a=0}^{d-1}$ be an orthonormal basis of $\mathbb{C}^d$, then for each $x \in [n_X]$, $b \in [n_B]$ and $y \in [n_Y]$ we define:

$$\rho_x = \sum_{a=0}^{d-1} p(a|x) |a\rangle\langle a| \, ,$$

and

$$M_{b|y} = \sum_{a=0}^{d-1} p(b|y,a) |a\rangle\langle a| \, .$$

Since $p(a|x)$ and $p(b|a,y)$ are probability distributions, it follows that ρ_x and $\{M_{b|y}\}$ are well-defined density matrices and POVMs, respectively. Hence,

$$\begin{aligned} p(b|x,y) &= \mathrm{Tr}\big(\rho_x M_{b|y}\big) && (1.8) \\ &= \mathrm{Tr}\left\{\left(\sum_{a=0}^{d-1} p(a|x) |a\rangle\langle a|\right)\left(\sum_{a'=0}^{d-1} p(b|y,a') |a'\rangle\langle a'|\right)\right\} && (1.9) \\ &= \sum_{a=0}^{d-1}\sum_{a'=0}^{d-1} p(a|x)p(b|y,a)\, \mathrm{Tr}(|a\rangle\langle a|\, |a'\rangle\langle a'|) && (1.10) \\ &= \sum_{a=0}^{d-1} p(a|x)p(b|y,a). && (1.11) \end{aligned}$$

So, every d-classical behavior is also a d-quantum behavior, *i.e.*, $C_d(n_X, n_Y, n_B) \subseteq Q_d(n_X, n_Y, n_B)$, for every choice of d, n_X, n_Y, n_B.

Remark 4. *If $X \subseteq Y$ and $\bar{X}, \bar{Y}$ are the convex hull of X and Y, respectively, then $\bar{X} \subseteq \bar{Y}$.*

By **Remark 4** and the already discussed inclusion $C_d(n_X, n_Y, n_B)$ is a subset of $Q_d(n_X, n_Y, n_B)$, it follows that $\bar{C}_d(n_X, n_Y, n_B) \subseteq \bar{Q}_d(n_X, n_Y, n_B)$, for every choice of d, n_X, n_Y, n_B. A stronger statement about the relationship between the classical and quantum sets will depend on the parameters d, n_X, n_Y, n_B. Indeed, in the next section (**Section 1.2**), we will see some examples where $\bar{C}_d(n_X, n_Y, n_B) \subsetneq \bar{Q}_d(n_X, n_Y, n_B)$. In other words, there are behaviors in prepare-and-measure scenarios that can be obtained by sending a qudit but cannot be obtained by sending a dit. On the other hand, in **Section 1.4** we will see non-trivial cases where $\bar{C}_d(n_X, n_Y, n_B) = \bar{Q}_d(n_X, n_Y, n_B)$, whereby non-trivial we mean cases where the sets $\bar{C}_d(n_X, n_Y, n_B), \bar{Q}_d(n_X, n_Y, n_B)$ do not contain all the behaviors of the scenario.

1.1.3 Geometric representation

Polytopes are geometric objects with flat sides (facets), being multidimensional generalizations of two-dimensional polygons and three-dimensional polyhedra [57]. Various branches of mathematics and applications rely upon a particular type of polytopes, the convex ones, which have the additional property of convexity (convex polytopes) [58, 59]. There are several ways to define a convex polytope, but in the quantum information context, two of them are the most frequent. The first consist of the intersection of half-spaces (half-space representation), while the second relies on the convex hull of a set of points (vertex representation).

The vertex representation (V-description) defines a convex polytope as every set, which is the convex hull (**Def. 4**) of a finite set of points [58]. Multiple V-descriptions exist for the same polytope, *i.e.*, the convex hull of different sets of points may result in the same set. The polytope's extremal points (vertices) give the minimal[4] V-description. An *extremal point* is a point that does not lie on any open line segment connecting two points.

Half-space representations define convex polytopes as intersections of finite numbers of half-spaces. A half-space, in $\mathbb{R}^n$, can be written as the whole set of vector $(x_1, \cdots, x_n)$ in $\mathbb{R}^n$ that satisfy a linear inequality:

$$\alpha_1 x_1 + \cdots + \alpha_n x_n \leqslant K,$$

where $\alpha_1, \cdots, \alpha_n$ are real number coefficients and the upper bound K is also a real number [58]. Similar to the V-description, several H-descriptions of a convex polytope may exist. However[5], there is a minimal H-description, which is unique, given by the set of facets of the polytope. *Faces* of convex polytopes are intersections of the polytope with half-spaces such that no interior points of the polytope lie on the half-space. In an n-dimensional polytope, vertices are 0-dimensional faces, edges are 1-dimensional faces, ..., *facets* are $(n-1)$-dimensional faces. In general, half-space intersections need not to be bound. However, for the case where these intersections result in a bounded set, the definitions provided by the V-description and the H-description are equivalent. This equivalence is proved in theorem 1.1 of [60].

The H-descriptions are essential in various correlation scenarios. For instance, these facets are known as Bell inequalities in Bell nonlocality scenarios [61]. There is still no general method to build H-descriptions of these polytopes directly. In contrast, enumerating the extremal points of classicality polytopes (the V-description) is conceptually straightforward. The process of casting one to the other (*i.e.*, enumerating the vertices or facets) can be carried out using specialized software such as PANDA [62]. Due to its computational complexity, this can only be done for the most uncomplicated cases.

Now that we have established some geometric concepts about convex sets, we can apply them in the study of quantum and classical behavior sets in PM scenarios. As

[4] in the sense that any other V-description of the polytope must contain all extreme points. In this way, the set of extreme points is a V-description and is contained in any other V-description.

[5] for a full-dimensional convex polytope

discussed in **Section 1.1.1**, it is straightforward to conclude that $D_d \subsetneq C_d \subseteq \bar{C}_d$. However, the relationship between D_d and $\bar{C}_d$ is even more profound. In fact, by a Fine-like theorem, one may conclude $\bar{C}_d$ is the convex hull of the set D_d.

Theorem 1 (Fine-like theorem for PM scenarios)**.** *The set $\bar{C}_d$ is the convex hull of the points in D_d. Moreover, the points of D_d are the extremal points of $\bar{C}_d$.*

As already discussed, D_d is a finite set of points, so $\bar{C}_d$ is a polytope, with extremal points given by the set D_d. The proof of the above theorem is provided in **Appendix A.2**.

It is straightforward to conclude that if $d' \leqslant d$, then $D_{d'} \subseteq D_d$. Moreover, if $d' < d \leqslant n_X$, $D_{d'}$ is a proper subset of D_d. In **Appendix A.3** is described a determinist behavior that belongs to D_d but does not belong to $D_{d'}$. Remembering that, by the PM Fine-like theorem (**Thrm. 1**) we know that D_d and $D_{d'}$ configure the set of extreme points of the polytopes $\bar{C}_d$ and $\bar{C}_{d'}$, respectively. Thus, putting together the previous remark and the PM Fine-like theorem, we conclude that $\bar{C}_{d'}$ is a proper subset of $\bar{C}_d$. In **Fig. 4** we summarize this information, where we pictorially illustrate the relationships between sets of behaviors in PM scenarios obtained with classical communication.

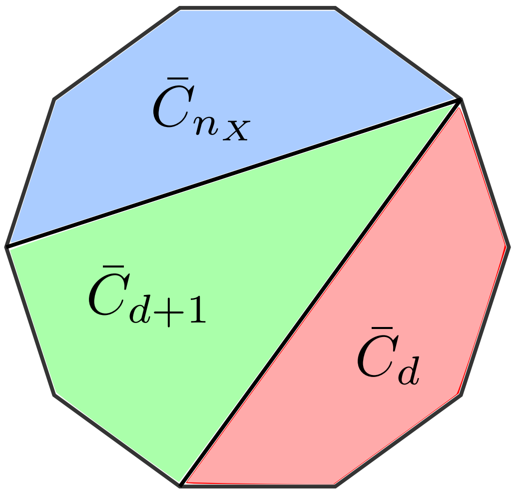


Figure 4 – Pictorial representation of the set of classical behaviors in a prepare-and-measure scenario, where $\bar{C}_d$ (obtained from d-dimensional classical communication with shared randomness) is contained within $\bar{C}_{d+1}$, and $\bar{C}_{n_X}$ represents the complete set of behaviors. The relative sizes of the sets depicted in the figure do not accurately reflect their actual relative volumes and have been altered for improved readability.

In contrast with the classical behaviors, describing the quantum behaviors $\bar{Q}_d$ is much more challenging. The boundaries are usually smooth curved surfaces rather than faces. One way to get a geometric view of quantum behavior sets is by comparing them with classical behavior sets.

As we saw in **Section 1.1.2**, every behavior obtained by sending a dit can also be obtained by sending a qudit, *i.e.*, $\bar{C}_d \subseteq \bar{Q}_d$. On the other hand, as we will see in **Section 1.2**, we can have an advantage in PM task if we send one qudit instead of a dit. Hence, not every behavior obtained by sending a qudit can be obtained by sending a dit. Indeed, in general, $\bar{C}_d$ is a proper subset of $\bar{Q}_d$.

Moreover, in **Section 1.3.1**, we will see that there are behaviors in $\bar{C}_{d+1}$ that are not contained in $\bar{Q}_d$. On the other hand, in general, behaviors obtained from sending a qudit cannot be simulated even with high-dimension classical systems [63, 64]. In fact, in [64] it is shown that there is a behavior in $\bar{Q}_2$ that does not belong to $\bar{C}_3$. Thus, in general, neither $\bar{Q}_d$ is a subset of $\bar{C}_{d+1}$ and vice versa. In **Fig. 5**, we pictorially represent these relationships between the sets $\bar{C}_d$, $\bar{Q}_d$, and $\bar{C}_{d+1}$.

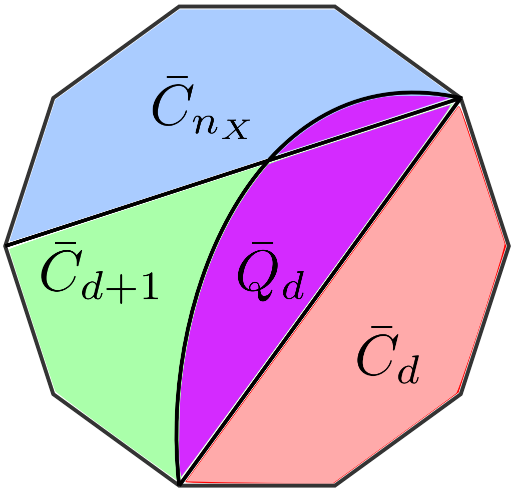


Figure 5 – Pictorial representation of the set of classical and quantum behaviors in a prepare-and-measure scenario. In general, $\bar{C}_d$ is a proper subset of $\bar{Q}_d$ which it's not a subset of $\bar{C}_{d+1}$. On the other hand, there are always behaviors in $\bar{C}_{d+1}$ which are not in $\bar{Q}_d$. Moreover, the proportions of the sets shown in the figure have been modified for ease of understanding, not reflecting their true relative volumes.

Another approach to estimating the set of quantum behaviors is through numerical methods. Internal approximations can be obtained using alternating convex searchers, SDPs[6] in see-saw interactions [66]. On the other hand, obtaining upper bounds valid for any quantum states and measurements is more demanding. Determining upper bounds that apply to any quantum states and measurements is a more challenging task. A hierarchy of semidefinite relaxations has been proposed in [67, 68] to bound quantum correlations on Hilbert spaces with limited dimensions. More efficient techniques exploiting the symmetries of the scenarios were introduced in [69].

[6] A semidefinite program (SDP) is a mathematical optimization problem that involves finding a linear function's maximum or minimum value, subject to constraints represented by a matrix of variables that must be positive semidefinite [65].

1.2 Quantum advantage in random access codes

The typical example of a PM protocol with quantum advantage is *random access codes* (RAC). A RAC is a category of communication tasks that are useful for various applications. In a RAC, a party, Alice, holds a set of randomly sampled data, and another party, Bob, aims to access an arbitrary subset of the information held by Alice. Alice is allowed to send a message to Bob. However, this communication is restricted[7]. Random access codes were originally described in [8] and later rediscovered and linked to quantum automata in [9]. The use of RACs is also prevalent in the foundations of quantum theory. Examples are the dimension witnessing [12], self-testing [18, 70], and attempts to characterize quantum probability distributions from information-theoretic principles [25]. As we will see in this section, RAC is part of the paradigm of a PM scenario.

We will describe here the simplest random access code protocol, namely, $2 \mapsto 1$ RAC. In this communication task, Alice is given a pair of bits $\boldsymbol{x} = x_0x_1 \in \{00, 01, 10, 11\}$ and Bob is queried with an input $y \in \{0, 1\}$. He must correspondingly guess what value the y-th bit of $\boldsymbol{x}$ was. For instance, suppose that Alice receives the bit pair $x_0x_1 = 01$. If Bob receives the input $y = 0$, they will succeed in the task if Bob outputs the bit $b = x_0 = 0$. On the other hand, if Bob receives the input $y = 1$, then the correct answer would be $b = x_1 = 1$. Note that such a protocol can be seen as a task within a PM scenario, having $n_X = 4$ (we can see two bits as a quart), $n_Y = 2$, and $n_B = 2$. Naturally, this task would be trivial if Alice was allowed to send her two input bits x_0x_1 to Bob. Nonetheless, the task becomes more interesting if Alice can send just one bit to Bob. Surprisingly, we will see that sending a qubit will be more advantageous than sending a bit.

Several figures of merit regarding how well Alice and Bob perform in a RAC have been proposed, such as the *worst case success probability* [9, 71], and the *average success probability* [44]. We will focus on the latter. Assuming both $\boldsymbol{x}$ and y are uniformly distributed, the average success probability of a behavior $\boldsymbol{p}$ is defined through.

$$\mathcal{P}_{\text{suc}}^{\text{avg}}(\boldsymbol{p}) = \frac{1}{8}\sum_{\boldsymbol{x},y} p(b = x_y \mid \boldsymbol{x}, y), \tag{1.12}$$

where $\frac{1}{8}$ is a normalization factor coming from our uniformity assumptions.

In the case where a single bit is sent, one of the strategies that maximize $\mathcal{P}_{\text{suc}}^{\text{avg}}$ is given by: (i) Alice always sends her first input bit x_0; and (ii) Bob always outputs this bit received from Alice, *i.e.*, $b = x_0$. Thus, they succeed in the task whenever Bob is queried with the input $y = 0$. On the other hand, since Alice's inputs are uniformly distributed, half of the time, both her bits are equal, so in this half of the time, they will also succeed in the task for $y = 1$. However, on the other half, if Alice's bits are distinct, they will

[7] Unlike the previous sections where the focus was on the set of behaviors, in this section, we are going to discuss a particular task.

fail the task for $y = 1$. Therefore, Alice and Bob reach the goal in three-quarters of the rounds. More formally, this is a deterministic strategy with $\mathcal{E}(x_0x_1) = x_0$ and $\mathcal{D}_i(x) = x$ for $i \in \{0, 1\}$. Indeed, the behavior obtained by them is given by:

$$p(b|\boldsymbol{x}, y) = \delta(b, (\mathcal{D}_y \circ \mathcal{E})(x_0x_1)) = \delta(b, x_0).$$

thus,

$$\mathcal{P}^{\text{avg}}_{\text{suc}}(\boldsymbol{p}) = \frac{1}{8}\sum_{\boldsymbol{x},y} p(b = x_y \mid \boldsymbol{x}, y) = \frac{1}{8}\sum_{\boldsymbol{x},y} \delta(x_y, x_0) \tag{1.13}$$

$$= \frac{1}{8}\sum_{x_0,x_1=0}^{1} \delta(x_0, x_0) + \frac{1}{8}\sum_{x_0,x_1=0}^{1} \delta(x_1, x_0) = \frac{4}{8} + \frac{2}{8} \tag{1.14}$$

$$= \frac{3}{4}. \tag{1.15}$$

One proof that this strategy is optimal can be found in [44]. Briefly, we have:

$$\sup_{\boldsymbol{p}\in\bar{C}_2} \mathcal{P}^{\text{avg}}_{\text{suc}}(\boldsymbol{p}) = \frac{3}{4}. \tag{1.16}$$

We will now see an advantage for Alice in communicating a qubit instead of a bit. Suppose Alice encodes her two bits x_0x_1 in a two-level system with the following state:

$$|\psi_{x_0x_1}\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + \frac{1}{\sqrt{2}}((-1)^{x_0} + i(-1)^{x_1})\,|1\rangle\right]. \tag{1.17}$$

Let us also consider that Bob's decoding process is performed with the following measurements: $\{M_{0|1}, M_{1|0}\}$ is a projective measurement on the basis

$$\left\{\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right), \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)\right\}, \tag{1.18}$$

and $\{M_{0|1}, M_{1|1}\}$ is also a projective measurement on the basis

$$\left\{\frac{1}{\sqrt{2}}\left(|0\rangle + i\,|1\rangle\right), \frac{1}{\sqrt{2}}\left(|0\rangle - i\,|1\rangle\right)\right\}. \tag{1.19}$$

From Born's rule, it is possible to show that:

$$p(x_y|x_0x_1, y) = \frac{1}{2} + \frac{1}{2\sqrt{2}} \approx 0.853553, \tag{1.20}$$

for all choices of x_0x_1 and y. Thus, substituting in (1.12), we have that following this quantum strategy, Alice and Bob reach an average probability of success in the RAC task given by

$$\begin{aligned}\mathcal{P}^{\text{avg}}_{\text{suc}}(\boldsymbol{p}) &= \frac{1}{8}\sum_{\boldsymbol{x},y} p(x_y \mid x_0x_1, y)\\ &= \frac{1}{8}\sum_{\boldsymbol{x},y}\left(\frac{1}{2} + \frac{1}{2\sqrt{2}}\right)\\ &= \frac{8}{8}\left(\frac{1}{2} + \frac{1}{2\sqrt{2}}\right)\\ &= \frac{1}{2} + \frac{1}{2\sqrt{2}} \approx 0.853553 \end{aligned} \tag{1.21}$$

The strategy described above is the optimal strategy with Alice sending a qubit to Bob [9]. Based on (1.16) and (1.21), we conclude that:

$$\sup_{\boldsymbol{p}\in\bar{C}_2} \mathcal{P}_{\text{suc}}^{\text{avg}}(\boldsymbol{p}) = \frac{3}{4} < \frac{1}{2} + \frac{1}{2\sqrt{2}} = \sup_{\boldsymbol{p}\in\bar{Q}_2} \mathcal{P}_{\text{suc}}^{\text{avg}}(\boldsymbol{p}) \tag{1.22}$$

Thus, as the function $\mathcal{P}_{\text{suc}}^{\text{avg}}$ has a greater maximum value in the set $\bar{Q}_2$ than in the set $\bar{C}_2$, it follows that $\bar{C}_2(4,2,2) \subsetneq \bar{Q}_2(4,2,2)$. Therefore, there is no equivalence between classical and quantum messages at the level of behaviors. In general, $\bar{C}_d$ is a proper subset of $\bar{Q}_d$ [72]. As we will see in **Section 1.3**, this result is surprising since the amount of information accessible in a qudit is only one dit.

It is important to note that in addition to showing that $\bar{C}_2(4,2,2) \subsetneq \bar{Q}_2(4,2,2)$, this example of RAC still gives us an exciting task where it is possible to obtain a quantum advantage over classical resources.

1.3 Holevo's bound

We saw in the previous section that sending a quantum bit can generate an advantage in communication tasks compared to sending a classical bit. Naively, such a result may not seem so surprising. In contrast to a classical bit, which has only two pure states, a qubit has infinitely many. Indeed, the generic description of a (pure) qubit requires two angles[8] [47]. In this way, it is natural to expect that it would be viable to transmit a large amount of classical information in a single quantum bit. However, the information encoded in a quantum state can only be accessed indirectly through a measurement process that often disturbs the observed state and makes a complete inference impossible. Holevo's theorem (also called Holevo's bound) gives one way to formalize this last statement [4].

Before establishing Holevo's bound, let us take an operational approach. In the following subsection, we will show that if Alice wants to send a ditstring[9] of size m to Bob, the number of qudits needed to perfectly encode this message is at least m. Thus, we see that quantum dits are equivalent to classical dits for this standard communication task.

1.3.1 A simple task: Guessing Alice's input

Let us discuss the most elementary task in a communication scenario: sending a classical message between two agents. The goal of the task is for Bob to output Alice's input. To fit it into a PM scenario, let us say that Alice is given a ditstring $\boldsymbol{x} = x_0 x_1 \cdots x_{m-1}$

[8] Any qubit pure state $|\psi\rangle$ can be uniquely represented as $cos(\theta/2)\,|0\rangle + e^{i\phi}\text{sen}(\theta/2)\,|1\rangle$, where $0 \leqslant \theta \leqslant \pi$ and $0 \leqslant \phi < 2\pi$.

[9] A ditstring is a sequence of dits.

and Bob wants to guess which ditstring was given to Alice[10]. Hence, his set of possible responses must be equal to Alice's set of inputs, *i.e.*, $n_B = d^m = n_X$. As Bob's goal is always the same (guess Alice's ditstring), we can consider that Bob is always getting the same input, *i.e.*, $n_Y = 1$.

To quantify the efficacy of Alice and Bob, we consider that this protocol is performed many times. At the end is estimated the set of probability distributions $p(\boldsymbol{b}|\boldsymbol{x})$ that describe their action[11]. For simplicity, suppose that the distribution for Alice's input ditstring is uniform. Thus, the *average probability of success* in the task, considering that the agents' behavior is $\boldsymbol{p}$, is given by:

$$\begin{aligned}\mathcal{P}_{\text{suc}}^{\text{avg}}(\boldsymbol{p}) &= \sum_{\boldsymbol{x}} p(B = \boldsymbol{x}, X = \boldsymbol{x}) = \sum_{\boldsymbol{x}} p(B = \boldsymbol{x}|X = \boldsymbol{x})p(X = \boldsymbol{x}) \\ &= \frac{1}{d^m}\sum_{\boldsymbol{x}} p(B = \boldsymbol{x} \mid X = \boldsymbol{x}) = \frac{1}{d^m}\sum_{\boldsymbol{x}} p(\boldsymbol{x}|\boldsymbol{x}), \end{aligned} \tag{1.23}$$

where B is the random variable of Bob's output and X is the random variable of Alice's input.

In the same way as in the RAC protocol, we will use the function $\mathcal{P}_{\text{suc}}^{\text{avg}}(\boldsymbol{p})$ as a figure of merit to evaluate the performance of Alice and Bob on the task. Mainly, we will be focused on comparing the maximum value of $\mathcal{P}_{\text{suc}}^{\text{avg}}(\boldsymbol{p})$ in cases of classical and quantum communication. Let us start discussing the situation where Alice can send k, where $k \leqslant m$, classical dits to Bob, and they are free to share any amount of classical pre-established correlations. This way, Alice and Bob will be performing the task with behaviors in the set $\bar{C}_{d^k}(d^m, 1, d^m)$. We are now going to introduce a strategy that, as we will see later, is the one that best performs the task considering these resources. For this strategy: (i) Alice sends her first k inputs dits to Bob, *i.e.*, $a = x_0x_1\cdots x_{k-1}$; (ii) The first k dits of Bob's outputs will be the k dits received from Alice. However, Bob randomly chooses his outputs' last $(m-k)$ dits. For simplicity, we can assume that Bob always chooses 0 for the missing $(m-k)$ dits. Thus, they succeed in the task when Alice's ditstring has a format $\boldsymbol{x} = x_0\cdots x_{k-1}0\cdots 0$. However, in all the other cases, they will fail. More formally, Alice and Bob perform a deterministic strategy (see **Def. 1**), where Alice's encoding function is given by:

$$\mathcal{E}(x_0\cdots x_{m-1}) = x_0\cdots x_{k-1}.$$

While Bob's decoding function is given by

$$\mathcal{D}(x_0\cdots x_{k-1}) = x_0\cdots x_{k-1}0\cdots 0.$$

[10]We should note here the difference of this protocol to the RAC protocol. We note that in the RAC protocol, Bob is interested in only one of Alice's bits. However, in this case, Bob is interested in finding the entire ditstring.

[11]Note that the behavior is now only denoted by $p(\boldsymbol{b}|\boldsymbol{x})$ instead of $p(\boldsymbol{b}|\boldsymbol{x}, y)$, this comes from the fact that since Bob has a single input, y is always the same value and therefore do not need to be mentioned.

Therefore, the behavior that describes this strategy is:

$$p(\boldsymbol{b}|\boldsymbol{x}) = \delta(\boldsymbol{b}, x_0 \cdots x_{k-1} 0 \cdots 0),$$

In this way, the average probability of success using this strategy is:

$$\begin{aligned} \mathcal{P}_{\text{suc}}^{\text{avg}}(\boldsymbol{p}) &= \frac{1}{d^m} \sum_{\boldsymbol{x}} p(\boldsymbol{x} \mid \boldsymbol{x}) = \frac{1}{d^m} \sum_{\boldsymbol{x}} \delta(x_0 \cdots x_{k-1} x_k \cdots x_{m-1},\ x_0 \cdots x_{k-1} 0 \cdots 0) \\ &= \frac{1}{d^m} \sum_{x_0, \cdots, x_{k-1}=0}^{d-1} 1 = \frac{d^k}{d^m}. \end{aligned} \tag{1.24}$$

As shown in [73], this is indeed the optimal strategy when Alice is sending k dits. Hence:

$$\max_{\boldsymbol{p} \in \bar{C}_{d^m}} \mathcal{P}_{\text{suc}}^{\text{avg}}(\boldsymbol{p}) = \frac{d^k}{d^m}. \tag{1.25}$$

A natural question is whether qudits can take advantage of dits in this task. If Alice sends k qudits to Bob, can they achieve an average probability of success greater than (1.25)? In fact, this is not the case. As proved by Nayak and Salzman [73][12], the optimal probability for $\mathcal{P}_{\text{suc}}^{\text{avg}}$, using qudits, is also equal to $\frac{d^k}{d^m}$, *i.e.*,

$$\max_{\boldsymbol{p} \in \bar{Q}_{d^k}} \mathcal{P}_{\text{suc}}^{\text{avg}}(\boldsymbol{p}) = \frac{d^k}{d^m}. \tag{1.26}$$

Thus, quantum dits are just as good as classical bits for the task where Alice wants to communicate her input to Bob. In the following subsection, we will see this result from a new perspective, that of mutual information. As we will see, the amount of information available about Alice's input is the same when comparing quantum dits with classical dits.

1.3.2 Holevo's bound

In order to establish the Holevo bound, we will briefly introduce some definitions of Shannon Information Theory. We recommend [42] to anyone interested in more details.

Definition 8 (Shannon Entropy)**.** *Let X be a random variable distributed according to a probability distribution $Pr(X = x) = p(x)$. The Shannon Entropy of X is given by*

$$H(X) = \sum_x p(x) \log_2 \left(\frac{1}{p(x)} \right) = - \sum_x p(x) \log_2 (p(x)).$$

In this definition, we are assuming

$$0 \log_2(1/0) \equiv \lim_{x \to 0^+} x \log_2(1/x) = 0.$$

[12] In [73], this result was discussed only for the case of bitstrings and qubits. However, the generalization for ditstrings and qudits is straightforward.

Roughly speaking, the Shannon entropy of a variable X is the greatest amount of information (in bits[13]) we could learn from seeing X [74]. We show below two examples that reinforce this interpretation of Shannon entropy.

Example 1

Let X be a random variable distributed according to a probability distribution $Pr(X = x) = p(x)$ on the set $\{0,1\}^n$. Then,

- If X is uniform, *i.e.*, $p(x) = 1/2^n$ for all $x \in \{0,1\}^n$, then

$$H(X) = \sum_{x\in\{0,1\}^n} p(x)\log_2\left(\frac{1}{p(x)}\right) = \sum_{x\in\{0,1\}^n} \frac{1}{2^n}\log_2(2^n) = \frac{n2^n}{2^n} = n.$$

  Therefore, if $p(x)$ is the uniform distribution, one gets n bits of information from seeing X.

- If $p(x)$ has all its probability on a single string $x_0 \in \{0,1\}^n$, *i.e.*, $X = x_0$ with probability 1 and $X = x$ with probability 0 for all $x \neq x_0$, then

$$H(X) = -\sum_{x\in\{0,1\}^n} p(x)\log_2(p(x)) = -1\log_2(1) = 0.$$

  Which is expected, since as X is a deterministic random variable, we are already sure of the value that X will obtain even before the event happens. Thus, such an event generates zero new information for the observers.

Suppose that X, Y are two, possibly correlated, random variables. Let $Pr(X = x, Y = y) = p(x,y)$ the joint probability distribution of the pair (X,Y). As the pair (X,Y) is a random variable, we can define the Shannon Entropy for (X,Y). In general, we note that if X and Y are independent, then $H(X,Y) = H(X) + H(Y)$, which is in line with our interpretation of Shannon entropy since we cannot tell anything about the other random variable by observing the first. Therefore, the amount of information we can learn by seeing the pair (X,Y) is equal to the sum of the amount of information we can learn by seeing variables X, Y separately.

Conversely, if X and Y are perfectly correlated, *i.e.*, $X = Y$, then $H(X,Y) = H(X) = H(Y)$. Seeing one of the variables (X,Y) immediately tells us what the other variable is. Hence, the amount of entropy in the pair is the same as in the variables themselves. We can formalize the notion of "how much does seeing one random variable tell me about the other" as follows:

[13] It should be noted that in this thesis, the term "bit" is used in two different ways, as the possible values assumed by a classical variable (the form used in the previous sections) and as a quantity of information defined from the Shannon entropy.

Definition 9 (Mutual information)**.** *The mutual information between two random variables X and Y is given by:*

$$I(X:Y) = H(X) + H(Y) - H(X,Y). \tag{1.27}$$

Another equivalent definition of mutual information is given by [42]:

$$I(X:Y) = \sum_x \sum_y p(x,y) \log_2 \left(\frac{p(b,x)}{p(b)p(x)} \right).$$

In the following example, we present two situations that allude to the interpretation of $I(X:Y)$ as the maximum amount of information we can learn from the variable Y based on knowledge of variable X (or vice versa, since $I(X:Y)$ is symmetric).

Example 2

Let X, Y be two random variables distributed according to a probability distribution $Pr(X = x, Y = y) = p(x,y)$. In this way,

- If X and Y are independent, then:

  $$I(X:Y) = H(X) + H(Y) - H(X,Y) = H(X) + H(Y) - H(X) - H(Y) = 0.$$

  This is consistent with our interpretation of mutual information since as X, and Y are independent, then seeing X tells us nothing about Y (and vice versa).

- If X and Y are perfectly correlated, then:

  $$I(X:Y) = H(X) + H(Y) - H(X,Y) = H(X) + H(Y) - H(X) = H(Y),$$

  which is also consistent with our interpretation of mutual information once as X and Y are equal, seeing X gives us the full information we can have seeing Y, *i.e.*, $H(Y)$.

Let's now introduce the classical capacity of a channel. To do so, we return to the scenario of PM where X is the random variable representing Alice's input and B is the random variable representing Bob's output[14]. We can interpret this PM scenario as being a classical channel, where with probability $p(b|x)$ returns the classical message b when it receives the classical input x. Let $p(x)$ be the probability distribution to which inputs x are given to Alice. Thus, the joint probability of the variables X, and B is given by

$$Pr(B = b, X = x) = Pr(X = x)Pr(B = b|X = x) = p(x)p(b|x).$$

The classical capacity of a channel $p(b|x)$ is defined as the supremum of the mutual information between variables X and B, where the supremum is concerning Alice's input distribution $p(x)$, i.e.,

$$C(\boldsymbol{p}) = \sup_{p(x)} I(B:X) = \sup_{p(x)} \sum_b \sum_x p(b,x) \log_2 \left(\frac{p(b,x)}{p(b)p(x)} \right). \tag{1.28}$$

[14]for simplicity, let's deal with the case that Bob does not receive any additional input, i.e., $n_Y = 1$.

Remembering that $I(B : X)$ represents the maximum amount of information we can learn from the variable X based on knowledge of variable B. Thus, $C(\boldsymbol{p})$ is a measure of the maximum amount of information that can be transmitted through the channel $\boldsymbol{p}$.

Let us now move to Quantum Information Theory and define the analog of Shannon entropy for quantum states.

Definition 10 (von Neumman Entropy)**.** *The von Neumann entropy of a quantum state $\rho \in \mathcal{L}(\mathbb{C}^d)$ is defined as:*

$$S(\rho) = -\sum_{i=0}^{d-1} p_i \log_2(p_i), \tag{1.29}$$

where $p_0, \cdots, p_{d-1}$ are the eigenvalues of ρ.

Remark 5. *It follows that $S(\rho) = H(X)$, where X is a random variable with a probability distribution given by the eigenvalues of ρ.*

We are now able to state the Holevo's bound. For such, let us consider the same situation as in the previous section, where Alice has a bitstring[15] $\boldsymbol{x} = x_0 \cdots x_{m-1}$ that she wants to transmit to Bob. Let us denote by X the random variable given by Alice's input bitstring and let $p(\boldsymbol{x})$ be the probability distribution of this random variable, *i.e.*, $Pr(X = \boldsymbol{x}) = p(\boldsymbol{x})$. We will assume that Alice sends k qubits to Bob, where $\rho_{\boldsymbol{x}}$ is the density matrix that describes this k qubits when Alice receives the bitstring $\boldsymbol{x}$ as input. Moreover, we will denote by ρ_X the average state sent from Alice to Bob, *i.e.*, $\rho_X = \sum_{\boldsymbol{x}} p(\boldsymbol{x})\rho_{\boldsymbol{x}}$. After receiving the k qubits from Alice, Bob performs a POVM measurement $\{M_b\}$ in this quantum system. Finally, let us denote by B the random variable describing the results of Bob's measurement. By Born's rule, $Pr(B = b|X = \boldsymbol{x}) = \mathrm{Tr}(\rho_{\boldsymbol{x}} M_b)$, thus, the joint probability distribution of variables (X, B) is equal to $Pr(B = b, X = \boldsymbol{x}) = Pr(B = b|X = \boldsymbol{x})Pr(X = \boldsymbol{x}) = \mathrm{Tr}(\rho_{\boldsymbol{x}} M_b)p(\boldsymbol{x})$. Holevo's theorem establishes an upper bound for the mutual information between Alice's inputs and Bob's measurement results.

Theorem 2 (Holevo's bound)**.**

$$I(X : B) \leqslant S(\rho_X) - \sum_{\boldsymbol{x}} p(\boldsymbol{x}) S(\rho_{\boldsymbol{x}}). \tag{1.30}$$

A proof of Holevo's bound can be found in section 6.2 of [75].

Given that the Von Neumann entropy is a nonnegative quantity, it immediately follows that

$$I(X : B) \leqslant S(\rho_X).$$

[15] As was done in the previous section, we could also consider the general case of ditstrings. However, we will restrict ourselves to bitstring to be consistent with the definition of Shannon Entropy presented in this thesis.

We also have that for any state ρ_X of k qubits, $S(\rho_X) \leqslant k$ holds. Thus, a corollary of Holevo's bound is that:

$$I(X:B) \leqslant k.$$

Remembering that the mutual information between two variables represents the amount of information available from one variable by observing the other. It then follows that the maximum amount of accessible information about Alice's input (variable X) that can be extracted from the result of Bob's measurement (variable B) is a maximum of k bits. This way,

k quantum bits can carry at most k bits of information.

More precisely, according to Holevo's bound, even though k qubits can "carry" a larger amount of (classical) information (as a consequence of quantum superposition), they can only retrieve a maximum of k classical bits of information. The appendix of [73] shows that Holevo's bound remains true under the condition that Alice and Bob share an arbitrary amount of randomness.

An equivalent way of stating this result, using the set notation introduced in this thesis and the classic channel capacity definition is:

$$\sup_{\boldsymbol{p}\in C_{2k}} C(\boldsymbol{p}) = k = \sup_{\boldsymbol{p}\in \bar{Q}_{2k}} C(\boldsymbol{p}). \tag{1.31}$$

This means that k bits of information is the maximum capacity of the channels obtained both by sending k bits and by sending k qubits.

At first sight, Holevo's bounds seem to be in contradiction with the quantum random access codes (**Sec. 1.2**), where we saw that Bob performs better in guessing one of Alice's two bits if she sends him a qubit instead of a bit. However, although in each run of the experiment, the available information about Alice's input from Bob's measurement is only one bit, he kind of chooses (from the choice of his measurement) which part of Alice's input he wants to get more information. In the following two sections, we will see that this possibility of choice is indeed the key to quantum advantage in PM scenarios.

1.4 Classical and quantum behaviors are equal when Bob has a single input

In **Section 1.2**, we saw the task of $2 \mapsto 1$ RAC. In each round of this task, Alice is given two bits, and Bob's goal is to guess one of those bits. Alice does not know which of the two bits Bob wants to guess, and their communication is restricted to just one (qu)bit from Alice to Bob. We saw that the agents perform better by sending a qubit rather than by sending a classical bit. In fact,

$$\sup_{\boldsymbol{p}\in\bar{C}_2} \mathcal{P}_{\text{suc}}^{\text{avg}}(\boldsymbol{p}) = \frac{3}{4} < \frac{1}{2} + \frac{1}{2\sqrt{2}} = \sup_{\boldsymbol{p}\in\bar{Q}_2} \mathcal{P}_{\text{suc}}^{\text{avg}}(\boldsymbol{p}),$$

where $\mathcal{P}_{\text{suc}}^{\text{avg}}$ is the average success probability. Hence, we have that:

$$\bar{C}_2(4,2,2) \subsetneq \bar{Q}_2(4,2,2).$$

On the other hand, Holevo's bound states that even with Alice sending one qubit, Bob can recover at most one bit of information about Alice's input[16]. Holevo's bound and quantum advantage in RAC do not contradict each other. In each round of the RAC protocol, Bob can choose to perform different measurements depending on which of the two bits of Alice he wants to recover. In this way, although the results of these measurements still reveal at most one bit of information about Alice's input, that information centers on which of Alice's two bits Bob is interested in[17]. It is worth noting that the term "bit" represents here both the basic unit of classical information in terms of Shannon entropy and the results of a two-state classical system.

Roughly speaking, in PM scenarios, the protocols in which quantum communication has an advantage over classical communication are those in which only part of Alice's input is interesting and, at the same time, that part varies at each round. Indeed, Frenkel and Weiner showed that in PM scenarios where Bob has a single input ($n_Y = 1$), the set of quantum and classical behaviors are identical [76]. Thus, in such scenarios, quantum communication cannot generate any advantage. More concretely, in [76] the following theorem is proved:

Theorem 3 (Frenkel-Weiner's theorem)**.** *For every choice of positive integers d, n_X, n_B, we have:*

$$\bar{C}_d(n_X, 1, n_B) = \bar{Q}_d(n_X, 1, n_B). \tag{1.32}$$

It should be noted that if shared randomness is not a resource available to the parties, it is possible to take advantage of quantum communication even in a scenario having $n_Y = 1$ [77].

1.5 Necessity of incompatibility for non-classicality

As seen in **Section 1.2**, in PM scenarios with Bob having inputs, we may have an advantage in performing communication tasks if the exchange of information between Alice

[16]This result is only valid in the absence of entanglement between the parts. More details will be given in section 2.2 of the next chapter.

[17]For *sharp measurements* (projective measurements), as measurements completely disturb the system, a second measurement does not enable him to extract additional information about Alice's input. Interestingly, even for *unsharp measurements*, Holevo's bound still remain.

and Bob occurs through quantum systems. A natural question is, what are the properties of these quantum systems that make such an advantage possible? This question applies to the quantum systems sent and the measurements performed. Similar analyses have already been done for several other correlation scenarios. Indeed, in Bell scenarios, we already know that both the quantum system shared between the parts must be entangled, and the measurements performed must be incompatible for nonclassicality to be achieved [61, 78]. Interestingly, although both previous conditions are necessary for Bell Nonlocality, neither is sufficient [79, 80]. In the case of EPR scenarios, although entanglement between the parties continues to be necessary (but not sufficient [81, 82]), incompatibility between Alice's measurements becomes a sufficient criterion for the existence of steering [80, 83]. This section will show that measurement incompatibility is also required for nonclassicality in PM scenarios. These results were published in [37].

For prepare-and-measure scenarios, it is already shown that if the quantum states prepared by Alice commute pairwise, then any behavior found from this preparation will be classically simulable [38]. More formally,

Proposition 1. *Let $\{\rho_x\}_{x\in n_X} \subset \mathcal{L}(\mathbb{C}^d)$ be a set of density matrices that commute pairwise, i.e., $[\rho_x, \rho_{x'}] = 0$ for all $x, x' \in [n_X]$. Let $\{M_{b|y}\}$ be any set of POVMs, then the behavior:*

$$p(b|x,y) = \mathrm{Tr}\big(\rho_x M_{b|y}\big) \tag{1.33}$$

belongs to $C_d(n_X, n_Y, n_B)$.

Before proving it formally, let us first discuss why this result is expected to be true. First, we recall that a set of commuting hermitian matrices are simultaneously diagonalizable [84]. Thus, let $\{|a\rangle\}_{a=0}^{d-1}$ be the orthonormal basis that diagonalizes the set of density matrices $\{\rho_x\}$. Since all these matrices have trace 1, it implies that for every x, there is $p(a|x)$, a probability distribution on a, such that

$$\rho_x = \sum_{a=0}^{d-1} p(a|x) \, |a\rangle\langle a| \, .$$

We observe that the communication of states ρ_x between the parts can be performed by Alice sending a classical dit and Bob applying a local transformation in his laboratory. In fact, given an input x, suppose that Alice randomly chooses a dit following the probability distribution $p(a|x)$. Let a be the result of this draw. Alice then sends this dit to Bob, whom then prepares a qudit $|a\rangle\langle a|$, $\{|a\rangle\}_{a=0}^{d-1}$ being the computational base of $\mathbb{C}^d$. Thus, when Alice receives input x, the average state that Bob prepares is given by mixing the states $|a\rangle\langle a|$ distributed with distribution $p(a|x)$. In other words, Bob will be preparing the state:

$$\sum_{a=0}^{d-1} p(a|x) \, |a\rangle\langle a| \, ,$$

which is equal to the state ρ_x. So, effectively the same type of communication is being carried out in both situations. Therefore, communication being quantum does not generate advantages in this circumstance.

Proof. Let $\{|a\rangle\}_{a=0}^{d-1}$ an orthonormal basis with respect to which the $\rho_x's$ are diagonal. Consequently, for every $x \in n_X$ there is $\alpha_0^x, \cdots, \alpha_{n_X-1}^x \in [0,1]$ s.t $\sum_a \alpha_a^x = 1$ and $\rho_x = \sum_a \alpha_a^x |a\rangle\langle a|$. Hence,

$$\operatorname{Tr}\big(\rho_x M_{b|y}\big) = \sum_{a=1}^{d} \alpha_a^x \operatorname{Tr}\big(|a\rangle\langle a| M_{b|y}\big) = \sum_{a=1}^{d} \alpha_a^x \langle a| M_{b|y} |a\rangle . \tag{1.34}$$

Let $p(a|x) = \alpha_a^x$ and $p(b|y,a) = \langle a| M_{b|y} |a\rangle$. Both $p(a|x)$ and $p(b|y,a)$ are probability distributions. Thus,

$$p(b|x,y) = \operatorname{Tr}\big(\rho_x M_{b|y}\big) = \sum_{a=1}^{d} \alpha_a^x \langle a| M_{b|y} |a\rangle = \sum_{a=1}^{d} p(a|x)p(b|y,a). \tag{1.35}$$

Therefore, $p(b|x,y)$ belongs to $C_d(n_X, n_Y, n_B)$. □

We can ask ourselves if this condition is sufficient, *i.e.*, whenever Alice's set of states does not commute pairwise, then such a set gives an advantage in a communication task. However, as shown in [27], there are sets of states that do not commute pairwise but cannot present nonclassicality in any PM scenario. Specifically, they develop a method that certifies that the behaviors arising from a preparation set are always classically reproducible regardless of the measurements used.

The natural next step is to look for constraints that nonclassicality in PM implies on Bob's measurement set. In [28], it was shown that the advantage in some RAC protocols require incompatibility of Bob's measurements. We will present here a generalization of this result, showing that the advantage in *any* PM scenario implies incompatibility in Bob's measurements. Before giving the details of this result, it is worth mentioning a recent work that connects nonclassicality in prepare-and-measure scenarios to quantum contextuality. In ref. [85], the authors demonstrate that any quantum state and observables that exhibit quantum contextuality also result in a communication task with a quantum advantage. Conversely, any quantum advantage in this task can be used as evidence for contextuality if an additional condition is met.

Let us start with the definition of joint measurability [86]:

Definition 11 (Joint measurability). *Let $\{M_{b|1}\}, \cdots, \{M_{b|n}\}$ be a family of POVM's. We say that this family of POVMs are jointly measurable if there exist a mother POVM $\{N_{z_1,\ldots,z_n}\}$, such that*

$$M_{b|y} = \sum_{\substack{z_1,\ldots,z_n \\ z_y=b}} N_{z_1,\ldots,z_n}, \quad \forall\ b, y. \tag{1.36}$$

We can now state the main result of this section.

Result 1. *Measurement incompatibility is necessary for nonclassicality in prepare-and-measure scenarios.*

In this way, any quantum advantage in communication tasks over all possible classical strategies with unlimited classical pre-established correlations implies that the quantum measurements performed by Bob to produce the outputs are incompatible. This statement may seem natural and straightforward, however, to the best of our knowledge, no formal proof of it has been published so far, except for the exceptional cases of some random access codes tasks [28, 78].

Proof. Let $p(b|x,y) \in Q_d(n_X, n_Y, n_B)$ be given by $p(b|x,y) = \mathrm{Tr}\big(\rho_x M_{b|y}\big)$, where the collection of Bob's POVMs we suppose to be jointly measurable. Thus, by **Definition 11**,

$$M_{b|y} = \sum_{\substack{z_1,\ldots,z_n=0 \\ z_y=b}}^{n_B-1} N_{z_1,\ldots,z_n}, \quad \forall\ b \in [n_B] \ \text{ and } \ y \in [n], \tag{1.37}$$

where for simplicity, we are denoting $n_Y = n$. Hence, the observed statistics are given by

$$p(b|x,y) = \mathrm{Tr}\big(\rho_x M_{b|y}\big) = \sum_{\substack{z_1,\ldots,z_n=0 \\ z_y=b}}^{n_B-1} \mathrm{Tr}(\rho_x N_{z_1,\ldots,z_n}). \tag{1.38}$$

Let's define $\boldsymbol{q}$ as the behavior:

$$q(z_1,\cdots,z_n|x) = \mathrm{Tr}(\rho_x N_{z_1,\ldots,z_n}) \ \ \forall z_1,\cdots,z_n \ \in \ [n_B]. \tag{1.39}$$

It follows that $\boldsymbol{q}$ is a behavior in $Q_d(n_X, 1, k)$ where $k = (n_B)^n$. We may regard the Frenkel-Weiner's theorem (**Thrm. 3**), which states that $\bar{C}_d(n_X,1,k) = \bar{Q}_d(n_X,1,k)$, that is, in a scenario where Bob has a single measurement, there is always a classical model that simulates the statistics attainable with quantum resources. Thus, by the definition of SR-d-classical behaviors,

$$\begin{aligned} \mathrm{Tr}(\rho_x N_{z_1,\ldots,z_n}) &= q(z_1,\ldots,z_n|x) \\ &= \sum_{a=0}^{d-1}\sum_{\lambda} \pi(\lambda) q_A(a|x,\lambda) q_B(z_1,\ldots,z_n|a,\lambda), \end{aligned} \tag{1.40}$$

for all $x \in [n_X]$ and $z_1,\ldots,z_n \in [n_B]$. Substituting (1.40) in (1.38), we have:

$$\begin{aligned} p(b|x,y) &= \sum_{\substack{z_1,\ldots,z_n=0 \\ z_y=b}}^{n_B-1} \mathrm{Tr}(\rho_x N_{z_1,\ldots,z_n}) \\ &= \sum_{\substack{z_1,\ldots,z_n=0 \\ z_y=b}}^{n_B-1} \sum_{a=0}^{d-1}\sum_{\lambda} \pi(\lambda) q_A(a|x,\lambda) q_B(z_1,\ldots,z_n|a,\lambda) \end{aligned} \tag{1.41}$$

Now, define

$$q_B(b|a,y,\lambda) = \sum_{\substack{z_1,\dots,z_n=0 \\ z_y=b}}^{n_B-1} q_B(z_1,\dots,z_n|a,\lambda), \tag{1.42}$$

for all b, a, y, λ. It is easy to see that $q_B(b|a,y,\lambda)$ are probability distributions. Substituting (1.42) in (1.41), we have:

$$\begin{aligned} p(b|x,y) &= \sum_{\substack{z_1,\dots,z_n=0 \\ z_y=b}}^{n_B-1} \sum_{a=0}^{d-1} \sum_{\lambda} \pi(\lambda) q_A(a|x,\lambda) q_B(z_1,\dots,z_n|a,\lambda) \\ &= \sum_{a=0}^{d-1} \sum_{\lambda} \pi(\lambda) q_A(a|x,\lambda) \left[\sum_{\substack{z_1,\dots,z_n=0 \\ z_y=b}}^{n_B-1} q_B((z_1,\dots,z_n)|a,\lambda) \right] \\ &= \sum_{a=0}^{d-1} \sum_{\lambda} \pi(\lambda) q_A(a|x,\lambda) q_B(b|a,y,\lambda). \end{aligned}$$

Therefore, $p(b|x,y)$ can be reproduced by preparing, sending, and then measuring classical instead of quantum states. Hence we conclude that measurement incompatibility is necessary for observing nonclassicality in PM scenarios. □

Naturally, we may wonder whether incompatibility is sufficient for nonclassicality in the prepare-and-measure scenario. However, this was proven to be false [27]. On the other hand, for PM scenarios where Alice receives quantum inputs, it was shown that measurement incompatibility is necessary and sufficient for nonclassicality [87].

2 Entanglement-assisted prepare and measure scenarios

In the previous chapter, we introduced the prepare-and-measure scenario, *i.e.*, a communication scenario where Alice encodes her message into a physical system sent to Bob, who extracts information from it through a measurement process. Previous correlations between Alice and Bob were assumed to be classical (shared randomness). Since the discussion of the EPR paradox [88] and of Bell's theory of non-locality [89], we know that entangled states can generate correlations that cannot be simulated using any classical latent variable. In this way, it is natural also to consider the situation in which Alice and Bob share prior entanglement in the prepare-and-measure scenario. Indeed, some of the essential quantum information protocols can be realized with this combination of quantum correlations and communication; among them stands out dense coding [7] and quantum teleportation [90]. We will denote prepare-and-measure scenarios with quantum correlations as *entanglement-assisted prepare-and-measure* scenarios (EA-PM) [30].

The use of entanglement as a resource in prepare-and-measure scenarios has only recently started to be systematically investigated, and many crucial questions remain open [32–36].

2.1 The different sets of behaviors

An *entanglement-assisted prepare-and-measure* scenario differs from a standard prepare-and-measure scenario only on the types of correlations the agents are allowed to share. Let us consider that Alice and Bob share an entangled state $\rho_{AB} \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$, where $\mathcal{H}_A$ is the Hilbert space of Alice's part and $\mathcal{H}_B$ is the Hilbert space of Bob's part. Alice chooses an input x from the set $[n_X]$ and encodes it, using her part of the entangled state ρ_{AB}, into a physical system. Then, Alice sends this physical system to Bob, who, based on an input $y \in [n_Y]$, performs a joint measurement on both the incoming system and his part of the initial entangled state ρ_{AB}. The outcome of this measurement will be denoted $b \in [n_B]$ and provided by Bob as output. Similar to the standard PM scenario, the characterization of the EA-PM is given by the conditional probability distributions $p(b|x, y)$, which we refer to as behaviors.

In the following two sub-sections, we will review the definitions of classical and quantum behaviors under the entanglement assistance paradigm. In contrast to the first chapter, we will begin by discussing the behaviors obtained when sending quantum systems.

2.1.1 EA Quantum behaviors

Let us consider the EA-PM paradigm with Alice communicating a qudit to Bob. In each round of this experiment, Alice receives as input the classical variable x and one part of the entangled state $\rho_{AB} \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$. She must provide a qudit as output. We can model all possible actions of Alice in the following way: based on the classical variable x, Alice chooses a *completely positive trace-preserving* (CPTP) map $\mathcal{C}_x : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathbb{C}^d)$ and applies it on her part of ρ_{AB}. Alice then sends her transformed state to Bob, who will have in hand the communicated d-dimensional quantum system from Alice and his part of the state ρ_{AB}. Therefore, Bob's quantum system is given by $\varrho^x_{AB} = (\mathcal{C}_x \otimes \mathcal{I})(\rho_{AB})$, where $\mathcal{I}$ is the identity channel, *i.e.*, $\mathcal{I} : \mathcal{L}(\mathcal{H}_B) \to \mathcal{L}(\mathcal{H}_B)$ with $\mathcal{I}(\sigma) = \sigma$ for every $\sigma \in \mathcal{L}(\mathcal{H}_B)$. Lastly, based on his input y, Bob performs a n_B-outcome POVM measurement $\{M_{b|y}\}$ on the state ϱ^x_{AB}. The result of Bob's measurement will be his output. We illustrate this procedure in **Fig. 6**.

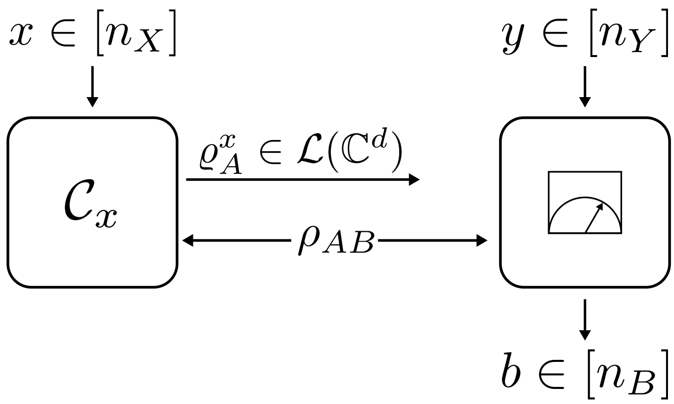


Figure 6 – In entanglement-assisted prepare-and-measure scenarios, a shared quantum resource ρ_{AB} may be exploited to enhance the performance of quantum and classical communication tasks. When communication is quantum, a general behavior is described by a local, possibly dimension-reducing CPTP transformation ($\mathcal{C}_x$) on one share of the resource, which is then sent and measured.

The set of all behaviors that can be achieved when Alice and Bob share the entangled state ρ_{AB} and Alice sends a qudit to Bob will be denoted as Q^ρ_d [33, 34]. More formally:

Definition 12 (EA$_\rho$-d-quantum behavior)**.** *Given a bipartite quantum state $\rho_{AB} \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$, a behavior $\boldsymbol{p}$ is a EA$_\rho$-d-quantum behavior, if there exists*

1. *for each $x \in [n_X]$, a CPTP map $\mathcal{C}_x : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathbb{C}^d)$;*
2. *for each $y \in [n_Y]$, a n_B-outcome POVM $\{M_{b|y}\}$ on $\mathcal{L}(\mathbb{C}^d \otimes \mathcal{H}_B)$,*

such that:

$$p(b|x,y) = \mathrm{Tr}\big[(\mathcal{C}_x \otimes \mathcal{I})(\rho_{AB}) \cdot M_{b|y}\big] \tag{2.1}$$

for all $x \in [n_X], y \in [n_Y]$ and $b \in [n_B]$. We denote the set of all EA$_\rho$*-d-quantum behaviors by $Q_d^\rho(n_X, n_Y, n_B)$.*

Remark 6. *For every state ρ_{AB}, we can show that $Q_d \subseteq Q_d^\rho$. Indeed, given $\boldsymbol{p} \in Q_d$, by the definition of a d-quantum behaviors (**Def. 6**), there is a set of quantum states $\{\rho_x\} \subset \mathcal{L}(\mathbb{C}^d)$ and a set of POVMs $\{M_{b|y}\} \subset \mathcal{L}(\mathbb{C}^d)$, such that*

$$p(b|x,y) = \operatorname{Tr}\big(\rho_x M_{b|y}\big). \tag{2.2}$$

So, for every $x \in [n_X]$, let's define the CPTP map $\mathcal{C}_x : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathbb{C}^d)$ given by:

$$\mathcal{C}_x(\sigma) = \operatorname{Tr}(\sigma)\rho_x.$$

Hence, $\mathcal{C}_x$ is a preparation map that gives ρ_x as output regardless of the input state. Applying this channel to part A of the state ρ_{AB}, we have:

$$(\mathcal{C}_x \otimes \mathcal{I})(\rho_{AB}) = \rho_x \otimes \operatorname{Tr}_A(\rho_{AB}) \tag{2.3}$$

On the other hand, for every $b \in [n_B]$ and $y \in [n_Y]$, let $N_{b|y} = M_{b|y} \otimes \mathbb{1}_B \in \mathcal{L}(\mathbb{C}^d \otimes \mathcal{H}_B)$. It follows that $\{N_{b|y}\}_{b\in[n_B]}$ are POVMs. Moreover, these measurements represent Bob performing the measurement $\{M_{b|y}\}_{b\in[n_B]}$ on the incoming qudit and the trivial measurement on his original part of ρ_{AB}. Thus, the behavior obtained with the application of channels $\mathcal{C}_x$ and measurements $\{N_{b|y}\}$, are given by:

$$\begin{aligned}\operatorname{Tr}\big[(\mathcal{C}_x \otimes \mathcal{I})(\rho_{AB}) \cdot N_{b|y}\big] &= \operatorname{Tr}\big[(\mathcal{C}_x \otimes \mathcal{I})(\rho_{AB}) \cdot (M_{b|y} \otimes \mathbb{1}_B)\big] \\ &= \operatorname{Tr}\big[\rho_x \otimes \operatorname{Tr}_A(\rho_{AB}) \cdot (M_{b|y} \otimes \mathbb{1}_B)\big],\end{aligned} \tag{2.4}$$

where in the last expression, we use eq. (2.3). Now, using the linear property from the trace of a tensor product:

$$\begin{aligned}\operatorname{Tr}\big[(\mathcal{C}_x \otimes \mathcal{I})(\rho_{AB}) \cdot N_{b|y}\big] &= \operatorname{Tr}\big[\rho_x \cdot M_{b|y}\big] \operatorname{Tr}[\operatorname{Tr}_A(\rho_{AB}) \cdot \mathbb{1}_B] \\ &= \operatorname{Tr}\big[\rho_x \cdot M_{b|y}\big] \operatorname{Tr}[\operatorname{Tr}_A(\rho_{AB})] \\ &= \operatorname{Tr}\big[\rho_x \cdot M_{b|y}\big] \operatorname{Tr}[\rho_{AB}] \\ &= \operatorname{Tr}\big[\rho_x \cdot M_{b|y}\big],\end{aligned} \tag{2.5}$$

*where we used that $\operatorname{Tr} \circ \operatorname{Tr}_A = \operatorname{Tr}$ and that the trace of the density matrix ρ_{AB} is equal to 1. Comparing them with eq. (2.2), we conclude that the behavior $\boldsymbol{p}$ also belongs to Q_d^ρ. Therefore, it follows that $Q_d \subseteq Q_d^\rho$, for every quantum state ρ. Furthermore, combining this result with **Remark 4**, it follows that $\bar{Q}_d \subseteq \bar{Q}_d^\rho$.*

*As we will see in **Section 2.2**, there are entangled states for which $\bar{Q}_d \subsetneq \bar{Q}_d^\rho$. In other words, we are going to show that some behaviors can be generated by sending a qudit aided by SR and an entangled state ρ_{AB}, but cannot be obtained by SR and a bare qudit*[1]*. In such cases, we will say that the bipartite state ρ_{AB} is useful for* EA-PM *with quantum communication.*

[1] By a "bare qudit" we mean a qudit that is *not* assisted with shared entanglement.

In the scenarios discussed in this thesis, the message sent from Alice to Bob is always a latent variable, where only the dimension of this message is restricted. It is natural to make a similar assumption for the quantum correlations that Alice and Bob share. We can assume that the entangled state shared by the agents has a bounded dimension, say D, however, the state itself is not specified. The behaviors obtained in this situation will be called *EA$_D$-d-quantum behavior*, and Q_d^D will denote the set of all these behaviors. The formal definition is given below:

Definition 13 (EA$_D$-d-quantum behavior)**.** *A behavior $\boldsymbol{p} = \{p(b|x,y)\}_{x,y,b}$ is a EA$_D$-d-quantum behavior if there are:*

1. *A quantum state $\rho_{AB} \in \mathcal{L}(\mathbb{C}^D \otimes \mathbb{C}^D)$;*
2. *for each $x \in [n_X]$, a CPTP map $\mathcal{C}_x : \mathcal{L}(\mathbb{C}^D) \to \mathcal{L}(\mathbb{C}^d)$;*
3. *for each $y \in [n_Y]$, a n_B-outcome POVM $\{M_{b|y}\}$ on $\mathcal{L}(\mathbb{C}^d \otimes \mathbb{C}^D)$;*

such that:

$$p(b|x,y) = \mathrm{Tr}\big[(\mathcal{C}_x \otimes \mathcal{I})(\rho_{AB}) \cdot M_{b|y}\big] \tag{2.6}$$

for all $x \in [n_X], y \in [n_Y]$ and $b \in [n_B]$. We denote the set of all EA$_D$-d-quantum behaviors by $Q_d^D(n_X, n_Y, n_B)$.

A more general case occurs when Alice and Bob share classical and quantum correlations. Let us consider the situation where, in addition to an arbitrary entangled state $\rho_{AB} \in \mathcal{L}(\mathbb{C}^D \otimes \mathbb{C}^D)$, they are free to share an arbitrary amount of shared randomness. In this circumstance, Alice and Bob can use this shared randomness to decide which quantum channel and which measurements to apply. The behaviors obtained with these resources will be denoted by *SR-EA$_D$-d-quantum behaviors*, and its whole set by $\bar{Q}_d^D$. More formally:

Definition 14 (SR-EA$_D$-d-quantum behavior)**.** *A family of probability distributions $\{p(b|x,y)\}_{b,x,y}$ is a SR-EA$_D$-d-quantum behavior if there exists:*

1. *a quantum state $\rho_{AB} \in \mathcal{L}(\mathbb{C}^D \otimes \mathbb{C}^D)$;*
2. *a probability distribution $\{\pi(\lambda)\}_{\lambda \in \Lambda}$;*
3. *for each $x \in [n_X]$ and $\lambda \in \Lambda$, a CPTP map $\mathcal{C}_x^\lambda : \mathcal{L}(\mathbb{C}^D) \to \mathcal{L}(\mathbb{C}^d)$;*
4. *for each $y \in [n_Y]$ and $\lambda \in \Lambda$, a n_B-outcome POVM $\{M_{b|y}^\lambda\}$ on $\mathcal{L}(\mathbb{C}^d \otimes \mathbb{C}^D)$;*

such that:

$$p(b|x,y) = \sum_\lambda \pi(\lambda)\,\mathrm{Tr}\big[(\mathcal{C}_x^\lambda \otimes \mathcal{I})(\rho_{AB}) \cdot M_{b|y}^\lambda\big] \tag{2.7}$$

for all $x \in [n_X], y \in [n_Y]$ and $b \in [n_B]$. We denote the set of all SR-EA$_D$-d-quantum behaviors by $\bar{Q}_d^D(n_X, n_Y, n_B)$.

A natural generalization occurs when Alice and Bob are allowed to share *any* entangled state ρ_{AB} of *any* (finite) dimension. We will denote the set of behaviors obtained by them in this situation by Q_d^E.

Summarizing, we denote the following set of behaviors: the case where ρ_{AB} is any D-dimensional entangled state defines the sets Q_d^D; where the shared resource ρ_{AB} is a specific state ρ (of local dimension D), we denote Q_d^ρ; and, finally, by Q_d^E we mean that ρ_{AB} has any finite dimension. Then,

$$Q_d \subseteq Q_d^\rho \subseteq Q_d^D \subseteq Q_d^E. \tag{2.8}$$

2.1.2 EA Classical Behaviors

The previous subsection dealt with quantum communication between Alice and Bob. There is, however, the possibility of restricting communication to be classical. In this circumstance, Alice receives as input the classical variable x and one part of the quantum state ρ_{AB} and needs to provide a classical output $a \in [d]$. There is a simple way to introduce this case using the previously presented formalism. To do this, we need to impose that the CPTP maps $\mathcal{C}_x$ output classical diagonal states[2]:

$$\mathcal{C}_x(\sigma) = \sum_{a=0}^{d-1} p(a|x,\sigma) |a\rangle\langle a|, \tag{2.9}$$

where $p(a|x,\sigma)$ is a probability distribution on a.

An equivalent way of describing this situation is by Alice measuring her part of the entangled state ρ_{AB} and sending the result to Bob. The Riesz representation theorem asserts that linear maps with the form of eq. (2.9) can be written in terms of the Born rule [33,91]. More concretely, any CPTP map in this form represents a *measure and prepare* channel[3] [92], *i.e.*,

$$\mathcal{C}_x(\sigma) = \sum_{a=0}^{d-1} p(a|x,\sigma) |a\rangle\langle a| = \sum_{a=0}^{d-1} \mathrm{Tr}\big(\sigma \cdot N_{a|x}\big) |a\rangle\langle a|,$$

where $\{N_{a|x}\}$ is a POVM. In this way, after Alice applies the channel $\mathcal{C}_x$ and sends the resulting state, Bob's system will be given by:

$$\sigma^x = (\mathcal{C}_x \otimes \mathbb{1}_B)(\rho_{AB}) = \sum_{a=0}^{d-1} |a\rangle\langle a| \otimes \sigma_B^{a,x},$$

[2] Remembering that a set of qudits that commute pairwise can be implemented by sending dits and applying local transformations, see Proposition 1 and the following comments.

[3] Also known as *entanglement breaking* channels.

where

$$\sigma_B^{a,x} = \mathrm{Tr}_A(\rho_{AB} \cdot (N_{a|x} \otimes \mathbb{1}_B)) \tag{2.10}$$

represents Bob's (sub-normalized) reduced state when Alice applies the POVM measurement $\{N_{a|x}\}$ to her part of ρ_{AB} and obtains output a. Thus, the behaviors obtained by the agents in a PM scenario have the following decomposition:

$$\begin{aligned} p(b|x,y) &= \mathrm{Tr}\big[(\mathcal{C}_x \otimes \mathcal{I})(\rho_{AB}) \cdot M_{b|y}\big] \\ &= \sum_{a=0}^{d-1} \mathrm{Tr}\big[(|a\rangle\langle a| \otimes \sigma_B^{a,x}) \cdot M_{b|y}\big]. \end{aligned} \tag{2.11}$$

Any joint measurement $\{M_{b|y}\} \subset \mathcal{L}(\mathbb{C}^d \otimes \mathbb{C}^D)$ applied by Bob on the state $\sigma_B^{a,x}$ is equivalent to reading the classical system a and then measuring the quantum system B according to a. Thus, without loss of generality, we can assume that $M_{b|y} = \sum_{a'=0}^{d-1} (|a'\rangle\langle a'| \otimes M_{b|a',y})$. Using this in eq. (2.11), we have:

$$\begin{aligned} p(b|x,y) &= \sum_{a=0}^{d-1} \mathrm{Tr}\left[(|a\rangle\langle a| \otimes \sigma_B^{a,x}) \cdot \sum_{a'=0}^{d-1} (|a'\rangle\langle a'| \otimes M_{b|a',y})\right] \\ &= \sum_{a=0}^{d-1} \sum_{a'=0}^{d-1} \delta(a,a')\, \mathrm{Tr}\big[\sigma_B^{a,x} \cdot M_{b|a',y}\big] \\ &= \sum_{a=0}^{d-1} \mathrm{Tr}\big[\sigma_B^{a,x} \cdot M_{b|a,y}\big]. \end{aligned}$$

Now using the definition of states $\sigma_B^{a,x}$ (eq. (2.10)) in the equation above, we have:

$$\begin{aligned} p(b|x,y) &= \sum_{a=0}^{d-1} \mathrm{Tr}\big[\sigma_B^{a,x} \cdot M_{b|a,y}\big] \\ &= \sum_{a=0}^{d-1} \mathrm{Tr}\big[\mathrm{Tr}_A(\rho_{AB} \cdot (N_{a|x} \otimes \mathbb{1}_B)) \cdot M_{b|a,y}\big] \\ &= \sum_{a=0}^{d-1} \mathrm{Tr}\big[\rho_{AB}(N_{a|x} \otimes M_{b|y,a})\big]. \end{aligned} \tag{2.12}$$

Therefore, for the classical communication case, assuming that Alice and Bob share an entangled state ρ_{AB}, any procedure is equivalent to a three-step process where: (i) Alice initially performs a d-outcome measurement conditioned by the input x on her share of ρ_{AB}; (ii) she communicates the result of her measurement to Bob over a classical channel; and (iii) based on his input y, and the message received, Bob measures his part of ρ_{AB} [33, 34]. We illustrate this procedure in Fig. 7. The set of all behaviors that can be achieved when Alice and Bob share the entangled state ρ_{AB} and Alice sends a dit to Bob will be denoted as $C_d^\rho(n_X, n_Y, n_B)$. A formal definition is provided below.

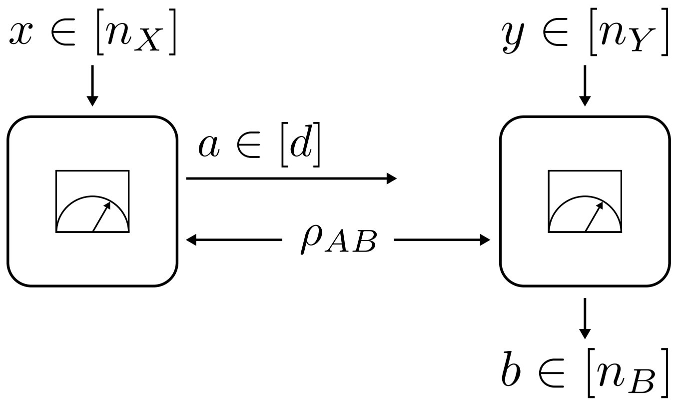


Figure 7 – In entanglement-assisted prepare-and-measure scenarios, a shared quantum resource ρ_{AB} may be exploited to enhance the performance of quantum and classical communication tasks. For classical communication, all behaviors can be described by Alice performing a local measurement on her share of ρ_{AB}. The output is then sent to the measurement device, which, informed by the classical dit and Bob's input y, measures his share of ρ_{AB}.

Definition 15 (EA$_\rho$-d-classical behavior)**.** *Given a bipartite quantum state $\rho_{AB} \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$, a behavior $\boldsymbol{p}$ is a* EA$_\rho$*-d-classical behavior, if there exists*

1. *for each $x \in [n_X]$, a d-outcome POVM $\{N_{a|x}\}$ on $\mathcal{L}(\mathbb{C}^D)$ for Alice;*
2. *for each $y \in [n_Y]$ and $a \in [d]$, a n_B-outcome POVM $\{M_{b|a,y}\}$ for Bob,*

such that:

$$p(b|x,y) = \sum_{a=0}^{d-1} \mathrm{Tr}\big[\rho_{AB}(N_{a|x} \otimes M_{b|y,a})\big], \tag{2.13}$$

for all $x \in [n_X], y \in [n_Y]$ and $b \in [n_B]$. We denote the set of all EA$_\rho$*-d-classical behaviors by $C_d^\rho(n_X, n_Y, n_B)$*

Remark 7. *Analogous to **Remark 6**, for every state ρ_{AB}, we can prove that $C_d \subseteq C_d^\rho$. Indeed, given $\mathbf{p} \in C_d$, by the definition of a d-classical behaviors (**Def. 2**), we have*

$$p(b|x,y) = \sum_{a=0}^{d-1} p(a|x)p(b|a,y).$$

So, for every $x \in [n_X]$ and $a \in [d]$, let's define $N_{a|x} = p(a|x)\mathbb{1}_A$. It follows that $\{N_{a|x}\}$ is a POVM. Analogously, we define $M_{b|a,y} = p(b|a,y)\mathbb{1}_B$. It also follows that $\{M_{b|a,y}\}$ is a

POVM. Moreover,

$$\begin{aligned}
\sum_{a=0}^{d-1} \mathrm{Tr}\big[\rho_{AB}(N_{a|x} \otimes M_{b|y,a})\big] &= \sum_{a=0}^{d-1} \mathrm{Tr}[\rho_{AB}(p(a|x)\mathbb{1}_A \otimes p(b|a,y)\mathbb{1}_B)] \\
&= \sum_{a=0}^{d-1} p(a|x)p(b|a,y)\,\mathrm{Tr}[\rho_{AB}(\mathbb{1}_A \otimes \mathbb{1}_B)] \\
&= \sum_{a=0}^{d-1} p(a|x)p(b|a,y) \\
&= p(b|x,y). \qquad (2.14)
\end{aligned}$$

Therefore, $\boldsymbol{p}$ *also belongs to* C_d^ρ*. Hence,* $C_d \subseteq C_d^\rho$*, for every state* ρ*. By* ***Remark 4****, if follows that* $\bar{C}_d \subseteq \bar{C}_d^\rho$*. In* ***Section 2.6.3****, we will see that there are entangled states for which* $\bar{C}_d \subsetneq \bar{C}_d^\rho$*.*

Similarly to the previous subsection, we can consider the situation where Alice and Bob can have *any* entangled state of local dimension D. This situation motivates the following definition:

Definition 16 (EA$_D$-d-classical behavior)**.** *A behavior* $\boldsymbol{p} = \{p(b|x,y)\}_{x,y,b}$ *is a* EA$_D$*-d-classical behavior if there are:*

1. *A quantum state* $\rho_{AB} \in \mathcal{L}(\mathbb{C}^D \otimes \mathbb{C}^D)$*;*
2. *for each* x *in* $[n_X]$*, a* d*-outcome POVM* $\{N_{a|x}\}$ *on* $\mathcal{L}(\mathbb{C}^D)$ *for Alice;*
3. *for each* $y \in [n_Y]$ *and* $a \in [d]$*, a* n_B*-outcome POVM* $\{M_{b|a,y}\}$ *on* $\mathcal{L}(\mathbb{C}^D)$ *for Bob*

such that:

$$p(b|x,y) = \sum_{a=0}^{d-1} \mathrm{Tr}\big[\rho_{AB}(N_{a|x} \otimes M_{b|y,a})\big] \qquad (2.15)$$

for all $x \in [n_X], y \in [n_Y]$ *and* $b \in [n_B]$*. We denote the set of all* EA$_D$*-d-classical behaviors by* $C_d^D(n_X, n_Y, n_B)$*.*

We may also consider when Alice and Bob share classical and quantum correlations. Let us suppose that they share an arbitrary entangled state $\rho_{AB} \in \mathcal{L}(\mathbb{C}^D \otimes \mathbb{C}^D)$ and also any amount of shared randomness. In this case, Alice and Bob can use this shared randomness to decide which measurements to apply. The behaviors obtained with these resources will be called *SR-*EA$_D$*-d-classical behaviors*, and its whole set by $\bar{C}_d^D$. More formally:

Definition 17 (SR-EA$_D$-d-classical behavior)**.** *A behavior* $\boldsymbol{p} = \{p(b|x,y)\}_{x,y,b}$ *is a SR-*EA$_D$*-d-classical behavior if there are:*

1. *A quantum state* $\rho_{AB} \in \mathcal{L}(\mathbb{C}^D \otimes \mathbb{C}^D)$*;*

2. *a probability distribution* $\{\pi(\lambda)\}_{\lambda\in\Lambda}$*;*

3. *for each* $x \in [n_X]$ *and* $\lambda \in \Lambda$*, a* d*-outcome POVM* $\{N^{\lambda}_{a|x}\}$ *on* $\mathcal{L}(\mathbb{C}^D)$ *for Alice;*

4. *for each* $y \in [n_Y]$*,* $a \in [d]$ *and* $\lambda \in \Lambda$*, a* n_B*-outcome POVM* $\{M^{\lambda}_{b|a,y}\}$ *on* $\mathcal{L}(\mathbb{C}^D)$ *for Bob,*

such that:

$$p(b|x,y) = \sum_{\lambda} \sum_{a=0}^{d-1} \pi(\lambda) \operatorname{Tr}\big[\rho_{AB}(N^{\lambda}_{a|x} \otimes M^{\lambda}_{b|y,a})\big], \tag{2.16}$$

for all $x \in [n_X], y \in [n_Y]$ *and* $b \in [n_B]$*. We denote the set of all SR-*EA_D*-d-classical behaviors by* $\bar{C}^D_d(n_X, n_Y, n_B)$*.*

As in the previous subsection, a natural generalization is given by Alice and Bob being allowed to share *any* entangled state ρ_{AB} (of finite local dimension). We will denote the set of all behaviors suitable to be simulated with these resources by C^E_d.

As a whole, we denote the following set of behaviors: the case where ρ_{AB} is any D-dimensional entangled state defines the sets C^D_d; in the case that ρ_{AB} is a particular state ρ, we denote C^{ρ}_d; and, finally, by C^E_d we mean that ρ_{AB} has any finite dimension. Then,

$$C_d \subseteq C^{\rho}_d \subseteq C^D_d \subseteq C^E_d. \tag{2.17}$$

2.2 Dense coding

Dense coding is a quantum communication protocol allowing one agent (Alice) to send two classical bits of information to another agent (Bob) using only one qubit. Charles Bennett and Stephen Wiesner proposed the idea of dense coding in 1992 [7]. Many subsequent studies extended the original study's results beyond the bi-dimensional noiseless setting [93–97]. Furthermore, dense coding was experimentally realized four years later, in 1996, by Klaus Mattle, Harald Weinfurter, Paul G. Kwiat, and Anton Zeilinger [98].

2.2.1 Standard formulation

The standard formulation of this protocol work in the following way:

1. The initial resource required for the protocol is a maximally entangled pair of qubits shared between Alice and Bob. Without loss of generality, suppose that Alice and Bob's quantum state description is given by:

$$|\Phi_{00}\rangle := \frac{|00\rangle + |11\rangle}{\sqrt{2}}.$$

2. Alice's qubit will be used to send two classical bits of information to Bob. She must first apply a unitary transformation to her qubit based on the pair of bits she intends to send. In the table below, we have an association between the pair of bits Alice wants to send and which unitary transformation she must apply. We also have what is the resulting state after the application of this unitary:

Intended Message	Applied Unitary	Resulting state
00	I	$\|\Phi_{00}\rangle := \dfrac{\|00\rangle + \|11\rangle}{\sqrt{2}}$
01	X	$\|\Phi_{01}\rangle := \dfrac{\|10\rangle + \|01\rangle}{\sqrt{2}}$
10	Z	$\|\Phi_{10}\rangle := \dfrac{\|00\rangle - \|11\rangle}{\sqrt{2}}$
11	XZ	$\|\Phi_{11}\rangle := \dfrac{-\|10\rangle + \|01\rangle}{\sqrt{2}}$

Here X, Y, Z are the Pauli matrices [99]. It's straightforward to show that $|\Phi_{00}\rangle$, $|\Phi_{01}\rangle$, $|\Phi_{10}\rangle$ and $|\Phi_{11}\rangle$ are normalized and mutually orthogonal. Thus,

$$\{|\Phi_{00}\rangle, |\Phi_{01}\rangle, |\Phi_{10}\rangle, |\Phi_{11}\rangle\}$$

is an orthonormal basis for $\mathbb{C}^2 \otimes \mathbb{C}^2$.

3. Alice sends her qubit to Bob;

4. Bob performs a projective measurement $\{M_{ij}\}_{i,j=0}^{1}$, where $M_{ij} = |\Phi_{ij}\rangle\langle\Phi_{ij}|$. Made this measurement, he can distinguish perfectly which of the four states he received. This way, if Bob knows the coding strategy used by Alice (Table 2), after measuring, he can find out which pair of bits Alice wishes to send him.

In this way, Alice can transmit[4] two classical bits if she shares a maximally entangled pair of qubits with Bob and sends a qubit to him. Furthermore, the dense coding protocol transmits the classical bits privately [100].

Several generalizations of this result have been demonstrated. In the next section, we will review in the EA-PM approach the generalization where two dits are transmitted sending a qudit assisted by a maximally entangled state of local dimension d.

2.2.2 Dense Coding as a task in a EA-PM

The dense coding protocol fits nicely as a task into an entanglement-assisted prepare-and-measure scenario with quantum communication. Indeed, let us consider the

[4]Remembering that we are using the word "transmit" for the abstract idea of information flow and the word "send" for the physical act of sending an object between agents.

same task of **Section 1.3.1**, where Alice receives two[5] dits ($n_X = d^2$) and Bob wants to guess which bitstring $\boldsymbol{x} = x_0 x_1$ was given to Alice ($n_B = d^2$). As Bob's goal is always the same (guess Alice's ditstring), we can consider that Bob is always getting the same input, *i.e.*, $n_Y = 1$. The figure of merit to be used to estimate how effective Alice and Bob are in fulfilling this task will be given by *average probability of success*, eq. (1.23), namely:

$$\mathcal{P}^{\text{avg}}_{\text{suc}}(\boldsymbol{p}) = \frac{1}{d^2} \sum_{x_0,x_1=0}^{d-1} p(x_0 x_1 | x_0 x_1). \tag{2.18}$$

Applying the result of eq. (1.26), we have that if Alice sends a bare qudit[6] to Bob, the maximum average probability of success will be $1/d$, *i.e.*,

$$\sup_{\boldsymbol{p} \in \bar{Q}_d} \mathcal{P}^{\text{avg}}_{\text{suc}}(\boldsymbol{p}) = \frac{d}{d^2} = \frac{1}{d}. \tag{2.19}$$

We will see that if Alice and Bob share a maximally entangled pair of qudits, they can accomplish this task with probability 1. In other words, Bob can always guess which two dits Alice has received, given only one qudit of communication between them (under the condition that this qudit is entanglement assisted).

In this section, we will use the definitions of high dimensional Bell states and the generalizations of Pauli matrices. These two objects are introduced in more detail in **Appendix A.4**. As the previous example, the main idea consists of Alice's preparation taking the shared state ρ_{AB} to elements of an orthogonal basis of $\mathbb{C}^d \otimes \mathbb{C}^d$. Thus, let us suppose that Alice and Bob share one of this higher-dimensional Bell states, $\rho_{AB} = |\Phi_{00}\rangle\langle\Phi_{00}|$, where

$$|\Phi_{00}\rangle := \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j\rangle_A \otimes |j\rangle_B \,, \tag{2.20}$$

For every pair $x_0, x_1 \in [d]$, we define:

$$\mathcal{C}_{x_0 x_1}(\rho) = U_{x_0 x_1} \rho U^{\dagger}_{x_0 x_1}, \tag{2.21}$$

where the unitaries $U_{x_0 x_1}$ were define in eq. (A.17), namely:

$$U_{x_0 x_1} := \sum_{k=0}^{d-1} e^{2\pi i k x_0 / d} |k\rangle_A \langle k \oplus_d x_1|_B \,. \tag{2.22}$$

On the other hand, let's assume that Bob's measurements are given by

$$M_{x_0 x_1} = |\Phi_{x_0 x_1}\rangle\langle\Phi_{x_0 x_1}| \,, \tag{2.23}$$

where

$$|\Phi_{x_0 x_1}\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} e^{2\pi i j x_0 / d} |j\rangle_{A'} \otimes |j \oplus_d x_1\rangle_{B'} \,, \tag{2.24}$$

[5] Unlike **Section 1.3.1**, here we will focus on the less general situation where the ditstring received as input by Alice has length equal to two.

[6] By a "bare qudit" we mean a qudit that is *not* assisted with shared entanglement.

see eq. (A.16). Hence, applying eq. (A.19), we have:

$$\begin{aligned}(\mathcal{C}_{x_0x_1} \otimes \mathcal{I})(\rho_{AB}) &= (U_{x_0x_1} \otimes \mathbb{1}_B) \left|\Phi_{00}\right\rangle\!\left\langle\Phi_{00}\right| (U^{\dagger}_{x_0x_1} \otimes \mathbb{1}_B) \\ &= \left|\Phi_{x_0x_1}\right\rangle\!\left\langle\Phi_{x_0x_1}\right| \end{aligned} \tag{2.25}$$

In this way, by the definition of EA_ρ-quantum behaviors (**Definition 12**):

$$\begin{aligned} p(\boldsymbol{b}|\boldsymbol{x}) &= \mathrm{Tr}[(\mathcal{C}_{x_0x_1} \otimes \mathcal{I})(\rho_{AB}) \cdot M_{b_0b_1}] \\ &= \mathrm{Tr}[\left|\Phi_{x_0x_1}\right\rangle\!\left\langle\Phi_{x_0x_1}\right| \cdot \left|\Phi_{b_0b_1}\right\rangle\!\left\langle\Phi_{b_0b_1}\right|] \\ &= |\left\langle\Phi_{x_0x_1}|\Phi_{b_0b_1}\right\rangle|^2 \\ &= \delta(x_0x_1, b_0b_1). \end{aligned} \tag{2.26}$$

Thus, substituting in eq. (2.18), we have:

$$\begin{aligned} \mathcal{P}^{\text{avg}}_{\text{suc}}(\boldsymbol{p}) &= \frac{1}{d^2} \sum_{x_0,x_1=0}^{d-1} p(x_0x_1|x_0x_1) \\ &= \frac{1}{d^2} \sum_{x_0,x_1=0}^{d-1} \delta(x_0x_1, x_0x_1) = \frac{1}{d^2} \sum_{x_0,x_1=0}^{d-1} 1 = \frac{d^2}{d^2} \\ &= 1. \end{aligned} \tag{2.27}$$

Therefore, with the aid of entanglement and by using only one qudit of communication, Bob can always correctly guess the two dits Alice has received.

Dense coding may seem to contradict Holevo's bound (**Section 1.3**) since sending a single qudit can transmit two dits of information. However, closer inspection only reveals that Holevo's bound does not apply to situations where the parties share quantum correlations. In the **Section 2.3**, we review the results of **Section 1.3.1** on the entanglement assistance paradigm.

2.2.3 Dense coding and behaviors sets

We have seen that transmitting two dits of information is possible using only one entanglement-assisted qudit. A consequence of this is that the behaviors obtained by sending two dits must be able to be performed by sending only one entanglement-assisted qudit. In this section, we will formalize this statement.

Proposition 2. *For every $d \geqslant 2$, $C_{d^2} \subseteq Q^d_d$.*

Proof. Let $\boldsymbol{p} \in C_{d^2}$, then by definition of a d^2-classical behavior (**Def. 2**) there exist two probability distributions $p(k, l|x)$ and $p(b|k, l, y)$ such that

$$p(b|x, y) = \sum_{k,l=0}^{d-1} p(k, l|x)p(b|k, l, y), \tag{2.28}$$

where we are using two dits (k, l) to represent a classical system of dimension d^2. Let us show that this behavior belongs to Q_d^d. To do so, we'll build a set of channels $\mathcal{C}_x$ and measurements $\{M_{b|y}\}$ that recover the behavior $\boldsymbol{p}$ from the decomposition of an EA_D-d-quantum behavior (eq. (2.6)). Let us first assume that Alice and Bob share the entangled state $\rho_{AB} = |\Phi_{00}\rangle\langle\Phi_{00}|$ (eq. (A.16)). Now, for every $x \in [n_X]$, let $\mathcal{C}_x : \mathcal{L}(\mathbb{C}^d) \to \mathcal{L}(\mathbb{C}^d)$ be given by:

$$\mathcal{C}_x(\rho) = \sum_{k,l=0}^{d-1} p(k, l|x) U_{kl} \rho U_{kl}^\dagger, \tag{2.29}$$

where the unitaries U_{kl} are the high-dimensional generalization of Pauli's matrices defined in eq. (A.17). It follows that $\mathcal{C}_x$ are CPTP maps. Let's now define the set of measurements to be implemented by Bob. For each $b \in [n_B]$ and $y \in [n_Y]$, let

$$M_{b|y} = \sum_{k,l=0}^{d-1} p(b|k, l, y) \, |\Phi_{kl}\rangle\langle\Phi_{kl}| \,, \tag{2.30}$$

where $\{|\Phi_{kl}\rangle\}$ is a orthonomal basis of maximally entangled states in $\mathbb{C}^d \otimes \mathbb{C}^d$ defined in eq. (A.16). By the relationship between the unitaries U_{kl} and the states $\{|\Phi_{kl}\rangle\}$ (eq. (A.19)), we have:

$$\begin{aligned}(\mathcal{C}_x \otimes \mathcal{I})(\rho_{AB}) &= \sum_{k,l=0}^{d-1} p(k, l|x)(U_{kl} \otimes \mathbb{1}_B) \, |\Phi_{00}\rangle\langle\Phi_{00}| \, (U_{kl}^\dagger \otimes \mathbb{1}_B) \\ &= \sum_{k,l=0}^{d-1} p(k, l|x) \, |\Phi_{kl}\rangle\langle\Phi_{kl}| \,. \end{aligned}\tag{2.31}$$

Hence,

$$\mathrm{Tr}\big[(\mathcal{C}_x \otimes \mathcal{I})(\rho_{AB}) \cdot M_{b|y}\big] = \sum_{k,l=0}^{d-1} \sum_{k',l'=0}^{d-1} p(k, l|x) p(b|k', l', y) \, \mathrm{Tr}[|\Phi_{kl}\rangle\langle\Phi_{kl}| \cdot |\Phi_{k'l'}\rangle\langle\Phi_{k'l'}|]$$

Remembering that $\{|\Phi_{kl}\rangle\}$ is an orthonormal basis, we have

$$\mathrm{Tr}[|\Phi_{kl}\rangle\langle\Phi_{kl}| \cdot |\Phi_{k'l'}\rangle\langle\Phi_{k'l'}|] = \delta((k, l), (k', l')).$$

Thus,

$$\begin{aligned}\mathrm{Tr}\big[(\mathcal{C}_x \otimes \mathcal{I})(\rho_{AB}) \cdot M_{b|y}\big] &= \sum_{k,l=0}^{d-1} \sum_{k',l'=0}^{d-1} p(k, l|x) p(b|k', l', y) \delta((k, l), (k', l')) \\ &= \sum_{k,l=0}^{d-1} p(k, l|x) p(b|k, l, y) = p(b|x, y). \end{aligned}\tag{2.32}$$

Thus, $\boldsymbol{p} \in Q_d^d$ and as $\boldsymbol{p}$ is a generic behavior of C_{d^2}, it follows that $C_{d^2} \subseteq Q_d^d$. $\square$

2.3 The optimality of dense coding

Previously, we learned that a single qudit can transmit two dit, as long as the parts share a maximally entangled pair of qudits. We question the efficiency of this method and if a protocol exists that can transmit more than two dits by sending a single qudit assisted by a pair of entangled qudits. We can also allow for the asymmetric situation where the dimension of the initial entangled state of Alice and Bob is higher than the communication between them. For example, suppose Alice and Bob share an entangled pair of ququarts, but Alice can only send one qubit to Bob. Can Alice, in this case, transmit more than two bits to Bob? Unfortunately, as shown in [101], and revised in this section, this is not the case. In fact, for the task of guessing Alice's input, the dense coding protocol is optimal, and the capacity of a quantum channel can be at most doubled under the presence of entanglement.

To discuss the optimality of the dense coding protocol, suppose that Alice receives two $n$its as input[7], being $n > d$. Bob wants to guess this pair of nits. However, Alice is constrained to send only one qudit to Bob. Let us further assume that Alice's inputs are uniformly chosen and that Alice and Bob share an entangled state ρ_{AB} of arbitrary dimension. To evaluate Alice and Bob's performance in this task, we will again use the *average probability of success* (eq. (1.23)):

$$\mathcal{P}_{\text{suc}}^{\text{avg}}(\boldsymbol{p}) = \frac{1}{n^2} \sum_{x_0, x_1 = 0}^{n-1} p(x_0 x_1 | x_0 x_1). \tag{2.33}$$

The main result of [73] (theorem 3.2) sets a bound for this average probability of success. Such a theorem, adapted to the scenario described here, can be stated as follows:

Theorem 4 (theorem 3.2 of [73])**.** *If Alice encodes two nits of information in her part of an arbitrary (but fixed) shared entangled state and sends one qudit to Bob, the average probability of correct decoding of a message chosen uniformly at random is bounded by* $\frac{d^2}{n^2}$.

The previous theorem can be alternatively translated as:

$$\max_{\boldsymbol{p} \in \bar{Q}_d^E} \mathcal{P}_{\text{suc}}^{\text{avg}}(\boldsymbol{p}) \leqslant \frac{d^2}{n^2}. \tag{2.34}$$

Such a bound is tight, and more than that, the dense coding protocol achieves such a value. In fact, suppose Alice and Bob share the state $\rho_{AB} = |\Phi_{00}\rangle\langle\Phi_{00}| \in \mathbb{C}^d \otimes \mathbb{C}^d$ and let $x_0 x_1$ be Alice's inputs. They will follow the following strategy:

[7] More generally, we could assume that the message is a $k$it, for whatever $k > d^2$. We use two $n$its to simplify the analysis.

1. If both x_0 and x_1 belong to $\{0, \cdots, d-1\}$, Alice will apply the channel $C_{x_0x_1}$ (eq. (2.21)) in her part of ρ_{AB}. By eq. (A.19), we have:

$$\begin{aligned}(\mathcal{C}_{x_0x_1} \otimes \mathcal{I})(\rho_{AB}) &= (U_{x_0x_1} \otimes \mathbb{1}_B)\, |\Phi_{00}\rangle\langle\Phi_{00}|\, (U^{\dagger}_{x_0x_1} \otimes \mathbb{1}_B) \\ &= |\Phi_{x_0x_1}\rangle\langle\Phi_{x_0x_1}| \end{aligned} \tag{2.35}$$

2. If x_0 or x_1 do not belong to $\{0, \cdots, d-1\}$, Alice will apply $\mathcal{C}_{00} = \mathcal{I}$ (eq. (2.21)) in her part of ρ_{AB};

3. Alice sends her result state to Bob;

4. Bob performs a projective measurement $\{M_{b_0b_1}\}$, where $M_{b_0b_1} = |\Phi_{b_0b_1}\rangle\langle\Phi_{b_0b_1}|$, on the two qudits in his possession.

The behavior obtained by them is given by:

$$\begin{aligned} p(b_0b_1|x_0x_1) &= \begin{cases} \mathrm{Tr}[(\mathcal{C}_{x_0x_1} \otimes \mathcal{I})(\rho_{AB}) \cdot M_{b_0b_1}] & \text{if } x_0, x_1 \in \{0, \cdots, d-1\} \\ \mathrm{Tr}[(\mathcal{C}_{00} \otimes \mathcal{I})(\rho_{AB}) \cdot M_{b_0b_1}] & \text{otherwise.} \end{cases} \\ &= \begin{cases} \mathrm{Tr}[|\Phi_{x_0x_1}\rangle\langle\Phi_{x_0x_1}| \cdot |\Phi_{b_0b_1}\rangle\langle\Phi_{b_0b_1}|] & \text{if } x_0, x_1 \in \{0, \cdots, d-1\} \\ \mathrm{Tr}[|\Phi_{00}\rangle\langle\Phi_{00}| \cdot |\Phi_{b_0b_1}\rangle\langle\Phi_{b_0b_1}|] & \text{otherwise.} \end{cases} \end{aligned}$$

Remembering that $\{|\Phi_{x_0x_1}\rangle\}$ is an orthonormal basis, we have

$$\mathrm{Tr}[|\Phi_{x_0x_1}\rangle\langle\Phi_{x_0x_1}| \cdot |\Phi_{b_0b_1}\rangle\langle\Phi_{b_0b_1}|] = |\langle\Phi_{x_0x_1}|\Phi_{b_0b_1}\rangle|^2 = \delta((x_0, x_1), (b_0, b_1)).$$

Thus,

$$\begin{aligned} p(b_0b_1|x_0x_1) &= \begin{cases} \mathrm{Tr}[(\mathcal{C}_{x_0x_1} \otimes \mathcal{I})(\rho_{AB}) \cdot M_{b_0b_1}] & \text{if } x_0, x_1 \in \{0, \cdots, d-1\} \\ \mathrm{Tr}[(\mathcal{C}_{00} \otimes \mathcal{I})(\rho_{AB}) \cdot M_{b_0b_1}] & \text{otherwise.} \end{cases} \\ &= \begin{cases} \delta(x_0x_1, b_0b_1) & \text{if } x_0, x_1 \in \{0, \cdots, d-1\} \\ \delta(00, b_0b_1) & \text{otherwise.} \end{cases} \end{aligned}$$

Therefore,

$$\begin{aligned} \mathcal{P}^{\text{avg}}_{\text{suc}}(\boldsymbol{p}) &= \frac{1}{n^2} \sum_{x_0,x_1=0}^{n-1} p(x_0x_1|x_0x_1) \\ &= \frac{1}{n^2} \sum_{x_0,x_1=0}^{d-1} \delta(x_0x_1, x_0x_1) = \frac{1}{n^2} \sum_{x_0,x_1=0}^{d-1} 1 \\ &= \frac{d^2}{n^2}. \end{aligned} \tag{2.36}$$

Thus, the dense coding strategy reaches the upper bound of the **Theorem 4**, being the optimal strategy for guessing Alice's inputs. Hence, by the previous Theorem and by the results of **Section 1.3.1**, we have:

$$\max_{\boldsymbol{p}\in Q_d^E} \mathcal{P}_{\text{suc}}^{\text{avg}}(\boldsymbol{p}) = \frac{d^2}{n^2} = \max_{\boldsymbol{p}\in C_{d^2}} \mathcal{P}_{\text{suc}}^{\text{avg}}(\boldsymbol{p})$$

We may ask, therefore, if in the prepare-and-measure scenario paradigm whether a qudit assisted by an entangled pair of qudits is equivalent to a pair of dits. More formally, we could ask ourselves whether $C_{d^2} = Q_d^E$ (recalling that by **Prop. 2**, we already have one of the inclusions). However, as recently shown, in both theory and experiment, qubit communication assisted by a shared pair of qubits is strictly more powerful than two classical bits [102]. Therefore, we have $C_4 \subsetneq Q_2^2$.

2.4 Geometrical Representation

This section will discuss a geometrical representation of classical and quantum sets of entanglement-assisted behaviors. We will put these sets in perspective with the polytopes of classical behaviors introduced in **Section 1.1.3**.

Starting with the EA_D-d-quantum behaviors, assuming $D \geqslant d$, by the dense coding protocol (**Prop. 2**), we saw that $\bar{C}_{d^2} \subseteq \bar{Q}_d^d$. One of the main results of this thesis will be to show that $\bar{Q}_d^d \subsetneq \bar{Q}_d^D$, for all d, D with $D \geqslant d+2$ (we will give more details in **Section 2.7**). Thus, as conclusion $\bar{C}_{d^2}$ is a proper subset of $\bar{Q}_d^D$, for $D \geqslant d+2$.

On the other hand, in **Section 2.3**, we have seen that it is impossible to transmit more than two dits of information by sending one qudit, regardless of how much entanglement is used. In this way, we have $\bar{C}_{d^2+1} \nsubseteq \bar{Q}_d^D$, for every D. To the best of our knowledge, there are no results in the literature that show $\bar{Q}_d^D$ is, or is not, a subset of $\bar{C}_{d^2+1}$. In **Fig. 8** we pictorially illustrate this facts.

Focusing now on EA_D-d-classical behaviors, it is well known that entanglement assistance is advantageous when communication is classical (see **Section 2.6** for a more detailed discussion). Thus, in general, $C_d \subsetneq C_d^D$. Moreover, a central result of this thesis is to show that such a relationship is true, regardless of the message dimension, *i.e.*, $C_d \subsetneq C_d^D$ for every d, D. On the other hand, unlike the quantum case, it is known that entanglement does not increase the channel capacity for classical communication [103]. Thus, transmitting more than one dit of information is impossible by sending only a classical entanglement-assisted dit. Hence, we have $C_{d+1} \nsubseteq C_d^D$. Finally, the opposite is also true. In fact, as was recently shown, behaviors coming from one qubit can be simulated by one EA bit, i.e., $\bar{Q}_2 \subseteq \bar{C}_2^2$ [36] . On the other hand, as mentioned earlier in this thesis, $\bar{Q}_2 \not\subset \bar{C}_3$ [64]. Thus, it follows that $\bar{C}_2^2 \not\subset \bar{C}_3$. In **Fig. 9** we pictorially illustrate this facts.

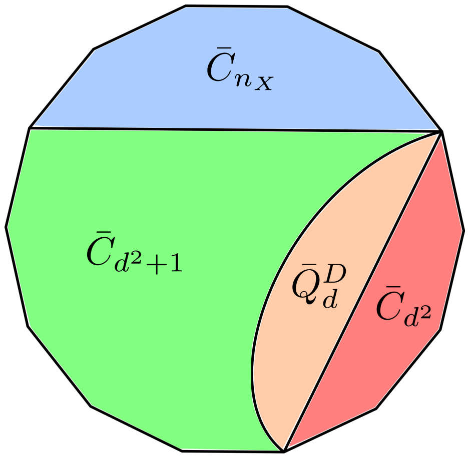


Figure 8 – Pictorial representation of the set of EA$_D$-d-quantum behaviors with respect to the classical polytopes. For $D \geqslant d+2$, in general, $\bar{C}_{d^2} \subsetneq \bar{Q}_d^D$. On the other hand, there are always behaviors in $\bar{C}_{d^2+1}$ that are not in $\bar{Q}_d^D$, *i.e.*, $\bar{C}_{d^2+1}$ is not a subset of $\bar{Q}_d^D$. Although the figure indicates, we don't know if $\bar{Q}_d^D$ is a subset of $\bar{C}_{d^2+1}$. Furthermore, the sizes of the sets illustrated in the figure have been adjusted for better comprehension, not representing their accurate relative volumes.

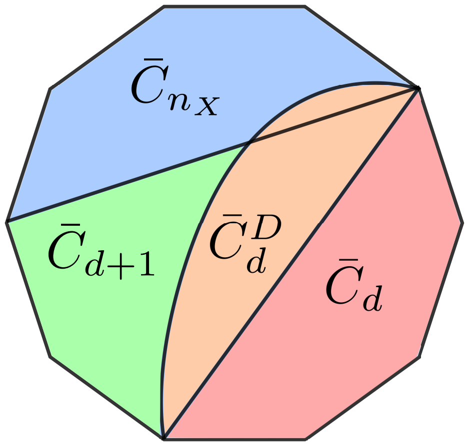


Figure 9 – The set of EA$_D$-d-classical behaviors is represented pictorially. In general, $\bar{C}_d$ is a subset of $\bar{C}_d^D$. Additionally, there are always behaviors in $\bar{C}_{d+1}$ that are not in $\bar{C}_d^D$. In spite of the figure, we do not know whether $\bar{C}_d^D$ is a subset of $\bar{C}_{d+1}$. Like Fig. 8, the representation of the sets' sizes have been changed for enhanced readability.

In [33], numerical methods are developed to approximate the sets of EA$_D$-d-classical and EA$_D$-d-quantum behaviors. The approximation of these sets is made both from inside and outside. Internal approximations are made through seesaw interactions. It is worth mentioning that it is necessary to fix the dimension of the shared entangled state. Therefore, the approximation is for the sets C_d^D and Q_d^D.

Two different methods are developed for external approximations. The first is performed through noncommutative polynomial optimization, to which an adaptation of the Navascués-Pironio-Acín (NPA) hierarchy of SDP relaxations is applied [104, 105]. However, in practice, with this method, the characterization of EA communication scenarios with more than a few inputs or outputs requires excessive computational resources. For this reason, a second approach is proposed where relaxations of the problem are introduced. Both of these methods are applicable in cases where the parties share an arbitrary amount of entanglement. Therefore, external approximations of the sets C_d^E and Q_d^E are performed.

2.5 Classical versus quantum communication in entanglement assisted prepare-and-measure scenarios

The results presented in this section have already been published in [37].

A central goal in entanglement-assisted communication scenarios is to comprehend the relationship between classical and quantum communication behaviors for various levels of entanglement and message dimension. Specifically, the goal is to identify connections between sets C_d^D and $Q_{d'}^{D'}$. The dense coding protocol shows that $C_{d^2} \subseteq Q_d^d$ (as proven in Prop. 2). Finding other meaningful relationships is a key objective in the field. We will list some of the results already discussed in the literature (see [34] for a more complete review).

- The teleportation protocol demonstrates that it is possible to transmit one qubit using two EA bits. However, if the focus is only on the behaviors obtained from sending a qubit, then one EA bit is sufficient [31,36]. Additionally, there are protocols in which an EA bit is more effective than an unassisted qubit[8]. In other words,

$$\bar{Q}_2 \subsetneq \bar{C}_2^2.$$

- In scenarios where Bob has a single measurement option ($n_Y = 1$), behaviors obtained by sending one bit assisted by a maximally entangled state of arbitrary dimension can also be obtained by sending two bare bits [35], *i.e.*,

$$\bar{C}_2^{E*} \subsetneq \bar{C}_4.$$

- There are behaviors obtained by sending one qubit assisted by an entangled pair of ququarts which can not be obtained by sending one qubit assisted by an entangled pair of qubits [33]. In other words,

$$\bar{Q}_2^2 \subsetneq \bar{Q}_2^4.$$

  This result will be generalized in Section 2.7.

[8] We may wonder if this result also holds for qudits and EA dits, however in [18] it is shown that $\bar{Q}_3 \nsubseteq \bar{C}_3^E$.

- This concept also applies to classical communication, where certain behaviors achieved by sending one bit with the assistance of an entangled pair of ququarts cannot be obtained by sending one bit with the assistance of an entangled pair of qubits [34]:

$$\bar{C}_2^2 \subsetneq \bar{C}_2^4.$$

- Since using an EA$_2$ qubit[9] effectively leads Bob to measure a four-dimensional quantum system, any behavior produced by it can also be obtained by a bare ququart. However, these resources are not equivalent, and the behaviors obtained with EA$_2$ qubit is strictly weaker than the one obtained with one unassisted ququart [34]:

$$\bar{Q}_2^2 \subsetneq \bar{Q}_4.$$

- Two bits of communication are necessary and sufficient in order to classically simulate the behaviors obtained by sending a qubit [64]:

$$\bar{Q}_2 \subsetneq \bar{C}_4 \quad \text{and} \quad \bar{Q}_2 \nsubseteq \bar{C}_3.$$

From the above list, we see that the cases involving bits and qubits are the most explored. However, the literature has few results with arbitrary dimension systems. Our following attainment goes directly in this direction. We show a non-trivial[10] family of inclusions between the EA classical set of behaviors with two dits of communication and the EA quantum set of behaviors with a single qudit of communication.

Result 2 (Behavior simulation cost)**.** *For any choice of D, d, n_X, n_Y, n_B we have*

$$C_{d^2}^{D}(n_X, n_Y, n_B) \subseteq Q_d^{D\cdot d}(n_X, n_Y, n_B), \tag{2.37}$$

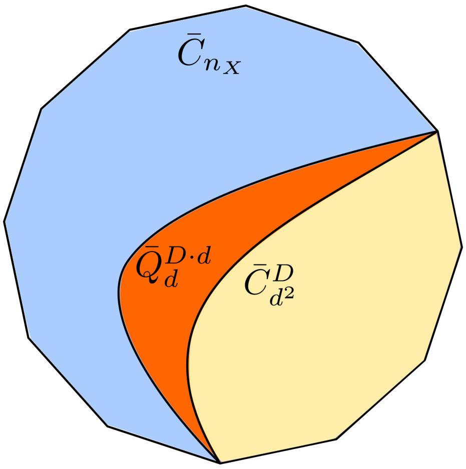
and

$$Q_d^{D}(n_X, n_Y, n_B) \subseteq C_{d^2}^{D\cdot d}(n_X, n_Y, n_B). \tag{2.38}$$

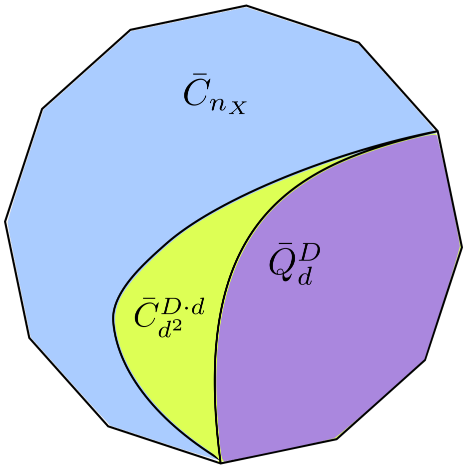
This result shows that in EA-PM scenarios, with an appropriate amount of additional entanglement, sending a single quantum dit, we can simulate the behaviors obtained by sending two EA classical dits, and *vice versa.* A pictorial illustration of this result is given in **Figure 10**. The proof of this result relies on the duality between quantum teleportation [90], and dense-coding protocols [7, 106].

[9] a EA$_2$ qubit is a qubit assisted by a pair of entangled qubits

[10] in the sense that both sets have the same channel capacity, *i.e.,* in both cases the maximum amount of information that it is possible to transmit of Alice's input are equal.



(a) Quantum simulation



(b) Classical simulation

Figure 10 – Pictorial representation of **Result 2**. In figure (a), we see that every behavior that can be obtained by sending two classical bits and being assisted by an entangled state of local dimension D can be simulated by sending a single qudit assisted by a state of local dimension $D \cdot d$. Complementarily, figure (b) shows that we can simulate every behavior obtained by sending a qubit assisted by a state of local dimension D by sending two bits assisted by an entangled state of local dimension $D \cdot d$.

Proof. Let us start with the inclusion $C_{d^2}^D \subseteq Q_d^{D\cdot d}$. Given a behavior $\boldsymbol{p} \in C_{d^2}^D$, by definition (**Def. 16**), there exist (i) a quantum state $\rho_{AB} \in \mathcal{L}(\mathbb{C}^D \otimes \mathbb{C}^D)$; (ii) for each $x \in [n_X]$, a POVM $\{N_{k,l|x}\}_{k,l\in[d]}$ for Alice[11]; and (iii), for each $k, l \in [d]$ and $y \in [n_Y]$ a POVM $\{M_{b|k,l,y}\}_{b\in[n_B]}$ for Bob, such that

$$p(b|x,y) = \sum_{k,l=0}^{d-1} \mathrm{Tr}\left[\rho_{AB}(N_{k,l|x} \otimes M_{b|k,l,y})\right]. \tag{2.39}$$

We are looking for a realization of this behavior in $Q_d^{D\cdot d}$. To find it, let us suppose that Alice and Bob share the following entangled state $\rho_{AB} \otimes |\Phi_{00}\rangle\langle\Phi_{00}|_{A'B'} \in \mathcal{L}(\mathbb{C}^{D\cdot d} \otimes \mathbb{C}^{D\cdot d})$, where $|\Phi_{00}\rangle_{A'B'} = \frac{1}{\sqrt{d}}\sum_{i=1}^{d^2} |ii\rangle$ is a maximally entangled state (see eq. (A.16)). Let us assume that systems A and A' are in Alice's lab, while systems B and B' are in Bob's lab. They use the following strategy to simulate the behavior $\boldsymbol{p}$.

1. For each $x \in [n_X]$, Alice performs measurement $\{N_{k,l|x}\}_{k,l=0}^{d-1}$ on her subsystem A. She gets the pair of dits k, l as a result, with probability $\mathrm{Tr}\big(\rho_{AB}(N_{k,l|x} \otimes \mathbb{1}_B)\big)$.

2. Alice wants to send her measurement result to Bob. For that, the agents will perform the dense coding protocol, using the state $|\Phi_{00}\rangle$ they share. Suppose then that the result of Alice's measurement was the pair of dits r, s. Alice then applies the

[11] we are using two dits to represent the d^2 possible results.

unitary U_{rs} (eq. (A.17)) over her share of the state $|\Phi_{00}\rangle$, resulting in the state $|\Phi_{rs}\rangle$ (eq. (A.19)). Since she can communicate a d-dimensional quantum system, she sends her part of the state $|\Phi_{rs}\rangle$ to Bob.

3. Bob applies a projective measurement on the basis $\{|\Phi_{kl}\rangle\}_{k,l=0}^{d-1}$ on the state $|\Phi_{rs}\rangle$, thus obtaining the dit pair r, s as the result.

4. Finally, he performs the POVM measurement $\{M_{b|r,s,y}\}_{b\in[n_B]}$ on his part of the state ρ_{AB}, where y is his choice of input.

Items 1 and 2 can be summarized as Alice applying the following quantum channel over the quantum state $\rho_{AB} \otimes |\Phi_{00}\rangle\langle\Phi_{00}|_{A'B'}$ shared between Alice and Bob:

$$\begin{aligned}(\mathcal{C}_x \otimes \mathcal{I}_{BB'})(\rho_{AB} \otimes |\Phi_{00}\rangle\langle\Phi_{00}|_{A'B'}) &= \sum_{k,l=0}^{d-1} \mathrm{Tr}_A(\rho_{AB} \cdot (N_{k,l|x} \otimes \mathbb{1}_B)) \otimes U_{k,l}\, |\Phi_{00}\rangle\langle\Phi_{00}|_{A'B'}\, U_{k,l}^\dagger \\ &= \sum_{k,l=0}^{d-1} \mathrm{Tr}_A(\rho_{AB} \cdot (N_{k,l|x} \otimes \mathbb{1}_B)) \otimes |\Phi_{kl}\rangle\langle\Phi_{kl}|_{A'B'} \end{aligned} \tag{2.40}$$

On the other hand, items 3 and 4 can be summarized as Bob applying the POVM measurement $\{M_{b|y}\}$ on the $A'BB'$ system that he owns (after Alice sends him the A' system), where

$$M_{b|y} = M_{b|y,r,s} \otimes |\Phi_{rs}\rangle\langle\Phi_{rs}|\,.$$

Thus, the behavior obtained by Alice and Bob following these steps is:

$$\begin{aligned}p(b|x,y) &= \mathrm{Tr}\big((\mathcal{C}_x \otimes \mathcal{I}_{BB'})(\rho_{AB} \otimes |\Phi_{00}\rangle\langle\Phi_{00}|_{A'B'}) \cdot M_{b|y}\big) \\ &= \sum_{k,l=0}^{d-1}\sum_{r,s=0}^{d-1} \mathrm{Tr}\left\{\big[\mathrm{Tr}_A(\rho_{AB} \cdot (N_{k,l|x} \otimes \mathbb{1}_B)) \otimes |\Phi_{kl}\rangle\langle\Phi_{kl}|_{A'B'}\big] \cdot \big[M_{b|y,r,s} \otimes |\Phi_{rs}\rangle\langle\Phi_{rs}|_{A'B'}\big]\right\} \\ &= \sum_{k,l=0}^{d-1}\sum_{r,s=0}^{d-1} \mathrm{Tr}\left[\mathrm{Tr}_A(\rho_{AB} \cdot (N_{k,l|x} \otimes \mathbb{1}_B)) \cdot M_{b|y,r,s}\right] \mathrm{Tr}\left[|\Phi_{kl}\rangle\langle\Phi_{kl}|_{A'B'} \cdot |\Phi_{rs}\rangle\langle\Phi_{rs}|_{A'B'}\right]\end{aligned}$$

As $\{\Phi_{kl}\}_{k,l=0}^{d-1}$ is an orthonormal basis, it holds that

$$\mathrm{Tr}\left[|\Phi_{kl}\rangle\langle\Phi_{kl}|_{A'B'} \cdot |\Phi_{rs}\rangle\langle\Phi_{rs}|_{A'B'}\right] = \delta[(k,l),(r,s)] = \delta(k,r)\delta(l,s).$$

On the other hand, by a trace property, namely $\mathrm{Tr}_A[X_{AB}(\mathbb{1}_A \otimes Y_B)] = \mathrm{Tr}_A(X_{AB})Y_B$, we have

$$\mathrm{Tr}_A(\rho_{AB} \cdot (N_{k,l|x} \otimes \mathbb{1}_B)) \cdot M_{b|y,r,s} = \mathrm{Tr}_A(\rho_{AB} \cdot (N_{k,l|x} \otimes M_{b|y,r,s}))$$

In this way,

$$\begin{aligned} p(b|x,y) &= \sum_{k,l=0}^{d-1}\sum_{r,s=0}^{d-1} \operatorname{Tr}\left[\operatorname{Tr}_A(\rho_{AB}\cdot(N_{k,l|x}\otimes \mathbb{1}_B))\cdot M_{b|y,r,s}\right]\operatorname{Tr}\left[|\Phi_{kl}\rangle\langle\Phi_{kl}|_{A'B'}\cdot|\Phi_{rs}\rangle\langle\Phi_{rs}|_{A'B'}\right] \\ &= \sum_{k,l=0}^{d-1}\sum_{r,s=0}^{d-1} \delta(k,r)\delta(l,s)\operatorname{Tr}_A(\rho_{AB}\cdot(N_{k,l|x}\otimes M_{b|y,r,s})) \\ &= \sum_{k,l=0}^{d-1}\operatorname{Tr}_A(\rho_{AB}\cdot(N_{k,l|x}\otimes M_{b|y,k,l})), \end{aligned}$$

which is equal to eq. (2.39). So, we can simulate any behavior $\boldsymbol{p}$ of $C_{d^2}^{D}$ by sending a single qudit assisted by an entangled state of local dimension $D\cdot d$. Therefore, $C_{d^2}^{D}\subseteq Q_d^{D\cdot d}$.

We are left with the inclusion $Q_d^D \subseteq C_{d^2}^{D\cdot d}$. From the definition of a EA_D-d-quantum behavior (**Def. 13**), given $\{p(b|x,y)\}\in Q_d^D$ there exist (i) a state $\rho_{AB}\in\mathcal{L}(\mathbb{C}^D\otimes\mathbb{C}^D)$; (ii) a set of CPTP maps $\{\mathcal{C}_x\}_{x\in[n_X]}$, with $\mathcal{C}_x:\mathcal{L}(\mathbb{C}^D)\to\mathcal{L}(\mathbb{C}^d)$; and (iii) a set of POVMs $\{M_{b|y}\}_b$, such that

$$p(b|x,y) = \operatorname{Tr}\left[(\mathcal{C}_x\otimes\mathcal{I})(\rho_{AB})M_{b|y}\right]. \tag{2.41}$$

To show that such behavior also belongs to $C_{d^2}^{D\cdot d}$, it is sufficient to show that Alice and Bob can remotely prepare the states $\varrho_{AB}^x = (\mathcal{C}_x\otimes\mathcal{I})(\rho_{AB})$ in Bob's lab, using only two dits of communication assisted by the entangled state $\rho_{AB}\otimes|\Phi_{00}\rangle\langle\Phi_{00}|_{A'B'}\in\mathcal{L}(\mathbb{C}^{D\cdot d}\otimes\mathbb{C}^{D\cdot d})$.

Let us start by fixing orthogonal bases for the spaces A and B, given by $\{|j\rangle_A\}_{j=0}^{d-1}$ and $\{|k\rangle_B\}_{k=0}^{D-1}$, respectively. For simplicity, let us initially assume that ϱ_{AB}^x is pure, *i.e.*, $\varrho_{AB}^x = |\varphi_x\rangle\langle\varphi_x|_{AB}$. Since $\{|j\rangle_A\otimes|k\rangle_B\}$ is a basis of the space $A\otimes B$, there is a set of coefficients $\{c_{jk}\}\subset\mathbb{C}$ such that

$$|\varphi_x\rangle_{AB} = \sum_{j=0}^{d-1}\sum_{k=0}^{D-1} c_{jk}\,|j\rangle_A\,|k\rangle_B\,. \tag{2.42}$$

Thus, the shared state between Alice and Bob after the application of the channel $\mathcal{C}_x$ on the systems A and B is given by

$$|\varphi_x\rangle_{AB}\,|\Phi_{00}\rangle_{A'B'} = \sum_{j,l=0}^{d-1}\sum_{k=0}^{D-1}\frac{c_{jk}}{\sqrt{d}}\,|j\rangle_A\,|k\rangle_B\,|l\rangle_{A'}\,|l\rangle_{B'}\,.$$

Following eq. (A.16), we can define an orthonormal basis of maximally entangled states for the subsystems in A and A',

$$|\Phi_{nm}\rangle_{AA'} = \frac{1}{\sqrt{d}}\sum_{r=0}^{d-1} e^{2\pi i rn/d}\,|r\rangle_A\otimes|r\oplus_d m\rangle_{A'}\,.$$

If Alice performs a measurement of this basis on her systems AA' and gets result the par of dits (n,m), Bob's post-measurement state (up normalization) becomes:

$$\begin{aligned}
&\langle\Phi_{nm}|_{AA'}\left(|\varphi_x\rangle_{AB}|\Phi_{00}\rangle_{A'B'}\right)\\
&=\sum_{j,l,r=0}^{d-1}\sum_{k=0}^{D-1}\frac{c_{jk}}{d}e^{-2\pi irn/d}\langle r|j\rangle_A\langle r\oplus_d m|l\rangle_{A'}|k\rangle_B|l\rangle_{B'}\\
&=\sum_{r=0}^{d-1}\sum_{k=0}^{D-1}\frac{c_{rk}}{d}e^{-2\pi irn/d}|k\rangle_B|r\oplus_d m\rangle_{B'}.
\end{aligned}$$

As the above state is a linear combination of orthogonal states, the square of its norm is $1/d^2$. Therefore, considering normalization, Bob's post-measurement state is given by:

$$\sum_{r=0}^{d-1}\sum_{k=0}^{D-1}c_{rk}e^{-2\pi irn/d}|k\rangle_B|r\oplus_d m\rangle_{B'}.$$

After getting the measurement result (dits n and m), Alice sends them to Bob through their classical communication channel. Informed by that, Bob applies the unitary $(\mathbb{1}_B\otimes U_{nm})$ over his pair of systems B,B', where U_{nm} is given by eq. (A.17). He ends up with the state:

$$\sum_{r,s=0}^{d-1}\sum_{k=0}^{D-1}c_{rk}e^{-2\pi i(r-s)n/d}\langle s\oplus_d m|r\oplus_d m\rangle_{B'}|k\rangle_B|s\rangle_{B'}=\sum_{r=0}^{d-1}\sum_{k=0}^{D-1}c_{rk}|k\rangle_B|r\rangle_{B'}$$

which is equal to the state $|\varphi_x\rangle_{AB}$ (eq. (2.42)). Hence, Alice and Bob carry out a teleportation from A to B'. At this point, to simulate the behavior (2.41), Bob can now apply the same set of measurements $\{M_{b|y}\}$ on the state obtained by him after the teleportation process. It is worth mentioning that the teleportation was done so that the correlations between systems A and B are preserved between systems B and B'. Bob's final state (up to a choice of basis) is equal to the state $|\varphi_x\rangle$. It is also worth noting that the bases are independent of Alice's input x, so Bob does not need to know Alice's input x to apply the unitary U_{nm}.

In the case where ϱ^x_{AB} is mixed, let $\varrho^x_{AB}=\sum_\alpha\lambda_\alpha|\varphi^x_\alpha\rangle\langle\varphi^x_\alpha|$ be a decomposition of ϱ^x_{AB} as a sum of pure states (spectral decomposition). For each α, there are coefficients $\{c^\alpha_{r,s}\}$ such that $|\varphi^x_\alpha\rangle=\sum_{j=0}^{d-1}\sum_{k=0}^{D-1}c^\alpha_{jk}|j\rangle_A|k\rangle_B$. Because all the transformations described in the pure state case are linear, the previous steps are also valid for the convex mixture of the states $|\varphi^x_\alpha\rangle$. So, even when the state $\varrho^x_{AB}=(\mathcal{C}_x\otimes\mathbb{1})(\rho_{AB})$ is mixed, Alice and Bob can teleport Alice's part of this state to Bob by sending only a couple of dits and being assisted by the state $\rho_{AB}\otimes|\Phi_{00}\rangle\langle\Phi_{00}|$. □

A direct application of **Result 2** leads us to:

Corollary 1 (Equivalence of classical/quantum behaviors). *For any choice of d, n_X, n_Y, n_B we have*

$$C^E_{d^2}(n_X, n_Y, n_B) = Q^E_d(n_X, n_Y, n_B). \tag{2.43}$$

Proof. By definition, we have that

$$C^E_{d^2} = \bigcup_{D=1}^{\infty} C^D_{d^2} \quad \text{and} \quad Q^E_d = \bigcup_{D=1}^{\infty} Q^D_d.$$

Therefore, given $\boldsymbol{p} \in C^E_{d^2}$, there exist $D \geqslant 1$ such that $\boldsymbol{p} \in C^D_{d^2}$. By result **Result 2**, $C^D_{d^2} \subseteq Q^{D\cdot d}_d$ and thus, $\boldsymbol{p} \in Q^{D\cdot d}_d \subseteq Q^E_d$. In this way, $C^E_{d^2} \subseteq Q^E_d$.

Similarly, given $\boldsymbol{p} \in Q^E_d$, there exist $D \geqslant 1$ such that $\boldsymbol{p} \in Q^D_d$. Again, by result **Result 2**, $Q^D_d \subseteq C^{D\cdot d}_{d^2}$. Thus, $\boldsymbol{p} \in C^{D\cdot d}_{d^2} \subseteq C^E_{d^2}$, which establishes the inverse inclusion $Q^E_d \subseteq C^E_{d^2}$. Therefore, $C^E_{d^2} = Q^E_d$. □

Hence, if entanglement is a free resource between the parties, in the paradigm of PM scenarios, the communication of one quantum dit is equivalent to the communication of two classical dits. This fact may seem like a trivial consequence of dense-coding [7]. However, it is worth emphasizing that the result presented here concerns the whole set of behaviors and not only the amount of information transmitted.

In Ref. [33], two algorithms are designed: one to limit the behaviors in C^E_d, and another for the behaviors in Q^E_d. Nonetheless, from the equivalence of EA classical/quantum behaviors (**Corollary 1**), $C^E_{d^2} = Q^E_d$, thus we can determine that both algorithms solve the same stimulating problem.

2.6 An entangled pair of qubits is advantageous regardless of message size

The results discussed in this section have already been published in [37].

In **Section 2.2**, we saw that entanglement assistance generates an advantage in quantum communication. An immediate question is whether EA is also advantageous for classical communication. This problem initially appeared in the context of *communication complexity problems*. In ref. [107], Clever and Buhrman exhibited the first entanglement-assisted protocol proved superior to any classical one. The role of EA in a communication complexity problem has been studied extensively, and several results have been obtained in recent decades, see [108] for a review.

In the context of prepare-and-measure scenarios, protocols where entanglement assistance is advantageous are also known. To cite an example, in [30], it is shown that EA-RAC performs better than classical RAC. In this section, we will build a family of protocols where we have an advantage in EA-PM with classical communication. Furthermore, we will show that an entangled pair of qubits is helpful in classical communication, regardless of message dimension. To do so, we start by considering the following inequality from the $C_2(3, 1, 4)$ scenario:

$$\mathcal{F}_2[\boldsymbol{p}] = 2\sum_{i=0}^{2} p(i|i) + \sum_{i=0}^{2} p(3|i) \leqslant 4. \tag{2.44}$$

We can interpret this linear functional $\mathcal{F}_2$ as a game where Alice is given an input out of three possibilities, $x \in [3]$, and she can send a bit to Bob, who must provide an answer $b \in [4]$. If Bob's answer is the same as Alice's input, *i.e.*, $b = x$, they get two points in the game. However, Bob also has an additional alternative answer, the output $b = 3$, which gives them one point regardless of Alice's input. We note that the game is not trivial since Alice is limited to sending only one bit to Bob. Thus, Bob doesn't know what Alice's input was in every round. We can view the distribution $p(b|x)$ as the probability that Bob gives b as an answer when Alice receives x as input. Therefore, the linear functional of eq. (2.44) is the average number of points Alice and Bob receive when they play the game following the strategy $\boldsymbol{p}$. In Ref. [109], a similar game was introduced and called by *ambiguous guessing games*.

Inequality (2.44) is a facet[12] of the polytope $\bar{C}_2(3, 1, 4)$, which means that no classical behavior, *i.e.*, the behaviors in $\bar{C}_2(3, 1, 4)$, violates it [110]. Nevertheless, the introduction of entanglement assistance enables one to violate such inequality. In fact, by employing the *see-saw algorithm* developed in [33], one can find a realization of a behavior $\boldsymbol{p}' \in C_2^2(3, 1, 4)$ such that

$$\mathcal{F}_2[\boldsymbol{p}'] = 4.155. \tag{2.45}$$

The code used to find such violations can be found in [110]. To extend this result, let us now scale inequality (2.44) to scenarios $C_d(d+1, 1, d+2)$ by defining the linear functional:

$$\mathcal{F}_d[\boldsymbol{p}] = 2\sum_{i=0}^{d} p(i|i) + \sum_{i=0}^{d} p(d+1|i). \tag{2.46}$$

Similarly to what has been done for the functional $\mathcal{F}_2$, we can interpret the functional $\mathcal{F}_d$ as the average score Alice and Bob achieve in a game in which Alice is given an input out of $(d+1)$ possibilities, *i.e.*, $x \in [d+1]$. Bob receives a dit from Alice and must provide an output $b \in [d+2]$. Bob and Alice get two points if their answers are the same, *i.e.*, $b = x$. In addition, Bob has an alternative response, $b = d+1$, which gives them one point disregarding Alice's input. Due to Alice receiving one between $(d+1)$ inputs but sending one dit to Bob (one between d symbols), Bob is incapable of knowing what Alice's input was in each round. Thus, if Alice and Bob play the game following the strategy $\boldsymbol{p}$, the linear functional of eq. (2.46) gives the average number of points they receive.

[12] In Section 1.1.3 the idea of facets and polytopes is introduced.

This section will explicitly describe a strategy in C_d that can reach $\mathcal{F}_d = 2d$. As also shown below, $2d$ is the maximum value of $\mathcal{F}_d$ for all behaviors in $\bar{C}_d$. On top of that, we also show that it is always possible to obtain a value for $\mathcal{F}_d$ greater than $2d$ when considering entanglement-assisted behaviors in C_d^2. This implies that there is always a C_d^2 behavior which is not in C_d, *i.e.*, that

Result 3. *For every integer $d \geqslant 2$, $C_d \subsetneq C_d^2$.*

2.6.1 Strategy to reach $\mathcal{F}_d = 2d$ in $C_d(d+1, 1, d+2)$

Consider the deterministic strategy where Alice applies the encoder function $\mathcal{E} : [d+1] \to [d]$ on the input received by her, where

$$\mathcal{E}(x) = \begin{cases} x, & \text{if } 0 \leqslant x \leqslant d-1 \\ 0, & \text{if } x = d, \end{cases}$$

and Bob decodes his received dit via

$$\begin{aligned} \mathcal{D} : [d] &\to [d+2] \\ k &\mapsto k \end{aligned} \tag{2.47}$$

The behavior obtained by them is given by:

$$q(b|x) = \delta[b, (\mathcal{D} \circ \mathcal{E})(x)] \tag{2.48}$$

$$= \begin{cases} \delta(b, x), & \text{if } 0 \leqslant x \leqslant d-1 \\ \delta(b, 0), & \text{if } x = d. \end{cases} \tag{2.49}$$

With this strategy, Alice and Bob reach the value of $2d$ for the functional (2.46). Indeed,

$$\begin{aligned} \mathcal{F}_d[\boldsymbol{q}] &= 2\sum_{i=0}^{d} q(i|i) + \sum_{i=0}^{d} q(d+1|i) \\ &= 2\left(\sum_{i=0}^{d-1} \delta(i,i) + \delta(d,0)\right) + \sum_{i=0}^{d-1} \delta(d+1,i) + \delta(d+1,0) \\ &= 2\sum_{i=0}^{d-1} 1 \\ &= 2d. \end{aligned} \tag{2.50}$$

2.6.2 Optimality of the strategy

We will now show that the maximum of $\mathcal{F}_d$ in $C_d(d+1, 1, d+2)$ is in fact equal to $2d$. As $C_d(d+1, 1, d+2)$ is a polytope and $\mathcal{F}_d$ is a linear functional, by the *fundamental theorem of linear programming*, the maximum value of $\mathcal{F}_d$ occurs at the extremal points of this polytope [111]. By the Fine-like theorem for PM scenarios (**Thrm. 1**), these extremal points are the deterministic strategies involving one dit of communication (**Def. 1**). Thus, it suffices to show that the maximum of $\mathcal{F}_d$ over the deterministic strategies is, in fact, $2d$.

Let $\boldsymbol{q}$ be a deterministic point of $C_d(d+1,1,d+2)$, *i.e.*, $\boldsymbol{q} \in D_d(d+1,1,d+2)$. Then, by definition, there is an enconding function $\mathcal{E} : [d+1] \to [d]$ and a decoding function $\mathcal{D} : [d] \to [d+2]$ such that $q(b|x) = \delta\left[b, (\mathcal{D} \circ \mathcal{E})(x)\right]$. Since the domain of $\mathcal{D}$ has cardinality d, it follows that the cardinality of the image of $\mathcal{D}$ is smaller or equal to d, *i.e.*, $|\mathrm{Im}(\mathcal{D})| \leqslant d$. On the other hand, $\mathrm{Im}(\mathcal{D} \circ \mathcal{E}) \subseteq \mathrm{Im}(\mathcal{D})$. Therefore, it follows that $|\mathrm{Im}(\mathcal{D} \circ \mathcal{E})| \leqslant d$.

At this point, we divide the problem into two situations. In the first one, we will assume that $(d+1) \in \mathrm{Im}(\mathcal{D} \circ \mathcal{E})$. As we also know that $|\mathrm{Im}(\mathcal{D} \circ \mathcal{E})| \leqslant d$ and $\mathrm{Im}(\mathcal{D} \circ \mathcal{E}) \subseteq [d+2]$[13], it follows that there are $k, l \in [d+1]$ such that $k, l \notin \mathrm{Im}(\mathcal{D} \circ \mathcal{E})$. Remembering that $q(b|x) = \delta\left[b, (\mathcal{D} \circ \mathcal{E})(x))\right]$, then it follows that $q(k|k) = 0 = q(l|l)$, so:

$$
\begin{aligned}
\mathcal{F}_d\left[\boldsymbol{q}\right] &= 2\sum_{i=0}^{d} q(i|i) + \sum_{i=0}^{d} q(d+1|i) \\
&= 2q(k|k) + 2q(l|l) + q(d+1|k) + q(d+1|l) + \sum_{\substack{i=0 \\ i\neq k,l}}^{d} \left[2q(i|i) + q(d+1|i)\right] \\
&= q(d+1|k) + q(d+1|l) + \sum_{\substack{i=0 \\ i\neq k,l}}^{d} \left[2q(i|i) + q(d+1|i)\right]. \qquad (2.51)
\end{aligned}
$$

Nevertheless, from normalization, we have the following upper bounds for the probabilities appearing in (2.51):

$$q(d+1|k) \leqslant 1, \qquad (2.52a)$$

$$q(d+1|l) \leqslant 1, \qquad (2.52b)$$

$$2q(i|i) + q(d+1|i) \leqslant 2[q(i|i) + q(d+1|i)] \leqslant 2. \qquad (2.52c)$$

Replacing them in (2.51), we reach

$$\mathcal{F}_d\left[\boldsymbol{q}\right] \leqslant 2 + 2(d-1) = 2d. \qquad (2.53)$$

Suppose now that $(d+1) \notin \mathrm{Im}(\mathcal{D} \circ \mathcal{E})$ which implies that $\mathrm{Im}(\mathcal{D} \circ \mathcal{E}) \subseteq [d+1]$. Therefore, $q(d+1|i) = \delta\left[d+1, (\mathcal{D} \circ \mathcal{E})(i)\right] = 0$ for all $i \in [d+1]$. Thus,

$$\mathcal{F}_d\left[\boldsymbol{q}\right] = 2\sum_{i=0}^{d} q(i|i).$$

[13] Remembering that $[d+2] = \{0, \cdots, d+1\}$.

On the other hand, as $|\mathrm{Im}(\mathcal{D} \circ \mathcal{E})| \leqslant d$, it follows that there is $k \in [d+1]$ such that $k \notin \mathrm{Im}(\mathcal{D} \circ \mathcal{E})$, which implies $q(k|k) = 0$. Then, using that $q(i|i) \leqslant 1$ for all $i \in [d+1]$,

$$\mathcal{F}_d\left[\boldsymbol{q}\right] = 2\sum_{\substack{i=0\\ i\neq k}}^{d} q(i|i) \leqslant 2d.$$

Hence, we have that no deterministic point in $D_d(d+1,1,d+2)$ reaches a value greater than $2d$ when the functional $\mathcal{F}_d$ is applied. Therefore,

$$\sup_{\boldsymbol{p}\in \bar{C}_d} \mathcal{F}_d\left[\boldsymbol{p}\right] = 2d. \tag{2.54}$$

2.6.3 Entanglement-assisted violation of $\mathcal{F}_d \leqslant 2d$

It is always possible to obtain a value for $\mathcal{F}_d$ greater than $2d$ when considering the entanglement-assisted behaviors in C_d^2. To prove it, we will use the behavior $\boldsymbol{p}' \in C_2^2(3,1,4)$ obtained in [110] — namely, the one such that $\mathcal{F}_2[\boldsymbol{p}'] = 4.155$— to construct a behavior that exceeds $2d$ for $\mathcal{F}_d$.

The distribution $\boldsymbol{p}^* \in C_d^2(d+1,1,d+2)$ is composed of

$$p^*(b|x) = \begin{cases} p'(b|x), & \text{if } 0 \leqslant x \leqslant 2 \text{ and } 0 \leqslant b \leqslant 3 \\ 0, & \text{if } 0 \leqslant x \leqslant 2 \text{ and } b > 3 \\ \delta(b,x), & \text{if } 3 \leqslant x \leqslant d. \end{cases} \tag{2.55}$$

We can see that $\boldsymbol{p}^*$ belongs to C_d^2 since as Alice can send a $d$it to Bob, she can use two symbols from this alphabet to simulate the $\boldsymbol{p}'$ part of the behavior. In contrast, the remaining $(d-2)$ symbols are used to simulate the other $\delta(b,x)$ part of the behavior $\boldsymbol{p}^*$. Therefore, substituting into eq. (2.46),

$$\mathcal{F}_d\left[\boldsymbol{p}^*\right] = \mathcal{F}_2[\boldsymbol{p}'] + 2\sum_{i=3}^{d} p^*(i|i) + \sum_{i=3}^{d} p^*(d+1|i).$$

But, from eq. (2.55),

$$2\sum_{i=3}^{d} p^*(i|i) = 2\sum_{i=3}^{d} \delta(i,i) = 2(d-2),$$

and

$$\sum_{i=3}^{d} p^*(d+1|i) = \sum_{i=3}^{d} \delta(d+1,i) = 0.$$

Thus,

$$\mathcal{F}_d\left[\boldsymbol{p}^*\right] = 4.155 + 2(d-2) = 2d + 0.155 > 2d. \tag{2.56}$$

Since the maximum of $\mathcal{F}_d$ in $\bar{C}_d$ is equal to $2d$, it follows that $\boldsymbol{p}^* \notin \bar{C}_d$.

2.7 Advantage in higher-dimensional entanglement assistance for every dimension

The results of this section were published in [37].

As we saw in the dense coding protocol (**Section 2.2**), if Alice and Bob share a maximally entangled pair of qudits, the communication of a single qudit from Alice to Bob can transmit two dits of information. Moreover, we saw that it is impossible to transmit more than two dits of information by sending just one qudit, regardless of how much entanglement Alice and Bob have at their disposal (**Section 2.3**). Thus, dense coding is considered optimal at the information capability level. Despite this optimality, some protocols require a higher amount of entanglement to reach optimality. As recently shown, Q_2^2 is a proper subset of Q_2^4 [33]. In other words, for scenarios where communication is limited, a maximally entangled state with the same local dimension as the communicated quantum system (Q_2^2) does not provide the whole set of behaviors Q_2^E. Therefore, even though high-dimensional entanglement does not increase a qudit's information capacity, it can be a helpful resource in other communication tasks. Whereby "high dimensional entanglement" means states whose local dimension is higher than the communicated quantum system.

In this section, we will extend the $Q_2^2 \subsetneq Q_2^4$ result, shown in Ref. [33]. More precisely, we are going to show that higher-dimensional entanglement is useful, irrespective of the dimension of the message.

Result 4. *For every $d \geqslant 2$, $Q_d^d \subsetneq Q_d^{2d}$.*

Proof. The proof is based on a concatenation of previous results. Indeed, as a consequence of dense coding (**Prop. 2**), $C_{d^2} \subseteq Q_d^d$. On the other hand, it is also straightforward that $Q_d^d \subseteq Q_{d^2}$. So, $C_{d^2} \subseteq Q_d^d \subseteq Q_{d^2}$. Focusing on scenarios with $n_Y = 1$, by Frenkel-Weiner's theorem (**Thrm. 3**), $C_{d^2} = Q_{d^2}$. Given that, $C_{d^2} \subseteq Q_d^d \subseteq Q_{d^2} = C_{d^2}$, thus

$$Q_d^d = C_{d^2}. \tag{2.57}$$

Then, combining eq. (2.57) with **Result 3**, we have:

$$Q_d^d = C_{d^2} \subsetneq C_{d^2}^2.$$

Applying then **Result 2**:

$$Q_d^d = C_{d^2} \subsetneq C_{d^2}^2 \subseteq Q_d^{2d}.$$

□

2.8 Unsteerable states are not valuable resources in classical communication tasks

The results of this section were published in [37].

Entanglement is necessary to achieve the best-known performance in several computational and cryptographic tasks. In a (semi)device-independent approach, this advantage is associated with more intrinsic properties, such as non-locality [61] and steering [112]. However, not all entangled states are nonlocal [79] nor are steerable [82]. A related and pivotal question can be made for EA-PM scenarios, *i.e.*, which entangled states are useful resources in EA-PM? This question was raised, for instance, in [34], in which the authors ask whether for any entangled state ρ there exists a d such that $\bar{C}_d^\rho \nsubseteq \bar{C}_d$. Naturally, the analogue for quantum communication scenarios, *i.e.*, whether $\bar{Q}_d^\rho \nsubseteq \bar{Q}_d$, is also of interest.

When ρ is a pure state, both results are true [77, 107, 113, 114]. Indeed, in [77], the authors present a game (4-cup & 2-ball game) in the scenario $C_2(6, 1, 4)$ where the advantage brought by an entanglement-assistance ρ grows at least linearly w.r.t. ρ's violation of the CHSH inequality [115]. As every entangled pure state violates the CHSH [116, 117], we conclude that every entangled pure state provides an advantage in this game. Similarly, for quantum communication, it has been shown that the optimal probability of success of Bob guessing two of Alice's bits, with Alice sending a single qubit but assisted by an entangled state, increases with the Schmidt number of the shared state between them [32].

When it comes to mixed states in entanglement-assisted classical communication scenarios, entanglement is not enough to achieve non-classicality. We will demonstrate that all unsteerable states result in classical behaviors, therefrom uncovering a tight connection between *steering* and prepare-and-measure scenarios.

Let us start introducing the ideas behind quantum steering [112, 118, 119]. An *assemblage* $\{\varrho_{a|x}\}$ is a collection of ensembles for a same state ρ_B, *i.e.*, $\sum_a \varrho_{a|x} = \rho_B, \forall x$. For a fixed bipartite state ρ_{AB}, every collection of measurements $\{N_{a|x}\}$ performed by Alice leads to an assemblage on Bob's side described by the elements

$$\varrho_{a|x} = \mathrm{Tr}_A[\rho_{AB}(N_{a|x} \otimes \mathbb{1}_B)]. \tag{2.58}$$

An assemblage $\{\varrho_{a|x}\}$ is *unsteerable* if it admits a *local hidden state model* [119], that is, if each element can be written as

$$\varrho_{a|x} = \sum_\lambda \pi(\lambda) p(a|x, \lambda) \sigma_\lambda, \tag{2.59}$$

with $\sigma_\lambda \in \mathcal{L}(\mathcal{H}_B)$ and $\{p(\lambda)\}_\lambda$, $\{p(a|x, \lambda)\}_a$ being probability distributions. Assemblages that are not unsteerable are said to be *steerable*.

More fundamentally, a *state* ρ_{AB} is unsteerable if for *every* possible choice of measurements $\{N_{a|x}\}$, the assemblage given by eq. (2.58) is unsteerable. Otherwise, ρ_{AB} is considered steerable. Interestingly, there are entangled states, such as some Werner and isotropic states, which are unsteerable for *any* set of POVMs [81, 82].

Suppose that ρ_{AB} is an unsteerable state. As stated in **Def. 15**, a behavior belongs to C_d^ρ if

$$p(b|x,y) = \sum_{a=0}^{d-1} \mathrm{Tr}\big[\rho_{AB}(N_{a|x} \otimes M_{b|a,y})\big]. \tag{2.60}$$

From a well-known property of the trace, $\mathrm{Tr}_A[X_{AB}(\mathbb{1}_A \otimes Y_B)] = \mathrm{Tr}_A(X_{AB})Y_B$, we get that

$$\begin{aligned} p(b|x,y) &= \sum_{a=0}^{d-1} \mathrm{Tr}\left\{\mathrm{Tr}_A[\rho_{AB}(N_{a|x} \otimes M_{b|a,y})]\right\} \\ &= \sum_{a=0}^{d-1} \mathrm{Tr}\left\{\mathrm{Tr}_A[\rho_{AB}(N_{a|x} \otimes \mathbb{1}_B)]M_{b|a,y}\right\} \\ &= \sum_{a=0}^{d-1} \mathrm{Tr}\left(\varrho_{a|x} M_{b|a,y}\right). \end{aligned} \tag{2.61}$$

As assemblages coming for unsteerable ρ_{AB} has a local hidden state model (eq. (2.59)), then

$$p(b|x,y) = \sum_{a=0}^{d-1} \sum_{\lambda} \pi(\lambda) p(a|x,\lambda) \mathrm{Tr}\left(\sigma_\lambda M_{b|a,y}\right)$$

Let $p(b|a,y,\lambda) = \mathrm{Tr}\left(\sigma_\lambda M_{b|a,y}\right)$, it follows that $\{p(b|a,y,\lambda)\}$ is a probability distribution. Therefore,

$$p(b|x,y) = \sum_{a=0}^{d-1} \sum_{\lambda} \pi(\lambda) p(a|x,\lambda) p(b|a,y,\lambda), \tag{2.62}$$

which is exactly the definition of a SR-d-classical behavior, *i.e.*, a behavior belonging to $\bar{C}_d$ (**Def 5**). Thus, any unsteerable resource ρ leads to $\bar{C}_d^\rho(n_X, n_Y, n_b) = \bar{C}_d(n_X, n_Y, n_B)$, for all choices of d, n_X, n_Y and n_B. As not all entangled states are steerable [82], we conclude that

Result 5. *Some entangled states are not advantageous in classical communication scenarios.*

While steering is necessary for classical communication advantages, it is not when communication is quantum (Result 3 from [32]).

Having connected steering with prepare-and-measure scenarios, some other results from the former readily translate to the latter. For instance, it is well known that, for an assemblage to be steerable, Alice must have used incompatible measurements [80, 83]. In this way, it immediately follows that

Corollary 2. *Alice's measurements must be incompatible to achieve advantages in entanglement-assisted classical communication tasks.*

Naturally, these observations can be applied as semi-device independent witnesses for steering and measurement incompatibility.

Inspired by this result, a subsequent work shows that states which are Bell local (*i.e.*, states that are incapable of generating Bell nonlocality) are also incapable of generating advantage in EA-PM scenarios with classical communication [36]. The argument goes as follows: Let ρ_{AB} be a Bell local state, then it is possible to show that this state will always be useless for EACC. In fact, given any tuple (n_X, n_Y, n_B, d) (i.e., given any prepare-and-measure scenario). Let $\boldsymbol{p}$ a behavior in $C_d^{\rho}(n_X, n_Y, n_B)$. So, by **Definition 15**, there exist POVMs $\{N_{a|x}\}$ and $\{M_{b|a,y}\}$ such that:

$$p(b|x,y) = \sum_{a=0}^{d-1} \mathrm{Tr}\big[\rho_{AB}(N_{a|x} \otimes M_{b|y,a})\big] \tag{2.63}$$

Let's now consider a Bell scenario where Alice takes an input $x \in [n_X]$ and outputs $a \in d$. On the other hand, Bob takes as input $(y,z) \in [n_Y] \times [d]$ and outputs $b \in [n_B]$. Therefore, the correlations resulting from this scenario can be written as $p(a,b|x,(y,z))$. A quantum implementation in this scenario takes place with Alice and Bob sharing the previously mentioned state ρ_{AB}, with Alice implementing the measurements $\{N_{a|x}\}$ and Bob the measurements $\{M_{b|y,z}\}$, that is:

$$p(a,b|x,(y,z)) = \mathrm{tr}\left[\rho_{AB}(N_{a|x} \otimes M_{b|y,z})\right]. \tag{2.64}$$

Since ρ_{AB} is a local Bell state, it follows that there is an LHV model to explain the behavior

$$p(a,b|x,(y,z)) = \sum_{\lambda} \pi(\lambda) p(a|x,\lambda) p(b|(y,z),\lambda). \tag{2.65}$$

The behaviors $p(b|x,y)$ of eq. (2.63) may be interpreted as a coarse graining of the behaviors $p(a,b|x,(y,z))$ of eq. (2.64). In fact:

$$\begin{aligned}\sum_{a=0}^{d-1} p(a,b|x,(y,a)) &= \sum_{a=0}^{d-1} \mathrm{tr}\left[\rho_{AB}(N_{a|x} \otimes M_{b|y,a})\right] \\ &= p(b|x,y).\end{aligned} \tag{2.66}$$

Therefore, substituting eq. (2.65) in the above equation, we have:

$$\begin{aligned}p(b|x,y) &= \sum_{a=0}^{d-1} p(a,b|x,(y,a)) \\ &= \sum_{a=0}^{d-1} \sum_{\lambda} \pi(\lambda) p(a|x,\lambda) p(b|(y,a),\lambda).\end{aligned} \tag{2.67}$$

This precisely defines an SR-d-classical behavior, as stated in **Def 5**. Thus, for any local resource ρ, $\bar{C}_d^{\rho}(n_X, n_Y, n_b) = \bar{C}_d(n_X, n_Y, n_B)$ for all values of d, n_X, n_Y, and n_B.

By adopting this approach, it is easy to see that if Alice's or Bob's measurements compatible, then $p(a, b|x, (y, a))$ will be a Bell local behavior and therefore $p(b|x, y)$ will be a classical behavior. This concept extends the finding in Corollary 2, indicating that non-classicality in EACC scenarios requires not only Alice's incompatible measurements but also Bob's.

It is essential to call attention to the results of Ref. [120]. There, it is shown that unsteerable states can be helpful for EA-RAC when the amount of randomness shared between the parties is bounded. Note that this does not contradict **Result 5**, as we are not bounding the amount of shared randomness here.

Concluding Remarks and Open Questions

This thesis contains an in-depth discussion of quantum and classical communication tasks under the prepare-and-measurement paradigm. Put another way, this work elaborates in more detail the results obtained in [37]. First, we formally showed that measurement incompatibility is necessary for quantum communication to outperform classical communication. It is widely accepted that this result is a direct corollary of a result from [76], however, as far as we are aware, no formal proof of it has been presented so far. It would also be interesting to find closer connections between measurement incompatibility and prepare-and-measure scenarios. Thenceforth, we focused on PM scenarios assisted by entangled quantum states — a setting that has only recently begun to be systematically investigated [33, 34]. To do so, we derived a chain of inclusions between classical and quantum sets of behaviors by increasing only the dimension of the assisted entanglement, see **Res. 2**. In turn, this result led to the observation that, in the limit of arbitrary entanglement, the sets of quantum and classical communication behaviors are identical — see **Corolary 1** for more details. We furthermore showed that there are classical communication protocols in which certain entangled pairs of qubits always lead to better performance, no matter what is the communication dimension (**Res. 3**).

Subsequently, by employing our chain of inclusions between classical and quantum sets of behaviors, we proved that increasing the entanglement dimension always leads to better performance, no matter the dimension of the system sent, **Res. 4**. This solves an open problem stated in Ref. [34]. Lastly, we discussed which properties entangled states must have to provide advantages in classical communication scenarios, showing that unsteerable states are useless in such tasks. Connecting this with previous results in quantum steering [82], we concluded that not all entangled states are useful resources in classical communication (**Res. 5**), this being another of the open problems listed in Ref. [34].

Several connections between ours and previous results can be highlighted. Firstly, Pauwels *et al.* [34] questioned whether there is an integer D such that $C_d^D = C_d^E$, and analogously for quantum communication, *i.e.*, $Q_d^D = Q_d^E$. From **Res. 2**, we see that these two questions are equivalent. Indeed, by answering one, our result will guarantee the same answer to the other. This can be of great help, as the case with classical communication can be more easily linked to Bell scenarios, in which we know that an arbitrary amount of entanglement is useful [121–123].

Secondly, in Ref. [33], two algorithms are developed: one to bound the behaviors in C_d^E, and another for the behaviors in Q_d^E. However, from **Corollary 1**, $C_{d^2}^E = Q_d^E$, therefore we conclude that, in essence, both algorithms solve the same problem.

Third, the proof of **Result 2** is based on the duality between dense coding and quantum teleportation protocols. In recent work, the concept of *entanglement-reversible channels* is introduced [124]. These are channels that are reversible with the aid of an entangled resource state. Teleportation and dense coding are standard examples of entanglement-reversible channels. These ideas can be helpful in EA-PM scenarios, with the possibility of generating relationships such as those in **Result 2**.

Lastly, Frenkel and Weiner [35] showed that, for communication channel scenarios (*i.e.*, PM scenarios with $n_Y = 1$), $C_2^{D*} \subseteq C_4$ for every D, where C_2^{D*} is the set of behaviors obtained by sending one bit and having assistance from a maximally entangled state of local dimension D. They also conjecture that $C_d^E \subseteq C_{d^2}$ for every d. If the conjecture ends up being true, we can readily apply Corollary 1, along with Frenkel-Weiner's theorem (**Thrm. 3**), to conclude that $Q_d^E \subseteq Q_{d^4}$ for every d. This can also be seen as an alternative way to prove this conjecture.

Further questions naturally arise from our results. For one, it would be interesting to find a characterization for the set of entangled states that are useful in EA-PM scenarios with classical communication. As discussed, this must be a subset of the steerable states. Inspired by this result, a subsequent work shows that states that are Bell local (*i.e.*, states that are incapable of generating Bell nonlocality) are also incapable of generating advantage in EA-PM scenarios with classical communication [36]. The question remains whether these two concepts are equivalent or if there are states that are Bell nonlocal but still unable to generate advantage in EA classical communication. Furthermore, this is connected with a long-standing open problem in communication complexity problems (CCP), namely, if a violation of a Bell inequality always implies quantum advantage in a CCP [114].

Further questions naturally arise from our results. Let us now enumerate some of them:

1. From the **Result 1** we saw that measurement incompatibility is necessary for nonclassicality in prepare-and-measure scenarios. Furthermore, it was shown in [27] that not every set of incompatible measurements generates nonclassicality in PM scenarios. It would be interesting to characterize this set and relate it to the set of nonclassical measurements for Bell scenarios [80].

2. It would be interesting to find a characterization for the set of entangled states that are useful in EA-PM scenarios with classical communication. As discussed in **Section 2.8**, this must be a subset of the nonlocal states. The question remains whether these two concepts are equivalent or if there are states that are Bell nonlocal but still unable to generate advantage in EA classical communication. Furthermore,

this is connected with a long-standing open problem in communication complexity problems (CCP), namely, if a violation of a Bell inequality always implies quantum advantage in a CCP [114].

3. From the **Corollary 1**, we saw that $C_{d^2}^E = Q_d^E$. It would be interesting to verify whether such a relationship also holds for the case where the communication channel is noisy.

4. Infinite dimensional entangled states give any advantage in EA-PM? In another words, is there some scenario where $C_d^E \subsetneq C_d^\infty$ or $Q_d^E \subsetneq Q_d^\infty$? where C_d^∞, Q_d^∞ represent the sets of behaviors obtained in a PM scenario assisted by any entangled state of infinite dimension, to classical and quantum communication respectively.

Bibliography

1 BENNETT, C. H.; BRASSARD, G. Quantum cryptography: Public key distribution and coin tossing. *Theoretical Computer Science*, v. 560, p. 7–11, 2014. ISSN 0304-3975. Theoretical Aspects of Quantum Cryptography – celebrating 30 years of BB84. Disponível em: https://www.sciencedirect.com/science/article/pii/S0304397514004241. Citado na página 15.

2 EKERT, A. K. Quantum cryptography based on bell's theorem. *Phys. Rev. Lett.*, American Physical Society, v. 67, p. 661–663, Aug 1991. Disponível em: https://link.aps.org/doi/10.1103/PhysRevLett.67.661. Citado na página 15.

3 SHOR, P. W. Polynomial-time algorithms for prime factorization and discrete logarithms on a quantum computer. *SIAM Journal on Computing*, v. 26, n. 5, p. 1484–1509, out. 1997. ISSN 0097-5397, 1095-7111. Disponível em: http://epubs.siam.org/doi/10.1137/S0097539795293172. Citado na página 15.

4 HOLEVO, A. S. Bounds for the quantity of information transmitted by a quantum communication channel. *Problemy Peredachi Informatsii*, Russian Academy of Sciences, Branch of Informatics, Computer Equipment and ..., v. 9, n. 3, p. 3–11, 1973. Citado 2 vezes nas páginas 15 and 32.

5 JOZSA, R.; SCHUMACHER, B. A new proof of the quantum noiseless coding theorem. *Journal of Modern Optics*, v. 41, n. 12, p. 2343–2349, dez. 1994. ISSN 0950-0340, 1362-3044. Disponível em: http://www.tandfonline.com/doi/abs/10.1080/09500349414552191. Citado na página 15.

6 SCHUMACHER, B. Quantum coding. *Physical Review A*, v. 51, n. 4, p. 2738–2747, abr. 1995. ISSN 1050-2947, 1094-1622. Disponível em: https://link.aps.org/doi/10.1103/PhysRevA.51.2738. Citado 2 vezes nas páginas 15 and 23.

7 BENNETT, C. H.; WIESNER, S. J. Communication via one- and two-particle operators on einstein-podolsky-rosen states. *Phys. Rev. Lett.*, American Physical Society, v. 69, p. 2881–2884, Nov 1992. Disponível em: https://link.aps.org/doi/10.1103/PhysRevLett.69.2881. Citado 5 vezes nas páginas 15, 44, 52, 62, and 67.

8 WIESNER, S. Conjugate coding. *ACM Sigact News*, ACM New York, NY, USA, v. 15, n. 1, p. 78–88, 1983. Citado 2 vezes nas páginas 15 and 30.

9 AMBAINIS, A.; NAYAK, A.; TA-SHMA, A.; VAZIRANI, U. Dense quantum coding and a lower bound for 1-way quantum automata. In: *Proceedings of the thirty-first annual ACM symposium on Theory of computing - STOC '99*. Atlanta, Georgia, United States: ACM Press, 1999. p. 376–383. ISBN 9781581130676. Disponível em: http://portal.acm.org/citation.cfm?doid=301250.301347. Citado 4 vezes nas páginas 15, 19, 30, and 32.

10 NAYAK, A. Optimal lower bounds for quantum automata and random access codes. In: *40th Annual Symposium on Foundations of Computer Science (Cat. No.99CB37039)*. New York City, NY, USA: IEEE Comput. Soc, 1999. p. 369–376. ISBN 9780769504094. Disponível em: http://ieeexplore.ieee.org/document/814608/. Citado na página 15.

11 GALLEGO, R.; BRUNNER, N.; HADLEY, C.; ACÍN, A. Device-independent tests of classical and quantum dimensions. *Phys. Rev. Lett.*, American Physical Society, v. 105, p. 230501, Nov 2010. Disponível em: https://link.aps.org/doi/10.1103/PhysRevLett.105.230501. Citado 4 vezes nas páginas 15, 19, 24, and 25.

12 WEHNER, S.; CHRISTANDL, M.; DOHERTY, A. C. Lower bound on the dimension of a quantum system given measured data. *Phys. Rev. A*, American Physical Society, v. 78, p. 062112, Dec 2008. Disponível em: https://link.aps.org/doi/10.1103/PhysRevA.78.062112. Citado 2 vezes nas páginas 15 and 30.

13 AHRENS, J.; BADZIAG, P.; CABELLO, A.; BOURENNANE, M. Experimental device-independent tests of classical and quantum dimensions. *Nature Physics*, v. 8, n. 8, p. 592–595, 2012. ISSN 1745-2473, 1745-2481. Disponível em: http://www.nature.com/articles/nphys2333. Citado na página 15.

14 HENDRYCH, M.; GALLEGO, R.; MIČUDA, M.; BRUNNER, N.; ACÍN, A.; TORRES, J. P. Experimental estimation of the dimension of classical and quantum systems. *Nature Physics*, v. 8, n. 8, p. 588–591, ago. 2012. ISSN 1745-2473, 1745-2481. Disponível em: http://www.nature.com/articles/nphys2334. Citado na página 15.

15 BOWLES, J.; QUINTINO, M. T.; BRUNNER, N. Certifying the dimension of classical and quantum systems in a prepare-and-measure scenario with independent devices. *Phys. Rev. Lett.*, American Physical Society, v. 112, p. 140407, Apr 2014. Disponível em: https://link.aps.org/doi/10.1103/PhysRevLett.112.140407. Citado na página 15.

16 BOWLES, J.; BRUNNER, N.; PAWŁOWSKI, M. Testing dimension and nonclassicality in communication networks. *Phys. Rev. A*, American Physical Society, v. 92, p. 022351, Aug 2015. Disponível em: https://link.aps.org/doi/10.1103/PhysRevA.92.022351. Citado na página 15.

17 WANG, Y.; PRIMAATMAJA, I. W.; LAVIE, E.; VARVITSIOTIS, A.; LIM, C. C. W. Characterising the correlations of prepare-and-measure quantum networks. *npj Quantum Information*, Nature Publishing Group, v. 5, n. 1, p. 1–6, 2019. Citado na página 15.

18 TAVAKOLI, A.; KANIEWSKI, J.; VÉRTESI, T.; ROSSET, D.; BRUNNER, N. Self-testing quantum states and measurements in the prepare-and-measure scenario. *Phys. Rev. A*, American Physical Society, v. 98, p. 062307, Dec 2018. Disponível em: https://link.aps.org/doi/10.1103/PhysRevA.98.062307. Citado 4 vezes nas páginas 15, 19, 30, and 61.

19 MIKLIN, N.; OSZMANIEC, M. A universal scheme for robust self-testing in the prepare-and-measure scenario. *Quantum*, Verein zur Forderung des Open Access Publizierens in den Quantenwissenschaften, v. 5, p. 424, abr. 2021. ISSN 2521-327X. Disponível em: https://doi.org/10.22331/q-2021-04-06-424. Citado na página 15.

20 PAWŁOWSKI, M.; BRUNNER, N. Semi-device-independent security of one-way quantum key distribution. *Phys. Rev. A*, American Physical Society, v. 84, p. 010302, Jul 2011. Disponível em: https://link.aps.org/doi/10.1103/PhysRevA.84.010302. Citado 2 vezes nas páginas 15 and 16.

21 WOODHEAD, E.; PIRONIO, S. Secrecy in prepare-and-measure clauser-horne-shimony-holt tests with a qubit bound. *Phys. Rev. Lett.*, American Physical Society, v. 115, p. 150501, Oct 2015. Disponível em: https://link.aps.org/doi/10.1103/PhysRevLett.115.150501. Citado na página 15.

22 LI, H.-W.; YIN, Z.-Q.; WU, Y.-C.; ZOU, X.-B.; WANG, S.; CHEN, W.; GUO, G.-C.; HAN, Z.-F. Semi-device-independent random-number expansion without entanglement. *Phys. Rev. A*, American Physical Society, v. 84, p. 034301, Sep 2011. Disponível em: https://link.aps.org/doi/10.1103/PhysRevA.84.034301. Citado na página 15.

23 LI, H.-W.; PAWŁOWSKI, M.; YIN, Z.-Q.; GUO, G.-C.; HAN, Z.-F. Semi-device-independent randomness certification using n to 1 quantum random access codes. *Phys. Rev. A*, American Physical Society, v. 85, p. 052308, May 2012. Disponível em: https://link.aps.org/doi/10.1103/PhysRevA.85.052308. Citado na página 15.

24 PASSARO, E.; CAVALCANTI, D.; SKRZYPCZYK, P.; ACÍN, A. Optimal randomness certification in the quantum steering and prepare-and-measure scenarios. *New Journal of Physics*, IOP Publishing, v. 17, n. 11, p. 113010, oct 2015. Disponível em: https://doi.org/10.1088/1367-2630/17/11/113010. Citado na página 15.

25 PAWŁOWSKI, M.; PATEREK, T.; KASZLIKOWSKI, D.; SCARANI, V.; WINTER, A.; ZUKOWSKI, M. Information causality as a physical principle. *Nature*, Springer Science and Business Media LLC, v. 461, n. 7267, p. 1101–1104, out. 2009. Disponível em: https://doi.org/10.1038/nature08400. Citado 2 vezes nas páginas 15 and 30.

26 CHAVES, R.; MAJENZ, C.; GROSS, D. Information–theoretic implications of quantum causal structures. *Nature communications*, Nature Publishing Group, v. 6, n. 1, p. 1–8, 2015. Citado na página 15.

27 GOIS, C. de; MORENO, G.; NERY, R.; BRITO, S.; CHAVES, R.; RABELO, R. General method for classicality certification in the prepare and measure scenario. *PRX Quantum*, American Physical Society, v. 2, p. 030311, Jul 2021. Disponível em: https://link.aps.org/doi/10.1103/PRXQuantum.2.030311. Citado 4 vezes nas páginas 16, 41, 43, and 78.

28 CARMELI, C.; HEINOSAARI, T.; TOIGO, A. Quantum random access codes and incompatibility of measurements. *EPL (Europhysics Letters)*, IOP Publishing, v. 130, n. 5, p. 50001, jun 2020. Disponível em: https://doi.org/10.1209/0295-5075/130/50001. Citado 3 vezes nas páginas 16, 41, and 42.

29 GAVINSKY, D. A note on shared randomness and shared entanglement in communication. arXiv, 2005. Disponível em: https://arxiv.org/abs/quant-ph/0505088. Citado na página 16.

30 PAWŁOWSKI, M.; ZUKOWSKI, M. Entanglement-assisted random access codes. *Phys. Rev. A*, American Physical Society, v. 81, p. 042326, Apr 2010. Disponível em: https://link.aps.org/doi/10.1103/PhysRevA.81.042326. Citado 3 vezes nas páginas 16, 44, and 67.

31 PAWŁOWSKI, M.; WINTER, A. “Hyperbits”: The information quasiparticles. *Physical Review A*, v. 85, n. 2, p. 022331, fev. 2012. ISSN 1050-2947, 1094-1622. Disponível em: https://link.aps.org/doi/10.1103/PhysRevA.85.022331. Citado 2 vezes nas páginas 16 and 61.

32 MORENO, G.; NERY, R.; GOIS, C. de; RABELO, R.; CHAVES, R. Semi-device-independent certification of entanglement in superdense coding. *Phys. Rev. A*, American Physical Society, v. 103, p. 022426, Feb 2021. Disponível em: https://link.aps.org/doi/10.1103/PhysRevA.103.022426. Citado 4 vezes nas páginas 16, 44, 73, and 74.

33 TAVAKOLI, A.; PAUWELS, J.; WOODHEAD, E.; PIRONIO, S. Correlations in entanglement-assisted prepare-and-measure scenarios. *PRX Quantum*, American Physical Society, v. 2, p. 040357, Dec 2021. Disponível em: https://link.aps.org/doi/10.1103/PRXQuantum.2.040357. Citado 12 vezes nas páginas 16, 17, 44, 45, 48, 49, 60, 61, 67, 68, 72, and 77.

34 PAUWELS, J.; TAVAKOLI, A.; WOODHEAD, E.; PIRONIO, S. *Entanglement in prepare-and-measure scenarios: many questions, a few answers*. arXiv, 2021. Disponível em: https://arxiv.org/abs/2108.00442. Citado 9 vezes nas páginas 16, 17, 44, 45, 49, 61, 62, 73, and 77.

35 FRENKEL, P. E.; WEINER, M. On entanglement assistance to a noiseless classical channel. *Quantum*, Verein zur Förderung des Open Access Publizierens in den Quantenwissenschaften, v. 6, p. 662, mar. 2022. ISSN 2521-327X. Disponível em: https://doi.org/10.22331/q-2022-03-01-662. Citado 4 vezes nas páginas 16, 44, 61, and 78.

36 PAUWELS, J.; PIRONIO, S.; CRUZEIRO, E. Z.; TAVAKOLI, A. Adaptive advantage in entanglement-assisted communications. *Physical Review Letters*, v. 129, n. 12, p. 120504, set. 2022. ISSN 0031-9007, 1079-7114. Disponível em: https://link.aps.org/doi/10.1103/PhysRevLett.129.120504. Citado 6 vezes nas páginas 16, 44, 59, 61, 75, and 78.

37 VIEIRA, C.; GOIS, C. de; POLLYCENO, L.; RABELO, R. *Interplays between classical and quantum entanglement-assisted communication scenarios*. arXiv, 2022. Disponível em: https://arxiv.org/abs/2205.05171. Citado 9 vezes nas páginas 16, 17, 18, 40, 61, 67, 72, 73, and 77.

38 DALL'ARNO, M.; PASSARO, E.; GALLEGO, R.; ACÍN, A. Robustness of device-independent dimension witnesses. *Phys. Rev. A*, American Physical Society, v. 86, p. 042312, Oct 2012. Disponível em: https://link.aps.org/doi/10.1103/PhysRevA.86.042312. Citado 5 vezes nas páginas 19, 24, 25, 26, and 40.

39 TAVAKOLI, A.; CRUZEIRO, E. Z.; BRASK, J. B.; GISIN, N.; BRUNNER, N. Informationally restricted quantum correlations. *Quantum*, v. 4, p. 332, set. 2020. ISSN 2521-327X. Disponível em: https://quantum-journal.org/papers/q-2020-09-24-332/. Citado na página 19.

40 TAVAKOLI, A.; CRUZEIRO, E. Z.; WOODHEAD, E.; PIRONIO, S. Informationally restricted correlations: a general framework for classical and quantum systems. *Quantum*, v. 6, p. 620, jan. 2022. ISSN 2521-327X. Disponível em: https://quantum-journal.org/papers/q-2022-01-05-620/. Citado 2 vezes nas páginas 19 and 26.

41 PAUWELS, J.; PIRONIO, S.; WOODHEAD, E.; TAVAKOLI, A. Almost qudits in the prepare-and-measure scenario. *Phys. Rev. Lett.*, American Physical Society, v. 129, p.

250504, Dec 2022. Disponível em: https://link.aps.org/doi/10.1103/PhysRevLett.129.250504. Citado na página 19.

42 COVER, T. M. *Elements of information theory.* [S.l.]: John Wiley & Sons, 1999. Citado 3 vezes nas páginas 19, 34, and 36.

43 WIKIPEDIA. *Bit — Wikipedia, The Free Encyclopedia.* 2022. http://en.wikipedia.org/w/index.php?title=Bit&oldid=1128534561. [Online; accessed 21-December-2022]. Citado na página 20.

44 AMBAINIS, A.; LEUNG, D.; MANCINSKA, L.; OZOLS, M. *Quantum Random Access Codes with Shared Randomness.* arXiv, 2008. Disponível em: https://arxiv.org/abs/0810.2937. Citado 3 vezes nas páginas 20, 30, and 31.

45 NEWMAN, I. Private vs. common random bits in communication complexity. *Information processing letters*, v. 39, n. 2, p. 67–71, 1991. Citado na página 21.

46 DIRAC, P. A. M. On the theory of quantum mechanics. *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*, The Royal Society London, v. 112, n. 762, p. 661–677, 1926. Citado na página 23.

47 NIELSEN, M. A.; CHUANG, I. *Quantum computation and quantum information.* [S.l.]: American Association of Physics Teachers, 2002. Citado 2 vezes nas páginas 23 and 32.

48 NEUMANN, J. V. *Mathematical foundations of quantum mechanics: New edition.* [S.l.]: Princeton university press, 2018. Citado na página 23.

49 LANDAU, L. D. *Collected papers of LD Landau.* [S.l.]: Pergamon, 1965. Citado na página 23.

50 FANO, U. Description of states in quantum mechanics by density matrix and operator techniques. *Rev. Mod. Phys.*, American Physical Society, v. 29, p. 74–93, Jan 1957. Disponível em: https://link.aps.org/doi/10.1103/RevModPhys.29.74. Citado na página 23.

51 GLEASON, A. M. Measures on the closed subspaces of a hilbert space. In: *The Logico-Algebraic Approach to Quantum Mechanics.* [S.l.]: Springer, 1975. p. 123–133. Citado na página 23.

52 PERES, A. *Quantum theory: concepts and methods.* [S.l.]: Springer, 1997. Citado na página 23.

53 BERBERIAN, S. K.; BERBERIAN, K. et al. Notes on spectral theory. Citeseer, 1966. Citado na página 24.

54 JAUCH, J.-M.; PIRON, C. Generalized localizability. *Helvetica Physica Acta*, v. 40, n. 5, p. 559–570, 1967. Citado na página 24.

55 BRANDT, H. E. Positive operator valued measure in quantum information processing. *American Journal of Physics*, American Association of Physics Teachers, v. 67, n. 5, p. 434–439, 1999. Citado na página 24.

56 BORN, M. Quantenmechanik der stobvorgange. *Zeitschrift fur physik*, Springer, v. 38, n. 11, p. 803–827, 1926. Citado na página 24.

57 COXETER, H. S. M. *Regular polytopes*. [S.l.]: Courier Corporation, 1973. Citado na página 26.

58 GRÜNBAUM, B.; KLEE, V.; PERLES, M. A.; SHEPHARD, G. C. *Convex polytopes*. [S.l.]: Springer, 1967. v. 16. Citado na página 27.

59 BOYD, S.; BOYD, S. P.; VANDENBERGHE, L. *Convex optimization*. [S.l.]: Cambridge university press, 2004. Citado na página 27.

60 ZIEGLER, G. M. *Lectures on polytopes*. [S.l.]: Springer Science & Business Media, 2012. v. 152. Citado na página 27.

61 BRUNNER, N.; CAVALCANTI, D.; PIRONIO, S.; SCARANI, V.; WEHNER, S. Bell nonlocality. *Rev. Mod. Phys.*, American Physical Society, v. 86, p. 419–478, Apr 2014. Disponível em: https://link.aps.org/doi/10.1103/RevModPhys.86.419. Citado 3 vezes nas páginas 27, 40, and 73.

62 LORWALD, S.; REINELT, G. Panda: a software for polyhedral transformations. *EURO Journal on Computational Optimization*, Elsevier, v. 3, n. 4, p. 297–308, 2015. Citado na página 27.

63 QUADEER, M.; BANIK, M.; AMBAINIS, A.; RAI, A. *Strong supremacy of quantum systems as communication resource*. arXiv, 2017. Disponível em: https://arxiv.org/abs/1703.01996. Citado na página 29.

64 RENNER, M. J.; TAVAKOLI, A.; QUINTINO, M. T. *The classical cost of transmitting a qubit*. arXiv, 2022. Disponível em: https://arxiv.org/abs/2207.02244. Citado 3 vezes nas páginas 29, 59, and 62.

65 WENDELL, R. E.; HURTER, A. P. Minimization of a non-separable objective function subject to disjoint constraints. *Operations Research*, v. 24, n. 4, p. 643–657, ago. 1976. ISSN 0030-364X, 1526-5463. Disponível em: http://pubsonline.informs.org/doi/10.1287/opre.24.4.643. Citado na página 29.

66 GOIS, C. A. B. de. *Classicality and dense coding in the prepare and measure scenario*. Dissertação (Mestrado) — Universidade Estadual de Campinas, Campinas-Brazil, 2021. Citado na página 29.

67 NAVASCUéS, M.; VéRTESI, T. Bounding the set of finite dimensional quantum correlations. *Physical Review Letters*, v. 115, n. 2, p. 020501, jul. 2015. ISSN 0031-9007, 1079-7114. Disponível em: https://link.aps.org/doi/10.1103/PhysRevLett.115.020501. Citado na página 29.

68 NAVASCUéS, M.; FEIX, A.; ARAúJO, M.; VéRTESI, T. Characterizing finite-dimensional quantum behavior. *Physical Review A*, v. 92, n. 4, p. 042117, out. 2015. ISSN 1050-2947, 1094-1622. Disponível em: https://link.aps.org/doi/10.1103/PhysRevA.92.042117. Citado na página 29.

69 TAVAKOLI, A.; ROSSET, D.; RENOU, M.-O. Enabling computation of correlation bounds for finite-dimensional quantum systems via symmetrization. *Physical Review Letters*, v. 122, n. 7, p. 070501, fev. 2019. ISSN 0031-9007, 1079-7114. Disponível em: https://link.aps.org/doi/10.1103/PhysRevLett.122.070501. Citado na página 29.

70 FARKAS, M.; KANIEWSKI, J. m. k. Self-testing mutually unbiased bases in the prepare-and-measure scenario. *Phys. Rev. A*, American Physical Society, v. 99, p. 032316, Mar 2019. Disponível em: https://link.aps.org/doi/10.1103/PhysRevA.99.032316. Citado na página 30.

71 AMBAINIS, A.; NAYAK, A.; TA-SHMA, A.; VAZIRANI, U. Dense quantum coding and quantum finite automata. *Journal of the ACM*, v. 49, n. 4, p. 496–511, 2002. ISSN 0004-5411, 1557-735X. Disponível em: https://dl.acm.org/doi/10.1145/581771.581773. Citado na página 30.

72 TAVAKOLI, A.; HAMEEDI, A.; MARQUES, B.; BOURENNANE, M. Quantum random access codes using single d-level systems. *Phys. Rev. Lett.*, American Physical Society, v. 114, p. 170502, Apr 2015. Disponível em: https://link.aps.org/doi/10.1103/PhysRevLett.114.170502. Citado na página 32.

73 NAYAK, A.; SALZMAN, J. Limits on the ability of quantum states to convey classical messages. *J. ACM*, Association for Computing Machinery, New York, NY, USA, v. 53, n. 1, p. 184–206, jan 2006. ISSN 0004-5411. Disponível em: https://doi.org/10.1145/1120582.1120587. Citado 3 vezes nas páginas 34, 38, and 57.

74 SHANNON, C. E. A mathematical theory of communication. *Bell System Technical Journal*, v. 27, n. 3, p. 379–423, jul. 1948. ISSN 00058580. Disponível em: https://ieeexplore.ieee.org/document/6773024. Citado na página 35.

75 VEDRAL, V. *Introduction to quantum information science*. Oxford: Oxford University Press, 2006. (Oxford graduate texts). OCLC: ocm70173203. ISBN 9780199215706 9780198571193. Citado na página 37.

76 FRENKEL, P. E.; WEINER, M. Classical Information Storage in an n-Level Quantum System. *Communications in Mathematical Physics*, v. 340, n. 2, p. 563–574, dez. 2015. ISSN 0010-3616, 1432-0916. Disponível em: http://link.springer.com/10.1007/s00220-015-2463-0. Citado 2 vezes nas páginas 39 and 77.

77 PATRA, R. K.; NAIK, S. G.; LOBO, E. P.; SEN, S.; GUHA, T.; BHATTACHARYA, S. S.; ALIMUDDIN, M.; BANIK, M. *Classical superdense coding and communication advantage of a single quantum*. arXiv, 2022. Disponível em: https://arxiv.org/abs/2202.06796. Citado 2 vezes nas páginas 39 and 73.

78 GÜHNE, O.; HAAPASALO, E.; KRAFT, T.; PELLONPÄÄ, J.-P.; UOLA, R. *Incompatible measurements in quantum information science*. arXiv, 2021. Disponível em: https://arxiv.org/abs/2112.06784. Citado 2 vezes nas páginas 40 and 42.

79 WERNER, R. F. Quantum states with einstein-podolsky-rosen correlations admitting a hidden-variable model. *Phys. Rev. A*, American Physical Society, v. 40, p. 4277–4281, Oct 1989. Disponível em: https://link.aps.org/doi/10.1103/PhysRevA.40.4277. Citado 2 vezes nas páginas 40 and 73.

80 QUINTINO, M. T.; VÉRTESI, T.; BRUNNER, N. Joint measurability, einstein-podolsky-rosen steering, and bell nonlocality. *Phys. Rev. Lett.*, American Physical Society, v. 113, p. 160402, Oct 2014. Disponível em: https://link.aps.org/doi/10.1103/PhysRevLett.113.160402. Citado 3 vezes nas páginas 40, 74, and 78.

81 QUINTINO, M. T.; VÉRTESI, T.; CAVALCANTI, D.; AUGUSIAK, R.; DEMIANOWICZ, M.; ACÍN, A.; BRUNNER, N. Inequivalence of entanglement, steering, and bell nonlocality for general measurements. *Phys. Rev. A*, American Physical Society, v. 92, p. 032107, Sep 2015. Disponível em: https://link.aps.org/doi/10.1103/PhysRevA.92.032107. Citado 2 vezes nas páginas 40 and 74.

82 NGUYEN, H. C.; GÜHNE, O. Some quantum measurements with three outcomes can reveal nonclassicality where all two-outcome measurements fail to do so. *Phys. Rev. Lett.*, American Physical Society, v. 125, p. 230402, Dec 2020. Disponível em: https://link.aps.org/doi/10.1103/PhysRevLett.125.230402. Citado 4 vezes nas páginas 40, 73, 74, and 77.

83 UOLA, R.; MORODER, T.; GÜHNE, O. Joint measurability of generalized measurements implies classicality. *Phys. Rev. Lett.*, American Physical Society, v. 113, p. 160403, Oct 2014. Disponível em: https://link.aps.org/doi/10.1103/PhysRevLett.113.160403. Citado 2 vezes nas páginas 40 and 74.

84 HORN, R. A.; JOHNSON, C. R. *Matrix analysis*. [S.l.]: Cambridge university press, 2012. Citado na página 40.

85 GUPTA, S.; SAHA, D.; XU, Z.-P.; CABELLO, A.; MAJUMDAR, A. S. *Quantum contextuality provides communication complexity advantage*. arXiv, 2022. Disponível em: https://arxiv.org/abs/2205.03308. Citado na página 41.

86 HEINOSAARI, T.; REITZNER, D.; STANO, P. Notes on joint measurability of quantum observables. *Foundations of Physics*, v. 38, n. 12, p. 1133–1147, dez. 2008. ISSN 0015-9018, 1572-9516. Disponível em: http://link.springer.com/10.1007/s10701-008-9256-7. Citado na página 41.

87 GUERINI, L.; QUINTINO, M. T.; AOLITA, L. Distributed sampling, quantum communication witnesses, and measurement incompatibility. *Phys. Rev. A*, American Physical Society, v. 100, p. 042308, Oct 2019. Disponível em: https://link.aps.org/doi/10.1103/PhysRevA.100.042308. Citado na página 43.

88 EINSTEIN, A.; PODOLSKY, B.; ROSEN, N. Can quantum-mechanical description of physical reality be considered complete? *Phys. Rev.*, American Physical Society, v. 47, p. 777–780, May 1935. Disponível em: https://link.aps.org/doi/10.1103/PhysRev.47.777. Citado na página 44.

89 BELL, J. S. On the einstein podolsky rosen paradox. *Physics Physique Fizika*, American Physical Society, v. 1, p. 195–200, Nov 1964. Disponível em: https://link.aps.org/doi/10.1103/PhysicsPhysiqueFizika.1.195. Citado na página 44.

90 BENNETT, C. H.; BRASSARD, G.; CRÉPEAU, C.; JOZSA, R.; PERES, A.; WOOTTERS, W. K. Teleporting an unknown quantum state via dual classical and einstein-podolsky-rosen channels. *Phys. Rev. Lett.*, American Physical Society, v. 70, p. 1895–1899, Mar 1993. Disponível em: https://link.aps.org/doi/10.1103/PhysRevLett.70.1895. Citado 4 vezes nas páginas 44, 62, 96, and 97.

91 RUDIN, W. Functional analysis 2nd ed. *International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York*, 1991. Citado na página 48.

92 HORODECKI, M.; SHOR, P. W.; RUSKAI, M. B. Entanglement breaking channels. *Reviews in Mathematical Physics*, v. 15, n. 06, p. 629–641, ago. 2003. ISSN 0129-055X, 1793-6659. Disponível em: https://www.worldscientific.com/doi/abs/10.1142/S0129055X03001709. Citado na página 48.

93 HIROSHIMA, T. Optimal dense coding with mixed state entanglement. *Journal of Physics A: Mathematical and General*, v. 34, n. 35, p. 6907–6912, set. 2001. ISSN 0305-4470, 1361-6447. Disponível em: https://iopscience.iop.org/article/10.1088/0305-4470/34/35/316. Citado na página 52.

94 HORODECKI, M.; HORODECKI, P.; HORODECKI, R. l.; LEUNG, D.; TERHA, B. Classical capacity of a noiseless quantum channel assisted by noisy entanglement. *Quantum Information and Computation*, v. 1, n. 3, p. 70–78, out. 2001. ISSN 15337146, 15337146. Disponível em: http://www.rintonpress.com/journals/doi/QIC1.3-6.html. Citado na página 52.

95 LIU, X. S.; LONG, G. L.; TONG, D. M.; LI, F. General scheme for superdense coding between multiparties. *Physical Review A*, v. 65, n. 2, p. 022304, jan. 2002. ISSN 1050-2947, 1094-1622. Disponível em: https://link.aps.org/doi/10.1103/PhysRevA.65.022304. Citado na página 52.

96 LIU, B.-H.; HU, X.-M.; HUANG, Y.-F.; LI, C.-F.; GUO, G.-C.; KARLSSON, A.; LAINE, E.-M.; MANISCALCO, S.; MACCHIAVELLO, C.; PIILO, J. Efficient superdense coding in the presence of non-Markovian noise. *EPL (Europhysics Letters)*, v. 114, n. 1, p. 10005, abr. 2016. ISSN 0295-5075, 1286-4854. Disponível em: https://iopscience.iop.org/article/10.1209/0295-5075/114/10005. Citado na página 52.

97 HAYASHI, M.; WANG, K. Dense coding with locality restriction on decoders: quantum encoders versus superquantum encoders. *PRX Quantum*, v. 3, n. 3, p. 030346, set. 2022. ISSN 2691-3399. Disponível em: https://link.aps.org/doi/10.1103/PRXQuantum.3.030346. Citado na página 52.

98 MATTLE, K.; WEINFURTER, H.; KWIAT, P. G.; ZEILINGER, A. Dense coding in experimental quantum communication. *Phys. Rev. Lett.*, American Physical Society, v. 76, p. 4656–4659, Jun 1996. Disponível em: https://link.aps.org/doi/10.1103/PhysRevLett.76.4656. Citado na página 52.

99 WIKIPEDIA. *Pauli matrices — Wikipedia, The Free Encyclopedia*. 2023. http://en.wikipedia.org/w/index.php?title=Pauli%20matrices&oldid=1127422850. [Online; accessed 06-January-2023]. Citado na página 53.

100 DENG, F.-G.; LONG, G. L.; LIU, X.-S. Two-step quantum direct communication protocol using the Einstein-Podolsky-Rosen pair block. *Physical Review A*, v. 68, n. 4, p. 042317, out. 2003. ISSN 1050-2947, 1094-1622. Disponível em: https://link.aps.org/doi/10.1103/PhysRevA.68.042317. Citado na página 53.

101 NAYAK, A.; SALZMAN, J. On communication over an entanglement-assisted quantum channel. In: *Proceedings of the thiry-fourth annual ACM symposium on Theory of computing*. [S.l.: s.n.], 2002. p. 698–704. Citado na página 57.

102 PIVETEAU, A.; PAUWELS, J.; HÅKANSSON, E.; MUHAMMAD, S.; BOURENNANE, M.; TAVAKOLI, A. Entanglement-assisted quantum communication with simple measurements. *Nature Communications*, v. 13, n. 1, p. 7878, dez. 2022. ISSN 2041-1723. Disponível em: https://www.nature.com/articles/s41467-022-33922-5. Citado na página 59.

103 CUBITT, T. S.; LEUNG, D.; MATTHEWS, W.; WINTER, A. Zero-error channel capacity and simulation assisted by non-local correlations. *IEEE Transactions on Information Theory*, v. 57, n. 8, p. 5509–5523, ago. 2011. ISSN 0018-9448, 1557-9654. Disponível em: http://ieeexplore.ieee.org/document/5961832/. Citado na página 59.

104 NAVASCUÉS, M.; PIRONIO, S.; ACÍN, A. Bounding the set of quantum correlations. *Physical Review Letters*, v. 98, n. 1, p. 010401, jan. 2007. ISSN 0031-9007, 1079-7114. Disponível em: https://link.aps.org/doi/10.1103/PhysRevLett.98.010401. Citado na página 61.

105 NAVASCUÉS, M.; PIRONIO, S.; ACÍN, A. A convergent hierarchy of semidefinite programs characterizing the set of quantum correlations. *New Journal of Physics*, v. 10, n. 7, p. 073013, jul. 2008. ISSN 1367-2630. Disponível em: https://iopscience.iop.org/article/10.1088/1367-2630/10/7/073013. Citado na página 61.

106 WERNER, R. F. All teleportation and dense coding schemes. *Journal of Physics A: Mathematical and General*, IOP Publishing, v. 34, n. 35, p. 7081–7094, aug 2001. Disponível em: https://doi.org/10.1088/0305-4470/34/35/332. Citado na página 62.

107 CLEVE, R.; BUHRMAN, H. Substituting quantum entanglement for communication. *Phys. Rev. A*, American Physical Society, v. 56, p. 1201–1204, Aug 1997. Disponível em: https://link.aps.org/doi/10.1103/PhysRevA.56.1201. Citado 2 vezes nas páginas 67 and 73.

108 BUHRMAN, H.; CLEVE, R.; MASSAR, S.; WOLF, R. de. Nonlocality and communication complexity. *Rev. Mod. Phys.*, American Physical Society, v. 82, p. 665–698, Mar 2010. Disponível em: https://link.aps.org/doi/10.1103/RevModPhys.82.665. Citado na página 67.

109 DOOLITTLE, B.; CHITAMBAR, E. Certifying the classical simulation cost of a quantum channel. *Phys. Rev. Research*, American Physical Society, v. 3, p. 043073, Oct 2021. Disponível em: https://link.aps.org/doi/10.1103/PhysRevResearch.3.043073. Citado na página 68.

110 GOIS, C. de. *Code to accompany "Interplays between classical and quantum entanglement-assisted communication scenarios"*. 2022. Disponível em: https://github.com/cgois/eapam_cc_vs_qc. Citado 2 vezes nas páginas 68 and 71.

111 VANDERBEI, R. J. et al. *Linear programming*. [S.l.]: Springer, 2020. Citado na página 70.

112 UOLA, R.; COSTA, A. C. S.; NGUYEN, H. C.; GÜHNE, O. Quantum steering. *Rev. Mod. Phys.*, American Physical Society, v. 92, p. 015001, Mar 2020. Disponível em: https://link.aps.org/doi/10.1103/RevModPhys.92.015001. Citado na página 73.

113 TROJEK, P.; SCHMID, C.; BOURENNANE, M.; BRUKNER, C.; ZUKOWSKI, M.; WEINFURTER, H. Experimental quantum communication complexity. *Phys. Rev. A*, American Physical Society, v. 72, p. 050305, Nov 2005. Disponível em: https://link.aps.org/doi/10.1103/PhysRevA.72.050305. Citado na página 73.

114 TAVAKOLI, A.; ZUKOWSKI, M.; BRUKNER, C. Does violation of a bell inequality always imply quantum advantage in a communication complexity problem? *Quantum*, Verein zur Forderung des Open Access Publizierens in den Quantenwissenschaften, v. 4, p. 316, set. 2020. ISSN 2521-327X. Disponível em: https://doi.org/10.22331/q-2020-09-07-316. Citado 3 vezes nas páginas 73, 78, and 79.

115 CLAUSER, J. F.; HORNE, M. A.; SHIMONY, A.; HOLT, R. A. Proposed experiment to test local hidden-variable theories. *Phys. Rev. Lett.*, v. 23, n. 15, p. 880–884, out. 1969. Citado na página 73.

116 GISIN, N. Bell's inequality holds for all non-product states. *Physics Letters A*, v. 154, n. 5, p. 201–202, 1991. ISSN 0375-9601. Disponível em: https://www.sciencedirect.com/science/article/pii/037596019190805I. Citado na página 73.

117 YU, S.; CHEN, Q.; ZHANG, C.; LAI, C. H.; OH, C. H. All entangled pure states violate a single bell's inequality. *Phys. Rev. Lett.*, American Physical Society, v. 109, p. 120402, Sep 2012. Disponível em: https://link.aps.org/doi/10.1103/PhysRevLett.109.120402. Citado na página 73.

118 SCHRÖDINGER, E. Discussion of Probability Relations between Separated Systems. *Mathematical Proceedings of the Cambridge Philosophical Society*, v. 31, n. 4, p. 555–563, out. 1935. ISSN 0305-0041, 1469-8064. Disponível em: https://www.cambridge.org/core/product/identifier/S0305004100013554/type/journal_article. Citado na página 73.

119 WISEMAN, H. M.; JONES, S. J.; DOHERTY, A. C. Steering, entanglement, nonlocality, and the einstein-podolsky-rosen paradox. *Phys. Rev. Lett.*, American Physical Society, v. 98, p. 140402, Apr 2007. Disponível em: https://link.aps.org/doi/10.1103/PhysRevLett.98.140402. Citado na página 73.

120 JEBARATHINAM, C.; DAS, D.; KANJILAL, S.; SRIKANTH, R.; SARKAR, D.; CHATTOPADHYAY, I.; MAJUMDAR, A. S. Superunsteerability as a quantifiable resource for random access codes assisted by bell-diagonal states. *Phys. Rev. A*, American Physical Society, v. 100, p. 012344, Jul 2019. Disponível em: https://link.aps.org/doi/10.1103/PhysRevA.100.012344. Citado na página 76.

121 PÁL, K. F.; VÉRTESI, T. Maximal violation of a bipartite three-setting, two-outcome bell inequality using infinite-dimensional quantum systems. *Phys. Rev. A*, American Physical Society, v. 82, p. 022116, Aug 2010. Disponível em: https://link.aps.org/doi/10.1103/PhysRevA.82.022116. Citado na página 77.

122 SLOFSTRA, W. The set of quantum correlations is not closed. *Forum of Mathematics, Pi*, v. 7, p. e1, 2019. ISSN 2050-5086. Disponível em: https://www.cambridge.org/core/product/identifier/S2050508618000033/type/journal_article. Citado na página 77.

123 COLADANGELO, A.; STARK, J. An inherently infinite-dimensional quantum correlation. *Nature Communications*, v. 11, n. 1, p. 3335, dec 2020. ISSN 2041-1723. Disponível em: http://www.nature.com/articles/s41467-020-17077-9. Citado na página 77.

124 VERDON, D. *Entanglement-invertible channels*. arXiv, 2022. Disponível em: https://arxiv.org/abs/2204.04493. Citado na página 78.

A Appendix

A.1 Equivalence between the two definitions of deterministic points

Let $p(b|x, y)$ be a behavior given by **Def. 1**, i.e,

$$p(b|x, y) = \delta(b, (\mathcal{D}_y \circ \mathcal{E})(x)).$$

So, let's take $p_D(a|x) = \delta(a, \mathcal{E}(x))$ and $p_D(b|y, a) = \delta(b, \mathcal{D}_y(a))$. It follows that $p_D(a|x)$ and $p_D(b|y, a)$ are deterministic. We also have the following:

$$\begin{aligned} \sum_{a=0}^{d-1} p_D(a|x)p_D(b|y, a) &= \sum_{a=0}^{d-1} \delta(a, \mathcal{E}(x))\delta(b, \mathcal{D}_y(a)) \\ &= \delta(b, (\mathcal{D}_y \circ \mathcal{E})(x)) \\ &= p(b|x, y). \end{aligned} \tag{A.1}$$

Therefore, every behavior that can be decomposed as eq. (1.1) can also be decomposed as eq. (1.2).

Now let $p(b|x, y)$ be a behavior of the form:

$$p(b|x, y) = \sum_{a=0}^{d-1} p_D(a|x)p_D(b|y, a),$$

where $p_D(a|x)$ and $p_D(b|y, a)$ are deterministic. Therefore, for each $x \in [n_X]$ there is $a_x \in [d]$, such that $p_D(a|x) = \delta(a, a_x)$. In the same way, for each $y \in [n_X]$ and $a \in [d]$, there is $b_{a,y} \in [d]$, such that $p_D(b|y, a) = \delta(b, b_{a,y})$. Therefore, let's define $\mathcal{E} : [n_X] \to [d]$ by $\mathcal{E}(x) = a_x$. It follows that $p_D(a|x) = \delta(a, \mathcal{E}(x))$. On the other hand, for every $y \in [n_Y]$, let $\mathcal{D}_y : [d] \to [n_B]$ given by $\mathcal{D}_y(a) = b_{a,y}$. So, it also follows that $p_D(b|y, a) = \delta(b, b_{a,y}) = \mathcal{D}_y(a)$. In this way:

$$\begin{aligned} \delta(b, (\mathcal{D}_y \circ \mathcal{E})(x)) &= \sum_{a=0}^{d-1} \delta(a, \mathcal{E}(x))\delta(b, \mathcal{D}_y(a)) \\ &= \sum_{a=0}^{d-1} p_D(a|x)p_D(b|y, a) \\ &= \delta(b, (\mathcal{D}_y \circ \mathcal{E})(x)). \end{aligned} \tag{A.2}$$

Thus, every behavior that can be decomposed as eq. (1.2) can also be decomposed as eq. (1.1). In this way, the decompositions given by eq. (1.2) and eq. (1.2) are equivalent.

A.2 Fine-like theorem for PM scenarios

Theorem 1 (Fine-like theorem for PM scenarios)**.** *The set $\bar{C}_d$ is the convex hull of the points in D_d. Moreover, the points of D_d are the extremal points of $\bar{C}_d$.*

Proof. We start with the straightforward "only if" condition. From **Definitions 1 and 5**, it follows that $D_d \subset \bar{C}_d$. Couple this with the fact that $\bar{C}_d$ is a convex set, then every convex combination of elements in D_d will be in $\bar{C}_d$, which completes the first half of the proof.

We will now tackle the "if" condition. Given $p(b|x,y) \in \bar{C}_d$, by **Definition 5**,

$$p(b|x,y) = \sum_{a=0}^{d-1} \sum_{\lambda} \pi(\lambda) p(a|x,\lambda) p(b|a,y,\lambda). \tag{A.3}$$

Let us define $f_{x,\lambda}(a) = \sum_{0 \leqslant \boldsymbol{\alpha} \leqslant a} p(\boldsymbol{\alpha}|x,\lambda)$. Given a random parameter $0 \leqslant \mu_1 \leqslant 1$, we define the deterministic distribution

$$p^d(a|x,\lambda,\mu_1) = \begin{cases} 1, & \text{if } f_{x,\lambda}(a-1) \leqslant \mu_1 < f_{x,\lambda}(a) \text{ [1]} \\ 0, & \text{otherwise.} \end{cases}$$

As we can see,

$$\int_0^1 p^d(a|x,\lambda,\mu_1) d\mu_1 = \int_{f_{x,\lambda}(a-1)}^{f_{x,\lambda}(a)} d\mu_1 = f_{x,\lambda}(a) - f_{x,\lambda}(a-1) = p(a|x,\lambda). \tag{A.4}$$

Let us also define $g_{x,\lambda} : [0,1] \to [0,1]$ by

$$g_{x,\lambda}(\mu_1) = \sum_a f_{x,\lambda}(a) \mathbb{1}_{\left[f_{x,\lambda}(a-1), f_{x,\lambda}(a)\right)}(\mu_1),$$

where $\mathbb{1}_{\left[f_{x,\lambda}(a-1), f_{x,\lambda}(a)\right)}$ is the characteristic function of the interval $[f_{x,\lambda}(a-1), f_{x,\lambda}(a))$. It follows from the definition that $g_{x,\lambda}(\mu_1) = f_{x,\lambda}(a) \iff f_{x,\lambda}(a-1) \leqslant \mu_1 < f_{x,\lambda}(a)$. Because of that, we can rewrite p as

$$p^d(a|x,\lambda,\mu_1) = \delta\left(f_{x,\lambda}(a), g_{x,\lambda}(\mu_1)\right). \tag{A.5}$$

On the other hand, when x, λ, μ_1 are fixed, there exist only one a' s.t $f_{x,\lambda}(a') = g_{x,\lambda}(\mu_1)$. Given that, let us define $\mathcal{E}^{\lambda,\mu_1} : \{0,\ldots,n_X\} \to \{0,\ldots,d-1\}$ as $\mathcal{E}^{\lambda,\mu_1}(x) = a'$, being a' the only element s.t $f_{x,\lambda}(a') = g_{x,\lambda}(\mu_1)$. Then, we can easily see that $\delta\left(f_{x,\lambda}(a), g_{x,\lambda}(\mu_1)\right) = \delta\left(a, \mathcal{E}^{\lambda,\mu_1}(x)\right)$. Therefore, combining with eq. (A.5) we have:

$$p^d(a|x,\lambda,\mu_1) = \delta\left(a, \mathcal{E}^{\lambda,\mu_1}(x)\right). \tag{A.6}$$

In a completely analogous way, let us define $h_{a,y,\lambda}(b) = \sum_{0 \leqslant \beta \leqslant b} p(\beta|a,y,\lambda)$. Given a random parameter μ_2 between 0 and 1, we define the deterministic distribution:

$$p^d(b|a,y,\lambda,\mu_2) = \begin{cases} 1 & \text{if } h_{a,y,\lambda}(b-1) \leqslant \mu_2 < h_{a,y,\lambda}(b) \text{ [2]} \\ 0 & \text{otherwise.} \end{cases}$$

As we can see

$$\int_0^1 p^d(b|a,y,\lambda,\mu_2)d\mu_2 = \int_{h_{a,y,\lambda}(b-1)}^{h_{a,y,\lambda}(b)} d\mu_2 = h_{a,y,\lambda}(b) - h_{a,y,\lambda}(b-1) = p(b|a,y,\lambda). \quad \text{(A.7)}$$

We also define $k_{a,y,\lambda} : [0,1] \to [0,1]$ by

$$k_{a,y,\lambda}(\mu_2) = \sum_b h_{a,y,\lambda}(b) \mathbb{1}_{\left[h_{a,y,\lambda}(b-1),h_{a,y,\lambda}(b)\right)}(\mu_2),$$

where $\mathbb{1}_{\left[h_{a,y,\lambda}(b-1),h_{a,y,\lambda}(b)\right)}$ is the characteristic function of the interval $[h_{a,y,\lambda}(b-1), h_{a,y,\lambda}(b))$. It follows from the definition that $k_{a,y,\lambda}(\mu_2) = h_{a,y,\lambda}(b) \iff h_{a,y,\lambda}(b-1) \leqslant \mu_2 < h_{a,y,\lambda}(b)$. Because of that, we can rewrite p as

$$p^d(b|a,y,\lambda,\mu_2) = \delta\left(h_{a,y,\lambda}(b), k_{a,y,\lambda}(\mu_2)\right). \quad \text{(A.8)}$$

On the other hand, when a, y, λ, μ_2 are fixed, there exist only one b' s.t $h_{a,y,\lambda}(b') = k_{a,y,\lambda}(\mu_2)$. Given that, let us define $\mathcal{D}_y^{\lambda,\mu_2} : \{0,\ldots,d-1\} \to \{0,\ldots,n_B-1\}$ as $\mathcal{D}_y^{\lambda,\mu_2}(a) = b'$, being b' the only point of $\{0,\ldots,d-1\}$ where $h_{a,y,\lambda}(b') = k_{a,y,\lambda}(\mu_2)$. Then, we can easily see that $\delta\left(h_{a,y,\lambda}(b), k_{a,y,\lambda}(\mu_2)\right) = \delta\left(b, \mathcal{D}_y^{\lambda,\mu_2}(a)\right)$. Therefore, combining with eq. (A.8) we have

$$p^d(b|a,y,\lambda,\mu_2) = \delta\left(b, \mathcal{D}_y^{\lambda,\mu_2}(a)\right). \quad \text{(A.9)}$$

Notice that, from equations (A.3), (A.4) and (A.7), we have:

$$\begin{aligned} p(b|x,y) &= \sum_a \sum_\lambda \pi(\lambda) \int_0^1 p^d(a \mid x,\lambda,\mu_1)d\mu_1 \int_0^1 p^d(b \mid a,y,\lambda,\mu_2)d\mu_2 \\ &= \sum_a \sum_\lambda \int_0^1 \int_0^1 \pi(\lambda) p^d(a \mid x,\lambda,\mu_1) p^d(b \mid a,y,\lambda,\mu_2) d\mu_1 d\mu_2. \end{aligned}$$

Applying now equations (A.6) and (A.9):

$$\begin{aligned} p(b|x,y) &= \sum_a \sum_\lambda \int_0^1 \int_0^1 \pi(\lambda)\delta\left(a, \mathcal{E}^{\lambda,\mu_1}(x)\right) \delta\left(b, \mathcal{D}_y^{\lambda,\mu_2}(a)\right) d\mu_1 d\mu_2 \\ &= \sum_\lambda \int_0^1 \int_0^1 \pi(\lambda) \sum_a \left(\delta\left(a, \mathcal{E}^{\lambda,\mu_1}(x)\right) \delta\left(b, \mathcal{D}_y^{\lambda,\mu_2}(a)\right)\right) d\mu_1 d\mu_2 \\ &= \sum_\lambda \int_0^1 \int_0^1 \pi(\lambda)\delta\left(b, \left(\mathcal{D}_y^{\lambda,\mu_2} \circ \mathcal{E}^{\lambda,\mu_1}\right)(x)\right) d\mu_1 d\mu_2. \end{aligned}$$

Let $\tilde{\Lambda} = \Lambda \times [0,1] \times [0,1]$, $\tilde{\lambda} = (\lambda, \mu_1, \mu_2)$, $\tilde{\pi}(\tilde{\lambda})d\tilde{\lambda} = \pi(\lambda)d\mu_1 d\mu_2$, $\mathcal{E}^{\tilde{\lambda}} = \mathcal{E}^{\lambda,\mu_1}$ and $\mathcal{D}^{\tilde{\lambda}} = \mathcal{D}^{\lambda,\mu_2}$ Then,

$$p(b|x,y) = \int_{\tilde{\Lambda}} \tilde{\pi}(\tilde{\lambda})\delta\left(b, (\mathcal{D}_y^{\tilde{\lambda}} \circ \mathcal{E}^{\tilde{\lambda}})(x)\right) d\tilde{\lambda} \quad \forall\, b,x,y. \quad \text{(A.10)}$$

whereas by construction, the behavior $p(b|x,y,\tilde{\lambda}) = \delta\left(b, (\mathcal{D}_y^{\tilde{\lambda}} \circ \mathcal{E}^{\tilde{\lambda}})(x)\right)$ belongs to D_d. Remembering that the set D_d is finite[3], let's say that N is the cardinality of this set.

[3] It follows from the fact that the set of functions $\mathcal{D} : \{0,\ldots,d-1\}^m \to \{0,\ldots,d-1\}$ and $\mathcal{E} : \{0,\ldots,d-1\}^n \to \{0,\ldots,d-1\}^m$ are both finite.

Labeling the elements of D_d by $\boldsymbol{p}_i$, i.e., $D_d = \{\boldsymbol{p}_1, \cdots, \boldsymbol{p}_N\}$, we will then partition the set $\tilde{\Lambda}$ from the elements of D_d. For this, we define:

$$\tilde{\Lambda}_i := \left\{\tilde{\lambda} \in \tilde{\Lambda} \mid \delta\left(b, (\mathcal{D}_y^{\tilde{\lambda}} \circ \mathcal{E}^{\tilde{\lambda}})(x)\right) = p_i(b|x,y)\right\}.$$

On the other hand, we define:

$$r(i) = \int_{\tilde{\Lambda}_i} \tilde{\pi}(\tilde{\lambda}) d\tilde{\lambda}. \tag{A.11}$$

It follows that $r(i) \geqslant 0$ and

$$\sum_{i=1}^{N} r(i) = \sum_{i=1}^{N} \int_{\tilde{\Lambda}_i} \tilde{\pi}(\tilde{\lambda}) d\tilde{\lambda} = \int_{\tilde{\Lambda}} \tilde{\pi}(\tilde{\lambda}) d\tilde{\lambda} = 1.$$

Thus, rearranging the integral of eq. (A.10) using the partition $\Lambda = \bigcup_{i=1}^{N} \Lambda_i$, we have:

$$\begin{aligned} p(b|x,y) &= \int_{\tilde{\Lambda}} \tilde{\pi}(\tilde{\lambda}) \delta\left(b, (\mathcal{D}_y^{\tilde{\lambda}} \circ \mathcal{E}^{\tilde{\lambda}})(x)\right) d\tilde{\lambda} \\ &= \sum_{i=1}^{N} \int_{\tilde{\Lambda}_i} \tilde{\pi}(\tilde{\lambda}) \delta\left(b, (\mathcal{D}_y^{\tilde{\lambda}} \circ \mathcal{E}^{\tilde{\lambda}})(x)\right) d\tilde{\lambda} \\ &= \sum_{i=1}^{N} \int_{\tilde{\Lambda}_i} \tilde{\pi}(\tilde{\lambda}) p_i(b|x,y) d\tilde{\lambda} \\ &= \sum_{i=1}^{N} p_i(b|x,y) \int_{\tilde{\Lambda}_i} \tilde{\pi}(\tilde{\lambda}) d\tilde{\lambda} \\ &= \sum_{i=1}^{N} r(i) p_i(b|x,y). \end{aligned} \tag{A.12}$$

Therefore, we have seen that any behavior in $\bar{C}_d$ can be seen as a convex combination of the elements of D_d. □

A.3 If $d' < d \leqslant n_X$, then $D_{d'} \subsetneq D_d$

In this appendix, we will build a behavior that belongs to D_d but does not belong to $D_{d'}$, where $d' < d \leqslant n_X$ (see **Def. 1**). For simplicity, we will assume that $n_Y \geqslant d$. Let's suppose that Alice applies the encoder function $\mathcal{E} : [n_X] \to [d]$ on the input received by her, where

$$\mathcal{E}(k) = \begin{cases} k, & \text{if } 0 \leqslant k \leqslant d-1 \\ 0, & \text{if } d \leqslant k \leqslant n_X - 1, \end{cases}$$

and Bob, regardless of his input y, decoding his received message via

$$\begin{aligned} \mathcal{D} : [d] &\to [n_B] \\ k &\mapsto k \end{aligned} \tag{A.13}$$

The behavior obtained by them is given by:

$$p(b|x,y) = \delta\left[b, (\mathcal{D}\circ\mathcal{E})(x)\right] = \begin{cases} \delta(b,x), & \text{if } 0 \leqslant x \leqslant d-1 \\ \delta(b,0), & \text{if } d \leqslant k \leqslant n_B - 1. \end{cases} \tag{A.14}$$

By **Definition 1**, this behavior belongs to D_d. Let's suppose, by contradiction, that $\boldsymbol{p}$ also belongs to $D_{d'}$. Thus, by **Definition 1**, there is an encoding function $\mathcal{E}' : [n_X] \to [d']$ and a set of decoding functions $\{\mathcal{D}'_y\}_{y\in[n_Y]}$, where $\mathcal{D}'_y : [d'] \to [n_B]$, such that

$$p(b|x,y) = \delta\left[b, (\mathcal{D}'_y\circ\mathcal{E}')(x)\right].$$

So, for every $y \in [n_Y]$, by eq. (A.14):

$$\delta\left[b, (\mathcal{D}'_y\circ\mathcal{E}')(x)\right] = \delta(b,x) \quad \text{if } 0 \leqslant x \leqslant d-1, \tag{A.15}$$

which implies in $(\mathcal{D}'_y\circ\mathcal{E}')(x) = x$ for $0 \leqslant x \leqslant d-1$. Therefore, $\{0,\cdots,d-1\} \subset \mathrm{Im}(\mathcal{D}'_y\circ\mathcal{E}')$ and thus $\mathrm{Im}(\mathcal{D}'_y\circ\mathcal{E}') \geqslant d$.

On the other hand, as $\mathrm{Im}(\mathcal{E}') \subseteq [d']$, it follows $|\,\mathrm{Im}(\mathcal{E}')| \leqslant d'$. We also have, $|Im(\mathcal{D}'_y\circ\mathcal{E}')| \leqslant |Im(\mathcal{E}')|$, so $|Im((\mathcal{E}'\circ\mathcal{E}'))| \leqslant d'$. Therefore, $d \leqslant \mathrm{Im}(\mathcal{D}'_y\circ\mathcal{E}') \leqslant d'$, which is an absurd, as $d' < d$, by hypothesis. Thus, $\boldsymbol{p}$ belongs to D_d but does not belong to $D_{d'}$.

A.4 Higher-dimensional Bell basis

Higher-dimensional Bell basis are orthogonal basis of maximally entangled states. They were introduced by Bennett *et al.* [90] in order to generalize quantum teleportation protocol. Let us consider a higher-dimensional Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, whereby simplicity, we will assume that $\mathcal{H}_A$ and $\mathcal{H}_B$ has the same dimension. Let $\{|i\rangle'_A\}$ and $\{|i\rangle'_B\}$ be orthonormal bases of $\mathcal{H}'_A$ and $\mathcal{H}'_B$, respectively. We will introduced the following set of pure quantum states $|\Phi_{nm}\rangle$, $0 \leqslant n,m \leqslant d-1$, where:

$$|\Phi_{nm}\rangle := \frac{1}{\sqrt{d}}\sum_{j=0}^{d-1} e^{2\pi i jn/d}\,|j\rangle_{A'} \otimes |j \oplus_d m\rangle_{B'}, \tag{A.16}$$

where $j \oplus_d m = (j+m) \bmod d$. The states $|\Phi_{nm}\rangle$ are easily seen to be orthonormal,

$$\begin{aligned}
&\langle\Phi_{n'm'}|\Phi_{nm}\rangle = \\
&= \frac{1}{d}\sum_{j,k=0}^{d-1} e^{2\pi i(jn-kn')/d}\,\langle k|j\rangle_{A'}\,\langle k\oplus_d m'|j\oplus_d m\rangle_{B'} \\
&= \frac{1}{d}\sum_{k=0}^{d-1} e^{2\pi i k(n-n')/d}\;\delta_{m,m'} = \delta_{n,n'}\delta_{m,m'}.
\end{aligned}$$

To generalize the Pauli matrices, Bennett et al. [90] introduce the unitary operators

$$U_{nm} := \sum_{k=0}^{d-1} e^{2\pi i k n/d} \left|k\right\rangle_{A'} \left\langle k \oplus_d m\right|_{A'} . \tag{A.17}$$

Wherefrom we can see that

$$\begin{aligned}
&(U_{nm} \otimes Id) \left|\Phi_{00}\right\rangle \\
&= \left(\sum_{k=0}^{d-1} e^{2\pi i k n/d} \left|k\right\rangle_{A'} \left\langle k \oplus_d m\right|_{A'} \otimes Id\right) \left(\frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} \left|j\right\rangle_{A'} \otimes \left|j\right\rangle_{B'}\right) \\
&= \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} \sum_{j=0}^{d-1} e^{2\pi i k n/d} \left\langle k \oplus_d m | j\right\rangle_{A'} \left|k\right\rangle_{A'} \otimes \left|j\right\rangle_{B'} \\
&= \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} \sum_{j=0}^{d-1} e^{2\pi i k n/d} \delta(k \oplus_d m, j) \left|k\right\rangle_{A'} \otimes \left|j\right\rangle_{B'} \\
&= \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} e^{2\pi i k n/d} \left|k\right\rangle_{A'} \otimes \left|k \oplus_d m\right\rangle_{B'} \\
&= \left|\phi_{nm}\right\rangle .
\end{aligned} \tag{A.18}$$

That is,

$$(U_{nm} \otimes Id) \left|\phi_{00}\right\rangle = \left|\phi_{nm}\right\rangle , \tag{A.19}$$

for all $0 \leqslant n, m \leqslant d-1$.