MODULE : 3

Limit Theorems

1. Markov's Inequality

If x is a r.v that takes only non -ve values, then for any value $a>0$, $P[x \geq a] \leq \frac{E[x]}{a}$.

2. Chebyshev's Inequality

If x is a r.v with mean μ and variance σ^2, then for any value $k>0$, $P[|x-\mu| \geq k] \leq \frac{\sigma^2}{k^2}$.

then, $P[|x-\mu|<k] = 1-\frac{\sigma^2}{k^2}$

Q.1) Suppose we know that the no. of items produced in a factory during a week is a r.v with mean 500.

a) what can be said about the probability that this week's production is atleast 1000?

b) If the variance of a week's production is ~~not~~ known to equal 100, then what can be said about the probability that this week's production will be b/w 400 and 600?

→ Given, $\mu = 500$

$E[x] = 500$

a) $P[x \geq a] \leq \frac{E[x]}{a}$

ie, $P[x \geq 1000] \leq \frac{500}{1000} = \frac{1}{2}$

b) $P[|x-\mu| \geq k] \leq \frac{\sigma^2}{k^2}$

$E[x^2]-(E[x])^2 = 100 = \sigma^2$

$P[400<x<600] = P[400-500 < x-500 < 600-500]$

~~The probability~~

$= P[-100 < x - \mu < 100]$

$= P[|x - \mu| < 100]$

$= \sigma^2 = 100 \qquad k^2 = 100^2$

$$1 - \frac{\sigma^2}{k^2} = 1 - \frac{1}{100} = \frac{99}{100} = \underline{\underline{0.99}}$$

$P[|X - \mu| < 100] \geq \underline{\underline{0.99}}$

The probability that this week's production will be b/w 400 and 600

Strong Law of Large Numbers

Central Limit Theorem (CLT)

we say that the r.vs $x_1, x_2 \dots$ is identicaly and independently distributed (iid) if they are all independent and all of them have the same probability distribution.

CLT : Let $X_1, X_2 \dots X_n$ be iid r.vs each having mean μ and variance σ^2 and let $S_n = X_1 + X_2 \dots + X_n$ and $\bar{X} = \frac{S_n}{n}$, then

$$Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}} = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$$ will follow standard normal distribution $[\sim N(0,1)]$.

Q.2) A computer generates 20 random numbers which are uniformly distributed b/w 0 and 1. Find approximately the probability that,

a) their sum is atleast 10

b) their average is b/w 0.4 and 0.6

→ Let, X_i denote i^{th} random number generated

$i = 1, 2, \dots 20.$

$X_i \sim u(0,1)$

$f(x_i) = 1 \quad , \quad 0 < x < 1$

- $\mu = \frac{a+b}{2} = \frac{1}{2} = \underline{0.05}$
- $\sigma^2 = \frac{(b-a)^2}{12} = \frac{1}{12}$

It is an iid

a) $P[X_1 + X_2 \dots + X_n \geq 10] = P[S_{20} \geq 10]$

$= \frac{S_{20} - 20 \times \frac{1}{2}}{\frac{1}{\sqrt{12}} \times \sqrt{20}} \sim N(0,1)$

$= \frac{S_{20} - 10}{\sqrt{\frac{20}{12}}}$ ~~$P[S_{20} \geq 10]$~~

$\Rightarrow P[S_{20} \geq 10] = 1 - P[S_{20} < 10]$

$= 1 - P\left[Z_n < \frac{10-10}{\sqrt{20/12}}\right]$

$= 1 - P[Z_n < 0]$

$= 1 - 0.5$

$= \underline{0.5}$

b) The avg is b/w 0.4 and 0.6

$P[0.4 < \bar{x} < 0.6] = P\left[\frac{0.4-0.5}{\frac{1}{\sqrt{12 \times 20}}} < Z_n < \frac{0.6-0.5}{\frac{1}{\sqrt{12 \times 20}}}\right]$

$= P[-1.54 < Z_n < 1.54]$

$= P[1.54] - P[-1.54]$

$= 0.93822 - 0.06178$

$= \underline{0.87644}$

8/8 Normal approximation to binomial.

~~Let $x_1, x_2 \dots x_n$~~

- If X is a binomialy distributed r.v with parameters n and p, then $\frac{X-np}{\sqrt{npq}}$ ~~then the~~ approaches standard normal distribution ie, $\sim N(0,1)$.

- If $npq \geq 10$, use normal distribution

Q. Let X be the no. of times that a fair coin flip 40 times lands heads. Find the probability that $X = 20$. Use the normal approximation and then compare it to exact solution.

→ $z = \frac{X-np}{npq} \sim N(0,1)$

~~P[X=20]~~ here, $X = 0, 1, 2 \dots 40$

$P[X=20] = P[19.5 < x < 20.5]$

here, $np = 40 \times \frac{1}{2} = 20$

$npq = 40 \times \frac{1}{2} \times \frac{1}{2} = 10$

$\therefore P[X=20] = P\left[\frac{19.5-20}{\sqrt{10}} < z \quad \frac{20.5-20}{\sqrt{10}}\right]$

$= P[-0.16 < z < 0.16]$

$= \underline{\underline{0.12712}}$

Q. ~~1~~. A fair coin is tossed 200 times. Find approximately the probability that no. of heads ~~occur~~ obtained is b/w 80 and 120 using central limit theorem.

→ $z = \frac{X-np}{\sqrt{npq}} \sim N(0,1)$

** $n = 200 \quad p = \frac{1}{2} \quad q = \frac{1}{2} \quad np = 100 \quad npq = 50$

$Z = \frac{x - np}{\sqrt{npq}} \qquad P[80 < x < 120]$

$$\frac{x-np}{\sqrt{npq}} = \frac{80-100}{\sqrt{50}} < x < \frac{120-100}{\sqrt{50}}$$

$$P\left[\frac{-20}{\sqrt{50}} < x < \frac{20}{\sqrt{50}}\right] = P[-2.828 < Z < 2.828]$$

$$= 0.99767 - 0.00233$$

$$= \underline{\underline{0.99534}}$$

12/8

Q In a game involving repeated throws of a balanced die, a person receives ₹3 if the resulting number is greater than or equal to 3 and loses ₹3 otherwise. Use central limit theorem to find the probability that in 25 trials his total earnings exceeds ~~is~~ ₹25.

→ x : be the ~~earning~~ profit/loss in single toss of the die.

$X = 3, -3$

pdf of X :

X	3	-3
p(x)	$\frac{4}{6} = \frac{2}{3}$	$\frac{2}{6} = \frac{1}{3}$

• Mean, $\mu = \sum_x x \cdot p(x) = 3 \times \frac{2}{3} - 3 \times \frac{1}{3} = 1$ //

• Variance, $\sigma^2 = E(x^2) - [E(x)]^2 = \sum_x x^2 p(x) - 1^2$

$$= 9 \times \frac{2}{3} + 9 \times \frac{1}{3} - 1 = 9 - 1 = 8 \,//$$

thus $\sigma = \sqrt{8} = 2\sqrt{2}$ //

$n = 25$

Total earnings $\rightarrow S_n = x_1 + \cdots + x_{25}$

$$\frac{S_n - n\mu}{\sigma\sqrt{n}} \sim N(0,1) \qquad (CLT)$$

$$P[S_n > 25] = P\left[Z > \frac{25 - 25\times 1}{2\sqrt{2}\times\sqrt{25}}\right]$$

$$= \frac{1}{2}$$

Q. If 30 fair dice are thrown, what is the probability that the sum obtained is b/w 95 & 125.

$\rightarrow$ $n = 30$

x: no. obtained after one throw of a fair dice.

pdf of x : $p(x) = 1/6$, $x = 1, \ldots 6$

- $\mu = \sum_x x \cdot p(x) = 1\times \frac{1}{6} + \cdots + 6\times\frac{1}{6}$

$$= \frac{1}{6}\times(1+2+3+4+5+6) = \frac{21}{6} = \frac{7}{2}$$

- $\sigma^2 = E(x^2) - \mu^2 = \sum_x x^2 p(x) - \left(\frac{7}{2}\right)^2$

$$= \frac{1}{6}\left[1^2+2^2+3^2+4^2+5^2+6^2\right] - \frac{49}{4}$$

$$= \frac{91}{6} - \frac{49}{4} = \frac{35}{12}$$

$\therefore \sigma = \sqrt{\frac{35}{12}}$

$n = 30$

$S_n \rightarrow x_1 + x_2 + \cdots + x_{30}$

$$\frac{S_n - n\mu}{\sigma\sqrt{n}} \sim N(0,1) \qquad (CLT)$$

$$P[95 < S_n < 125] = P\left[\frac{95 - 30\times 7/2}{\sqrt{35/12}\times\sqrt{30}} < Z < \frac{125 - 30\times 7/2}{\sqrt{35/12}\times\sqrt{30}}\right]$$

$= P[-1.06 < z < 2.13]$

$= 0.98341 - 0.14457$

$= \underline{\underline{0.83884}}$

Q. A random sample of size hundred is taken from a population whose mean is 80 and σ^2 is 400. Using CLT, with what probability can we assert that the μ of the sample will not differ by population μ by more than 6?

→ Given, $n = 100$, $\mu = 80$, $\sigma^2 = 400$

$\sigma = \sqrt{400} = 20$

By CLT, $\dfrac{\bar{x} - \mu}{\sigma\sqrt{n}} \sim N(0,1)$

$\therefore P[|\bar{x} - \mu| < 6] = P[-6 < \bar{x} - \mu < 6]$

$= P\left[\dfrac{-6}{20/\sqrt{100}} < z < \dfrac{6}{20/\sqrt{100}}\right]$

$= P[-3 < z < 3]$

$= P[3] - P[-3]$

$= 0.99813 - 0.00135$

$= \underline{\underline{0.9973}}$

Revision

13/7 Q. The pdf of a r.v is given by $p(x) = \dfrac{k}{2x}$, $x = 1, 2, 3, 4$. Find k and $P[x \neq 3]$

→

X	1	2	3	4
P(x)	$k/2$	$k/4$	$k/6$	$k/8$

Total probability: $\dfrac{k}{2} + \dfrac{k}{4} + \dfrac{k}{6} + \dfrac{k}{8} = 1$

$k\left(\dfrac{25}{24}\right) = 1$

$k = \dfrac{24}{25}$ //

$P[X \neq 3] = \frac{k}{2} + \frac{k}{4} + \frac{k}{8} = \frac{24}{25} \cancel{+} \left(\frac{7}{8}\right)$

$= \frac{21}{25}$

Mean, $\mu = \sum_{x} x\, p(x) = 1 \times \frac{k}{2} + 2 \times \frac{k}{4} + 3 \times \frac{k}{6} + 4 \times \frac{k}{8}$

$= k\left(\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}\right)$

$= \frac{24}{25} \times 2$

$= \frac{48}{25} = 1.92$

Variance, $\sigma^2 = \sum_{x} x^2 p(x) - \mu^2 = k\left[1^2/2 + 2^2/4 + 3^2/6 + 4^2/8\right] - \mu^2$

$= \frac{24}{25}\left[\frac{1}{2} + 1 + \frac{3}{2} + 2\right] - \mu^2 = \frac{24}{25} \times 5 - \left(\frac{48}{25}\right)^2$

$= \frac{24}{5} - \frac{2304}{625}$

$= \frac{696}{625}$

Q.2) A communication set up consists of 5 subsystems each of which may fail with probability 0.15. The system will work if atleast 3 of the subsystems work correctly. What is the probability that the system will work and what is the expected no. of correctly ~~no~~ working subsystems?

$\rightarrow$ $n = 5$, $p = 1 - 0.15 = 0.85$

$b[x; n, p] = {}^nC_x\, p^x q^{n-x}$, $x = 0, 1, 2 \ldots n$

pdf : $p(x) = {}^5C_x (0.85)^x (0.15)^{5-x}$

a) Probability that the system will work $= P[X \geq 3]$

$= P(3) + P(4) + P(5)$

$= {}^5C_3 (0.85)^3 (0.15)^2 + {}^5C_4 (0.85)^4 (0.15)^1 +$

${}^5C_5 (0.85)^5 (0.15)^0$

$= \cancel{0.1384}$ 0.97335

b) $\mu = E[X] = np = 5 \times 0.85$

$= 4.25$

Q.3) If a computer generates 1000 non -ve integers following poisson distribution with mean 2.5, how many of them would 2, 4 & 8?

→ Mean, $\lambda = 2.5$

$P(x) = \frac{e^{-2.5}(2.5)^x}{x!}$, $x = 0, 1, 2, \ldots$

$P(2) = \frac{e^{-2.5}(2.5)^2}{2!} = 0.25651$

Expected no. of 2's in 1000s trials $= 1000 \times 0.2565$

$= 256.5$

≈ 257

$P(4) = \frac{e^{-2.5}(2.5)^4}{4!} = 0.1336$

Expected no. of 4's in 1000 trials $= 1000 \times 0.1336$

$= 133.6$

≈ 134

$P(8) = \frac{e^{-2.5}(2.5)^8}{8!} = 3.10 \times 10^{-3}$

Expected no. of 8's in 1000 trials $= 1000 \times 3.10 \times 10^{-3}$

≈ 3

Q 4) The joint pdf of x and y is given by $P(x,y) = kxy$, $x = 1,2,3$ and $y = 1,2,3$. Find k and check whether x and y are independent.

→ $\sum_x \sum_y kxy = 1$

$k[1+2+3+2+4+6+3+6+9] = 1$

$$k = \frac{1}{36}$$

$$f_X(x) = \sum_Y \cancel{k} \frac{xy}{36} = \frac{x}{36}[1+2+3] = \frac{6x}{36} = \frac{x}{6}$$

$$f_Y(y) = \sum_X \frac{xy}{36} = \frac{y}{36}[1+2+3] = \frac{6y}{36} = \frac{y}{6}$$

$$f_X(x) \cdot f_Y(y) = \frac{x}{6} \times \frac{y}{6} = \frac{xy}{36} = p(x,y)$$

Hence it is independent.

9/9/25

Random Process or Stochastic Process

- A random process is a collection of random variables the set $\{X(t,s)\}$ that are functions of real variable t and s belongs to sample space.
- The set of possible values of any individual member of a ~~~~ random process is called state space.

Note

* If t and s are fixed, then $X(t,s)$ is a constant
* when t is fixed, $X(t,s)$ is a random variable.
* If s is fixed, $X(t,s)$ is a time function.
* when both t and s are variables, $X(t,s)$ will be collection of random variables.

Types of Random Process

1. Discrete ~~type~~ time random process

A discrete random process or a random sequence $\{X_n, n \in T\}$ where T is a countable set, usually the set of Integers. This usually represents a process which is observed at regular intervals of time.

Eg: • Let X_n be the outcome of nth toss of a die

2. Continuous time Random Process

A random process $\{X(t), t \in T\}$ where T is an interval usually time. It represents a process which is observed continuously over a period time.

Eg: • Let X_n be the temperature in °C measured continuoutsly

23/9

Poisson Process

$$P[N(t) = n] = e^{-\lambda t} \cdot \frac{(\lambda t)^n}{n!} \qquad n = 0, 1, 2 \ldots$$

- Poisson random variable is used to model rare events occuring over a fixed period of time

- For example, the no. of accidents in a particular period, at a particular place
- No. of hits on a website

- It is a counting process

Counting Process

- A counting process $\{N(t), t \geq 0\}$ is a random process which counts the no. of times an event occurs from time zero upto an including time t.

- The possible values of N(t) are non negative integers

- For $0 \leq s \leq t$, $N(t) - N(s)$ = the no. of times the event occurs for interval $(s, t]$

- A continuous time random process is said to have independent increments if, for $0 \le t_1 < t_2 < \dots < t_n$; $X(t_2) - X(t_1)$, $X(t_3) - X(t_2)$ ... are independent.

- A random process $X(t)$ is said to have stationary increments if

$$X(t) - X(s) \stackrel{d}{=} X(t+h) - X(s+h)$$

- For a fixed $\lambda > 0$, a counting process $N(t)$ is said to be a poison process with rate λ if it satisfies the following conditions:

1. $N(0) = 0$
2. $N(t)$ has independent and stationery increments
3. For any t, $N(t)$ follows poisson distribution with parameter λt

ie, $\boxed{P[N(t) = n] = e^{-\lambda t} \cdot \frac{(\lambda t)^n}{n!}}$ $n = 0, 1, 2, 3, \dots$

Note

1. If $N(t)$ is a poisson process, the no. of occu-rances in any time interval of length τ follows a poisson distribution with parameter $\lambda\tau$. ie, $P[N(\tau) = n] = e^{-\lambda\tau} \frac{(\lambda\tau)^n}{n!}$, $n = 0, 1, 2, 3 \dots$

Q. A cell phone calls process ~~by~~ time certain time

a) what is the probability that more than 3 calls arrive in an interval of length 20secs?

b) what is the probability that more than 2 calls arrive in an interval of length 30 secs?

ans: Given $\lambda = 12$

$$P[N(\tau)=n] = e^{-12\tau}\frac{(12\tau)^n}{n!} \quad n = 0, 1, 2 \ldots$$

a) $\tau = 20\,sec = \frac{1}{3}\,min$

$$P[N(\tfrac{1}{3}) > 3] = 1 - (P[N(\tfrac{1}{3}) \le 3])$$

$$= 1 - P(N(\tfrac{1}{3}) = 0, 1, 2, 3]$$

$$= 1 - e^{-12/3}\left[\frac{(12/3)^0}{0!} + \frac{(12/3)^1}{1!} + \frac{(12/3)^2}{2!} + \frac{(12/3)^3}{3!}\right]$$

$$= 1 - e^{-4}\left(1 + 4 + 8 + \frac{64}{2\times3}\right)$$

$$= 1 - e^{-4} \times$$

$$= \underline{\underline{0.5665}}$$

b) $\tau = 30\,sec = \frac{1}{2}\,min$

$$P[N(\tfrac{1}{2}) > 2] = 1 - P[N(\tfrac{1}{2}) \le 2]$$

$$= 1 - P[N(\tfrac{1}{2}) = 0, 1, 2]$$

$$= 1 - e^{-12/2}\left[\frac{(12/2)^0}{0!} + \frac{(12/2)^1}{1!} + \frac{(12/2)^2}{2!}\right]$$

$$=$$