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Mathematics

Module 1 : Discrete Probability Distribution

- Let S be a sample space associated with a random experiment. Then a random variable X, $X: S \to R$ which associates to every outcome $s \in S$ a real $x(s)$.
- The probability distribution of a random variable X is a rule which allows us to compute probability of $P[X=x]$ for every real number x.
- The set of possible values of a random variable is finite/countable then the probability distribution is said to be discrete.

Axioms

Let $P(x)$ denote the probability distribution function of discrete random variable X, then

i) $0 \le p(x) \le 1$ (positivity)

ii) $\sum_x p(x) = 1$ (total probability)

Q.1) Find the value of k so that the following is a valid probability distribution of a discrete random variable.

X	0	1	2	3
P(x)	$\frac{k}{2}$	$\frac{k}{3}$	$\frac{k+1}{3}$	$\frac{2k-1}{6}$

→ By total probability,

$$\frac{k}{2}+\frac{k}{3}+\frac{k+1}{3}+\frac{2k-1}{6}=1 \Rightarrow \frac{k}{2}+\frac{2k+1}{3}+\frac{2k-1}{6}=1$$

$$3k+4k+2+2k-1=6 \Rightarrow 9k=5$$

$$\therefore k = 5/9$$

Find $P[0 < x \leq 2]$

ie, $P[0 < x \leq 2] = P[x=1] + P[x=2]$

$$= \frac{k}{3} + \frac{k+1}{3} = \frac{2k+1}{3}$$

$$= \frac{\frac{10}{9}+1}{3} = \frac{19}{27}$$

Q.2 Probability distribution of random variable is given by $P(x) = \frac{k}{2x}$, $x = 1, 2, 3, 4$. Find k. and $P[x \neq 3]$

→

X	1	2	3	4
$P(x)$	$\frac{k}{2}$	$\frac{k}{4}$	$\frac{k}{6}$	$\frac{k}{8}$

- By total probability, $\frac{k}{2} + \frac{k}{4} + \frac{k}{6} + \frac{k}{8} = 1$

$$\frac{4k+2k+k}{8} + \frac{k}{6} = 1 \Rightarrow 7k \times 6 + 8k = 48$$

$$42k + 8k = 48 \Rightarrow 50k = 48$$

$$\therefore k = \frac{24}{25}$$

- $P[X \neq 3] = P[x=1] + P[x=2] + P[x=4]$

$$= \frac{k}{2} + \frac{k}{4} + \frac{k}{8} = \frac{7k}{8}$$

$$= \frac{7}{8} \times \frac{24}{25} = \frac{21}{25}$$

Q.3 Possible values of a discrete random variable X are 1, 2, 3, 4, 5. If all the values are equally likely, find $P[X \text{ is even}]$.

→ Since equally likely, $P(x) = \frac{1}{5}$, $x = 1, 2, 3, 4, 5$.

then $P[X \text{ is even}] = P(X=2) + P(X=4)$

$$= \frac{1}{5} + \frac{1}{5} = \frac{2}{5}$$

⇒ If X is discrete, then the probability distribution function (pdf) is also called probability mass function (pmf).

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Cumulative distribution function (CDF)

- CDF of a random variable x,

$$\boxed{F\cancel{P}[x] = P[X \le x]}$$

eg:- $F[1] = P[X \le 1]$

Mean and Variance of r.v x.

- For a r.v x, Mean, $\boxed{\mu = \sum_{x} x \cdot p(x)}$

- Variance, $\sigma^2 = \sum_{x} (x-\mu)^2 \, p(x)$

$$= \sum_{x} (x^2 - 2\mu x + \mu^2)\, p(x)$$

$$= \sum_{x} x^2 p(x) - \sum_{x} 2\mu x \, p(x) + \sum_{x} \mu^2 p(x)$$

$$= \sum_{x} x^2 p(x) - 2\mu x \sum x\, p(x) + \mu^2 \sum_{x} p(x)$$

$$\boxed{\sigma^2 = \left[\sum_{x} x^2 p(x)\right] - \mu^2}$$

Q.4 - Pmf of a rv x is $F(x) = \frac{x}{6}$, $x = 1, 2, 3$

Find the mean and variance of x

→ Mean, $\mu = \sum_{x} x f(x)$

$$= 1 \cdot \frac{1}{6} + 2 \cdot \frac{2}{6} + 3 \cdot \frac{3}{6}$$

$$= \frac{1}{6} + \frac{4}{6} + \frac{9}{6} = \frac{14}{6}$$

$$= \frac{7}{3}$$

Variance, $\sigma^2 = \left[\sum_{x} x^2 p(x)\right] - \mu^2$

$$= \left(1^2 \cdot \frac{1}{6} + 2^2 \cdot \frac{2}{6} + 3^2 \cdot \frac{3}{6}\right) - \frac{49}{9}$$

$$= \frac{1 + 8 + 27}{6} - \frac{49}{9} = \frac{36}{6} - \frac{49}{9}$$

$$= 6 - \frac{49}{9} = \frac{5}{9}$$

4
54 14
49
5

Q.5 An individual who has automobile insurance ~~is~~ from a certain company is randomly selected. Let y be the number of moving violations for which the individual was sighted for the last 3 yrs. The pmf of y is

y	0	1	2	3
$P(y)$	0.6	0.25	0.1	0.05

Find ① the cdf of y,

② $P[y>2]$

③ $P[\frac{1}{2} \le y \le \frac{5}{2}]$

→ ① $F(y) = P(Y \le y]$

$$F(y) = \begin{cases} 0 & , y<0 \\ 0.60 & , y=0 \\ 0.85 & , y=1 \\ 0.95 & , y=2 \\ 1 & , y \geq 3 \end{cases}$$

$P[y \le 0]$

$P(y \le 1]$

② $P[y>2] = 0.05$

③ $P(\frac{1}{2} \le y \le \frac{5}{2}] = 0.35$

Expectation

$\boxed{E[x] = \mu}$. For a function $g(x)$,

$$\boxed{E(g(x)) = \sum_{g(x)} g(x) \cdot p(x)}$$

$$\boxed{\begin{aligned} \sigma^2 &= E(x^2) - \mu^2 \\ \sigma^2 &= E(x^2) - [E(x)]^2 \end{aligned}}$$

Q. Show that

1. $E[aX+b] = aE[X]+b$
2. $Var[aX+b] = a^2 Var(X)$

$\rightarrow$ 1) $$E[ax+b] = \sum_x (ax+b)\, p(x)$$
$$= \sum_x ax \cdot p(x) + \sum_x b \cdot p(x)$$
$$= a \underbrace{\sum_x x \cdot p(x)} + b \underbrace{\sum_x p(x)}$$
$$= a \cdot E(x) + b \cdot 1$$
$$= \underline{\underline{aE(x)+b}}$$

2) $$Var[ax+b] = E[(ax+b)^2] - [E(ax+b)]^2$$
$$= E[a^2x^2 + 2axb + b^2] - [aE(x)+b]^2$$
$$= a^2 E[x^2] + 2abE(x) + b^2 - [a^2E(x)^2 + 2abE(x) + b^2]$$
$$= a^2 [E(x^2) - [E(x)]^2]$$
$$= \underline{\underline{a^2 Var(x)}}$$

- For a constant 'a', $E(a) = a$
$Var(a) = 0$

Q. Consider the following game that involves tossing a fair dice. If the outcome of a toss is an even number, you will gain ₹2
If the outcome is 1 or 3, you will loss ₹ 1
If the outcome is 5, you loss ₹ 3.
What are the expected winnings?

$\rightarrow$ X : Profit or loss

X	-3	-1	2
$P[X]$	$1/6$	$2/6$	$3/6$

$$E[X] = \mu = \sum_x x \cdot p(x) = -3 \times \frac{1}{6} + -1 \times \frac{2}{6} + 2 \times \frac{3}{6}$$
$$= -\frac{1}{2} - \frac{1}{3} + 1 = -\frac{5}{6} + 1 = \frac{1}{6}$$

Binomial Distribution.

- Experiment with 'n' no. of trials.
- Each trial consists of two outcomes, namely success and failure.
- $P[\text{success}] = p$ and p is fixed.
- $P[\text{failure}] = 1-p = q$
- Each trial is independent.
- $p(x) = b[x; n, p]$

$$\boxed{p(x) = {}^nC_x\, p^x q^{n-x}}$$

$x = 0, 1, 2, 3 \dots n$

probability of 'x' success in 'n' trials:

Q. 10 coins are tossed simultaneously. Find the probability of getting seven tails.

→ $n = 10$, $p = \frac{1}{2}$, $q = \frac{1}{2}$

(P: prob of getting a tail.)

pdf, $p(x; 10, \frac{1}{2}] = {}^{10}C_x (\frac{1}{2})^x (\frac{1}{2})^{10-x}$

$\frac{10 \times \cancel{9}^3 \times \cancel{8}^4}{1 \times \cancel{2} \times \cancel{3}}$

where, $x = 0, 1, 2, \dots 10$

then, $P[7] = {}^{10}C_7 (\frac{1}{2})^7 (\frac{1}{2})^3$

$= \frac{10!}{7!\,3!} \times \frac{1}{2^{10}}$

$= 0.11718 \approx \frac{15}{128}$

Q. Find the mean ~~&~~ and variance of binomial distributions with parameters n and p.

→ $p(x) = b[x; n, p] = {}^nC_x\, p^x q^{n-x}$, $x = 0, 1 \dots n$

Mean, $\mu = \sum_x x \cdot p(x)$

$= \sum_x x \cdot {}^nC_x\, p^x q^{n-x}$

$$= \sum_x \cancel{x} \frac{n!}{(n-x)!\,\cancel{x!}\,(x-1)!} p^x q^{n-x} \qquad n-x = (n-1)-(x-1)$$

$$= np \sum_x \frac{(n-1)!}{(n-1)-(x-1)!\,(x-1)!} p^{x-1} q^{(n-1)-(x-1)}$$

$$= np \sum_x {}^{n-1}C_{x-1}\, p^{x-1} q^{(n-1)-(x-1)}$$

$$= np\,(p+q)^{n-1} \qquad \text{(Binomial theorem)}$$

$$\boxed{\mu = np}$$

Variance

$$[x^2 = x(x-1)+x]$$

$$\sigma^2 = E(x^2) - [E(x)]^2$$

$$= \sum_x x^2\, {}^nC_x\, p^x q^{n-x} - (np)^2$$

$$= \sum_x (x(x-1)+x)\, {}^nC_x\, p^x q^{n-x} - n^2p^2$$

$$= \sum_x x(x-1)\, {}^nC_x\, p^x q^{n-x} + \underbrace{\sum_x x\, {}^nC_x\, p^x q^{n-x}}_{\mu = np} - n^2p^2$$

$$= \sum_x \frac{x(x-1)\, n!}{(n-x)!\,x!} p^x q^{n-x} + np - n^2p^2$$

$$= \sum_x \frac{n!}{(n-x)!\,(x-2)!} p^x q^{n-x} + np - n^2p^2$$

$$= n(n-1)p^2(p+q)^{n-2} + np - n^2p^2$$

$$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$$

$$= np(1-p) = \underline{\underline{npq}}$$

$$\boxed{\sigma^2 = npq}$$

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Q. The mean and variance of binomial distribution are 16 and 8 respectively. Find $P[X=0]$ and $P[x=1]$

→ $\mu = np$, $\sigma^2 = npq$

Given, $\mu = np = 16$

$$\sigma^2 = npq = 8$$
$$16q = 8$$
$$q = \frac{1}{2} \qquad \therefore p = 1/2$$

then, $n = \frac{8}{pq} = 8 \times 2 \times 2 = 32$

$$b[x; 32, 1/2] = {}^{32}C_x \left(\tfrac{1}{2}\right)^x \left(\tfrac{1}{2}\right)^{32-x}$$

- $P[X=0] = {}^{32}C_0 \left(\tfrac{1}{2}\right)^{32} = \left(\tfrac{1}{2}\right)^{32}$
- $P[X=1] = {}^{32}C_1 \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)^{31} = \frac{32}{2^{32}}$

Q. Suppose that 20% of all copies of a particular textbook fail a certain binding strength test. Let X denote the number among 15 randomly selected copies that fail the test. Find probability that atmost 4 fail the test.

→ $P = 0.2$, $q = 0.8$, $n = 15$

$$b[x; 15; 0.2] = {}^{15}C_x (0.2)^x (0.8)^{15-x}$$

- $P[X \leq 4] = P(0) + P(1) + P(2) + P(3) + P(4)$

$$= (0.8)^{15} + 15(0.1)(0.8)^{14} + 105(0.2)^2(0.8)^{13} + 455(0.2)^3(0.8)^{12} + 1365(0.2)^4(0.8)^{11}$$
$$= 0.8336$$

Q. The sum of mean and variance of B.D for 5 trials is 1.8. Find pdf.

→ $\mu + \sigma^2 = 1.8$

$$np + npq = 1.8$$
$$5p(1+q) = 1.8$$
$$5p(1+1-p) = 1.8 \Rightarrow 5p(2-p) = 1.8$$
$$\Rightarrow 10p - 5p^2 - 1.8 = 0$$

ie, $p = 9/5$, $p = 1/5$

Since p cannot be more than 1, $p = 1/5$

$p = 1/5$, hence $q = 1 - 1/5 = 4/5$

$\therefore \quad \underline{\underline{{}^5C_x (1/5)^x (4/5)^{5-x}}} \quad , \quad x = 0, 1, 2 \ldots$

Q. 6 dice are thrown 729 times. How many times do you expect atleast 3 dice to throw 5 or 6?

$\rightarrow \quad n = 6 \quad , \quad p = \frac{2}{6} = \frac{1}{3} \quad , \quad q = \frac{2}{3}$

$P[x \geq 3] = {}^6C_3 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^3 + P(4) + P(5) + P(6)$

$\boxed{b[x; 6; 1/3] = {}^6C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{6-x}} \quad x = 0, 1, 2, 3 \ldots 6$

$$= {}^6C_3 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^3 + {}^6C_4 \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^2 +$$

$$ {}^6C_5 \left(\frac{1}{3}\right)^5 \left(\frac{2}{3}\right)^1 + {}^6C_6 \left(\frac{1}{3}\right)^6$$

$$= 20 \left(\frac{1}{27}\right)\left(\frac{8}{27}\right) + 15 \left(\frac{1}{27}\right)\left(\frac{4}{9}\right) + 6\left(\frac{1}{27 \times 9}\right)\left(\frac{2}{3}\right) + \frac{1}{3^6}$$

$P[x \geq 3] = \frac{233}{729}$

$\therefore$ Required ans $= \cancel{729} \times \frac{233}{\cancel{729}}$

$= \underline{\underline{233}}$

11/7/25 **Poisson Distribution Theorem**

The probability distribution of a binomial random variable 'X' approaches that of a poisson random variable as $n \to \infty$, $p \to 0$ in such a way that $np = \lambda$ stays a constant.

$$b[x; n, p] = {}^nC_x\, p^x q^{n-x}, \quad x = 0, 1, 2 \dots n$$

* for $n > 25$, use poisson theorem.

Proof

$$b[x; n, p] = {}^nC_x\, p^x q^{n-x}$$

$$= \frac{n!}{(n-x)!\, x!} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{(n-x+1)(n-x+2)\dots(n-1)(n)}{x!} \frac{\lambda^x}{n^x} \left(1 - \frac{\lambda}{n}\right)^{n-x}$$

As $n \to \infty$,

$$= \lim_{n\to\infty} \frac{(n-x+1)\dots(n-1)(n)}{x! \times n^x} \cdot \lambda^x \cdot \left(1 - \frac{\lambda}{n}\right)^{n-x}$$

$$= \lim_{n\to\infty} \frac{\cancel{n^x}\left(1 - \frac{x-1}{n}\right)\dots\left(1 - \frac{1}{n}\right)(1)}{x!\ \cancel{n^x}} \cdot \lambda^x \cdot \left(1 - \frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{\lambda^x}{x!} \lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^n \underbrace{\left(1 - \frac{\lambda}{n}\right)^{-x}}_{1}$$

$$\lim_{n\to\infty} \left(1 - \frac{x}{n}\right)^n = e^{-x}$$

$$= \boxed{\frac{\lambda^x}{x!} \cdot e^{-\lambda}}, \quad x = 0, 1, 2 \dots$$

Q.1 Find the mean and variance of poisson distribution.

$\rightarrow$ $$p(x) = e^{-\lambda} \frac{\lambda^x}{x!}, \quad x = 0, 1, 2 \dots$$

$$\text{Mean, } M = \sum x\, p(x) = \sum_x x \cdot e^{-\lambda} \frac{\lambda^x}{x!}$$

$$= \sum_x e^{-\lambda} \frac{\lambda^x}{(x-1)!}$$

$$= e^{-\lambda} \cdot \lambda \cdot \sum_{x} \frac{\lambda^{x-1}}{(x-1)!}$$

$$= \lambda e^{-\lambda} \left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots \right]$$

$$\boxed{1 + \frac{x}{1} + \frac{x^2}{2!} + \cdots = e^x}$$

$$= \lambda e^{-\lambda} \cdot e^{\lambda}$$

Mean $= \lambda$

Variance, $\sigma^2 = E(x^2) - [E(x)]^2$

$$= \sum_{x} x^2 e^{-\lambda} \frac{\lambda^x}{x!} - \lambda^2$$

$$= e^{-\lambda} \sum_{x} [x(x-1)+x] \frac{\lambda^x}{x!} - \lambda^2$$

$$= e^{-\lambda} \left[\sum_{x} x(x-1) \frac{\lambda^x}{x!} + \sum_{x} x \cdot \frac{\lambda^x}{x!}\right] - \lambda^2$$

$$= e^{-\lambda} \left[\lambda^2 \underbrace{\sum_{x} \frac{\lambda^{x-2}}{(x-2)!}}_{e^{\lambda}} + \lambda \underbrace{\sum_{x} \frac{\lambda^{x-1}}{(x-1)!}}_{e^{\lambda}}\right] - \lambda^2$$

$$= \left[e^{-\lambda} \cdot e^{\lambda} (\lambda^2 + \lambda)\right] - \lambda^2$$

$$= \cancel{\lambda^2} + \lambda - \cancel{\lambda^2}$$

$$\sigma^2 = \lambda$$

Q. It is ~~note~~ known that 2% of the bolts produced by a company are defective. The bolts are supplied in boxes of two hundred bolts. What is the probability that a randomly chosen box contains not more than 5 defective bolts?

→ Given, $n = 200$, $p = 0.02$ (2%)

$$\lambda = np = 200 \times 0.02 = 4$$

$$p(x) = e^{-\lambda} \frac{\lambda^x}{x!}$$

$$= e^{-4} \frac{4^x}{x!}, \quad x = 0, 1, 2 \ldots$$

here, $P[X \leq 5] = P(0) + P(1) + \dots P(5)$

$$= e^{-4}\left[1 + \frac{4}{1!} + \frac{4^2}{2!} + \frac{4^3}{3!} + \frac{4^4}{4!} + \frac{4^5}{5!}\right]$$

$$= e^{-4} \times \frac{643}{15}$$

$$= \underline{\underline{0.785}}$$

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Q. If X follows poisson distribution such that the probability of $X = 1 = 3/10$ and $P[X=2] = 1/5$. Find $P[X=0]$ and $P[X=4]$

→ we know, $P(x) = e^{-\lambda}\frac{\lambda^x}{x!}$

$$P(1) = e^{-\lambda}\frac{\lambda}{1!} = \frac{3}{10} \quad (1) \qquad \text{and} \qquad P(2) = e^{-\lambda}\frac{\lambda^2}{2!} = \frac{1}{5} \quad (2)$$

ie, $\frac{(1)}{(2)} \Rightarrow \frac{\cancel{e^{-\lambda}}\,\cancel{\lambda}}{\cancel{e^{-\lambda}}\,\frac{\lambda^2}{2}} = \frac{3}{\cancel{10}_2} \times \frac{\cancel{5}}{1}$

$$\frac{2}{\lambda} = \frac{3}{2} \qquad \therefore \lambda = \frac{4}{3}$$

- $P(X=0) = e^{-\lambda}\frac{\lambda^0}{0!} = e^{-4/3} = \underline{\underline{0.263}}$
- $P(X=4) = e^{-\lambda}\frac{\lambda^4}{4!} = e^{-4/3}\frac{(4/3)^4}{4!} = \underline{\underline{0.034}}$

Joint pdf of two discrete random variables

Let X and Y be two discrete ~~ran~~ rVs.

Then the joint pdf of (X, Y) is

$$P[X = x, Y = y] = P(x, y)$$

Eg: Consider the experiment of rolling ~~t~~ dice.

X: sum of the numbers, $x = 2, 3 \dots 12$

Y: difference of the numbers, $y = 0, 1, 2, 3 \dots 5$

X \ Y	0	1	2	3	4	5	
2	1/36	0	0	0	0	0	(1,1)
3	0	2/36	0	0	0	0	
4	1/36	0	2/36	0	0	0	
5	0	2/36	0	2/36	0	0	(1,4) (2,3)...
6	1/36	0	2/36	0	2/36	0	(1,5) (2,4) (3,3)...
7	0	2/36	0	2/36	0	2/36	
8	1/36	0	2/36	0	2/36	0	
9	0	2/36	0	2/36	0	0	
10	1/36	0	2/36	0	0	0	
11	0	2/36	0	0	0	0	
12	1/36	0	0	0	0	0	

The total probability :

$$\boxed{\sum_X \sum_Y P(x,y) = 1}$$

Q. The joint probability of X and Y is given by $P(x,y) = cxy$, $x = 1,2,3$ and $y = 1,2,3$. Find value of c.

→ $$\sum_x \sum_y P(x,y) = 1$$

$$\sum_x \sum_y cxy = 1$$

ie, $c\,[1+2+3+2+4+6+3+6+9] = 1$

$$c = 1/36$$

$$\therefore \quad P(x,y) = \frac{xy}{36}$$

pdf :

X \ Y	1	2	3	f(x)
1	1/36	2/36	3/36	6/36
2	2/36	4/36	6/36	12/36
3	3/36	6/36	9/36	18/36
f(y)	6/36	12/36	18/36	

$P[X=1] = 6/36$

$P[X=2] = 12/36$

$P[X=3] = 18/36$

here, $f(x)$: marginal pdf of x
$f(y)$: marginal pdf of y

Marginal pdf of X, $\boxed{f_X(x) = \sum_{Y} P(x,y)}$

Marginal pdf of Y, $\boxed{f_Y(y) = \sum_{X} P(x,y)}$

Independence

Two random variables ~~of~~ X and y are said to be statistically independent if its joint pdf $P(x,y) = f_X(x) \cdot f_Y(y)$

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Q. A box contains 3 balls : ~~lt,~~ labeled 1, 2 and 3. Two balls are drawn from it without replacement in succession. Let X denote the number on 1^{st} ball and Y on the II^{nd} ball. Find
1) Joint pdf of (x, y)
2) $p[X \geq 2, Y \geq 2]$
3) Probability distribution of $Z = X + Y$
4) Marginal distributions.

→ 1)

X \ Y	1	2	3
1	0	1/6	1/6
2	1/6	0	1/6
3	1/6	1/6	0

2) $P[X \geq 2, Y \geq 2] = P(2,3) + P(3,2)$
$= 1/6 + 1/6 = 2/6$
$= 1/3$

3) $Z = X + Y$

Z	3	4	5
P(Z)	1/3	1/3	1/3

$P(3) = P(1,2) + P(2,1)$
$= 1/6 + 1/6 = 1/3$

4) $f_X(x)$

X	1	2	3
P(x)	1/3	1/3	1/3

$f_Y(y)$

Y	1	2	3
P(y)	1/3	1/3	1/3

Q. Joint pdf of (x,y) is given by $P(x,y) = kxy$ where $x = 1,2,3$ and $y = 1,2,3$.

Find (a) k (b) Are x and y independent

→ (a) $\sum_{x=1,2,3} \sum_{Y=1,2,3} kxy = 1$ (Total prob = 1)

$k[1+2+3+2+4+6+3+6+9] = 1$

$$k = 1/36$$

$\therefore P(x,y) = \frac{xy}{36}$

(b) Marginal pdfs

$$f_X(x) = \sum_{Y=1,2,3} \left(\frac{xy}{36}\right) = \frac{x}{36} + \frac{2x}{36} + \frac{3x}{36} = \frac{6x}{36} = \frac{x}{6}$$

$\boxed{f_X(x) = \frac{x}{6}}$, $x = 1,2,3$

$$f_Y(y) = \sum_{X=1,2,3} \left(\frac{xy}{36}\right) = \frac{y}{36} + \frac{2y}{36} + \frac{3y}{36} = \frac{6y}{36} = \frac{y}{6}$$

$\boxed{f_Y(y) = \frac{y}{6}}$, $y = 1,2,3$

X	1	2	3
$f_X(x)$	$1/6$	$1/3$	$1/2$

$$P(x,y) = \frac{xy}{36} = \frac{x}{6} \cdot \frac{y}{6}$$

$$= f_X(x) \cdot f_Y(y)$$

$\therefore X$ and Y are independent

Q. Joint pdf of (x,y) is given by $P(x,y) = k(x+y)$ where $x = 0,1,2$ and $y = 1,2,3$. Find k and check whethere x and y are independent or not.

→ (a) $\sum_{X=0,1,2} \sum_{Y=1,2,3} k(x+y) = 1$

$k[1+2+3+2+3+4+3+4+5] = 1$

$$k = 1/27$$

ⓑ $f_X(x) = \sum_{Y=1,2,3} \frac{(x+y)}{27} = \frac{x+1}{27} + \frac{x+2}{27} + \frac{x+3}{27}$

$$= \frac{3x+6}{27} = \frac{x+2}{9}$$

$f_Y(y) = \sum_{X=0,1,2} \frac{(x+y)}{27} = \frac{y}{27} + \frac{1+y}{27} + \frac{2+y}{27}$

$$= \frac{3y+3}{27} = \frac{y+1}{9}$$

$f_X(x) \cdot f_Y(y) = \frac{(x+2)}{9} \times \frac{(y+1)}{9} \neq P(x,y)$

$\therefore$ X and Y are not independent.

Expectation involving more than one r.v

$$\boxed{E(g(x,y)) = \sum_X \sum_Y g(x,y)\, p(x,y)}$$

- $g(x,y) = XY$

$E[XY] = \sum_x \sum_y xy\, p(x,y)$

- if X and Y are independent,

$P(x,y) = f_X(x) \cdot f_Y(y)$

$\therefore \quad E[XY] = \sum_x \sum_y xy\, f_X(x)\, f_Y(y)$

$$= \sum_X x \cdot f_X(x) \times \sum_Y y\, f_Y(y)$$

$$\{ E[XY] = E[X] \cdot E[Y] \}$$

Q. Joint pdf of (x,y) is $p(x,y) = \frac{x(x+y)}{70}$ where $x = 1,2,3$ and $y = 3,4$. Find $E(x)$ and $E(y)$

→ $E[x] = \sum_X x \cdot f_X(x)$

$E[Y] = \sum_Y y \cdot f_Y(y)$

- $f_X(x) = \sum_{Y=3,4} \frac{x(x+y)}{70} = \frac{x(x+3)}{70} + \frac{x(x+4)}{70}$

$$= \frac{2x^2 + 7x}{70}$$

$$f_Y(y) = \sum_{x=1,2,3} \frac{x(x+y)}{70} = \frac{1+y}{70} + \frac{4+2y}{70} + \frac{9+3y}{70}$$

$$= \frac{14+6y}{70}$$

- $$E[X] = \sum_{x=1,2,3} x \cdot f_X(x) = \sum_{x=1,2,3} x \cdot \frac{2x^2+7x}{70}$$

$$= \frac{2+7}{70} + \frac{2(8+14)}{70} + \frac{3(18+21)}{70}$$

$$= \frac{9+16+28+54+63}{70} = \frac{170}{70} = \frac{17}{7}$$

18 × 3 = 54, 63

- $$E[Y] = \sum_{y=\cancel{1,2},3,4} y \cdot f_Y(y) = \sum_{y=3,4} y \cdot \frac{14+6y}{70}$$

$$= \frac{14+18}{70} + \frac{14+24}{70} = \frac{32}{70} + \frac{38}{70} = \frac{70}{70} = 1$$

Q. Joint pdf of x and y given by p(x,y)

X \ Y	0	1	2	$f_X(x)$
0	1/6	1/4	1/8	13/24
1	1/8	1/6	1/6	11/24
$f_Y(y)$	7/24	10/24	7/24	

Find: ① Var (X)
② Var (Y)
③ E[XY]

① $Var(X) = E[X^2] - [E(X)]^2$

$$E[X^2] = \sum_x x^2 f_X(x) = \sum_{x=0,1} x^2 f_X(x)$$

$$= 0 \times \frac{13}{24} + 1^2 \times \frac{11}{24} = \frac{11}{24}$$

$$[E(X)]^2 = \left(\sum_{x=0,1} x \cdot f_X(x)\right)^2 = \left(0 \times \frac{13}{24} + 1 \times \frac{11}{24}\right)^2$$

$$= \frac{121}{576}$$

$$\therefore Var(X) = \frac{11}{24} - \frac{121}{576} = \frac{143}{576}$$

② $Var(Y) = E(Y^2) - [E(Y)]^2$

$$E(Y^2) = \sum_{y=1,2,3} y^2 \cdot f_Y(y) = 0 \times 7/24 + 1^2 \times 10/24 + 2^2 \times 7/24$$

$$= \frac{38}{24}$$

$$[E(Y)]^2 = \left(0 \times \frac{7}{24} + 1 \times \frac{10}{24} + 2 \times \frac{7}{24}\right)^2 = 1$$

$$\therefore \ Var(y) = E(Y^2) - (E[Y])^2 = \frac{38}{24} - 1$$

$$= \frac{14}{24} = \frac{7}{12}$$

5 $$E[XY] = \sum_{x=0,1} \sum_{y=1,2,0} xy\, P(x,y)$$

$$= 0 \times 0 \times 1/6 + 1 \times 0 \times 1/4 + 0 \times 1/8 + 0 + 1 \times 1/6 + 2 \times 1/6$$

$$= \frac{3}{6} = \frac{1}{2}$$