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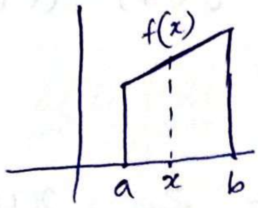
Module : 2

Continuous Probability Distribution

~~Mean~~

* Total probability = 1
* $\int_{-\infty}^{\infty} f(x) \cdot dx = 1$



* A random variable X is said to be continuous random variable if
  1. The set of possible values of x is an interval or a union of intervals.
  2. There is a f , $f : R \to R$ such that for any (a,b)
     $P[a < x < b] = \int_a^b f(x) \cdot dx$.

* <u>Mean</u> : $\mu = \int_{-\infty}^{\infty} x \cdot f(x) \cdot dx$

* <u>Expectation</u> : $E[g(x)] = \int_{-\infty}^{\infty} g(x) \cdot f(x)\, dx$

* <u>Variance</u> : $\sigma^2 = E[x^2] - (E(x))^2$
  $= \int_{-\infty}^{\infty} x^2 f(x) \cdot dx - \mu^2$

* <u>CDF</u> , $F(x) = P[X \leq x]$
  $= \int_{-\infty}^{\infty} f(x) dx$

  $F(a) = \int_{-\infty}^{a} f(x) \cdot dx$

Q.1 The diameter of an electric cable is a continuous random variable with density function pdf :

$$f(x) = \begin{cases} kx(1-x) & , 0 \leq x < 1 \\ 0 & , \text{otherwise} \end{cases}$$

① Find k ② $P[X \leq 1/2]$ ③ CDF ④ Mean ⑤ Variance.

$\rightarrow$ $\int_0^1 f(x) \cdot dx = 1$ $\Rightarrow$ $\int_0^1 kx(1-x)\, dx = k\int_0^1 (x - x^2)\, dx$

$$\Rightarrow k\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^1 = 1$$

$$k\left[\tfrac{1}{2}-\tfrac{1}{3}\right] = 1$$

$$\underline{\underline{k = 6}}$$

② $P(X \le \tfrac{1}{2}) = \int_0^{1/2} k(x)(1-x)\cdot dx$

$$= 6\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^{1/2} = 6\left[\frac{1}{8}-\frac{1}{24}\right] = \frac{6\times 2}{24}$$

$$= \frac{1}{2}$$

③ CDF : $F(x) = \begin{cases} 0 & , x<0 \\ \int_0^x f(x)\cdot dx & , 0<x<1 \\ 1 & , x \ge 1 \end{cases} = \begin{cases} 0 & , x<0 \\ 3x^2-2x^3 & , 0\le x \le 1 \\ 1 & , x\ge 1 \end{cases}$

④ $\mu = \int x f(x)\cdot dx = \int_0^1 x\cdot 6x(1-x)\cdot dx$

$$= 6\left[\frac{x^3}{3}-\frac{x^4}{4}\right]_0^1 = 6\left[\frac{1}{3}-\frac{1}{4}\right] = 2-\tfrac{3}{2}$$

$$= \frac{1}{2}$$

⑤ $\sigma^2 = \int_0^1 x^2 6x(1-x)\cdot dx - \mu^2 = 6\left[\frac{x^4}{4}-\frac{x^5}{5}\right]_0^1 - \frac{1}{4}$

$$= 6\left[\frac{1}{4}-\frac{1}{5}\right] - \frac{1}{4}$$

$$= \frac{6}{20}-\frac{1}{4} = \frac{1}{20}$$

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Q.2 Find the value of c so that $f(x) = cxe^{-x}$, $x \ge 0$ is a valid pdf.

→ Total probability = 1

ie, $\int_0^\infty cxe^{-x} = 1$

$$c\left[x\int e^{-x}dx - \int 1 \int e^{-x}\cdot dx\right] = 1$$

$$c\left[x\frac{e^{-x}}{(-1)} - \int\frac{e^{-x}}{(-1)}dx\right] = 1 \Rightarrow \left[c - xe^{-x} - e^{-x}\right]_0^1 = 1$$

$$\Rightarrow c\left[(0-0)-(0-1)\right] = 1 \Rightarrow \underline{\underline{c=1}} \qquad \therefore \boxed{f(x) = xe^{-x},\ x\ge 0}$$

3) Find the value b so that $f(x) = 2x$, $0 \le x \le b$
Also find the CDF, mean and variance.

→ ⓐ $\int_0^b 2x \cdot dx = 1 \Rightarrow 2\left[\frac{x^2}{2}\right]_0^b = 1 \Rightarrow [x^2]_0^b = 1$

$b^2 = 1 \Rightarrow b = 1$ becoz 'b' can't be -ve.

ⓑ CDF

$$F(x) = \begin{cases} 0 & , x < 0 \\ \int_0^x 2x \cdot dx & , 0 \le x \le 1 \\ 1 & , x > 1 \end{cases} = \begin{cases} 0 & , x < 0 \\ x^2 & , 0 \le x \le 1 \\ 1 & , x > 1 \end{cases}$$

ⓒ Mean

$$\mu = \int_0^1 x \cdot 2x \, dx = 2\left[\frac{x^3}{3}\right]_0^1 = \frac{2}{3}$$

ⓓ Variance

$$\sigma^2 = \int_0^1 2x^3 \cdot dx - \mu^2 = 2\left[\frac{x^4}{4}\right]_0^1 - \left(\frac{2}{3}\right)^2$$

$$= \cancel{\frac{2}{3}} \cancel{2} \frac{1}{2} - \frac{4}{9} = \frac{1}{18}$$

Uniform Probability Distribution

Continuous r.v X, $x \in [a, b]$

$f(x) = k$, $a \le x \le b$

- Total probability = 1

$\int_a^b k \cdot dx = 1$

$k \int_a^b 1 \cdot dx = 1 \Rightarrow k\,[x]_a^b = 1$

$k\,[b-a] = 1 \Rightarrow k = \frac{1}{b-a}$

$\therefore f(x) = \frac{1}{b-a}$, $a \le x \le b$

- Mean, $\mu = \int_a^b x \frac{1}{b-a} \cdot dx = \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b$

$= \frac{1}{2(b-a)} [x^2]_a^b = \frac{b^2 - a^2}{2(b-a)} = \frac{\cancel{(b-a)}(b+a)}{2\cancel{(b-a)}}$

$\mu = \frac{b+a}{2}$

- Variance, $\sigma^2 = E(x^2) - \mu^2$

$$= \int_a^b x^2 \frac{1}{b-a}\, dx - \frac{(a+b)^2}{4}$$

$$= \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b - \frac{(a+b)^2}{4}$$

$$= \frac{b^3-a^3}{3(b-a)} - \frac{a^2+2ab+b^2}{4}$$

$$= \frac{\cancel{(b-a)}(a^2+ab+b^2)}{3\cancel{(b-a)}} - \frac{a^2+2ab+b^2}{4}$$

$$= \frac{a^2-2ab+b^2}{12}$$

$$= \frac{(a-b)^2}{12}$$

Q. X is uniformly distributed with $\mu = 1$ and $\sigma^2 = 4/3$. Find pdf of X.

→ Given, $\frac{b+a}{2} = 1$ and $\frac{(a-b)^2}{12} = \frac{4}{3}$

$a+b = 2$ $\quad (a-b)^2 = 16$

$a+b = 2$ $\quad a-b = \pm 4$

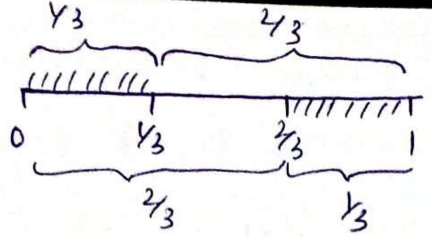
$a+b = 2$ -- ① $\quad a-b = -4$ -- ② $\Big\}$ $a<b$, $(a-b) = -ve$

① + ② ⇒ $2a = -2$, $a = -1$ so, $b = 2-3$... $b = 3$

ie, $f(x) = \frac{1}{4}$, $-1 \le x \le 3$

Q. A string 1m long is cut into two pieces at a random point b/w its ends. What is the probability that the length of one piece is atleast twice the length of the other?

→ $X \sim U(0,1)$, $f(x) = 1$, $0 < x < 1$



$x \in (0, 1/3) \cup (2/3, 1)$
$\downarrow$
given condition

Required probability $= P[x \in (0, 1/3) \cup (2/3, 1)]$

$$= \int_0^{1/3} 1 \cdot dx + \int_{2/3}^{1} 1 \cdot dx$$

$$= (x]_0^{1/3} + [x]_{2/3}^{1} = \frac{1}{3} + [1 - \frac{2}{3})$$

$$= \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

Q. A random variable x has uniform distribution over the interval $(-3, 3)$. Find $P[|x-2| < 2]$.

$\rightarrow$ $X \sim U(-3, 3)$

$$f(x) = \frac{1}{3-(-3)} = \frac{1}{6}, \quad -3 < x < 3$$

$$P[|x-2| < 2] = P[-2 < x - 2 < 2]$$

$$= P[0 < x < 4]$$

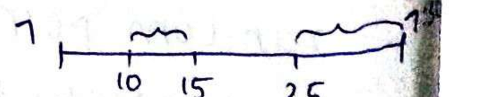
$$= \int_0^3 \frac{1}{6} \cdot dx = \frac{1}{6} \times (3)$$

$$= \frac{1}{2}$$

Q. Buses arrives at a specific stop at ~~random~~ 15 min interval, starting at 7am. If a passenger arrives at the stop at a random ~~tym~~ time, ie, uniformly distributed b/w 7 & 7:30 am. Find the probability that he waits

a) Less than 5 mins b) Atleast 6 mins

ans:- a) $P[10 < x < 15] + P[25 < x < 30]$



$$f(x) = \frac{1}{30}, \quad 0 < x < 30$$

ie, $$\int_{10}^{15} \frac{1}{30} dx + \int_{25}^{30} \frac{1}{30} dx = \frac{5}{30} + \frac{5}{30} = \frac{1}{3}$$

b) $P[0<x<9] + P[15<x<24] = \frac{9}{30} + \frac{9}{30}$

$= \frac{3}{5}$ //

1) $P[X<20] = 1-e^{-20/50} = 1-e^{-2/5} = 0.\underline{3297}$

2) $P[X \geq 60] = e^{-60/50} = e^{-6/5} = 0.\underline{3011}$

Q) Suppose a new machine is put into operation at time 0. It's life time is exp. r.v with mean life 12 hrs. If the machine has not failed by the end of the first day. What is probability that it will work of the whole of not day.

→ $\mu = \beta = 12$

$f(x) = \frac{1}{12} e^{-x/12}$, $x>0$

$P(x>48 / x>24) = P(x>24+24 / x>24)$

$= P[x>24] = e^{-24/12} = e^{-2}$

$= 0.\underline{1353}$

Exponential Distribution

A continuous random variable x is said to be follow exponential distribution with parameter β if,

$$f(x) = \frac{1}{\beta} e^{-x/\beta} , \quad x>0 \qquad \lambda = \frac{-1}{\beta}$$

- CDF

$F[x] = P[X \leq x]$

$= \int_0^x \frac{1}{\beta} e^{-x/\beta} . dx = \frac{1}{\beta} \left[\frac{e^{-x/\beta}}{-1/\beta} \right]_0^x$

$\begin{cases} F[x] = 0 \\ \text{if } x \leq 0 \end{cases}$

$= -\left[e^{-x/\beta}\right]_0^x = -\left[e^{-x/\beta} - 1\right]$

$$\boxed{CDF = 1 - e^{-x/\beta}}$$

For a point $a>0$,

$P[X \le a] = 1 - e^{-a/\beta}$

$P[X > a] = e^{-a/\beta}$

- Mean

$$\mu = \int_{-\infty}^{\infty} x \cdot f(x) \cdot dx = \int_{0}^{\infty} x \cdot \frac{1}{\beta} e^{-x/\beta} dx$$

$$= \frac{1}{\beta}\left[x \int e^{-x/\beta} dx - \int 1 \int e^{-x/\beta} dx\, dx\right]$$

$$= \frac{1}{\beta}\left[x \frac{e^{-x/\beta}}{-1/\beta} - \frac{e^{-x/\beta}}{(-1/\beta)^2}\right]_0^\infty$$

$$= \frac{1}{\beta}\left[(0-0) - (0-\beta^2)\right]$$

ie $\boxed{\mu = \beta}$

- Variance

$$\sigma^2 = E[x^2] - (E[x])^2$$

$$= \int_0^\infty x^2 \frac{1}{\beta} e^{-x/\beta} dx - \beta^2$$

$$= \frac{1}{\beta}\left[x^2 \frac{e^{-x/\beta}}{-1/\beta} - 2x \frac{e^{-x/\beta}}{(-1/\beta)^2} + 2 \frac{e^{-x/\beta}}{(-1/\beta)^3} - 0\right]_0^\infty - \beta^2$$

$$\sigma^2 = \frac{1}{\beta}\left[(0-0-0) - (0-0+2(-\beta)^3 \cdot 1)\right] - \beta^2$$

$$= \frac{1}{\cancel{\beta}} \times 2\beta^{\cancel{3}\,2} - \beta^2 = 2\beta^2 - \beta^2$$

$$= \beta^2(2-1)$$

ie, $\boxed{\sigma^2 = \beta^2}$

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$$f(x) = \frac{1}{\beta} e^{-x/\beta}, \quad x > 0$$

- $CDF = 1 - e^{-x/\beta}$
- Mean, $\mu = \beta$
- Variance, $\sigma^2 = \beta^2$

<u>Memoryless Property</u>

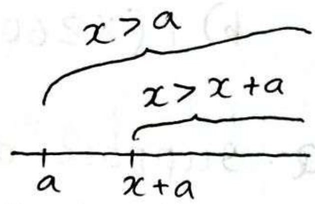
Let X be an exponential r.v with parameter β. Then probability of $P[X > x+a / x > a] = P[X > x]$

<u>Proof</u>

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$P[X > x+a / x > a] = \frac{P(X > (x+a) \cap (x > a))}{P(x > a)}$$

$$= \frac{P[X > x+a]}{P[X > a]}$$



$$= \frac{e^{-(x+a)/\beta}}{e^{-a/\beta}}$$

$$= \frac{e^{-x/\beta} \cdot \cancel{e^{-a/\beta}}}{\cancel{e^{-a/\beta}}} = e^{-x/\beta}$$

$$= P[X > x]$$

Q If the distance that a car can run bebore its battery wears out is exponentially distributed with an average value 5000 kms. If the owner decides to take a 2000 km ~~trata~~ trip, what is the probability that he will be able to complete the trip without having to replace the car battery.

→ Given, $\mu = \beta = 5000$

$$f(x) = \frac{1}{5000} e^{-x/5000}, \quad x > 0$$

then, $P[X > 2000] = e^{\frac{-2000}{5000}} = e^{-2/5}$

$$= \underline{0.6703}$$

Q. Amount of time that a surveilance camera will run without having to be reset is a r.v having exponential distribution with mean 50 days. Find the probability that such a camera will a) have to be reset in less than 20 days b) not have to be reset ~~a~~ in atleast 60 days.

→ $\beta = 50$

$$f(x) = \frac{1}{50} e^{-x/50}, \quad x > 0$$

a) $P[X < 20] = 1 - e^{-20/50} = 1 - e^{-2/5} = 1 - \underline{0.6703}$

$$= \underline{0.3297}$$

b) $P[X \geq 60] = e^{-60/50} = e^{-6/5} = \underline{0.3011}$

Q. Suppose a new machine is put into operation at time 0. Its life time is an exponential r.v. with mean life 12 hrs. If the machine has not failed ~~at~~ by the end of the first day, what is the probability that it will work for the whole of the next day?

→ $\beta = 12$

$$f(x) = \frac{1}{12} e^{-x/12}, \quad x > 0$$

$P[X > 48 / X > 24] = P[X > 24 + 24 / X > 24]$

$= P[X > 24]$ (memoryless property)

$= e^{-24/12} = e^{-2}$

$= \underline{0.1353}$

Uniform Distribution

The normal r.v or gaussian r.v X is a continuous r.v with pdf $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$, $-\infty < x < \infty$.

Where the constants μ and σ are the parameters of normal distribution.

- when $\mu = 0$ and $\sigma = 1$, then the distribution becomes the standard normal distibution and the random variable of standard normal distribution is denoted by Z and

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2\sigma^2}, \quad -\infty \leq z < \infty$$

- For a normal distribution with parameters μ and σ, Mean $= \mu$ and Variance $= \sigma^2$
- The graph of normal distribution is a bell shaped curve and is perfectly symmetric w.r.t μ.

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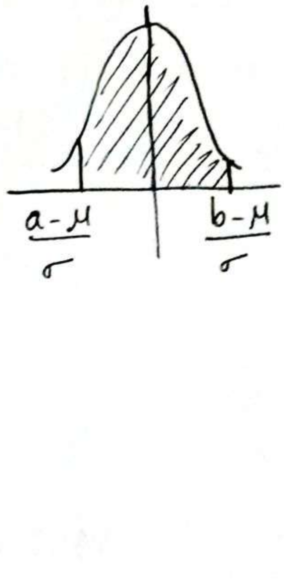
- If $x \sim N(\mu, \sigma^2)$, then $z = \frac{x-\mu}{\sigma} \sim N(0,1)$
- $P[a \leq x \leq b] = P\left[\frac{a-\mu}{\sigma} \leq z \leq \frac{b-\mu}{\sigma}\right]$

$$= F\left[\frac{b-\mu}{\sigma}\right] - F\left[\frac{a-\mu}{\sigma}\right]$$



Q. ~~Find~~ $X \sim (-3, 2^2)$. Find

1) $P[1 \leq x \leq 2]$

here, $z = \frac{x-\mu}{\sigma} = \frac{x-(-3)}{2} = \frac{x+3}{2}$

then, $P(1 \leq x \leq 2) = P\left[\frac{1+3}{2} \leq z \leq \frac{2+3}{2}\right]$

$$= P[2 \leq z \leq 2.5]$$

$$= F[2.5] - F[2]$$

$= .99379 - .97725$

$= 0.01654$

2) $P[x \geq -1.5] = P[z \geq \frac{-1.5+3}{2}]$

$= P[z \geq 0.75]$

$= [1 - P[z < 0.75]$

$= 1 - P[0.75] = 1 - 0.77337$

$= .22663$

3) $P[x < -3] = P[z < \frac{-3+3}{2}]$

$= P[0] = 0.5$

4) $P[|x+3| < 2] = P[-2 < x+3 < 2]$

$= P[\frac{-2}{2} < z < \frac{2}{2}]$

$= P(-1 < z < 1]$

$= F[1] - F[-1] = .84134 - .15866$

$= 0.68268$

5) $P[|x+5| > 1] = 1 - P[|x+5| \leq 1]$

$= 1 - P[-1 \leq x+5 \leq 1]$

$= 1 - P[\frac{-1-2}{2} \leq z \leq \frac{1-2}{2}]$

$= 1 - P[-1.5 \leq z \leq -0.5]$

$= 1 - (F[-0.5] - F(-1.5])$

$= 1 - [.30854 - .06681]$

$= 1 - 0.24173$

$= .75827$

Q. Suppose that the lifetimes of light bulbs produced by a company are normally distributed with mean 1000 hrs and standard deviation 100 hrs. Is this company correct when it claims that 95%

of its light bulbs last atleast 900 hrs?

→ $X \sim N(1000, 100^2)$

$Z = \frac{x - 1000}{100} \sim N(0, 1)$

claim to check : $P[X \geq 900] = 0.95$

$P[X \geq 900] = 1 - P(X < 900)$

$= 1 - P[Z < \frac{900 - 1000}{100}] = 1 - P[Z < -\frac{100}{100}]$

$= 1 - P[-1]$

$= 1 - .15866$

$= \underline{0.84134}$

ie, 84% hence claim is wrong

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Q. X is a r.v with mean 50 and standard deviation 10. Find α and β such that $P[X < \alpha] = 0.10$ and $P(X > \beta] = 0.05$.

→ Here, $X \sim N(50, 10^2)$

$Z = \frac{x - 50}{10}$

• $P(X < \alpha] = 0.10 \Rightarrow P[Z < \frac{\alpha - 50}{10}] = 0.10$

from table, $P[Z < -1.28] = 0.10$

ie, $\frac{\alpha - 50}{10} = -1.28$

$\alpha = -1.28 \times 10 + 50$

$\alpha = \underline{37.2}$

• $P(X > \beta] = 0.05 \Rightarrow 1 - P[Z \leq \frac{\beta - 50}{10}] = 0.05$

from table, $P[Z \leq 1.645] = 0.95$

ie, $\frac{\beta - 50}{10} = 1.645$

$B = 1.645 \times 10 + 50$
$= \underline{66.45}$

Q. The marks obtained by a batch of students in mathematics is normally distributed with mean 65 and S.D 5. What is the probability that the score of a student is above 70?

→ $X \sim N(65, 5^2)$, $z = \frac{x-65}{5}$

To find $P[x \geq 70]$

ie, $= 1 - P[z < 70] = 1 - P[z < 5/5]$

$= 1 - P[z < 1]$

$= 1 - 0.84134$

$= \underline{0.15866}$

Q. The weekly wages of 1000 workmen are normally distributed about a mean of ₹ 500 and standard deviation 50. Estimate the number of workers whose weekly wages will be
(a) b/w 400 and 600 (b) less than 400
(c) more than 600.

→ $X \sim N(500, 50^2)$, $z = \frac{x-500}{50}$

(a) $P[400 < x < 600] = P\left[\frac{400-500}{50} < z < \frac{600-500}{50}\right]$

$= P[-2 < z < 2] = P[z < 2] - P[z \leq -2]$

$= .97725 - .02275$

$= \underline{0.9545}$

(b) $P[x < 400] = P\left[z < \frac{400-500}{50}\right]$

$= P[z < -2]$

$= 0.02275$

no. of workmen whose weekly wage b/w 400 and 600 is $\Big\} = 0.9545 \times 1000$

$= 954.5$

$\simeq \underline{955}$

no. of workmen whose daily wage is less than 400 is $\Big\} = 0.02275 \times 1000$

$= 22.75$

$\approx \underline{23}$

ⓒ $P[x > 600] = 1 - P[x \le 600]$

$= 1 - P[z \le 2]$

$= 1 - 0.97725 = 0.02275$

no. of workmen $= 0.02275 \times 1000 = 22.75$

$\simeq \underline{23}$

Q. The marks obtained by a batch of students in a certain subject are normally distributed. 10% of students got less than 45 marks while 5% got more than 75. Find the % of students which score b/w 45 and 60.

→ $X \sim N(\mu, \sigma^2)$, $\mu = ?$ $\sigma^2 = ?$

- $P[X < 45] = 0.1$

$P[z < \frac{45 - \mu}{\sigma}] = 0.1$

ie, $P[z < -1.28] = 0.1$

$\therefore \frac{45 - \mu}{\sigma} = -1.28$

$45 - \mu = -1.28\sigma$

ie, $\mu - 1.28\sigma = 45$ -- ①

- $P(X > 75) = 0.05$

$1 - P(X \le 75] = 0.05$

$P(X \le 75 = 0.95$

ie, $P[z \le \frac{75 - \mu}{\sigma}] = 0.95$

$\therefore \frac{75 - \mu}{\sigma} = 1.645$

$75 - \mu = 1.645\sigma$

$\mu - 1.645\sigma = 75$ .. ②

from ① and ② $\Rightarrow \mu = 58.128$, $\sigma = 10.256$

then, $P(45 < x < 60) = P(\frac{45 - 58.128}{10.256} < z < \frac{60 - 58.128}{10.256}]$

$= P(-1.28 < z < 0.1825] = P[0.1825] - P[z \le -1.28]$

$= 0.47115 = \underline{45\%}$

Joint pdf

- $f(x,y)$ = function of x & y

X, Y are continuous r.v

Marginal pdf

- $f_X(x) = \int_Y f(x,y)\cdot dy$

- $f_Y(y) = \int_X f(x,y)\cdot dx$

Independence

- X and Y are independent

then, $f(x,y) = f_x \cdot f_y$

Q. The joint pdf of two continuous r.v X, Y is given by $f(x,y) = kxy$, $0<x<4$ and $1<y\leq 5$. Find

1) k

2) $P[x \geq 3, y \leq 4]$

3) $P[1<x<2, 2<y<3]$

4) Marginal pdf, check independency

5) $P[X+Y<3]$

ans : 1) Total probability = 1

$$\int_0^4 \int_1^5 kxy \, dy\, dx = 1$$

$$k \int_1^5 y\,dy \left[\frac{x^2}{2}\right]_0^4 = 1 \Rightarrow k\left[\frac{y^2}{2}\right]_1^5 \left[\frac{x^2}{2}\right]_0^4 = 1$$

$$k\left[\frac{\cancel{24}^{12}}{\cancel{2}}\right][8] = 1$$

$$\therefore \; k = \frac{1}{96}$$

ie, $f(x,y) = \dfrac{xy}{96}$, $0<x<4$, $1<y<5$

2) $P[x \geq 3, y \leq 4] = \cancel{\int_{y=4}^{4}\int_{x=3}^{4}} \int_{y=1}^{4}\int_{x=3}^{4} \dfrac{xy}{96}\cdot dx\,dy$

$$= \frac{1}{96}\int_1^4 y \cdot dy \cdot \left[\frac{x^2}{2}\right]_3^4 = \frac{1}{96}\left[\frac{y^2}{2}\right]_1^4 \left[\frac{x^2}{2}\right]_3^4$$

$$= \frac{1}{96} \times \left(8 - \frac{1}{2}\right) \times \left(8 - \frac{9}{2}\right) = \frac{1}{96} \times \frac{15}{2} \times \frac{7}{2}$$

$$= \frac{35}{128}$$

3) $$P[1 < x < 2,\ 2 < y < 3] = \int_{x=1}^{2}\int_{y=2}^{3} \frac{xy}{96} \cdot dx\, dy$$

$$= \frac{1}{96}\left[\frac{x^2}{2}\right]_1^2 \left[\frac{y^2}{2}\right]_2^3 = \frac{1}{96}\left(2 - \frac{1}{2}\right)\left(\frac{9}{2} - 2\right)$$

$$= \frac{1}{96} \times \frac{3}{2} \times \frac{5}{2} = \frac{5}{128}$$

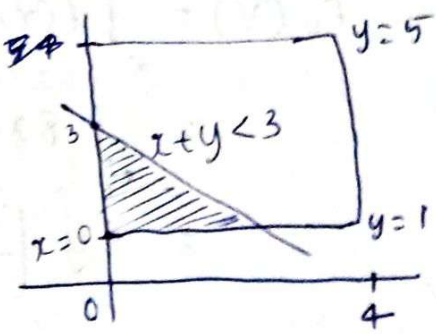
4) $$f_x = \int_1^5 \frac{xy}{96}\, dy = \frac{x}{96}\left[\frac{y^2}{2}\right]_1^5 = \frac{x}{96 \times 2} \times 24 = \frac{x}{8}$$

$$f_y = \int_0^4 \frac{xy}{96}\, dx = \frac{y}{96}\left[\frac{x^2}{2}\right]_0^4 = \frac{y}{96} \times 8 = \frac{y}{12}$$

here, $\frac{x}{8} \times \frac{y}{12} = \frac{xy}{96}$

ie, $f_x \cdot f_y = f(x,y)$. hence independent

5) $$P(X + Y < 3] = \int_{y=1}^{3}\int_{x=0}^{x=3-y} \frac{xy}{96}\, dx\, dy$$



$$\cancel{= \frac{1}{96} \times \frac{1}{2}\int_1^3 (9y - 6y^2 + y^3)\, dy}$$

$$= \frac{1}{96}\int_1^3 y\left[\frac{x^2}{2}\right]_0^{3-y} dy$$

$$= \frac{1}{96 \times 2}\int_1^3 y\left((3-y)^2 - 0\right) dy = \frac{1}{96 \times 2}\int_1^3 (9y - 6y^2 + y^3)\, dy$$

$$= \frac{1}{96 \times 2}\left[\frac{9y^2}{2} - \frac{6y^3}{3} + \frac{y^4}{4}\right]_1^3$$

$$= \frac{1}{96 \times 2}\left[\frac{9}{2}(9-1) - 2(27-1) + \frac{1}{4}(81-1)\right]$$

$$= \frac{1}{96 \times 2}\left[36 - 52 + 20\right] = \frac{4}{192} = \frac{1}{48}$$

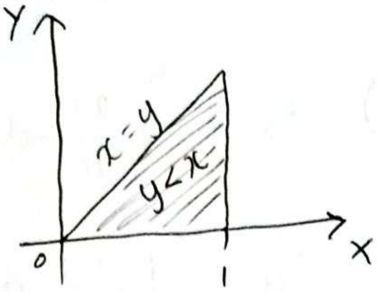
5/8 <u>Expectation</u>

$$E[g(x,y)] = \int_X \int_Y g(x,y) f(x,y)\, dy\, dx$$

$$g(x,y) = xy$$

$$E[X,Y] = \int_X \int_Y xy\, f(x,y)\, dy\, dx$$

Q. The joint pdf of two continuous r.v is x and y is $f(x,y) = 8xy$, $0<y<x<1$. Check whether x and y are independent



ans: Marginal pdf:

$$f_X(X) = \int_Y f(x,y)\cdot dy$$

$$= \int_0^x 8xy\cdot dy$$

$$= 8x\left[\frac{y^2}{2}\right]_0^x = 4x \times x^2$$

$$= \underline{\underline{4x^3}}, \quad 0<x<1$$

$$f_Y(Y) = \int_X f(x,y)\cdot dx$$

$$= \int_y^1 8xy\cdot dx$$

$$= 8y\left[\frac{x^2}{2}\right]_y^1 = \frac{8y}{2}(1-y^2)$$

$$\cancel{= 4y^3, \ 0<y<1}$$

$$= 4y\,\underline{\underline{[1-y^2]}}, \quad 0<y<1$$

Q) R.V x and y have a joint pdf given by $f(x,y) = x+y$, $0<x<1$, $0<y<1$. Find

a) The marginal pdfs b) $E[XY]$

c) Variance of X & Y.

ans ÷ a) $f_X(x) = \int_0^1 (x+y)\,dy = \int_0^1 x\,dy + \int_0^1 y\,dy$

$= \left[xy + \frac{y^2}{2}\right]_{0=y}^{1} = \underline{\underline{x + \frac{1}{2}}}, \quad 0 < x < 1$

$f_Y(y) = \int_0^1 (x+y)\,dx = \left[\frac{x^2}{2} + yx\right]_{0=x}^{1}$

$= \underline{\underline{y + \frac{1}{2}}}, \quad 0 < \cancel{x}\, y < 1$

b) $E[XY] = \int_0^1\int_0^1 xy(x+y)\,dx\,dy$

$= \int_0^1 y\left[\frac{x^3}{3} + \frac{x^2 y}{2}\right]_0^1 dy = \int_0^1 y\left(\frac{1}{3} + \frac{y}{2}\right) dy$

$= \left(\frac{y^2}{6} + \frac{y^3}{6}\right)_0^1 = \underline{\underline{\frac{1}{3}}}$

c) $\sigma_X^2 = E[X^2] - (E[X])^2$

$E[X^2] = \int_X x^2 f_X(x)\,dx = \int_0^1 x^2\left(x + \frac{1}{2}\right) dx$

$= \left[\frac{x^4}{4} + \frac{x^3}{6}\right]_0^1 = \frac{1}{4} + \frac{1}{6} = \frac{5}{12}$ //

$E[X] = \int_X x f_X(x)\,dx = \int_0^1 x\left(x + \frac{1}{2}\right) dx$ $\frac{11}{144}$

$= \left[\frac{x^3}{3} + \frac{x^2}{4}\right]_0^1 = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$

$\sigma_X^2 = E[X^2] - (E[X])^2 = \frac{5}{12} - \left(\frac{7}{12}\right)^2 = \frac{11}{144}$