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MODULE : IV

MARKOV PROCESS

Markov chain

- Markov process is a random process in which ~~depends in wh~~ the value of the rp depends only upon the most recent previous value and is independent of all the other values.

ie, a random process $\{X(t), t \geq 0\}$ is called a Markov process if

$$P[X(t_{n+1}) = x_{n+1} / X(t_n) = x_n, X(t_{n-1}) = x_{n-1} \dots, X(t_0) = x_0]$$
$$= P[X(t_{n+1}) = x_{n+1} / X(t_n) = x_n]$$

where $t_0, t_1 \dots t_{n-1}, t_{n+1}$ are known as states of the process

- A discrete state Markov process is called Markov's chain

- The conditional probability : $P[X_{n+1} = j / X_n = i]$ is called one step transition probability from state i to state j and is denoted by P_{ij}

- The one step transition probability is independent of n ie,

$P[X_{n+1} = j / X_n = i] = P[X_{m+1} = j / X_m = i]$ then the ~~Markov~~ process is known as homogeneous Markov chain.

- When Markov chain is homogeneous, the one step transition probability ~~matrix~~ is denoted by P_{ij} ~~TPM~~ and transition probability matrix is denoted by

$P = [P_{ij}]$ ~~is~~

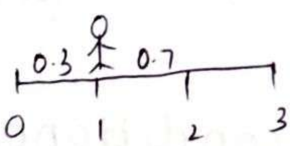
- Also, $P[X_n = j / X_0 = i]$

<u>n-step transition</u> ~~process~~ probability : probability that the process is in state j at step n given that it was in state i at step 0. Denoted by " $P_{ij}^{(n)}$ "

- In a TPM, sum of the values in each row is equal to 1.

Eg: Assume that a man is at an integral part of the x-axis b/w the origin and the point $x=3$. He takes a unit step either to the right with $p = 0.7$ or to the left with $p = 0.3$ unless he is at the origin when he takes a step to the right to reach $x=1$ or he is at the point ~~x=~~ 3 when he takes a step to the left to reach $x=2$.

$$\begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \;\; X_{n+1} \\ P = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0.3 & 0 & 0.7 & 0 \\ 0 & 0.3 & 0 & 0.7 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\ X_n & \end{array}$$



Q.1) A housewife buys three kinds of cereals A, B and C. She never buys the same cereal in successive weeks. If she buys A, the next week she buys B. However if she buys B or C, the next week she is three times as likely to buy A. Find TPM.

$$\rightarrow \begin{array}{cc} & \begin{array}{ccc} A & B & C \end{array} \\ P = \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 3/4 & 0 & 1/4 \\ 3/4 & 1/4 & 0 \end{bmatrix} \end{array}$$

A	C
$3x$	x

$3x + x = 1$

$x = 1/4$

2) A man tosses a fair coin until 3 heads occur in a row. Let X_n be the longest string of heads ending at the nth trial. Find ~~Ivt~~ TPM.

→ $$P = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0.5 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

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3) A fair die is tossed repeatedly. If X_n denotes the max no. of occuring in 1st n tosses, then find TPM

ans:

$$P = \begin{array}{c c} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{bmatrix} 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 2/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 3/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 4/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 0 & 5/6 & 1/6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

4. A msg transmission system is found to be Markovian with a transition probability of current msg to next msg is given by TPM

$$P = \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.1 & 0.2 & 0.7 \\ 0.6 & 0.3 & 0.1 \end{bmatrix}, \quad P(0) = [0.4 \quad 0.3 \quad 0.3].$$

Find the probabilities of the next msg.

ans: $P(0) = [0.29 \quad 0.27 \quad 0.44]$

5. The TPM of a Markov chain X_n having 3 states 1, 2 and 3 is

$$P = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}, \quad P(0) = [0.7 \quad 0.2 \quad 0.1]$$

Find a) $P[X_2 = 3]$ b) $P[X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2]$

ans: $P(0)P = [P[X_1=1] \quad P[X_1=2] \quad P[X_1=3]]$

$P(0)P \cdot P = P(0) \cdot P^2$

$= [P[X_2=1] \quad P[X_2=2] \quad P[X_2=3]]$

0.7+
0.12
0.3
1.12.28
.4
.3
.98

here, ~~$P(0)P = P[X_1=3] = 0.$~~

$P(0)P =$

$P(0)P^2 = [0.385 \quad 0.336 \quad 0.279]$

a) $P[X_2 = 3] = 0.279$

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$= P[X_3 = 2 / X_2 = 3, \underbrace{X_1 = 3, X_0 = 2}_{\times}] \times P[X_2 = 3, X_1 = 3, X_0 = 2]$

becoz $P(A \cap B) = P(A/B) \times P(B)$

By memoryless property:

$= P[X_3 = 2/X_2 = 3] \times P[X_2 = 3 / X_1 = 3, \underbrace{X_0 = 2}_{\times}] \times P[X_1 = 3, X_0 = 2]$

$= P_{32} \times P[X_2 = 3/X_1 = 3] \times P[X_1 = 3/X_0 = 2] \times P[X_0 = 2]$

$= P_{32} \times P_{33} \times P_{23} \times P_2(0)$

$= 0.4 \times 0.3 \times 0.2 \times 0.2$

$= \underline{0.0048}$

6. The TPM of a Markov chain having three states 1, 2, 3 is given by $P = \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.1 & 0.6 & 0.3 \\ 0.4 & 0.3 & 0.3 \end{bmatrix}$

and the initial pd is $P(0) = [0.5 \quad 0.3 \quad 0.2]$.

Find a) $P[X_2 = 2]$

b) $P[X_3 = 3, X_2 = 2, X_1 = 1, X_0 = 3]$

$0.1 + 0.03$ 0.15
0.08 0.18
21 0.06
39

→ a) $P(0) \cdot P = [0.21 \quad 0.39 \quad \cancel{0.38} \; 0.4]$

$\cancel{P(0) \cdot P^2 = 0.584 \quad 0.411 \quad 0.336}$

$P(0) \cdot P^2 = [\quad 0.419 \quad]$

$\therefore P[X_2 = 2] = \underline{\underline{0.4197}}$

b) $P[X_3 = 3, X_2 = 2, X_1 = 1, X_0 = 3] = P[P_{23} \times P_{12} \times P_{31} \times P[X_0 = 3]]$

$= 0.3 \times 0.3 \times 0.4 \times 0.2$

$= \underline{\underline{0.0072}}$

Long Run Proportions

Let X be P be a homogeneous Markov Chain with TPM T. If there exist a probability matrix π,

$\boxed{\pi P = \pi}$

then π is called a stationary distribution or a steady state distribution.

→ Any Markov chain P for which P^n has all ~~the~~ +ve entries for some +ve integer n is called a Regular matrix.

→ A Markov chain is said to be regular if some power of it's transition matrix has all positive entries.

Q. A housewife buys 3 kinds of cereals in successive weeks. If she buys A, then she buys C in the next week. If she buys B or C, then the next week she is 3 times as likely to buy A.

In the long run, how often she buys each?

$$P = \begin{array}{c|c} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 3/4 & 0 & 1/4 \\ 3/4 & 1/4 & 0 \end{bmatrix} \end{array}$$

$$\pi = [\pi_1 \quad \pi_2 \quad \pi_3]$$

$$\pi P = \pi$$

$$\Rightarrow 0\pi_1 + \tfrac{3}{2}\pi_2 + \tfrac{3}{4}\pi_3 = \pi_1$$

$$\pi_1 + 0\pi_2 + \tfrac{1}{4}\pi_3 = \pi_2$$

$$0\pi_1 + \tfrac{1}{4}\pi_2 + 0\pi_3 = \pi_3$$

$$\Rightarrow \quad \pi_1 - \tfrac{3}{4}\pi_2 - \tfrac{3}{4}\pi_3 = 0$$

$$\pi_1 - \pi_2 + \tfrac{1}{4}\pi_3 = 0$$

$$\tfrac{1}{4}\pi_2 - \pi_3 = 0$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 4 & -3 & -3 & 0 \\ 4 & -4 & 1 & 0 \\ 0 & 1 & -4 & 0 \end{array}\right]$$

$\pi_1 = 15/35$

$\pi_2 = 16/35$

$\pi_3 = 4/35$