- $$\underset{\tilde{x} \in \mathbf{R}^2}{\text{minimize}} \quad \max_k f_k(\tilde{x})$$

  where $f_k(\tilde{x}) = \frac{1}{2}(\tilde{x} - y_k)^T P_k (\tilde{x} - y_k) + r_k$ with

  $P_1 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},\ P_2 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix},\ P_3 = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix},\ y_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ y_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix},\ y_1 = \begin{bmatrix} -1 \\ -2 \end{bmatrix},\ r_1 = 0,\ r_2 = 1,\ r_3 = -1$

- Since the objective function above is not differentiable, the Newton's method cannot be applied directly.

- This problem can be transformed into an equivalent problem:

$$\begin{aligned} \underset{\tilde{x} \in \mathbf{R}^2,\, w \in \mathbf{R}}{\text{minimize}} \quad & w \\ \text{subject to} \quad & f_1(\tilde{x}) - w \leq 0 \\ & f_2(\tilde{x}) - w \leq 0 \\ & f_3(\tilde{x}) - w \leq 0 \end{aligned}$$

- To solve this problem, we will apply the barrier method.

- Let $x = \begin{bmatrix} \tilde{x} \\ w \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Then we use the log barrier $\varphi(x)$ to rewrite the problem.

$$\rightarrow \quad \underset{x \in \mathbf{R}^3}{\text{minimize}} \quad t f_0(x) + \varphi(x)$$

where $f_0(x) = w = x_3$ and $\varphi(x) = -\sum_{i=1}^{3} \log(-f_i(x) + w)$

- Initial point $x^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}$ and $t = 1$.

- Newton step $\Delta x_{nt} = -(\nabla^2 f(x))^{-1} \nabla f(x)$ → $\Delta x_{nt}^{(l)}$

- Newton decrement $\lambda(x) = (-\nabla f(x)^T \Delta x_{nt})^{1/2}$ → $\lambda^{(l)}(x)$

- Backtracking line search along search direction using $\beta = 0.7$ starting from $s = 1$ until $x^{(l)} + s\Delta x_{nt}^{(l)} \in \mathbf{dom}\, f$ (Note: $t^+ \coloneqq \beta t$)

- Continue the backtracking line search until $f(x + s\Delta x_{nt}^{(l)}) \leq \alpha s \lambda\left(x^{(l)}\right)^2$, where $\alpha = 0.1$

- Update $x^{(l+1)} = x^{(l)} + s\Delta x_{nt}^{(l)}$

- If $\lambda(x)^2 / 2 < \epsilon_{inner}$, then break the Newton interation.

- Updating $t^+ \coloneqq \mu t$. If $\frac{m}{t} < \epsilon_{outer}$, then break the loop.