CHAPTER 1

VECTORS & FIELDS

For lecture only, no distribution

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OUTLINE

- Vector Algebra
- Cartesian Coordinate System
- Cylindrical and Spherical Coordinate Systems
- Scalar and Vector Fields
- The Electric Field
- The Magnetic Field
- Lorentz Force Equation

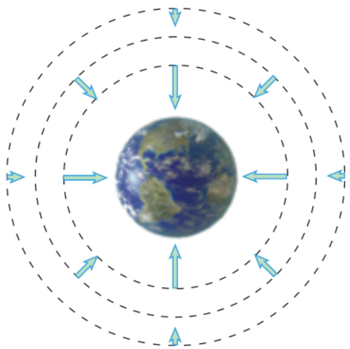
INTRODUCTION

- Key goals:
  - Deal with the problems related to electric and magnetic fields (EM1) and waves (EM2)
  - These fields are vector quantities and governed by Maxwell's equations.
  - Mathematical manipulations are required due to the large volume of vector calculation.
  - Different coordinates will be introduced for different geometrical objects
  - Electrical and magnetic fields are introduced with basic laws: Coulomb (**E**, involving charges), Ampere (**M**, involving current), and Lorentz force (**E** + **M**).

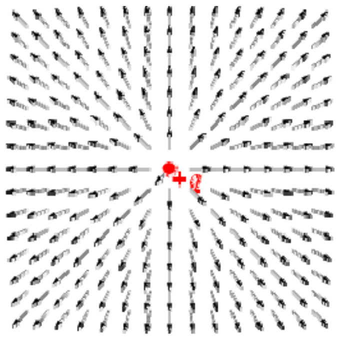
WHAT ARE FIELDS?

- **Features:**
  - A physical quantity which is characterized in terms of space (x,y,z) coordinates. Ex: Gravity or electric (magnetic) field

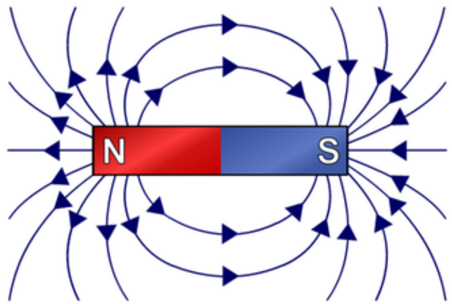
Gravity



Electric

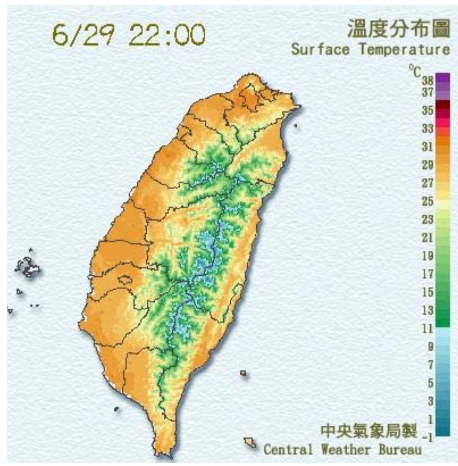


Magnetic



  - Scalar vs. vector:
    - ✓ Vector: electric (gravity, magnetic) field
    - ✓ Scalar: temperature
  - Classical vs. quantum fields
    - ✓ Classical: gravity, EM
    - ✓ Quantum: quantum mechanics can represent the classical fields, which generate the “quantum field theory”.

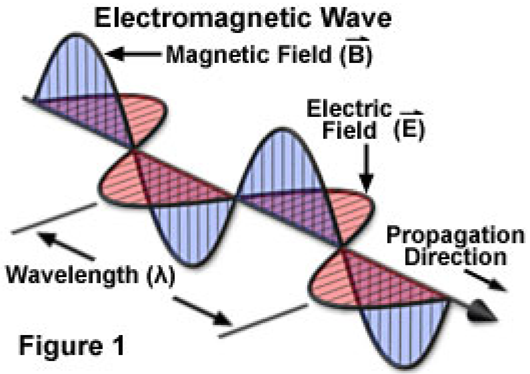
Temperature



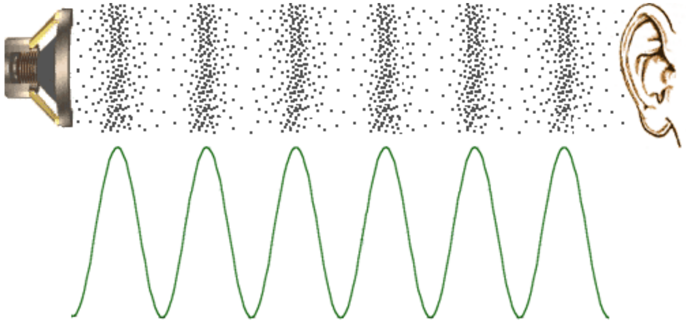
WHAT ARE WAVES?

- Features:
  - Definition: oscillations with energy transferred through space or mass
    - ✓ Water wave (by water molecules)
    - ✓ EM wave (through space)
    - ✓ Sound wave (via mass of air molecules)

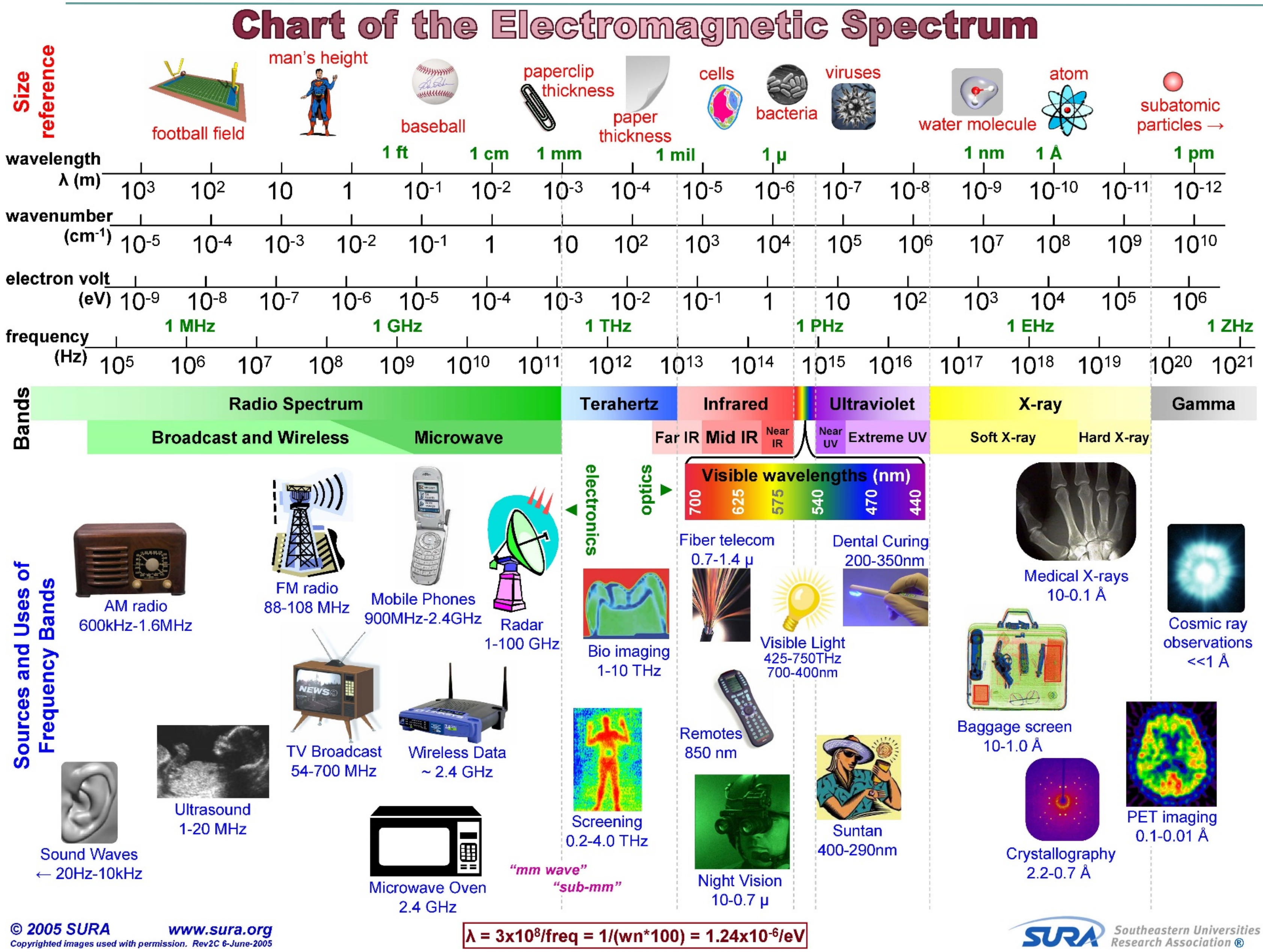
  - Natures:
    - ✓ Transverse: oscillation perpendicular to energy propagation; e.g. water wave, EM wave
    - ✓ Longitudinal: oscillation parallel to propagation; e.g. sound wave



  - Applications:
    - ✓ Laser: EM wave
    - ✓ Mobile communication: EM wave



WHAT ARE WAVES?



ELECTROMAGNETIC MODEL

- Essential steps for learning EM theory:
  - Basic quantities:
    - ✓ Ex: circuit theory $\Rightarrow$ V (voltages), I (current), (R) resistances, (L) inductances, and (C) capacitances
    - ✓ For EM theory, there exist several basic quantities $\Rightarrow$ ρ (charge density), **J** (current density), **E** (electric field), **D** (electric displacement flux), **B** (magnetic flux density), and **H** (magnetic field)
  - Postulate some fundamental relations (physics) and specify rules of operation for those quantities (mathematics)
    - ✓ Ex: circuit theory $\Rightarrow$ Kirchhoff's voltage and current laws (physics); Laplace transformation and differential equations (mathematics).
    - ✓ For EM theory, Gauss' Law, Faraday's Law, and Ampère's Law (physics); divergence of a vector field, gradient of a scalar field (mathematics)
  - We first introduce the mathematical tools first (chapter 1 in Rao) followed by physical parts later (chapter 2 ~ 5 in Rao).

I. VECTOR ALGEBRA

- Physical quantities: scalar vs. vector
  - Scalar:
    - ✓ characterized by magnitude only, e.g. mass, temperature, charge
    - ✓ Symbol: “lightface italic type” (e.g. *A*)
  - Vector:
    - ✓ characterized by magnitude and specific directions, e.g. velocity, acceleration, force, electric and magnetic fields
    - ✓ Symbol: “boldface roman type” (e.g. **A**)

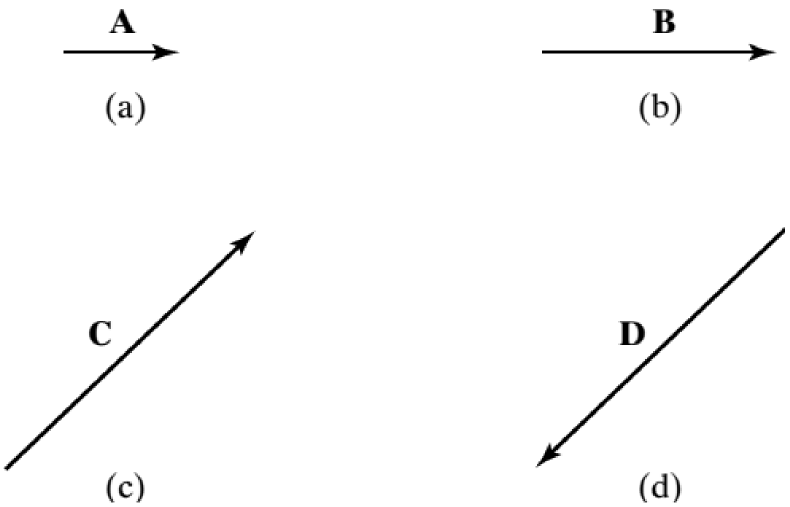
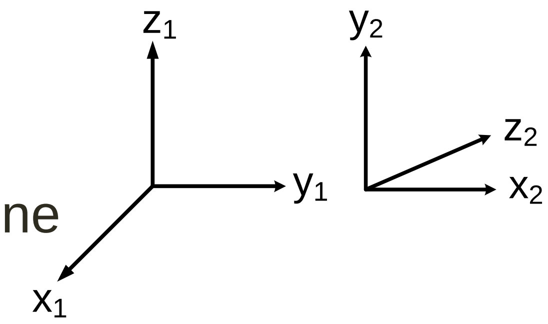


FIGURE 1.1
Graphical representation of vectors.

I. VECTOR ALGEBRA

Coordinates and orientations



- Frame of reference:
  - ✓ 3 directions (e.g. x, y, z) are needed to define a "point" in space
  - ✓ There are many sets of frames of reference i.e. you can define your (x_1, y_1, z_1) which is different from others (x_2, y_2, z_2). Assume there exists a point (a, b, c), they can always be expressed in terms of these two frames of reference.
- Right-hand vs. left-hand:
  - ✓ For a fixed orientation such as $\mathbf{a}_1$, we have 2 choices to place $\mathbf{a}_2$ and $\mathbf{a}_3$.
  - ✓ Use your right-hand fingers to rotate (from $\mathbf{a}_1$ to $\mathbf{a}_2$) "counter-clock-wise you got your thumb pointing along $\mathbf{a}_3$.
  - ✓ Similarly, you will get "clock-wise" by left-hand fingers if $\mathbf{a}_2$ and $\mathbf{a}_3$ are swapped.

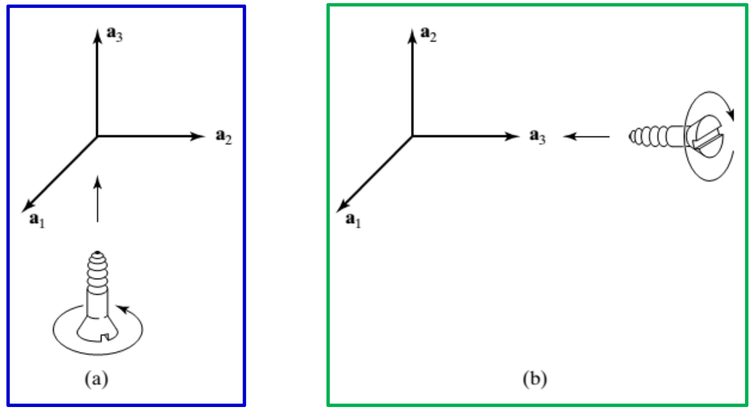


FIGURE 1.2

(a) Set of three orthogonal unit vectors in a right-handed system. (b) Set of three orthogonal unit vectors in a left-handed system.

I. VECTOR ALGEBRA

■ Unit vectors

- ➢ Features:
  - ✓ A unit vector has magnitude of unity and usually written as $\mathbf{a}_1$ (bold) with a subscript 1 (or 2, 3).

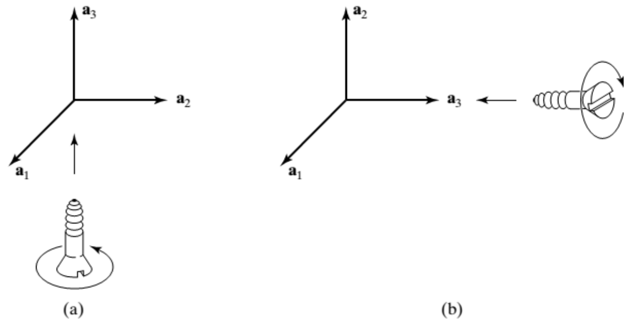


FIGURE 1.2
(a) Set of three orthogonal unit vectors in a right-handed system. (b) Set of three orthogonal unit vectors in a left-handed system.

  - ✓ For all vectors $\mathbf{A}_1$ (or $\mathbf{A}_2$, $\mathbf{A}_3$) along the direction of $\mathbf{a}_1$ (or $\mathbf{a}_2$, $\mathbf{a}_3$), they can be expressed in terms of multiples of $\mathbf{a}_2$ (or $\mathbf{a}_2$, $\mathbf{a}_3$):

$$\mathbf{A}_1 = n_1 \text{ x } \mathbf{a}_1 \text{ (or } \mathbf{A}_2 = n_2 \text{ x } \mathbf{a}_2,\ \mathbf{A}_3 = n_3 \text{ x } \mathbf{a}_3)$$

where n_1, n_2, and n_3 are integers.

  - ✓ For an arbitrary vector $\mathbf{A}$, it's written as $\mathbf{A} = A_1\mathbf{a}_1 + A_2\mathbf{a}_2 + A_3\mathbf{a}_3$, where A_1, A_2, and A_3 are multiples of the projection of $\mathbf{A}$ in $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ components (axes)

- ➢ ***Example***:

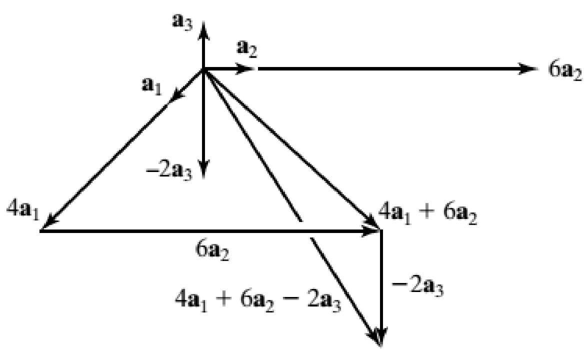


FIGURE 1.3
Graphical addition of vectors.

I. VECTOR ALGEBRA

Vector math:

- Magnitude of a vector: (where $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ are unit vectors)

$$\boxed{|\mathbf{A}| = |A_1\mathbf{a}_1 + A_2\mathbf{a}_2 + A_3\mathbf{a}_3| = \sqrt{A_1^2 + A_2^2 + A_3^2}}$$

- Unit vector ($\mathbf{a}_A$) along a arbitrary direction:

$$\boxed{a_A = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{A_1\mathbf{a}_1}{|\mathbf{A}|} + \frac{A_2\mathbf{a}_2}{|\mathbf{A}|} + \frac{A_3\mathbf{a}_3}{|\mathbf{A}|}}$$

- Addition/subtraction: (both A and B in terms of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$)

$$\mathbf{A} \pm \mathbf{B} = (A_1\mathbf{a}_1 + A_2\mathbf{a}_2 + A_3\mathbf{a}_3) \pm (B_1\mathbf{a}_1 + B_2\mathbf{a}_2 + B_3\mathbf{a}_3)$$

$$= (A_1 \pm B_1)\mathbf{a}_1 + (A_2 \pm B_2)\mathbf{a}_2 + (A_3 \pm B_3)\mathbf{a}_3$$

- Multiplication and division by scalars (*m* or *n*):

$$m\mathbf{A} = m(A_1\mathbf{a}_1 + A_2\mathbf{a}_2 + A_3\mathbf{a}_3) = mA_1\mathbf{a}_1 + mA_2\mathbf{a}_2 + mA_3\mathbf{a}_3$$

$$\frac{\mathbf{B}}{n} = \frac{1}{n}(B_1\mathbf{a}_1 + B_2\mathbf{a}_2 + B_3\mathbf{a}_3) = \frac{B_1}{n}\mathbf{a}_1 + \frac{B_2}{n}\mathbf{a}_2 + \frac{B_3}{n}\mathbf{a}_3$$

I. VECTOR ALGEBRA

Scalar (dot or inner) product:

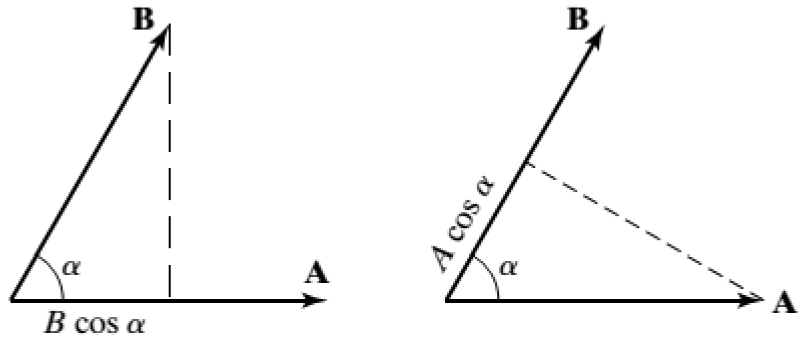
- Scalar quantity equal to the amplitude of two vectors **A** and **B**, which are *A* and *B*, and its cosine of the angle between them.

$$\boxed{\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos\alpha = AB\cos\alpha}$$

- Example: dot products of unit vectors

$$\begin{aligned} a_1\cdot a_1 &= 1 & a_1\cdot a_2 &= 0 & a_1\cdot a_3 &= 0\\ a_2\cdot a_1 &= 0 & a_2\cdot a_2 &= 1 & a_2\cdot a_3 &= 0\\ a_3\cdot a_1 &= 0 & a_3\cdot a_2 &= 0 & a_3\cdot a_3 &= 1 \end{aligned}$$

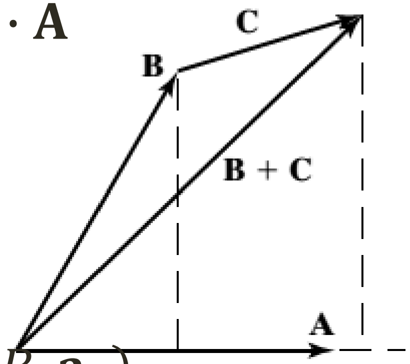
- Alternative: the projection of one vector onto another vector
  - ✓ **B** on **A**: $|\mathbf{B}|\cos\alpha \times |\mathbf{A}|$
  - ✓ **A** on **B**: $|\mathbf{A}|\cos\alpha \times |\mathbf{B}|$



I. VECTOR ALGEBRA

- Commutative:

$$\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos\alpha = AB\cos\alpha = BA\cos\alpha = \mathbf{B}\cdot\mathbf{A}$$



- Distributive:

$$\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C}$$

and

$$\begin{aligned}
\mathbf{A}\cdot\mathbf{B} &= (A_1\mathbf{a}_1 + A_2\mathbf{a}_2 + A_3\mathbf{a}_3)\cdot(B_1\mathbf{a}_1 + B_2\mathbf{a}_2 + B_3\mathbf{a}_3) \\
&= A_1\mathbf{a}_1\cdot B_1\mathbf{a}_1 + \cancel{A_1\mathbf{a}_1\cdot B_2\mathbf{a}_2} + \cancel{A_1\mathbf{a}_1\cdot B_3\mathbf{a}_3} \\
&\quad + \cancel{A_2\mathbf{a}_2\cdot B_1\mathbf{a}_1} + A_2\mathbf{a}_2\cdot B_2\mathbf{a}_2 + \cancel{A_2\mathbf{a}_2\cdot B_3\mathbf{a}_3} \\
&\quad + \cancel{A_3\mathbf{a}_3\cdot B_1\mathbf{a}_1} + \cancel{A_3\mathbf{a}_3\cdot B_2\mathbf{a}_2} + A_3\mathbf{a}_3\cdot B_3\mathbf{a}_3 \\
&\boxed{= A_1B_1 + A_2B_2 + A_3B_3}
\end{aligned}$$

The angle α can be written

$$\boxed{\alpha = \cos^{-1}\left(\frac{\mathbf{A}\cdot\mathbf{B}}{AB}\right) = \cos^{-1}\left(\frac{A_1B_1 + A_2B_2 + A_3B_3}{\sqrt{A_1^2 + A_2^2 + A_3^2}\sqrt{B_1^2 + B_2^2 + B_3^2}}\right)}$$

I. VECTOR ALGEBRA

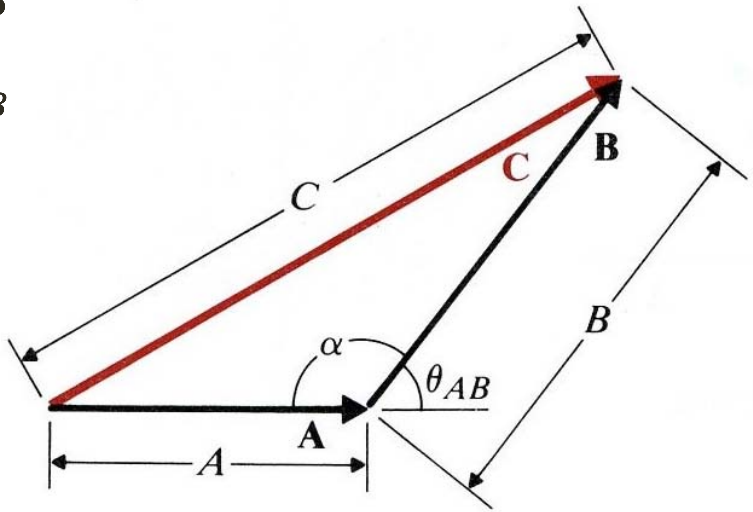
- ***Example***: Prove the law of cosines for a triangle

  - ✓ The law of cosines states that $C = \sqrt{A^2 + B^2 - 2AB\cos\alpha}$ and by considering all sides are vectors, we can write

$$\mathbf{C} = \mathbf{A} + \mathbf{B}$$

  - ✓ By taking inner product of C itself, we have

$$\begin{aligned} C^2 = \mathbf{C}\cdot\mathbf{C} &= (\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}+\mathbf{B}) \\ &= \mathbf{A}\cdot\mathbf{A} + \mathbf{A}\cdot\mathbf{B} + \mathbf{B}\cdot\mathbf{A} + \mathbf{B}\cdot\mathbf{B} \\ &= \mathbf{A}\cdot\mathbf{A} + 2\mathbf{A}\cdot\mathbf{B} + \mathbf{B}\cdot\mathbf{B} \\ &= A^2 + B^2 + 2AB\cos\theta_{AB} \\ &= A^2 + B^2 - 2AB\cos\alpha \end{aligned}$$



I. VECTOR ALGEBRA

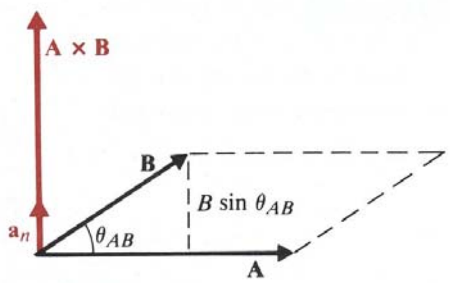
Vector (cross) product:

- Vector quantity:
  - ✓ Amplitude: equal to the product of the magnitudes of two vectors **A** and **B** and the since of the smaller angle between **A** and **B** (since there are 2 angles between any 2 arbitrary vectors and their sum is 180°)
  - ✓ Direction: normal to the plane containing vectors **A** and **B**

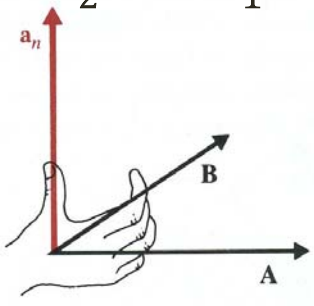
$$\boxed{\mathbf{A}\times\mathbf{B} = |\mathbf{A}||\mathbf{B}|\sin\alpha\,\mathbf{a}_N = AB\sin\alpha\,\mathbf{a}_N}$$

- Example: cross products of unit vectors

$$a_1 \times a_1 = 0 \qquad a_1 \times a_2 = a_3 \qquad a_1 \times a_3 = -a_2$$
$$a_2 \times a_1 = -a_3 \qquad a_2 \times a_2 = 0 \qquad a_2 \times a_3 = a_1$$
$$a_3 \times a_1 = a_2 \qquad a_3 \times a_2 = -a_1 \qquad a_3 \times a_3 = 0$$



(a) $\mathbf{A} \times \mathbf{B} = \mathbf{a}_n|AB \sin \theta_{AB}|$.



(b) The right-hand rule.

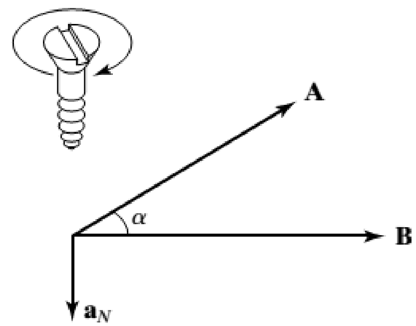


FIGURE 1.5
Cross product operation $\mathbf{A} \times \mathbf{B}$.

I. VECTOR ALGEBRA

- Commutative:

$$\mathbf{A}\times\mathbf{B} = |\mathbf{A}||\mathbf{B}|\sin\alpha\,\mathbf{a}_N = AB\sin\alpha\,\mathbf{a}_N = -BA\sin\alpha(-\,\mathbf{a}_N) = -\mathbf{B}\times\mathbf{A}$$

- Distributive:

$$\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = \mathbf{A}\times\mathbf{B}+\mathbf{A}\times\mathbf{C}$$

and

$$\begin{aligned}\mathbf{A}\times\mathbf{B} &= (A_1\mathbf{a}_1 + A_2\mathbf{a}_2 + A_3\mathbf{a}_3)\times(B_1\mathbf{a}_1 + B_2\mathbf{a}_2 + B_3\mathbf{a}_3)\\ &= \cancel{A_1\mathbf{a}_1\times B_1\mathbf{a}_1} + A_1\mathbf{a}_1\times B_2\mathbf{a}_2 + A_1\mathbf{a}_1\times B_3\mathbf{a}_3\\ &\quad + A_2\mathbf{a}_2\times B_1\mathbf{a}_1 + \cancel{A_2\mathbf{a}_2\times B_2\mathbf{a}_2} + A_2\mathbf{a}_2\times B_3\mathbf{a}_3\\ &\quad + A_3\mathbf{a}_3\times B_1\mathbf{a}_1 + A_3\mathbf{a}_3\times B_2\mathbf{a}_2 + \cancel{A_3\mathbf{a}_3\times B_3\mathbf{a}_3}\\ &\boxed{= \mathbf{a}_1(A_2B_3 - A_3B_2) + \mathbf{a}_2(A_3B_1 - A_1B_3) + \mathbf{a}_3(A_1B_2 - A_2B_1)}\end{aligned}$$

or

$$\boxed{\mathbf{A}\times\mathbf{B} = \begin{vmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3\\ A_1 & A_2 & A_3\\ B_1 & B_2 & B_3\end{vmatrix}}$$

- The unit normal vector $\mathbf{a}_N$ can be expressed as $\mathbf{a}_N = \dfrac{\mathbf{A}\times\mathbf{B}}{AB\sin\alpha} = \dfrac{\mathbf{A}\times\mathbf{B}}{|AB\sin\alpha|}$

I. VECTOR ALGEBRA

Triple products:

- Vector triple cross product: such as $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ or $(\mathbf{A}\times\mathbf{B})\times\mathbf{C}$
  - ✓ Generally , the above two vectors are NOT equal if we take $\mathbf{A}=\mathbf{a}_1$, $\mathbf{B}=\mathbf{a}_1$, and $\mathbf{C}=\mathbf{a}_2$, we have

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\mathbf{a}_1\times(\mathbf{a}_1\times\mathbf{a}_2)=\mathbf{a}_1\times\mathbf{a}_3=-\mathbf{a}_2$$

and $\neq$, but not always

$$(\mathbf{A}\times\mathbf{B})\times\mathbf{C}=(\mathbf{a}_1\times\mathbf{a}_1)\times\mathbf{a}_2=0\times\mathbf{a}_2=0$$

- Scalar triple product:
  - ✓ Involving three vectors with a dot and a cross operations, ex: $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$ or $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}$ (are they the same?)

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}=(A_1\mathbf{a}_1+A_2\mathbf{a}_2+A_3\mathbf{a}_3)\cdot\begin{vmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3\\ B_1 & B_2 & B_3\\ C_1 & C_2 & C_3\end{vmatrix}=\boxed{\begin{vmatrix}A_1 & A_2 & A_3\\ B_1 & B_2 & B_3\\ C_1 & C_2 & C_3\end{vmatrix}}$$

  - ✓ Why? Since the second term can be written as $R_1\mathbf{a}_1+R_2\mathbf{a}_2+R_3\mathbf{a}_3$ and the dot product action will take out all unit vectors, thus, we can replace $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ with A_1, A_2, and A_3.

I. VECTOR ALGEBRA

- Interchange the order of **A**, **B**, and **C**:

$$\because \mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

and we know changing the order in a cyclical manner wouldn't change the value, we have

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = \begin{vmatrix} B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \\ A_1 & A_2 & A_3 \end{vmatrix} = \mathbf{B}\cdot\mathbf{C}\times\mathbf{A}$$

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = \begin{vmatrix} C_1 & C_2 & C_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B \end{vmatrix} = \mathbf{C}\cdot\mathbf{A}\times\mathbf{B}$$

- Distributive law works for triple vector product:
  - ✓ For any D,

$$\mathbf{D}\cdot\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = (\mathbf{B}+\mathbf{C})\cdot\mathbf{D}\times\mathbf{A} = \mathbf{B}\cdot\mathbf{D}\times\mathbf{A}+\mathbf{C}\cdot\mathbf{D}\times\mathbf{A}$$

$$= \mathbf{D}\cdot\mathbf{A}\times\mathbf{B}+\mathbf{D}\cdot\mathbf{A}\times\mathbf{C} = \mathbf{D}\cdot(\mathbf{A}\times\mathbf{B}+\mathbf{A}\times\mathbf{C})$$

which follows that

$$\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = \mathbf{A}\times\mathbf{B}+\mathbf{A}\times\mathbf{C}$$

I. VECTOR ALGEBRA

Example 1.1:

$$\mathbf{A} = \mathbf{a}_1 + \mathbf{a}_2$$
$$\mathbf{B} = \mathbf{a}_1 + 2\mathbf{a}_2 - 2\mathbf{a}_3$$
$$\mathbf{C} = \mathbf{a}_2 + 2\mathbf{a}_3$$

(a) $\mathbf{A} + \mathbf{B} = (\mathbf{a}_1 + \mathbf{a}_2) + (\mathbf{a}_1 + 2\mathbf{a}_2 - 2\mathbf{a}_3) = 2\mathbf{a}_1 + 3\mathbf{a}_2 - 2\mathbf{a}_3$

(b) $\mathbf{B} - \mathbf{C} = (\mathbf{a}_1 + 2\mathbf{a}_2 - 2\mathbf{a}_3) - (\mathbf{a}_2 + 2\mathbf{a}_3) = \mathbf{a}_1 + \mathbf{a}_2 - 4\mathbf{a}_3$

(c) $4\mathbf{C} = 4(\mathbf{a}_2 + 2\mathbf{a}_3) = 4\mathbf{a}_2 + 8\mathbf{a}_3$

(d) $|\mathbf{B}| = |\mathbf{a}_1 + 2\mathbf{a}_2 - 2\mathbf{a}_3| = \sqrt{(1)^2 + (2)^2 + (-2)^2} = 3$

(e) $\mathbf{i}_B = \dfrac{\mathbf{B}}{|\mathbf{B}|} = \dfrac{\mathbf{a}_1 + 2\mathbf{a}_2 - 2\mathbf{a}_3}{3} = \dfrac{1}{3}\mathbf{a}_1 + \dfrac{2}{3}\mathbf{a}_2 - \dfrac{2}{3}\mathbf{a}_3$

(f) $\mathbf{A}\cdot\mathbf{B} = (\mathbf{a}_1 + \mathbf{a}_2)\cdot(\mathbf{a}_1 + 2\mathbf{a}_2 - 2\mathbf{a}_3) = (1)(1) + (1)(2) + (0)(-2) = 3$

(g) Angle between $\mathbf{A}$ and $\mathbf{B}$ $= \cos^{-1}\dfrac{\mathbf{A}\cdot\mathbf{B}}{AB} = \cos^{-1}\dfrac{3}{(\sqrt{2})(3)} = 45^\circ$

(h)
$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \\ 1 & 1 & 0 \\ 1 & 2 & -2 \end{vmatrix} = (-2 - 0)\mathbf{a}_1 + (0 + 2)\mathbf{a}_2 + (2 - 1)\mathbf{a}_3$$
$$= -2\mathbf{a}_1 + 2\mathbf{a}_2 + \mathbf{a}_3$$

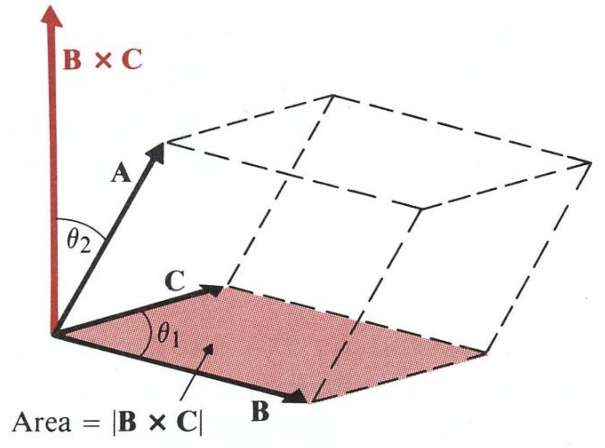
(i) Unit vector normal to $\mathbf{A}$ and $\mathbf{B}$ $= \dfrac{\mathbf{A} \times \mathbf{B}}{|\mathbf{A} \times \mathbf{B}|} = -\dfrac{2}{3}\mathbf{a}_1 + \dfrac{2}{3}\mathbf{a}_2 + \dfrac{1}{3}\mathbf{a}_3$

(j)
$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = \begin{vmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \\ -2 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 3\mathbf{a}_1 + 4\mathbf{a}_2 - 2\mathbf{a}_3$$

(k)
$$\mathbf{A}\cdot\mathbf{B} \times \mathbf{C} = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & -2 \\ 0 & 1 & 2 \end{vmatrix} = (1)(6) + (1)(-2) + (0)(1) = 4$$

I. VECTOR ALGEBRA

- Physical meaning of $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}$ (scalar triple product) is the volume spanned by three vectors



  - ✓ First, $\mathbf{B} \times \mathbf{C}$ is the vector of $(BC\sin\theta_1 \mathbf{a}_N)$ which includes the amplitude $AB\sin\alpha$ with a unit vector $\mathbf{a}_N = \frac{\mathbf{B}\times\mathbf{C}}{|\mathbf{B}\times\mathbf{C}|}$.
  - ✓ The area of red shaded is $BC\sin\theta_1$
  - ✓ The volume is equal to the area times the height, which is equivalent to the projection of vector A onto unit vector $\mathbf{a}_N$. The height is

$$|\mathbf{A} \cdot \mathbf{a}_N|$$

  - ✓ Thus, the volume is

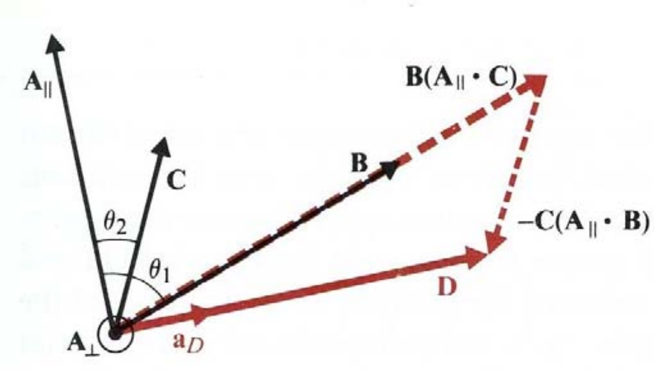
$$|\mathbf{A} \cdot \mathbf{a}_N| BC\sin\theta_1 = \mathbf{A} \cdot \frac{\mathbf{B} \times \mathbf{C}}{|\mathbf{B} \times \mathbf{C}|} BC\sin\theta_1$$

$$= \mathbf{A} \cdot \frac{\mathbf{B}\times\mathbf{C}}{BC\sin\theta_1} BC\sin\theta_1 = \mathbf{A} \cdot \mathbf{B} \times \mathbf{C}$$

I. VECTOR ALGEBRA

- ***Example:***

  Prove $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\mathbf{B}(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}\cdot\mathbf{B})$



  - Distributive law works for triple vector product:

$$\mathbf{A}=\mathbf{A}_{||}+\mathbf{A}_{\perp}$$

where $\mathbf{A}_{||}$ and $\mathbf{A}_{\perp}$ are parallel and perpendicular to the plane containing **B** and **C**. Then

$$\mathbf{D}=\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=(\mathbf{A}_{||}+\mathbf{A}_{\perp})\times(\mathbf{B}\times\mathbf{C})=\mathbf{A}_{||}\times(\mathbf{B}\times\mathbf{C})$$

since $\mathbf{A}_{\perp}$ is parallel to $(\mathbf{B}\times\mathbf{C})$ such that $\mathbf{A}_{\perp}\times(\mathbf{B}\times\mathbf{C})=\mathbf{0}$

  - Then we get

$$\boxed{\mathbf{D}[=\mathbf{A}_{||}\times(\mathbf{B}\times\mathbf{C})]}\cdot\mathbf{a}_D=|\mathbf{A}_{||}|[|\mathbf{B}||\mathbf{C}|\sin(\theta_1-\theta_2)]$$

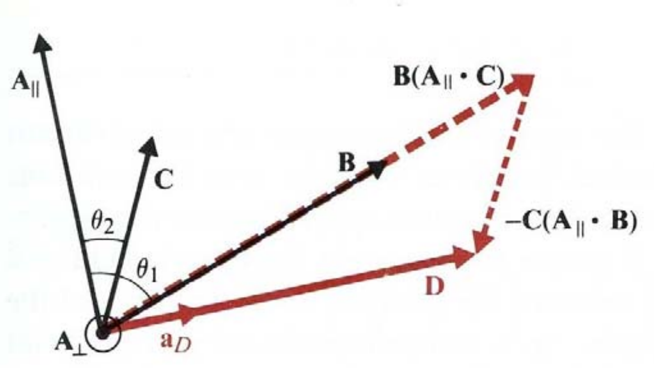
$$=(|\mathbf{B}|\sin\theta_1)\big(|\mathbf{A}_{||}||\mathbf{C}|\cos\theta_2\big)-(|\mathbf{C}|\sin\theta_2)\big(|\mathbf{A}_{||}||\mathbf{B}|\cos\theta_1\big)$$

$$=\boxed{[\mathbf{B}(\mathbf{A}_{||}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}_{||}\cdot\mathbf{B})]}\cdot\mathbf{a}_D$$

$\Rightarrow \mathbf{D}=\mathbf{A}_{||}\times(\mathbf{B}\times\mathbf{C})\overset{?}{=}[\mathbf{B}(\mathbf{A}_{||}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}_{||}\cdot\mathbf{B})]$ true?? Not really!!

I. VECTOR ALGEBRA

- If any of the above two terms has a component along the direction of $\mathbf{A}_{||}$, we still get the above equation, but they could be different. For example, if



$$\mathbf{D} = k\mathbf{A}_{||} + [\mathbf{B}(\mathbf{A}_{||}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}_{||}\cdot\mathbf{B})]$$

where *k* is a scalar. Then we will have

$$\begin{aligned}\mathbf{D}\cdot\boldsymbol{a_D} &= \{k\mathbf{A}_{||} + [\mathbf{B}(\mathbf{A}_{||}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}_{||}\cdot\mathbf{B})]\ \}\cdot\boldsymbol{a_D}\\ &= \{k\mathbf{A}_{||} + [\mathbf{B}(\mathbf{A}_{||}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}_{||}\cdot\mathbf{B})]\ \}\cdot\boldsymbol{a_D}\\ &= [\mathbf{B}(\mathbf{A}_{||}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}_{||}\cdot\mathbf{B})]\cdot\boldsymbol{a_D}\end{aligned}$$

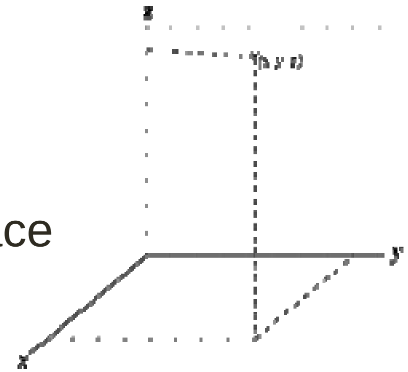
- To determine *k*, we can scalar multiply $\mathbf{A}_{||}$ for both sides **D**

$$\begin{aligned}\mathbf{A}_{||}\cdot\mathbf{D} = 0 &= \mathbf{A}_{||}\cdot\{k\mathbf{A}_{||} + [\mathbf{B}(\mathbf{A}_{||}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}_{||}\cdot\mathbf{B})]\}\\ &= kA_{||}^2 + [\cancel{(\mathbf{A}_{||}\cdot\mathbf{B})(\mathbf{A}_{||}\cdot\mathbf{C}) - (\mathbf{A}_{||}\cdot\mathbf{C})(\mathbf{A}_{||}\cdot\mathbf{B})}] = kA_{||}^2\end{aligned}$$

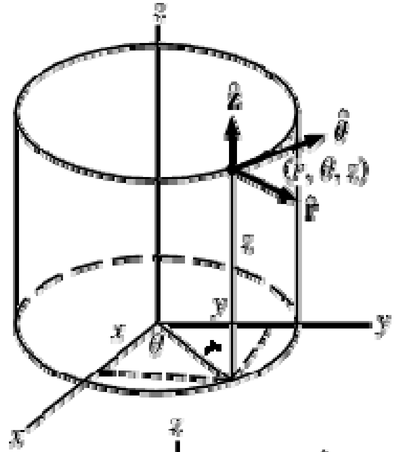
$$\rightarrow k = 0 \ \text{ and } \ \mathbf{D} = [\mathbf{B}(\mathbf{A}_{||}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}_{||}\cdot\mathbf{B})] = \mathbf{A}_{||}\times(\mathbf{B}\times\mathbf{C})$$

II. COORDINATE SYSTEMS

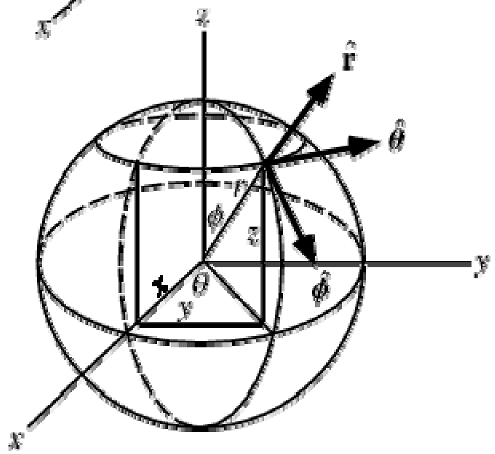
- Three major coordinate systems:
  - Cartesian:
    - ✓ using x, y, z to represent each point in the space
    - ✓ Most common and useful system



  - Cylindrical:
    - ✓ using r, θ (angle to +x-axis), z to represent…..
    - ✓ **Full space**: $r = 0 \rightarrow \infty;\ \theta = 0 \rightarrow 2\pi;$
      $z = -\infty \rightarrow \infty.$
    - ✓ Usually for a system with a moving direction (say z-axis) and it's symmetrical in the x-y plane such as light propagation



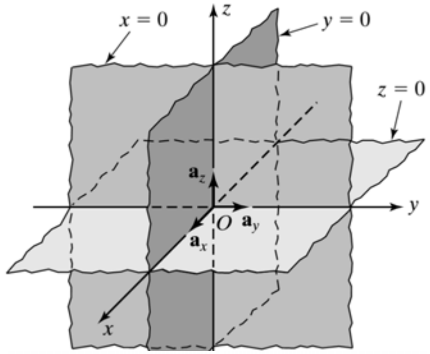
  - Spherical:
    - ✓ Using r, θ, ϕ (angle to +z-axis) to represent…
    - ✓ **Full space**: $R = 0 \rightarrow \infty;\ \theta = 0 \rightarrow 2\pi;$
      $\phi = -\frac{\pi}{2} \rightarrow \frac{\pi}{2}$ (or $0 \rightarrow \pi$).
    - ✓ Most important for a "point" question such as point charge

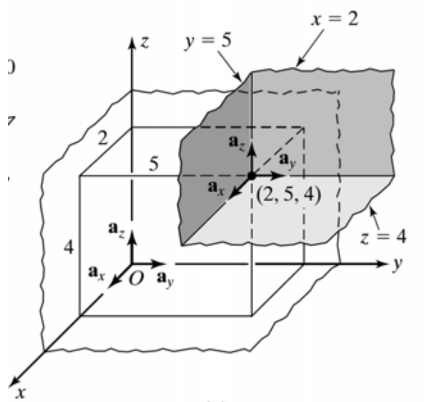


II. CARTESIAN COORDINATE SYSTEM

■ Features:

- Three mutual orthogonal planes:
  - ✓ top (a) – (x=0, y=0, z=0);
  - ✓ bottom (b) – (x=2, y=5, z=4)
- For top case (a), two planes (e.g. x=0, y=0) intersect to form a line; then this line intersect with the last plane (z=0) to form a point (x=0, y=0, z=0); similar for bottom





- Position vector:

$$\mathbf{r}_1 = x_1\mathbf{a}_x + y_1\mathbf{a}_y + z_1\mathbf{a}_z \quad \mathbf{r}_2 = x_2\mathbf{a}_x + y_2\mathbf{a}_y + z_2\mathbf{a}_z$$

$$\boxed{\begin{aligned}\mathbf{R}_{12} &= \mathbf{r}_2 - \mathbf{r}_1 \\ &= (x_2 - x_1)\mathbf{a}_x + (y_2 - y_1)\mathbf{a}_y + (z_2 - z_1)\mathbf{a}_z\end{aligned}}$$

$$\boxed{\mathbf{a}_{12} = \frac{\mathbf{R}_{12}}{R_{12}} = \frac{(x_2 - x_1)\mathbf{a}_x + (y_2 - y_1)\mathbf{a}_y + (z_2 - z_1)\mathbf{a}_z}{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{1/2}}}$$

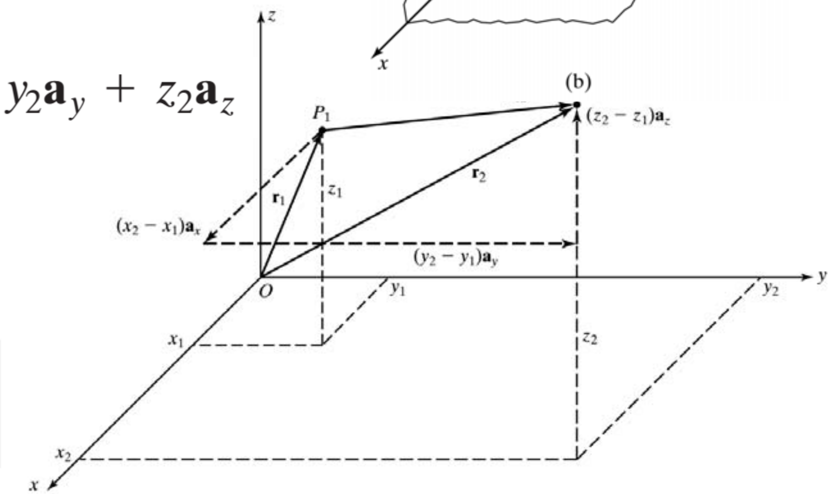
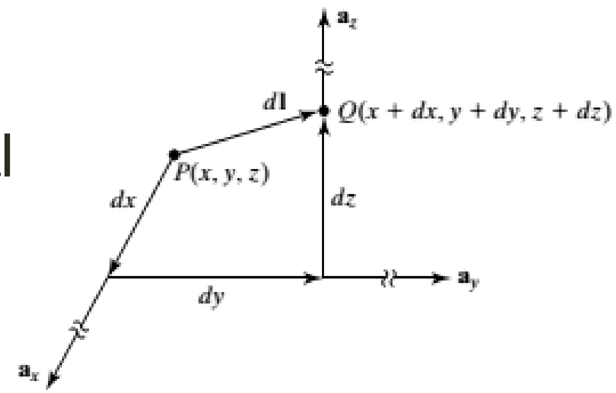


FIGURE 1.7

For obtaining the expression for the vector $\mathbf{R}_{12}$ from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$.

II. CARTESIAN COORDINATE SYSTEM

- **Differential length vector:**
  - defined as a vector from a point to a neighboring point with an infinitesimal distance

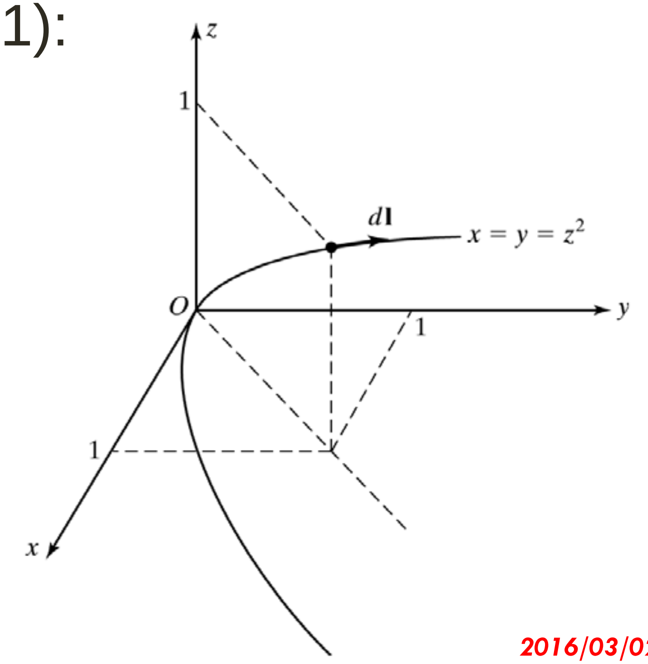


$$\boxed{d\mathbf{l} = dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z}$$

- ***Example 1.2***: Find the differential length vector along a curve of $x = y = z^2$ at the point (1,1,1):

$$x = y = z^2 \qquad dx = dy = 2z\,dz$$

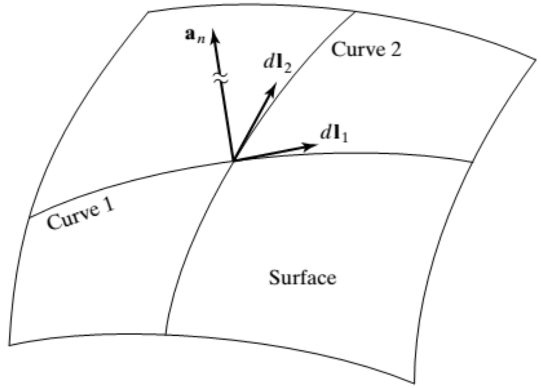
$$\begin{aligned} d\mathbf{l} &= dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z \\ &= 2\,dz\,\mathbf{a}_x + 2\,dz\,\mathbf{a}_y + dz\,\mathbf{a}_z \\ &= (2\mathbf{a}_x + 2\mathbf{a}_y + \mathbf{a}_z)\,dz \end{aligned}$$



  - ✓ Useful to find a unit vector normal to a surface

II. CARTESIAN COORDINATE SYSTEM

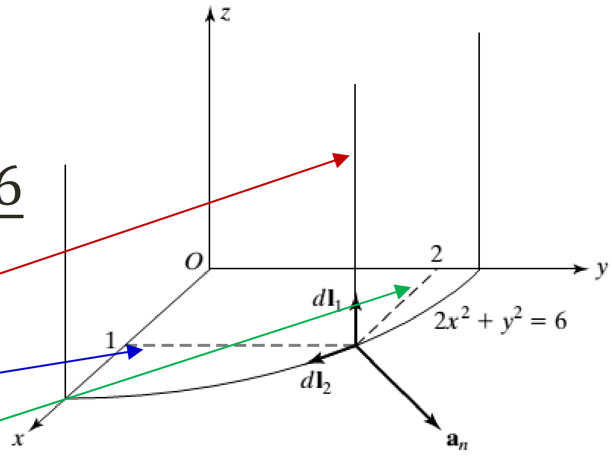
- Assume for two curves 1 and 2, d$\mathbf{l}_1$ and d$\mathbf{l}_2$ are the differential length vectors, respectively.



- Then a unit vector at point Q which is normal to the plane containing two curves is given by

$$\mathbf{a}_n = \frac{d\mathbf{l}_1 \times d\mathbf{l}_2}{|d\mathbf{l}_1 \times d\mathbf{l}_2|}$$

- ***Example 1.3***: for a surface $2x^2 + y^2 = 6$ at point (1,2,0), find the unit vector normal to the surface.



  - ✓ d$\mathbf{l}_1$: along the straight line of x=1/y=2 (where z is arbitrary). So d$\mathbf{l}_1$ = dz$\mathbf{a}_z$
  - ✓ d$\mathbf{l}_2$: tangent to the curve of $2x^2 + y^2 = 6$ & $z = 0$ (note: it's a line). Then we got

$$4xdx + 2ydy = 0 = dz$$

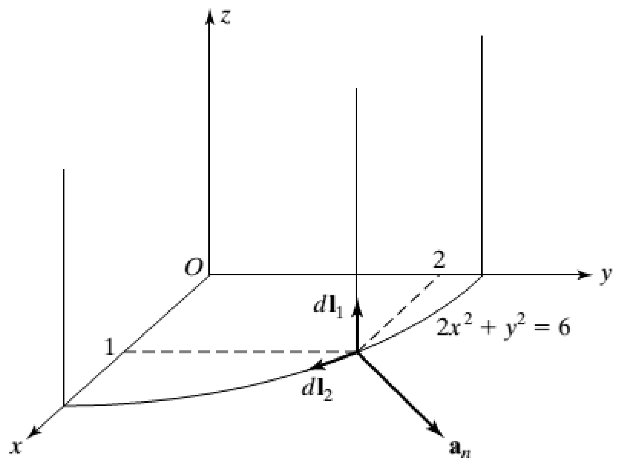
II. CARTESIAN COORDINATE SYSTEM

- ✓ So at (1,2,0), we have

$$4dx + 4dy = 0 \rightarrow dx = -dy \;\&\; dz = 0$$

Then d$\mathbf{l}_2$ becomes

$$d\mathbf{l}_2 = dx\boldsymbol{a}_x + dy\boldsymbol{a}_y + dz\boldsymbol{a}_y$$
$$= dx\boldsymbol{a}_x - dx\boldsymbol{a}_y$$

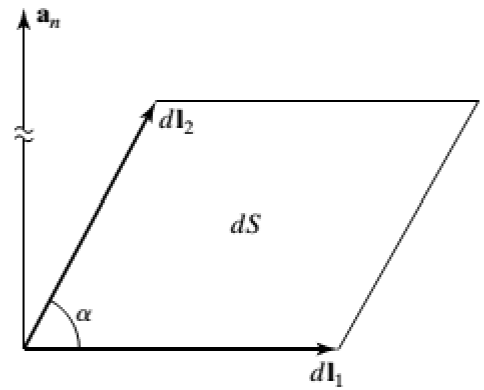


- ✓ So the unit normal vector is given by

$$\mathbf{a}_n = \frac{dz\,\mathbf{a}_z \times dx\,(\mathbf{a}_x - \mathbf{a}_y)}{|dz\,\mathbf{a}_z \times dx\,(\mathbf{a}_x - \mathbf{a}_y)|} = \frac{1}{\sqrt{2}}(\mathbf{a}_x + \mathbf{a}_y)$$

■ Differential surface vector:

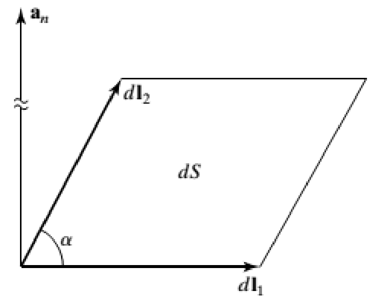
- ➢ 2 differential length vectors at a point define a differential surface (note: not a vector yet)



II. CARTESIAN COORDINATE SYSTEM

- The differential surface area is

$$dS = dl_1\, dl_2 \sin\alpha = |d\mathbf{l}_1 \times d\mathbf{l}_2|$$

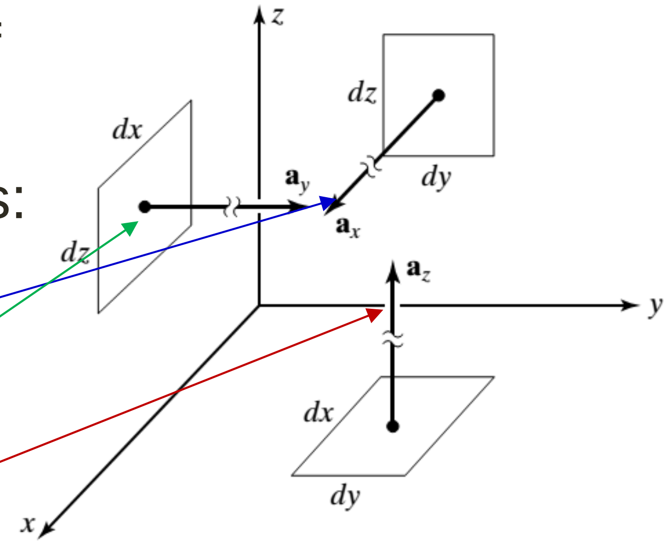


- The corresponding differential surface vector *d***S** is

$$d\mathbf{S} = \pm dS\, \mathbf{a}_n = \pm|d\mathbf{l}_1 \times d\mathbf{l}_2|\mathbf{a}_n = \pm d\mathbf{l}_1 \times d\mathbf{l}_2$$

- For 3 differential length elements of $dx\mathbf{a}_x,\ dy\mathbf{a}_y,\ dz\mathbf{a}_y$, we have the following differential surface vectors:

$$\pm dy\, \mathbf{a}_y \times dz\, \mathbf{a}_z = \pm dy\, dz\, \mathbf{a}_x$$
$$\pm dz\, \mathbf{a}_z \times dx\, \mathbf{a}_x = \pm dz\, dx\, \mathbf{a}_y$$
$$\pm dx\, \mathbf{a}_x \times dy\, \mathbf{a}_y = \pm dx\, dy\, \mathbf{a}_z$$



II. CARTESIAN COORDINATE SYSTEM

- Differential volume:
  - 3 differential length vectors at a point define a differential volume dv (note: not a vector yet)

$$dv = \boxed{area\ of\ the\ base\ of\ the\ parallelepiped} \times \boxed{height}$$

$$= |d\mathbf{l}_1 \times d\mathbf{l}_2||d\mathbf{l}_3 \cdot \mathbf{a}_n|$$

$$= |d\mathbf{l}_1 \times d\mathbf{l}_2| \frac{|d\mathbf{l}_3 \cdot d\mathbf{l}_1 \times d\mathbf{l}_2|}{|d\mathbf{l}_1 \times d\mathbf{l}_2|} = |d\mathbf{l}_3 \cdot d\mathbf{l}_1 \times d\mathbf{l}_2|$$

$$= |d\mathbf{l}_1 \cdot d\mathbf{l}_2 \times d\mathbf{l}_3| \text{ or } = d\mathbf{l}_2 \cdot d\mathbf{l}_3 \times d\mathbf{l}_1$$

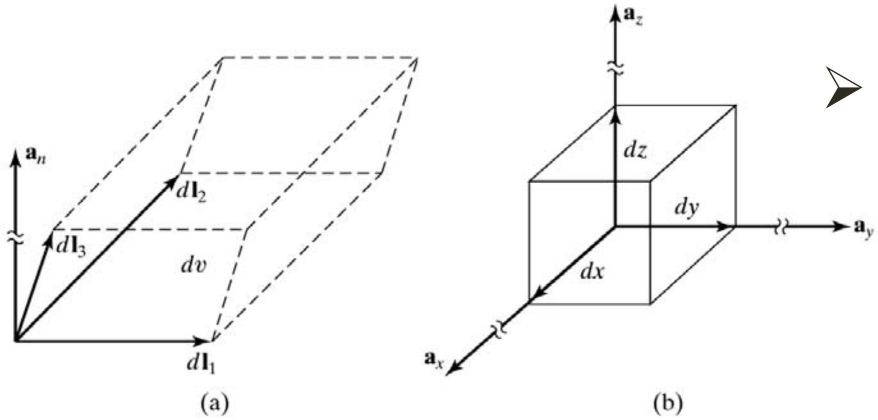


FIGURE 1.13

(a) Parallelepiped defined by three differential length vectors originating at a point.
(b) Differential volume in the Cartesian coordinate system.

- For differential vectors of

$$dx\,\mathbf{a}_x,\ dy\,\mathbf{a}_y, \text{ and } dz\,\mathbf{a}_z$$

The volume is

$$\boxed{dv = dx\,dy\,dz}$$

II. CARTESIAN COORDINATE SYSTEM

- Arbitrary surface
  - If a surface is defined with a function of $f(x, y, z) = 0$, particularly for a plane with the intercepts on x, y, and z axes can be written as

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} - 1 = 0$$

  - A curve is the intersection of two surfaces, so an arbitrary curve is defined by two surface equations

$$f(x, y, z) = 0 \qquad \text{and} \qquad g(x, y, z) = 0$$

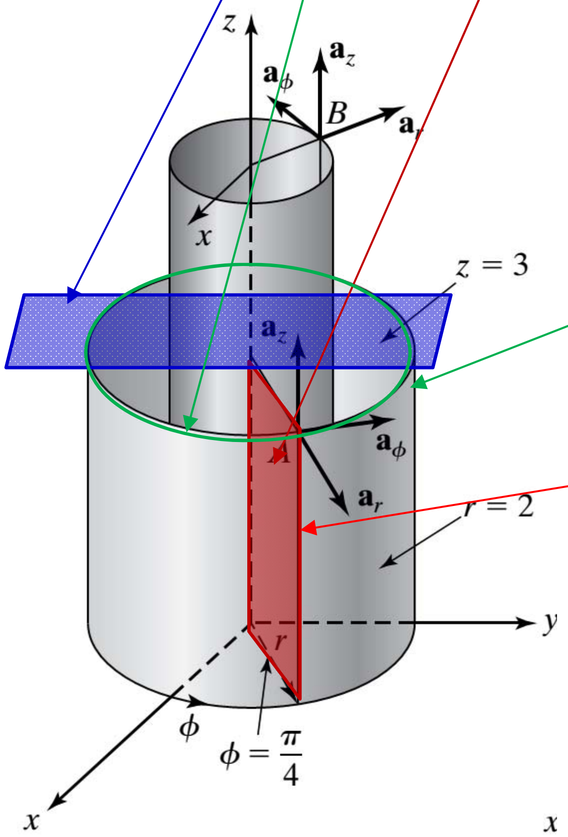
  - One can represent a curve by specifying the coordinate trace vs. time such as

$$x = x(t), \qquad y = y(t), \qquad z = z(t)$$

  - Example: a straight line passing through the origin with same angles to x, y, and z axes.
    - ✓ y = x (plane) & z = x (plane) intersecting to form a line
    - ✓ x = t, y = t, z = t, which can also refer to the same thing

III. CYLINDRICAL & SPHERICAL COORDINATE SYSTEMS

- Two other coordinate systems with sets of 3 orthogonal surfaces: **cylindrical & spherical**
  - Cylindrical: a point can be depicted by 3 surfaces
    - ✓ z = constant (Cartesian surface as before)
    - ✓ r = constant (cylinder surface)
    - ✓ ϕ = constant (a plane normal to x-y plane)



- Example:
  - r = 2, z = 3 intersect to form a curve (orange circle)
  - Then this green curve intersects with $\phi = \frac{\pi}{4}$ (red plane) to get a point A.
  - You have another way to get point A:
    1. r = 2 and $\phi = \frac{\pi}{4}$ intersect to form a vertical line
    2. Then this vertical line intersects with z = 3 to get point A

III. CYLINDRICAL & SPHERICAL COORDINATE SYSTEMS

- Unit vectors and volume, etc.
  - ✓ Unit vectors: $\boxed{\mathbf{a}_r,\ \mathbf{a}_\phi,\ \text{and}\ \mathbf{a}_z}$

    along r, ϕ, and z directions, where $\mathbf{a}_\phi$ is normal to $\mathbf{a}_r$ and along the tangent of the arc.
  - ✓ Orthonormality:

    $\mathbf{a}_r \times \mathbf{a}_\phi = \mathbf{a}_z,\ \ \mathbf{a}_\phi \times \mathbf{a}_z = \mathbf{a}_r,\ \mathbf{a}_z \times \mathbf{a}_r = \mathbf{a}_\phi$

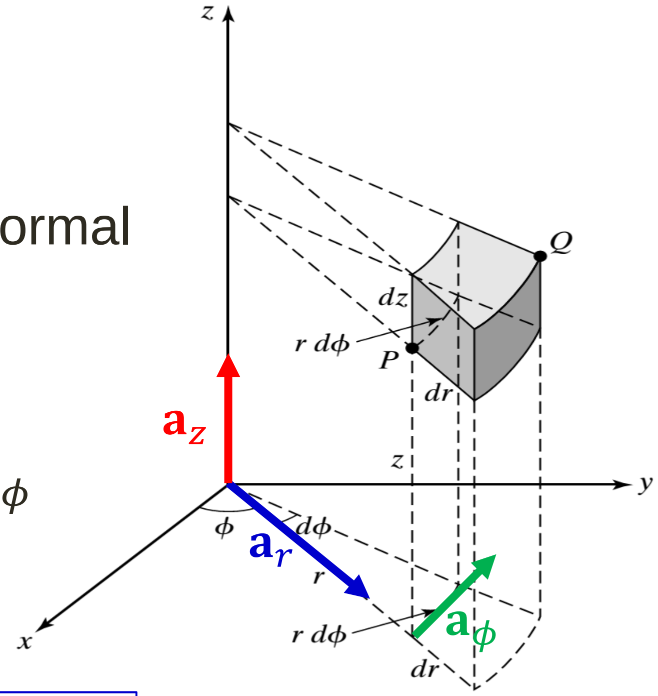
    based on right-hand rule.
  - ✓ Differential length vectors:

$$\boxed{d\mathbf{l} = dr\,\mathbf{a}_r + r\,d\phi\,\mathbf{a}_\phi + dz\,\mathbf{a}_z}$$

  - ✓ Differential surface vectors

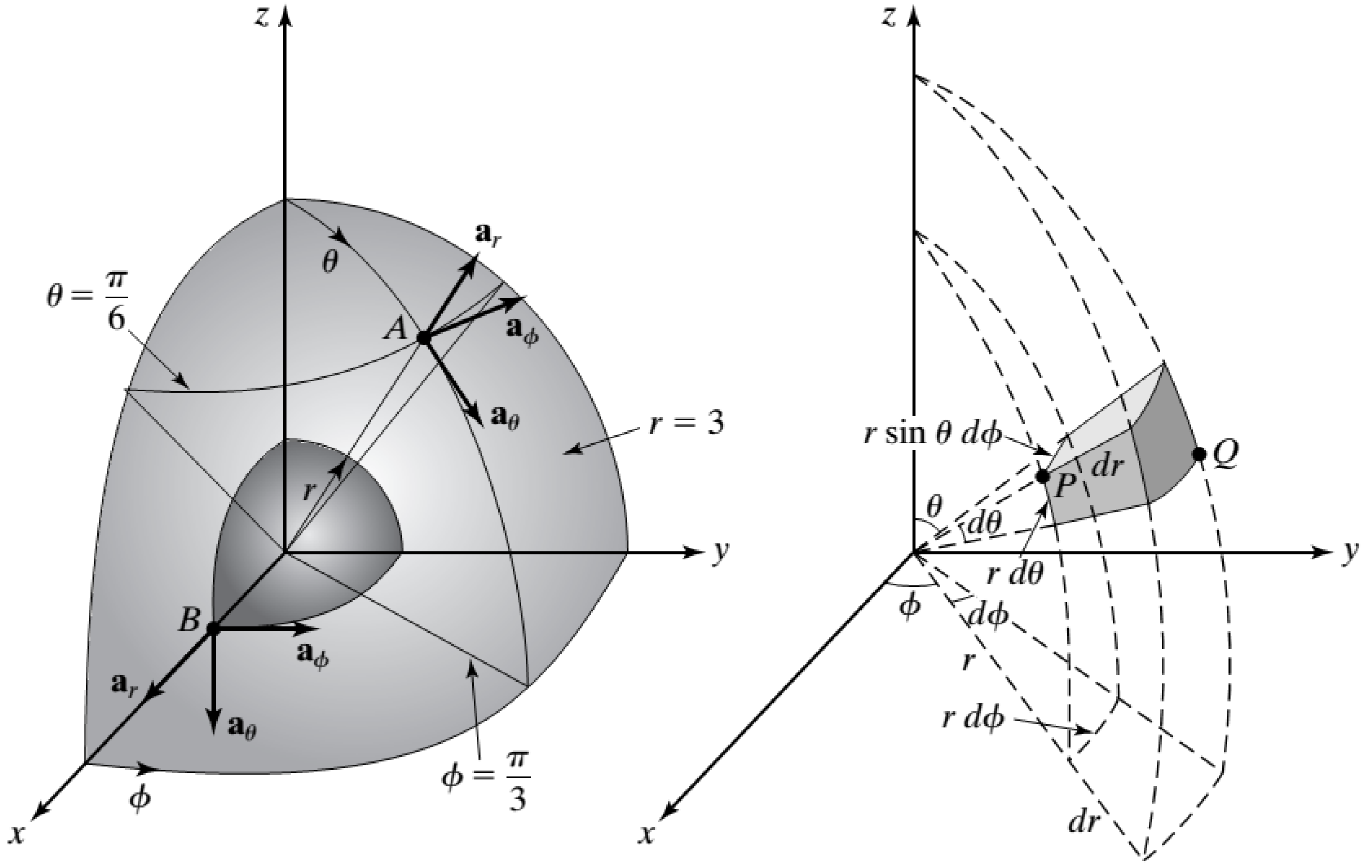
$$\boxed{\begin{aligned} \pm r\,d\phi\,\mathbf{a}_\phi \times dz\,\mathbf{a} &= \pm r\,d\phi\,dz\,\mathbf{a}_r \\ \pm dz\,\mathbf{a}_z \times dr\,\mathbf{a}_r &= \pm dr\,dz\,\mathbf{a}_\phi \\ \pm dr\,\mathbf{a}_r \times r\,d\phi\,\mathbf{a}_\phi &= \pm r\,dr\,d\phi\,\mathbf{a}_z \end{aligned}}$$

  - ✓ Volume: $\boxed{dv = (dr)(r\,d\phi)(dz) = r\,dr\,d\phi\,dz}$



III. CYLINDRICAL & SPHERICAL COORDINATE SYSTEMS

- Spherical system: a point can be depicted by 3 surfaces
  - ✓ r = constant (a sphere with distance from origin)
  - ✓ ϕ = constant (a plane normal to x-y plane)
  - ✓ θ = constant (a cone with an angle relative to z-axis)



III. CYLINDRICAL & SPHERICAL COORDINATE SYSTEMS

- Coordinate conversion:
  - Cartesian in terms of cylindrical or spherical system

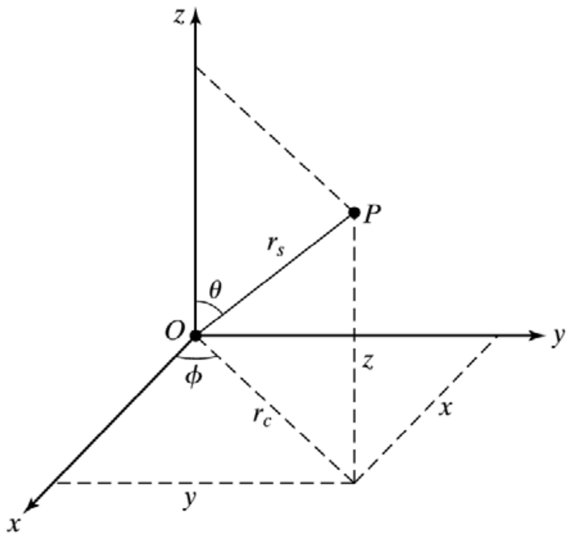
$$x = r_c \cos\phi \qquad y = r_c \sin\phi \qquad z = z$$

$$x = r_s \sin\theta\cos\phi \qquad y = r_s \sin\theta \sin\phi \qquad z = r_s \cos\theta$$

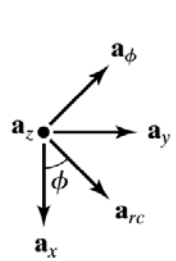
  - Cylindrical or Spherical in terms of Cartesian system

$$r_c = \sqrt{x^2 + y^2} \qquad \phi = \tan^{-1}\frac{y}{x} \qquad z = z$$

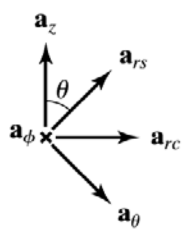
$$r_s = \sqrt{x^2 + y^2 + z^2} \qquad \theta = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z} \qquad \phi = \tan^{-1}\frac{y}{x}$$



Cartesian



Cylindrical



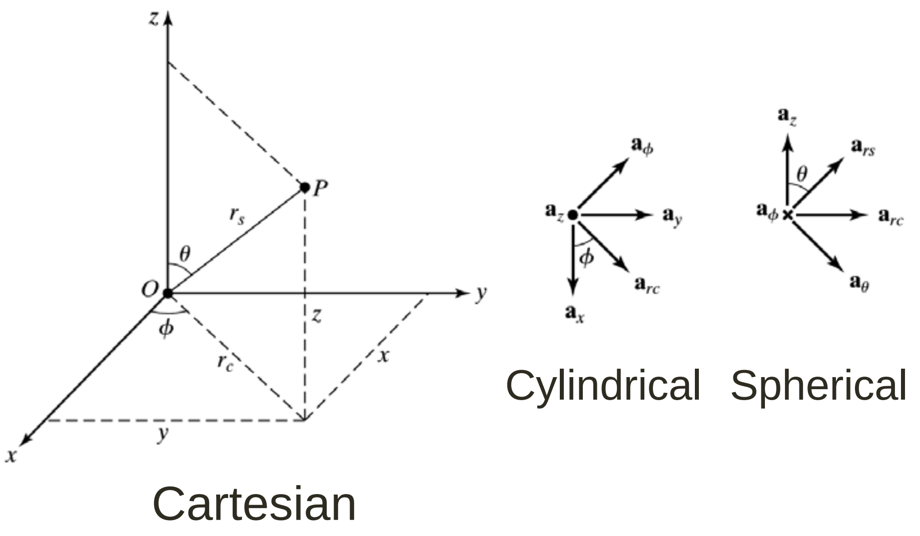
Spherical

III. CYLINDRICAL & SPHERICAL COORDINATE SYSTEMS

- Coordinate conversion:

$$x = r_c \cos\phi \qquad y = r_c \sin\phi \qquad z = z$$
$$x = r_s \sin\theta\cos\phi \qquad y = r_s \sin\theta \sin\phi \qquad z = r_s\cos\theta$$

$$r_c = \sqrt{x^2+y^2} \qquad \phi = \tan^{-1}\frac{y}{x} \qquad z = z$$
$$r_s = \sqrt{x^2+y^2+z^2} \qquad \theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z} \qquad \phi = \tan^{-1}\frac{y}{x}$$



$$\mathbf{a}_{rc}\cdot\mathbf{a}_x = \cos\phi \qquad \mathbf{a}_{rc}\cdot\mathbf{a}_y = \sin\phi \qquad \mathbf{a}_{rc}\cdot\mathbf{a}_z = 0$$
$$\mathbf{a}_{\phi}\cdot\mathbf{a}_x = -\sin\phi \qquad \mathbf{a}_{\phi}\cdot\mathbf{a}_y = \cos\phi \qquad \mathbf{a}_{\phi}\cdot\mathbf{a}_z = 0$$
$$\mathbf{a}_{z}\cdot\mathbf{a}_x = 0 \qquad \mathbf{a}_{z}\cdot\mathbf{a}_y = 0 \qquad \mathbf{a}_{z}\cdot\mathbf{a}_z = 1$$

$$\mathbf{a}_{rs}\cdot\mathbf{a}_x = \sin\theta\cos\phi \qquad \mathbf{a}_{rs}\cdot\mathbf{a}_y = \sin\theta\sin\phi \qquad \mathbf{a}_{rs}\cdot\mathbf{a}_z = \cos\theta$$
$$\mathbf{a}_{\theta}\cdot\mathbf{a}_x = \cos\theta\cos\phi \qquad \mathbf{a}_{\theta}\cdot\mathbf{a}_y = \cos\theta\sin\phi \qquad \mathbf{a}_{\theta}\cdot\mathbf{a}_z = -\sin\theta$$
$$\mathbf{a}_{\phi}\cdot\mathbf{a}_x = -\sin\phi \qquad \mathbf{a}_{\phi}\cdot\mathbf{a}_y = \cos\phi \qquad \mathbf{a}_{\phi}\cdot\mathbf{a}_z = 0$$

III. CYLINDRICAL & SPHERICAL COORDINATE SYSTEMS

- ***Example 1.4***: Convert $3\mathbf{a}_x + 4\mathbf{a}_y + 5\mathbf{a}_z$ into spherical coordinate

$$r_s = \sqrt{x^2 + y^2 + z^2}$$
$$\theta = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z}$$
$$\phi = \tan^{-1}\frac{y}{x}$$

$$\boxed{\begin{aligned} r_s &= \sqrt{3^2 + 4^2 + 5^2} = 5\sqrt{2} \\ \theta &= \tan^{-1}\frac{\sqrt{3^2 + 4^2}}{5} = \tan^{-1} 1 = 45^\circ \\ \phi &= \tan^{-1}\tfrac{4}{3} = 53.13^\circ \end{aligned}}$$

$$\begin{bmatrix} \mathbf{a}_x \\ \mathbf{a}_y \\ \mathbf{a}_z \end{bmatrix} = \begin{bmatrix} (\mathbf{a}_x \cdot \mathbf{a}_{rs}) & (\mathbf{a}_x \cdot \mathbf{a}_\theta) & (\mathbf{a}_x \cdot \mathbf{a}_\phi) \\ (\mathbf{a}_y \cdot \mathbf{a}_{rs}) & (\mathbf{a}_y \cdot \mathbf{a}_\theta) & (\mathbf{a}_y \cdot \mathbf{a}_\phi) \\ (\mathbf{a}_z \cdot \mathbf{a}_{rs}) & (\mathbf{a}_z \cdot \mathbf{a}_\theta) & (\mathbf{a}_z \cdot \mathbf{a}_\phi) \end{bmatrix} \begin{bmatrix} \mathbf{a}_{rs} \\ \mathbf{a}_\theta \\ \mathbf{a}_\phi \end{bmatrix}$$

$$\boxed{\begin{array}{lll} \mathbf{a}_{rs} \cdot \mathbf{a}_x = \sin\theta\cos\phi & \mathbf{a}_{rs} \cdot \mathbf{a}_y = \sin\theta\sin\phi & \mathbf{a}_{rs} \cdot \mathbf{a}_z = \cos\theta \\ \mathbf{a}_\theta \cdot \mathbf{a}_x = \cos\theta\cos\phi & \mathbf{a}_\theta \cdot \mathbf{a}_y = \cos\theta\sin\phi & \mathbf{a}_\theta \cdot \mathbf{a}_z = -\sin\theta \\ \mathbf{a}_\phi \cdot \mathbf{a}_x = -\sin\phi & \mathbf{a}_\phi \cdot \mathbf{a}_y = \cos\phi & \mathbf{a}_\phi \cdot \mathbf{a}_z = 0 \end{array}}$$

$$= \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} \mathbf{a}_{rs} \\ \mathbf{a}_\theta \\ \mathbf{a}_\phi \end{bmatrix} = \begin{bmatrix} 0.3\sqrt{2} & 0.3\sqrt{2} & -0.8 \\ 0.4\sqrt{2} & 0.4\sqrt{2} & 0.6 \\ 0.5\sqrt{2} & -0.5\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{a}_{rs} \\ \mathbf{a}_\theta \\ \mathbf{a}_\phi \end{bmatrix}$$

$$\begin{aligned} 3\mathbf{a}_x + 4\mathbf{a}_y + 5\mathbf{a}_z &= 3(0.3\sqrt{2}\mathbf{a}_{rs} + 0.3\sqrt{2}\mathbf{a}_\theta - 0.8\mathbf{a}_\phi) \\ &\quad + 4(0.4\sqrt{2}\mathbf{a}_{rs} + 0.4\sqrt{2}\mathbf{a}_\theta + 0.6\mathbf{a}_\phi) \\ &\quad + 5(0.5\sqrt{2}\mathbf{a}_{rs} - 0.5\sqrt{2}\mathbf{a}_\theta) \\ &= 5\sqrt{2}\mathbf{a}_{rs} \end{aligned}$$

IV. SCALAR AND VECTOR FIELDS

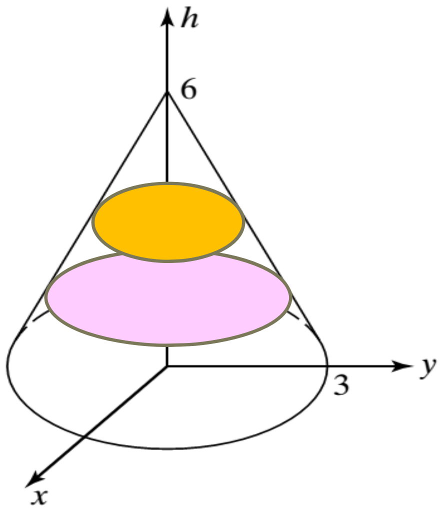
- Field:
  - A physical phenomenon associated with any points in a region in space (such as gravitational field), usually unseen. But light (EM) field can been seen (not all spectrum).
  - A physical quantity varies from point to point (in space) and in time.
  - Scalar vs. vector: if the direction matters called vector filed (EM); otherwise is scalar field (temperature).
  - Static vs. time-varying:
    - ✓ as a charge exists in a fixed point, it will exert an electric field without changing its direction/magnitude with time.
    - ✓ Temperature is changing day and nights in Taiwan, so we say it's time-varying.

IV. SCALAR AND VECTOR FIELDS

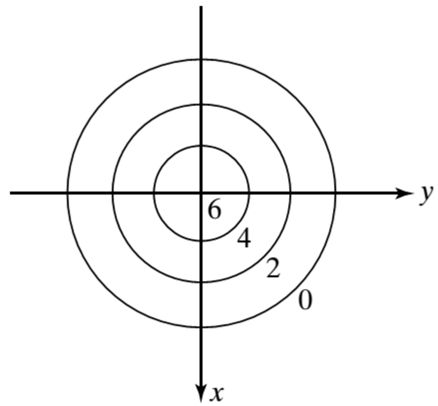
Scalar field:

- Relationship between height (h) and base position (x,y) can be written as

$$h(x, y) = 6 - 2\sqrt{x^2 + y^2}$$

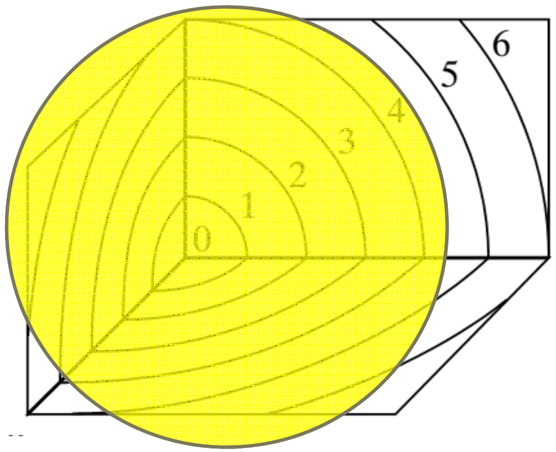


- In a circular contour in x-y plan, one can determine its height, which does not matter with the direction, but only the magnitude of height. So it's a **two-dimensional scalar field** with its field strength in terms of z (h, in this case).



- In a rectangular system, a "distance field" expressed in terms of r of points from the origin O is

$$r(x, y, z) = \sqrt{x^2 + y^2 + z^2}$$



- A constant-distance surface is a surface with the same r, which is a **three-dimensional scalar and static field** (not changing with time)

IV. SCALAR AND VECTOR FIELDS

Time-Varying Field:

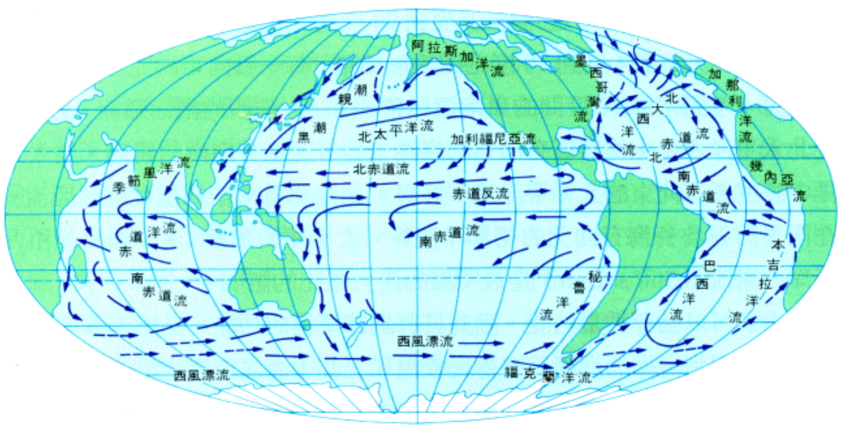
- Scalar field is generally written as $F_s(x, y, z, t)$, where *x*, *y*, and *z* denote the space and *t* represents time.
- Vector field is written with directions:

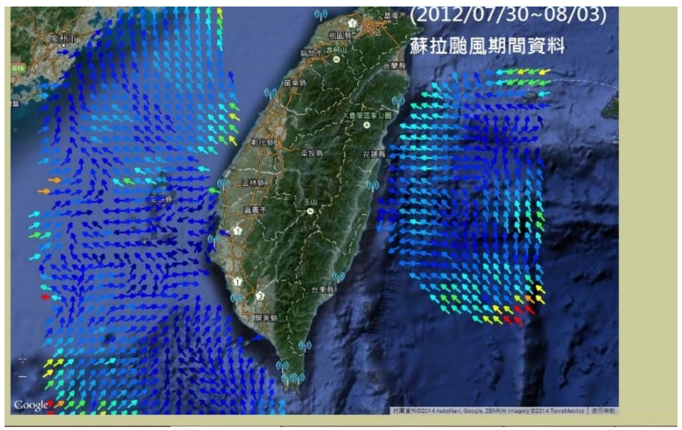
Cartesian $\quad \mathbf{F}(x, y, z, t) = F_x(x, y, z, t)\mathbf{a}_x + F_y(x, y, z, t)\mathbf{a}_y + F_z(x, y, z, t)\mathbf{a}_z$

Cylindrical $\quad \mathbf{F}(r, \phi, z, t) = F_r(r, \phi, z, t)\mathbf{a}_r + F_\phi(r, \phi, z, t)\mathbf{a}_\phi + F_z(r, \phi, z, t)\mathbf{a}_z$

Spherical $\quad \mathbf{F}(r, \theta, \phi, t) = F_r(r, \theta, \phi, t)\mathbf{a}_r + F_\theta(r, \theta, \phi, t)\mathbf{a}_\theta + F_\phi(r, \theta, \phi, t)\mathbf{a}_\phi$

- Vector fields can be depicted by "**directional**" lines or called **streamlines** or **flux** lines.





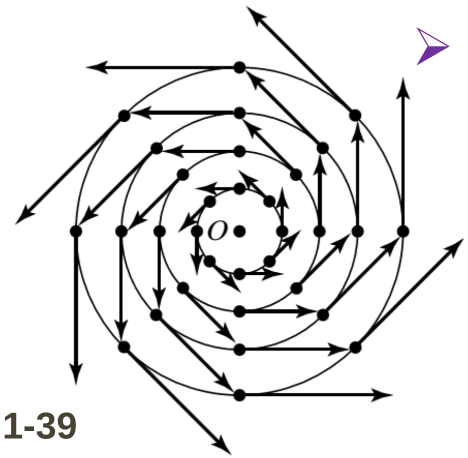
IV. SCALAR AND VECTOR FIELDS

- A direction line is a curve constructed such that the field is tangential to the curve for all points. So it means at any point, there exists a direction line (or vector) which describes field direction (by arrows) with certain magnitude (by colors or arrow length).

- To quantify this, we consider a vector **F** with a differential length vector *d***l** tangential to the curve. Since **F** // *d***l**, we have

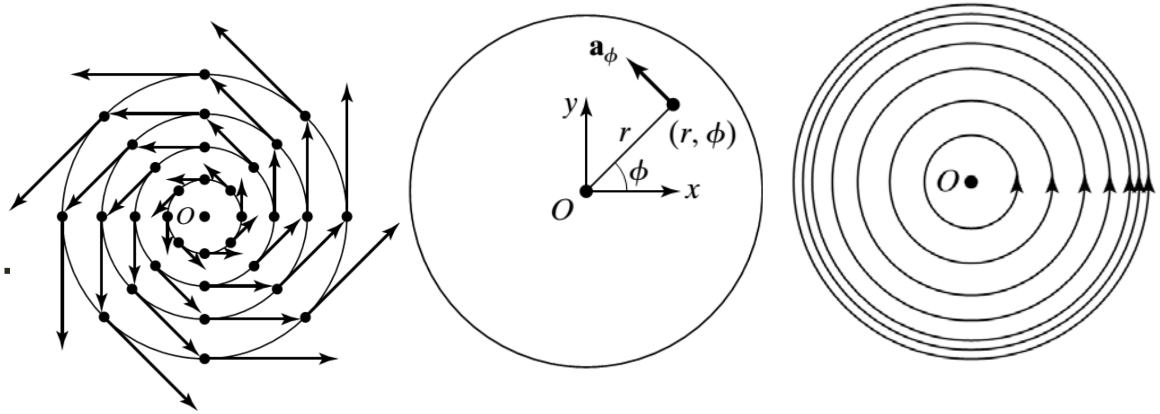
$$\boxed{\frac{dx}{F_x} = \frac{dy}{F_y} = \frac{dz}{F_z}} \qquad \boxed{\frac{dr}{F_r} = \frac{r\,d\phi}{F_\phi} = \frac{dz}{F_z}} \qquad \boxed{\frac{dr}{F_r} = \frac{r\,d\theta}{F_\theta} = \frac{r\sin\theta\,d\phi}{F_\phi}}$$

Cartesian **Cylindrical** **Spherical**



- ***Example 1.5***: Linear velocity vector field on a rotating disk
  - ✓ A circular disk of radius a rotating with constant angular velocity ω along an axis normal to the disk.
  - ✓ Now let's describe its motion by denoting linear velocity vector field.

IV. SCALAR AND VECTOR FIELDS



- ✓ Let's select x-y as its coordinate and set up as a 2-D problem with its center as the origin.
- ✓ Since it's a centrifuge motion, we prefer to convert x-y into (r, ϕ) to conserve its rotational symmetry.
- ✓ The linear tangential velocity is given as $\mathbf{v}(r, \phi) = \omega r \mathbf{a}_\phi$. Then based on slide 1-39, we have

$$\frac{dr}{0} = \frac{r\,d\phi}{\omega r} \quad \text{or} \quad \boxed{\begin{aligned} dr &= 0 \\ r &= \text{constant} \end{aligned}}$$

- ✓ To obtain the equations for directional lines, we can write

$$\begin{aligned} \mathbf{v}(x, y) &= \omega r(\mathbf{a}_\phi \cdot \mathbf{a}_x)\mathbf{a}_x + \omega r(\mathbf{a}_\phi \cdot \mathbf{a}_y)\mathbf{a}_y \\ &= \omega r(-\sin\phi\, \mathbf{a}_x + \cos\phi\, \mathbf{a}_y) \\ &= \omega(-y\mathbf{a}_x + x\mathbf{a}_y) \end{aligned} \quad \Rightarrow \quad \frac{dx}{-y} = \frac{dy}{x}$$

$$x\,dx + y\,dy = 0$$

$$\boxed{x^2 + y^2 = \text{constant}}$$

V. ELECTRIC FIELD

- Analogy to gravitational field
  - In a gravitational field, two bodies are attracted by the force exerted associated with their masses m_1 and m_2 and the distance *d* in between, and last a constant *A* to depict the relationship ($F_G = A\frac{m_1 m_2}{d^2}$).
  - For an electric (E) field, the associated physical quantity becomes electric charges (q), which can be positive or negative. In SI unit, an electron owns a charge of $\mathbf{-1.602 \times 10^{-19}}$ **Coulomb (or C).** **<u>The body can only have charges with its magnitude equal to positive (or negative) multiples (integer) electrons.</u>**
  - Some facts on E field:
    - ✓ Magnitude of force $\propto$ products of the associated charges
    - ✓ Magnitude of force inversely $\propto$ the square of distance between charges
    - ✓ Depending on medium (associated with dielectric constant)
    - ✓ Force direction along the line connecting the charges
    - ✓ +/+ and -/- repel; +/- and -/+ attract

V. ELECTRIC FIELD

Electric force

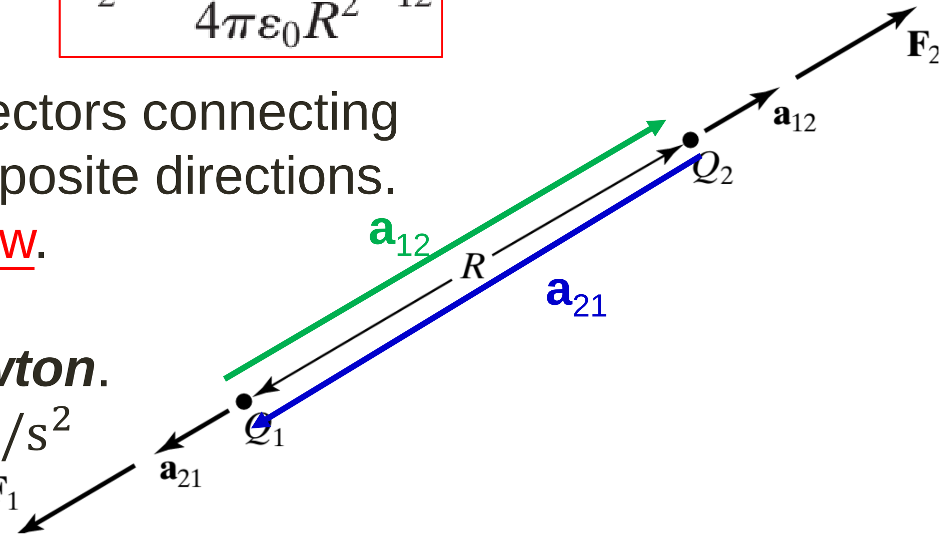
- Gravitational force: $F_G = A_G \frac{m_1 m_2}{d^2}$
- Electric force in vacuum: $F_E = A_E \frac{Q_1 Q_2}{d^2}$ where $A_E = \frac{1}{4\pi\varepsilon_0}$

  ε_0 is the permittivity in free space and equal to 8.854×10^{-12} Farads/meter or Coulombs/(Volt-meter)

- Assume two charges Q_1 and Q_2 separated by R, the force experienced (or exerted/applied on) by Q_1/Q_2 are

$$\boxed{\mathbf{F}_1 = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^2}\mathbf{a}_{21}} \qquad \boxed{\mathbf{F}_2 = \frac{Q_2 Q_1}{4\pi\varepsilon_0 R^2}\mathbf{a}_{12}}$$

  where $\mathbf{a}_{21}$ and $\mathbf{a}_{12}$ are unit vectors connecting point 2 to 1 (1 to 2) along opposite directions. which is called Coulomb's law.



- ***The unit of the force is newton***. , which is equal to $\mathrm{kg} \cdot \mathrm{meter/s}^2$

V. ELECTRIC FIELD

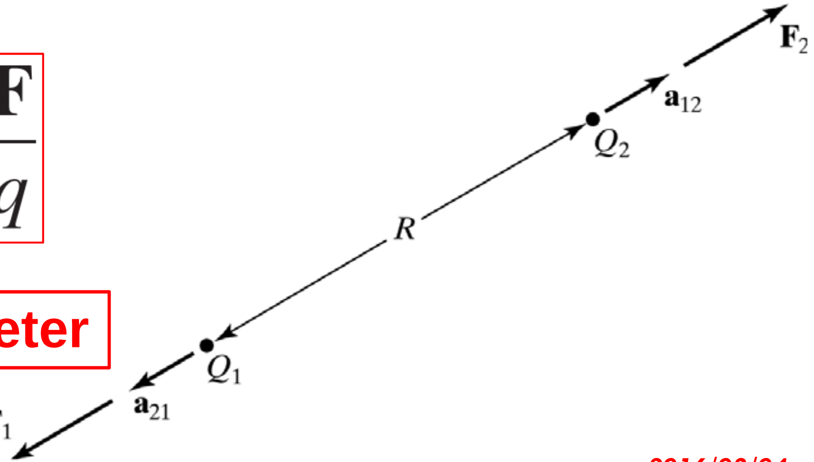
- Electric field
  - Gravitation field: the gravitational field created by body 1 (with mass m_1) is $\boxed{GF = A_G \frac{m_1}{d^2} = \frac{F_G}{m_2}}$
  - Electric field: the magnitude of electric field created by body 1 (with charge Q_1) is

$$\boxed{E = \frac{1}{4\pi\varepsilon_0}\frac{Q_1}{d^2} = \frac{F}{Q_2}}$$

  - More rigorous definition: a force experienced by a testing charge with extremely small quantity of charge placed in an electric field which is created by whatever other charges divided by its charge magnitude.

$$\boxed{\mathbf{E} = \lim_{q \to 0} \frac{\mathbf{F}}{q}}$$

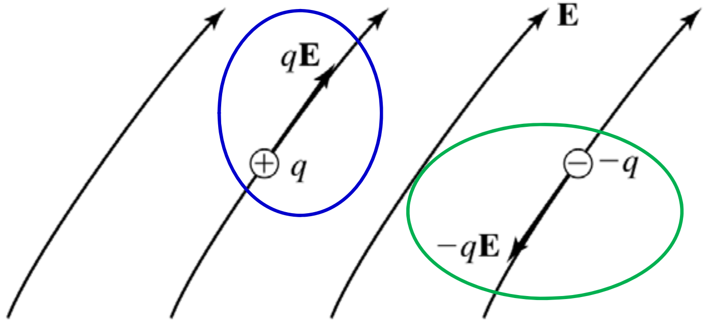
  - Unit: **Newton/Coulomb** or **Volt/meter**



V. ELECTRIC FIELD

- Field vs. force: depending on the charge type, if positive, the direction is parallel; if negative, the direction is anti-parallel.

$$\boxed{\mathbf{E} = \lim_{q \to 0} \frac{\mathbf{F}}{q}}$$



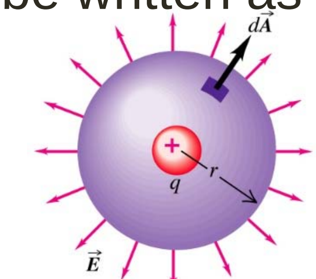
- If a testing charge of q is placed in a field created by a charge Q_1, one can write the force on testing charge q and field created by Q_1 as follows: (note: the direction is $\mathbf{a}_{12}$ from Q_1 to q)

$$\mathbf{F}_2 = \frac{Q_1 q}{4\pi\varepsilon_0 R^2}\mathbf{a}_{12} \qquad \mathbf{E}_2 = \frac{\mathbf{F}_2}{q} = \frac{Q_1}{4\pi\varepsilon_0 R^2}\mathbf{a}_{12}$$

- A general form of an **E** field by a charge Q can be written as

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_0 R^2}\mathbf{a}_R$$

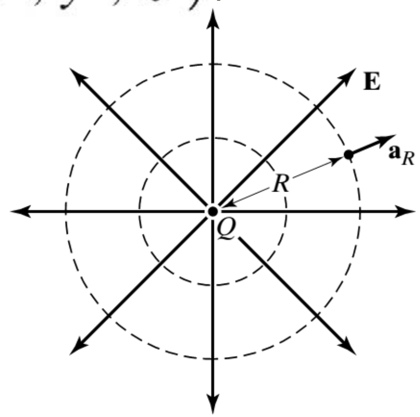
where $\mathbf{a}_R$ represents the outward unit vector



http://www.physics.sjsu.edu/becker/physics51/images/23_09Flux_Sphere.JPG

V. ELECTRIC FIELD

- At point $P(x, y, z)$ due to a point charge *Q* at $P'(x', y', z')$, the distance vector **R** from *P*' to *P* is given



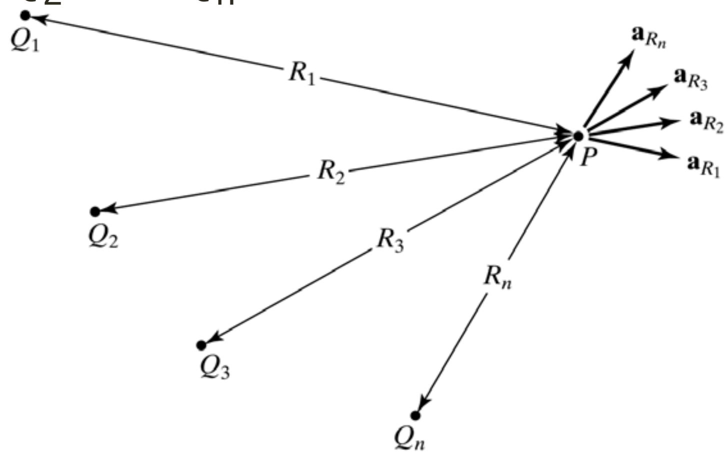
$$[(x - x')\mathbf{a}_x + (y - y')\mathbf{a}_y + (z - z')\mathbf{a}_z]$$

with unit vector of $\mathbf{a}_R = \mathbf{R}/R$.

- The **E** field can be written as follows:

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_0 R^2}\mathbf{a}_R = \frac{Q\mathbf{R}}{4\pi\varepsilon_0 R^3} = \frac{Q}{4\pi\varepsilon_0}\frac{(x - x')\mathbf{a}_x + (y - y')\mathbf{a}_y + (z - z')\mathbf{a}_z}{[(x - x')^2 + (y - y')^2 + (z - z')^2]^{3/2}}$$

- For a point *P* with multiple charges Q_1, Q_2, …, Q_n, the electric field is



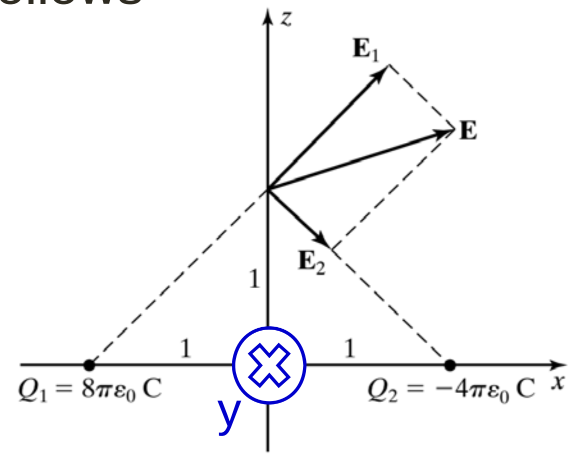
$$\mathbf{E} = \frac{Q_1}{4\pi\varepsilon_0 R_1^2}\mathbf{a}_{R_1} + \frac{Q_2}{4\pi\varepsilon_0 R_2^2}\mathbf{a}_{R_2} + \cdots + \frac{Q_n}{4\pi\varepsilon_0 R_n^2}\mathbf{a}_{R_n}$$

V. ELECTRIC FIELD

- ***Example 1.6***: 2 point charges $Q_1 = 8\pi\varepsilon_0$ C & $Q_2 = -4\pi\varepsilon_0$ C at (-1,0,0) and (1,0,0), respectively. We'd like to a) find E field at (0,0,1) and b) discuss streamlines of E field through (0,0,1).

a) At (0,0,1), the E field can be calculated as follows based on slide (1-44):

$$[\mathbf{E}]_{(0,0,1)} = [\mathbf{E}_1]_{(0,0,1)} + [\mathbf{E}_2]_{(0,0,1)}$$
$$= \frac{8\pi\varepsilon_0}{4\pi\varepsilon_0}\frac{(\mathbf{a}_x + \mathbf{a}_z)}{2^{3/2}} - \frac{4\pi\varepsilon_0}{4\pi\varepsilon_0}\frac{(-\mathbf{a}_x + \mathbf{a}_z)}{2^{3/2}}$$
$$= 1.118\left(\frac{3\mathbf{a}_x + \mathbf{a}_z}{\sqrt{10}}\right)$$

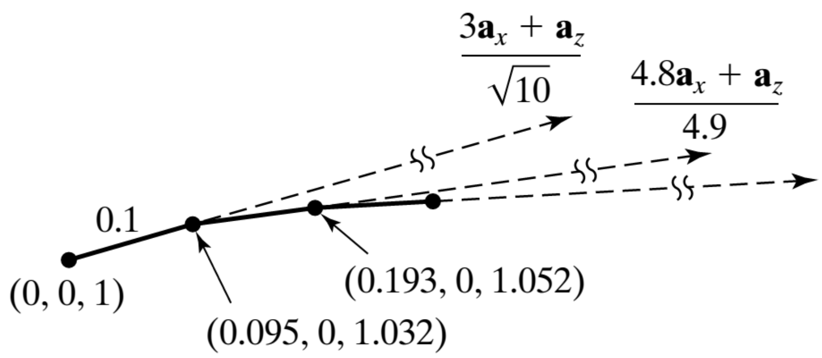


b) For streamlines, we recall they are curves such that at any given point on the curves, the E-field is tangential to the curve with the magnitude labeled by the arrow length or color.

Let's pick an increment vector $(3\mathbf{a}_x + \mathbf{a}_z)/\sqrt{10}$ with magnitude of 0.1 along the unit vector of E field from (0,0,1), we got (0.095,0,1.032) $= (0,0,1) + 0.1 \times \left(\frac{3}{\sqrt{10}}, 0, \frac{1}{\sqrt{10}}\right)$

V. ELECTRIC FIELD

Then at this point, the E field is



$$[\mathbf{E}]_{(0.095,\,0,\,1.032)} = \frac{8\pi\varepsilon_0}{4\pi\varepsilon_0}\frac{(1.095\mathbf{a}_x + 1.032\mathbf{a}_z)}{(1.095^2 + 1.032^2)^{3/2}} - \frac{4\pi\varepsilon_0}{4\pi\varepsilon_0}\frac{(-0.905\mathbf{a}_x + 1.032\mathbf{a}_z)}{(0.905^2 + 1.032^2)^{3/2}}$$

$$= 1.015\left(\frac{4.8\mathbf{a}_x + \mathbf{a}_z}{4.9}\right)$$

Then we repeat a new unit vector $(4.8\mathbf{a}_x + \mathbf{a}_z)/4.9$ with magnitude of 0.1 to find the ending coordinate of (0.193,0,1.052)= $(0.095,0,1.032) + 0.1 \times \left(\frac{4.8}{4.9}, 0, \frac{1}{4.9}\right)$

Coordinates

$X = 0.095$ $\quad Z = 1.032$

Field strength

$E = 1.015$

Field components

$UX = 0.979$ $\quad UZ = 0.204$

V. ELECTRIC FIELD

- ✓ Moving toward Q_2

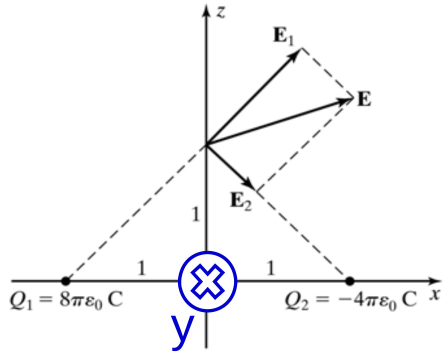


TABLE 1.1 Values of Parameters Pertinent to the Steps in the Computer Generation of the Direction Line of **E** in Fig. 1.25 for (a) the Segment from (0, 0, 1) Toward the Charge Q_2 and (b) the Segment from (0, 0, 1) Back Toward the Charge Q_1.

X = 0.000	Z = 1.000	E = 1.118	UX = 0.949	UZ = 0.316
X = 0.095	Z = 1.032	E = 1.015	UX = 0.979	UZ = 0.204
X = 0.193	Z = 1.052	E = 0.942	UX = 0.997	UZ = 0.076
X = 0.292	Z = 1.060	E = 0.898	UX = 0.998	UZ = −0.065
X = 0.392	Z = 1.053	E = 0.882	UX = 0.977	UZ = −0.215
X = 0.490	Z = 1.032	E = 0.898	UX = 0.930	UZ = −0.368
X = 0.583	Z = 0.995	E = 0.951	UX = 0.858	UZ = −0.513
X = 0.669	Z = 0.944	E = 1.051	UX = 0.766	UZ = −0.643
X = 0.745	Z = 0.879	E = 1.212	UX = 0.660	UZ = −0.751
X = 0.811	Z = 0.804	E = 1.459	UX = 0.548	UZ = −0.836
X = 0.866	Z = 0.721	E = 1.837	UX = 0.439	UZ = −0.899
X = 0.910	Z = 0.631	E = 2.426	UX = 0.337	UZ = −0.942
X = 0.944	Z = 0.536	E = 3.391	UX = 0.246	UZ = −0.969
X = 0.968	Z = 0.440	E = 5.100	UX = 0.167	UZ = −0.986
X = 0.985	Z = 0.341	E = 8.537	UX = 0.101	UZ = −0.995
X = 0.995	Z = 0.241	E = 17.101	UX = 0.049	UZ = −0.999
X = 1.000	Z = 0.142	E = 49.846	UX = 0.010	UZ = −1.000
X = 1.001	Z = 0.042	E = 577.540	UX = −0.023	UZ = −1.000

Number of steps = 17

(a)

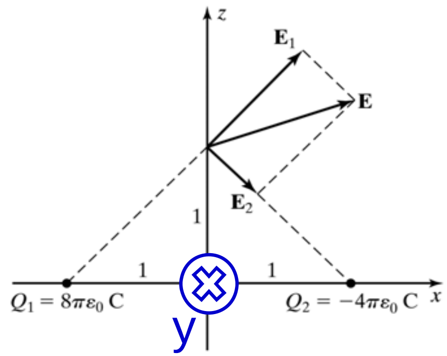
V. ELECTRIC FIELD

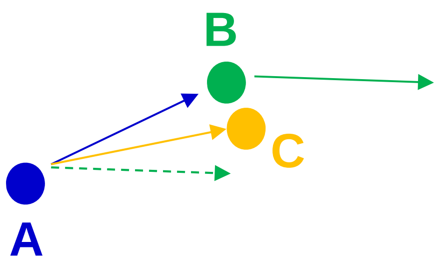
- ✓ Moving toward Q_1

X = 0.000	Z = 1.000	E = 1.118	UX = 0.949	UZ = 0.316
X = −.095	Z = 0.968	E = 1.243	UX = 0.908	UZ = 0.420
X = −.186	Z = 0.926	E = 1.411	UX = 0.862	UZ = 0.507
X = −.272	Z = 0.876	E = 1.634	UX = 0.815	UZ = 0.580
X = −.353	Z = 0.818	E = 1.931	UX = 0.768	UZ = 0.640
X = −.430	Z = 0.754	E = 2.333	UX = 0.724	UZ = 0.689
X = −.503	Z = 0.685	E = 2.888	UX = 0.684	UZ = 0.730
X = −.571	Z = 0.612	E = 3.681	UX = 0.648	UZ = 0.762
X = −.636	Z = 0.536	E = 4.871	UX = 0.616	UZ = 0.788
X = −.697	Z = 0.457	E = 6.769	UX = 0.590	UZ = 0.808
X = −.756	Z = 0.376	E = 10.074	UX = 0.568	UZ = 0.823
X = −.813	Z = 0.294	E = 16.616	UX = 0.551	UZ = 0.835
X = −.868	Z = 0.210	E = 32.588	UX = 0.538	UZ = 0.843
X = −.922	Z = 0.126	E = 91.176	UX = 0.529	UZ = 0.849
X = −.975	Z = 0.041	E = 860.610	UX = 0.522	UZ = 0.853

Number of steps = 14

(b)





- ✓ Cumulative errors
  - For points A to B with 0.1 x length of unit vector, we assume the **E** field is constant along the path, which is not the case. In reality, **E** field is changing across the entire space.
  - To Improve this cumulative error, we could insert a point, say C. The magnitude of the new vector is still 0.1 x length of unit vector, but along a new vector which is the bi-sect of E field at points A and B. not along the **E** vector at point A.

V. ELECTRIC FIELD

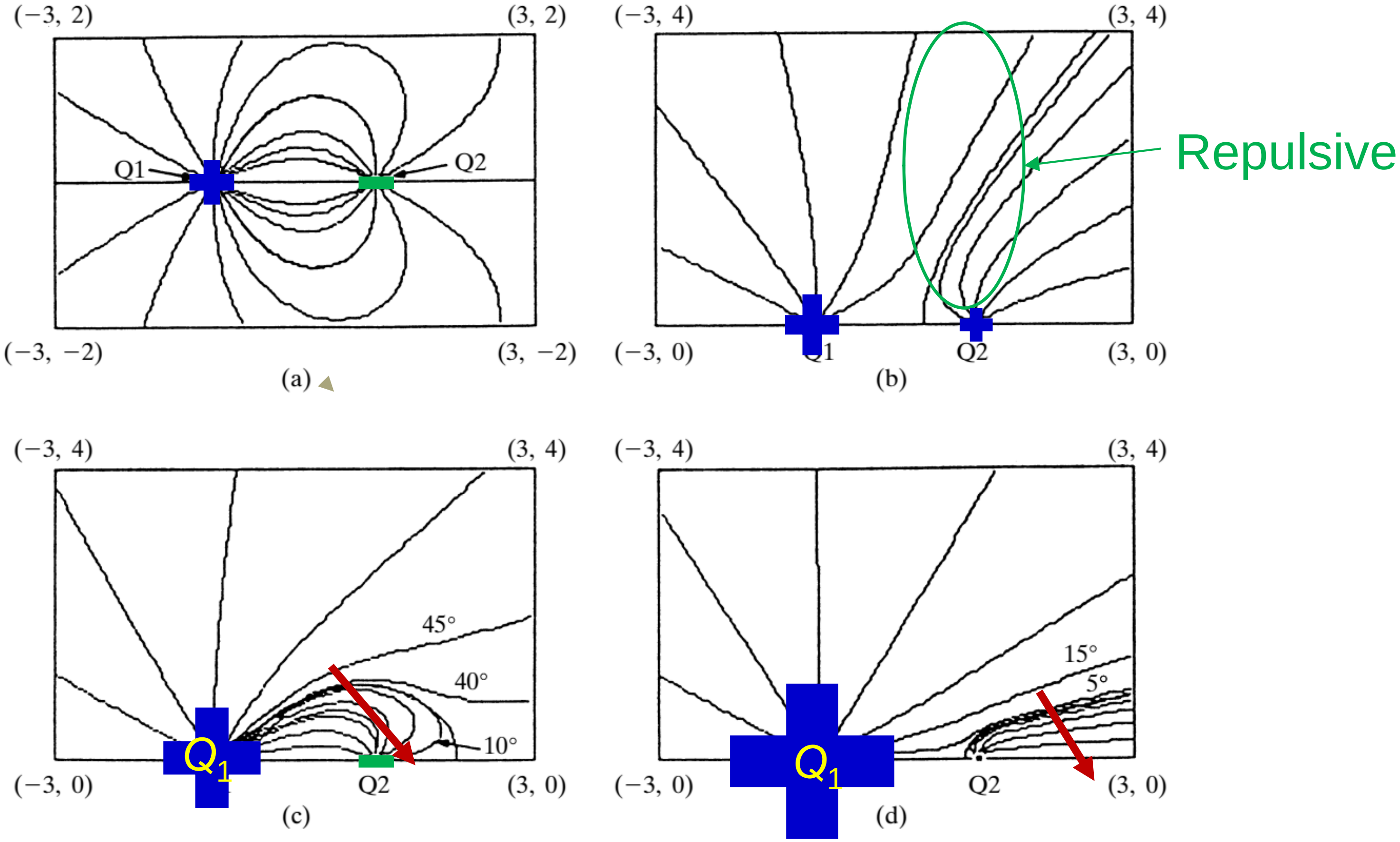
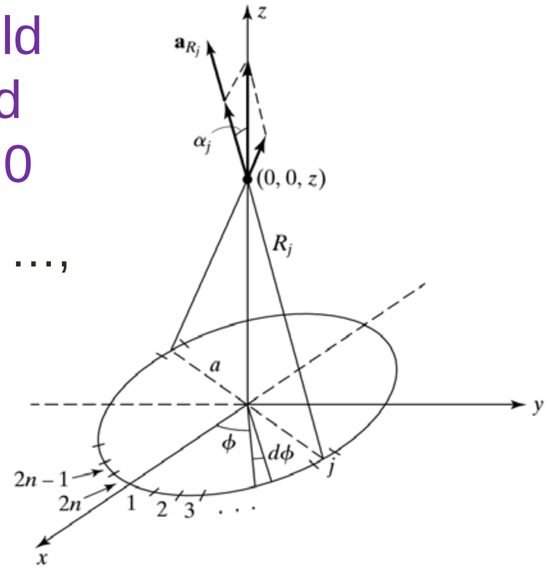


FIGURE 1.26

Computer-generated maps of direction lines of electric field for pairs of point charges Q_1 and Q_2 at $(-1, 0)$ and $(1, 0)$, respectively, in the xz-plane. (a) $Q_1 = 2Q, Q_2 = -Q$; (b) $Q_1 = 4Q, Q_2 = Q$; (c) $Q_1 = 9Q, Q_2 = -Q$; and (d) $Q_1 = 81Q, Q_2 = Q$.

V. ELECTRIC FIELD

➢ ***Example 1.7***: Find the intensity of electric field at z-axis for charge *Q* (C) uniformly distributed along the circle of radius *a* in x-y plane at *z* = 0



✓ Let's divide the ring into 2n segments (1, 2, 3, …, 2n). The electric field at j^{th} segment is

$$\mathbf{E}_j = \frac{Q_j}{4\pi\varepsilon_0 R_j^2}\mathbf{a}_{R_j}$$

✓ Since (total) Q is uniformly distributed, one can write Q_j as follows:

$$Q_j = \left(\frac{Q}{2\pi a}\right)\left(\frac{2\pi a}{2n}\right) = \frac{Q}{2n}$$

✓ And Rj is $R_j = \sqrt{z^2 + a^2}$

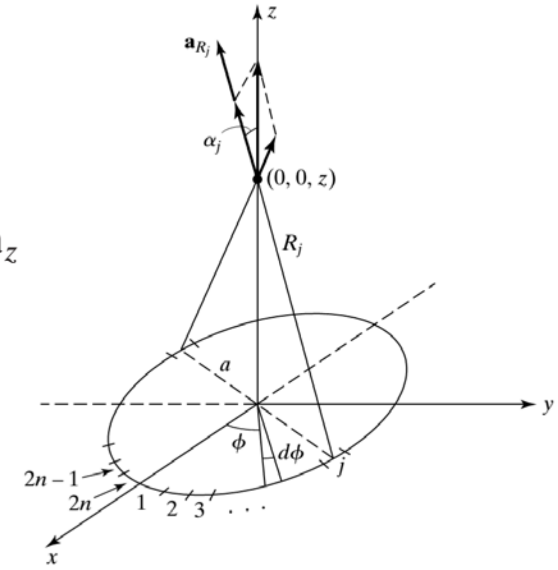
✓ Due to the circular symmetry, components of E fields lying on x-y plane canceled with each other, but only z-component left. We can sum up E fields as follows

$$[\mathbf{E}]_{(0,\,0,\,z)} = \sum_{j=1}^{n} \frac{2Q_j}{4\pi\varepsilon_0 R_j^2}(\mathbf{a}_{R_j}\cdot\mathbf{a}_z)\mathbf{a}_z$$

V. ELECTRIC FIELD

$$[\mathbf{E}]_{(0,0,z)} = \sum_{j=1}^{n} \frac{Q_j}{2\pi\varepsilon_0 R_j^2}\cos\alpha_j\,\mathbf{a}_z = \sum_{j=1}^{n} \frac{Q_j z}{2\pi\varepsilon_0 R_j^3}\mathbf{a}_z$$

$$= \sum_{j=1}^{n} \frac{Qz}{4\pi\varepsilon_0 n(z^2+a^2)^{3/2}}\mathbf{a}_z = \frac{Qz}{4\pi\varepsilon_0 (z^2+a^2)^{3/2}}\mathbf{a}_z$$



- ✓ The E field points toward +z if z > 0; toward –z direction if z < 0.

- ✓ Another way to summation is by integration as follows:

$$[\mathbf{E}]_{(0,0,z)} = \sum_{j=1}^{n} \frac{2Q_j}{4\pi\varepsilon_0 \boxed{R_j^2}} \boxed{(\mathbf{a}_{R_j}\cdot\mathbf{a}_z)}\mathbf{a}_z$$

$$[E_z]_{(0,0,z)} = \int_{\phi=0}^{\pi} \frac{2(Q/2\pi a)a\,d\phi}{4\pi\varepsilon_0 \boxed{(a^2+z^2)}} \boxed{\frac{z}{(a^2+z^2)^{1/2}}}$$

$$= \frac{Qz}{4\pi^2\varepsilon_0(a^2+z^2)^{3/2}}\int_{\phi=0}^{\pi} d\phi$$

$$= \frac{Qz}{4\pi\varepsilon_0(a^2+z^2)^{3/2}}$$

1. Uniformly charge distribution:

   $$\sum_1^{2n}.. = 2\times\sum_1^{n}.. = \int_0^{2\pi}.. = \int_0^{\pi} 2\times\cdots$$

2. Charge density:

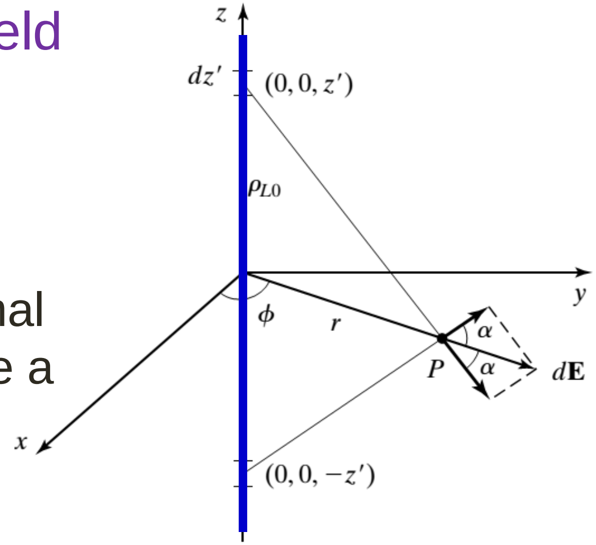
   $$R_j^2 \Rightarrow (a^2+z^2)$$

3. z-axis component

   $$(\mathbf{a}_{R_j}\cdot\mathbf{a}_z) \Rightarrow \frac{z}{(a^2+z^2)^{1/2}}$$

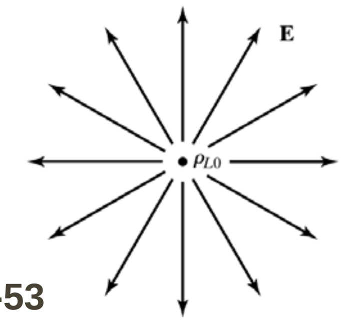
V. ELECTRIC FIELD

- ***Example 1.8***: Find the intensity of electric field everywhere for an infinitely long line charge (blue) along z –axis with uniform line density ρ_{L0} C/m.



  - ✓ Let's divide the line into a series of infinitesimal segments by considering each segment to be a point charge. Then superimpose them.
  - ✓ Consider at point P(r,ϕ,0):
    - Since point P lies at x-y plane (z = 0), exists a pair of segments on the charge line: (0,0, z') and (0,0,-z'). Both have the same length dz. The component along r direction of E field due to the two segments (points) can be written as

$$[d\mathbf{E}]_{(r,\phi,0)} = \underbrace{2}_{\text{z \& -z'}} \frac{\overbrace{\rho_{L0}\, dz'}^{\text{charge}}}{4\pi\varepsilon_0 \underbrace{[r^2 + (z')^2]}_{\text{distance}}} \overbrace{\cos\alpha\, \mathbf{a}_r}^{\mathbf{a}_r\text{ component}} = \frac{\rho_{L0} r\, dz'}{2\pi\varepsilon_0 [r^2 + (z')^2]^{3/2}} \mathbf{a}_r$$



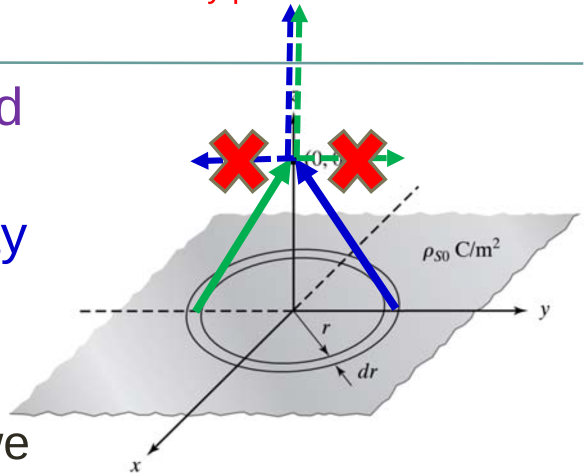
$$[\mathbf{E}]_{(r,\phi,0)} = \int_{z'=0}^{\infty} [d\mathbf{E}]_{(r,\phi,0)} = \int_{z'=0}^{\infty} \frac{\rho_{L0} r\, dz'}{2\pi\varepsilon_0 [r^2 + (z')^2]^{3/2}} \mathbf{a}_r = \frac{\rho_{L0}}{2\pi\varepsilon_0 r} \int_{\alpha=0}^{\pi/2} \cos\alpha\, d\alpha$$

$$\boxed{= \frac{\rho_{L0}}{2\pi\varepsilon_0 r} \mathbf{a}_r}$$

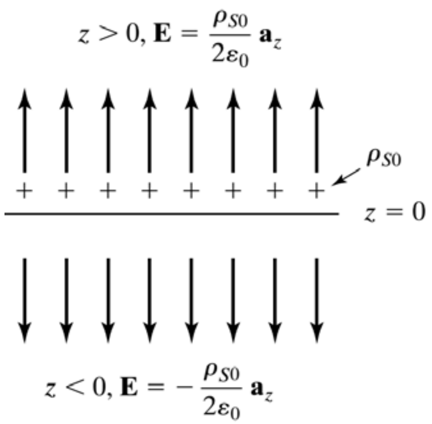
since the E_z components by –z and z cancelled each other.

V. ELECTRIC FIELD

the components with azimuthal angle
0 and π in x-y plane cancel each other



- ***Example 1.9***: Find the intensity of electric field everywhere for an infinite plane sheet of charge in x-y plane with uniform charge density ρ_{S0} C/m^2.
  - ✓ Similar to the above example, one can divide the charge sheet into very small parts. Then we select a point at z-axis (0,0,z).
  - ✓ Assuming the sheet can be divided and represented by a "circular plate" composed of many rings. At certain angle, the components of E fields are along z and x-y plane. At the opposite angle, there will be a x-y plane component which cancelled each other with only z component left.



charge

distance+z-component

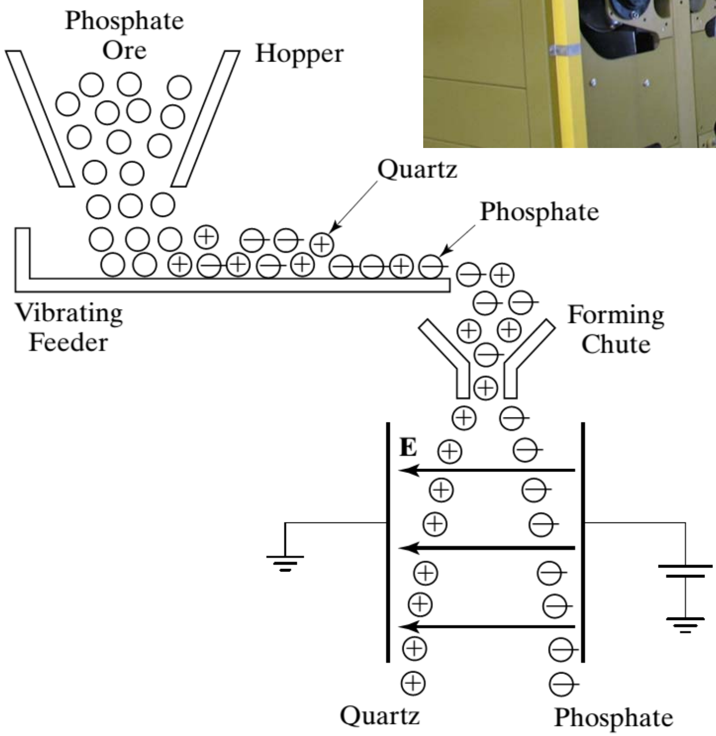
$$[d\mathbf{E}]_{(0,\,0,\,z)} = \frac{(\rho_{S0} 2\pi r\, dr)z}{4\pi\varepsilon_0(r^2 + z^2)^{3/2}}\mathbf{a}_z$$

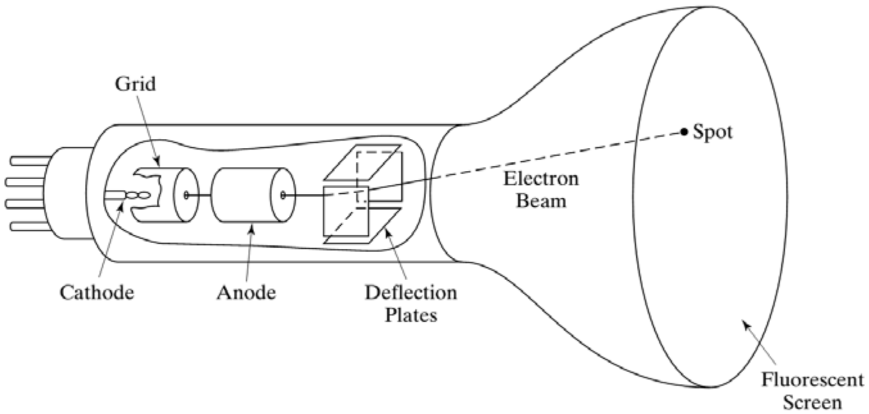
$$[\mathbf{E}]_{(0,\,0,\,z)} = \int_{r=0}^{\infty}[d\mathbf{E}]_{(0,0,z)} = \int_{r=0}^{\infty}\frac{\rho_{S0} r z\, dr}{2\varepsilon_0(r^2 + z^2)^{3/2}}\mathbf{a}_z$$

$$= \frac{\rho_{S0} z}{2\varepsilon_0}\left[-\frac{1}{\sqrt{r^2 + z^2}}\right]_{r=0}^{\infty}\mathbf{a}_z = \frac{\rho_{S0} z}{2\varepsilon_0|z|}\mathbf{a}_z = \pm\frac{\rho_{S0}}{2\varepsilon_0}\mathbf{a}_z$$

V. ELECTRIC FIELD

- **Applications**
  - Cathode Ray Tube (陰極射線管) for old TV technology (1934)
  - Mineral separator

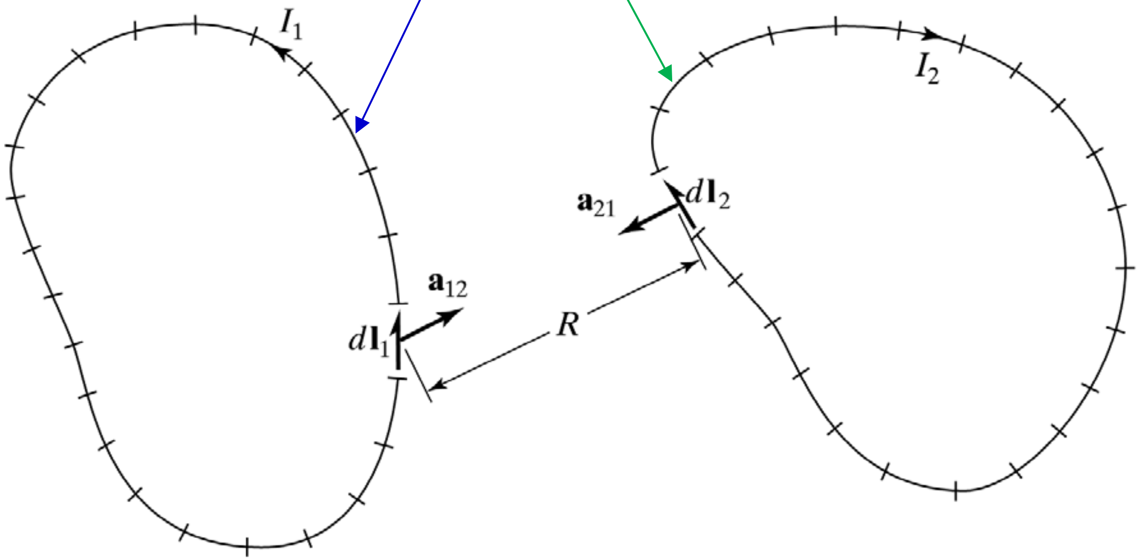




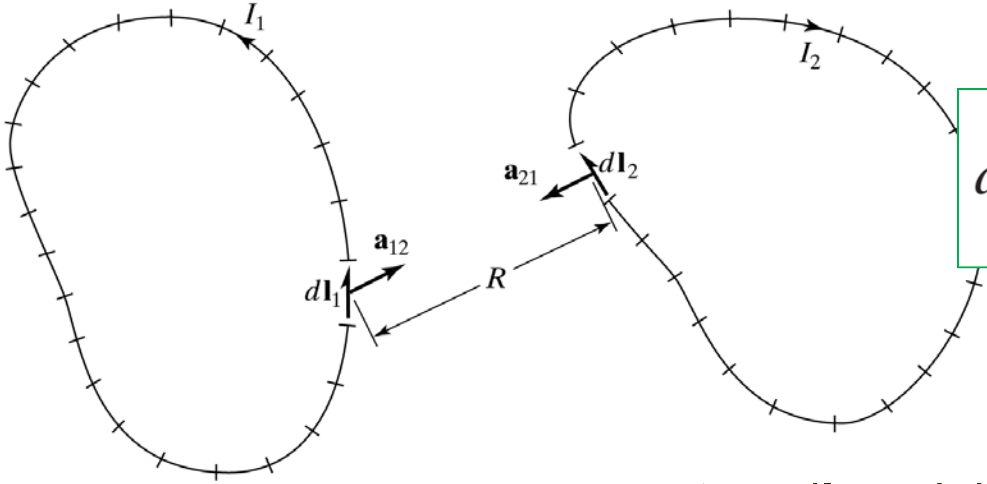
VI. MAGNETIC FIELD

Ampère's law

- Similar to Coulomb's law for electric field, Ampère's law depicts the nature of magnetic objects in magnetic field.
- Two circulating current loops generate magnetic force in between.
  - ✓ The total force experienced by a loop is the vector sum of forces experienced by the infinitesimal current elements (e.g. $d\mathbf{l}_1$ and $d\mathbf{l}_2$).
  - ✓ The force experienced by each current element is the vector sum of forces exerted from the infinitesimal elements in the second loop



VI. MAGNETIC FIELD



$$d\mathbf{F}_2 = I_2\, d\mathbf{l}_2 \times \left(\frac{k I_1\, d\mathbf{l}_1 \times \mathbf{a}_{12}}{R^2} \right)$$

Force on loop 2

$$d\mathbf{F}_1 = I_1\, d\mathbf{l}_1 \times \left(\frac{k I_2\, d\mathbf{l}_2 \times \mathbf{a}_{21}}{R^2} \right)$$

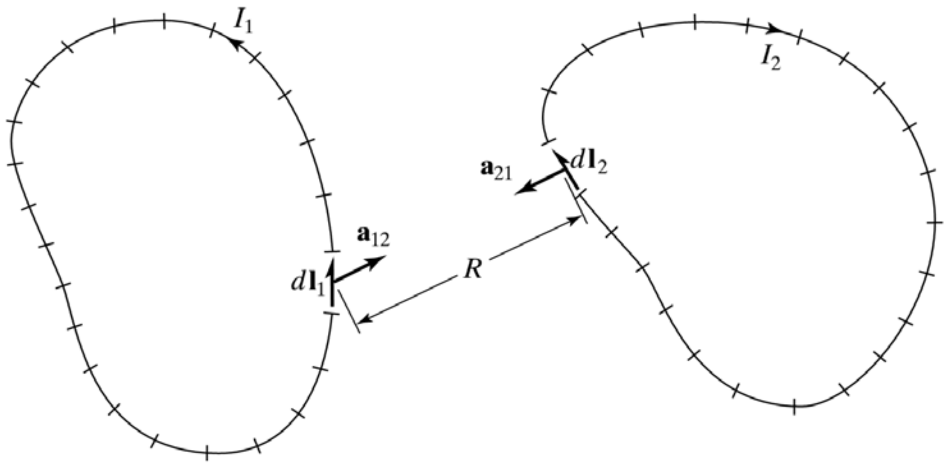
Force on loop 1

- d$\mathbf{l}_1$ and d$\mathbf{l}_2$: tangential vectors along the current loops
- $\mathbf{a}_{12}$ and $\mathbf{a}_{21}$: unit vectors joining the current elements in two loops (away from the current elements)
- k $(= \frac{\mu_0}{4\pi} = 10^{-7}$ H/m in free space): proportional constant, where μ_0 is the permeability of free space (unit: newton/A or henry/m and henry = newton-m/A); R: distance
- Force $\propto$ current (I_1 and I_2), length (d$\mathbf{l}_1$ and d$\mathbf{l}_2$), $\frac{1}{R^2}$;
- Force direction follows right-hand rule: if I_1 and I_2 circular with the opposite (same) orientation, the force will be attractive (repulsive)

VI. MAGNETIC FIELD

- From dF_1 and dF_2, one can observe that each force depends on the other current loop and its current. Thus, a magnetic field (B1) provided by I1 loop on current loop I2 can be written as

$$d\mathbf{F}_2 = I_2\, d\mathbf{l}_2 \times \boxed{\left(\frac{kI_1\, d\mathbf{l}_1 \times \mathbf{a}_{12}}{R^2}\right)} = I_2\, d\mathbf{l}_2 \times \mathbf{B}_1$$



$$\boxed{\mathbf{B}_1 = \frac{\mu_0}{4\pi}\frac{I_1\, d\mathbf{l}_1 \times \mathbf{a}_{12}}{R^2}}$$

$$\mathbf{B}_2 = \frac{\mu_0}{4\pi}\frac{I_2\, d\mathbf{l}_2 \times \mathbf{a}_{21}}{R^2}$$

- So the force can be written as $\boxed{d\mathbf{F} = I\, d\mathbf{l} \times \mathbf{B}}$

where B is called **<u>magnetic flux density vector</u>**

$$\boxed{\mathbf{B} = \frac{\mu_0}{4\pi}\frac{I\, d\mathbf{l} \times \mathbf{a}_R}{R^2}}$$ ***Biot-Savart's law***

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- ***Example 1.10***: Magnetic flux density

An infinitesimal length of 1 mm wire located at (1,0,0) with current of 2A in $\mathbf{a}_x$ direction. Find out the magnetic flux density at (0,2,2).

- Current element is given by

$$\boxed{I\,d\mathbf{l} = (2)(10^{-3})\mathbf{a}_x = 0.002\mathbf{a}_x}$$

- The distance **R** is

$$\mathbf{R} = (0-1)\mathbf{a}_x + (2-0)\mathbf{a}_y + (2-0)\mathbf{a}_z = -\mathbf{a}_x + 2\mathbf{a}_y + 2\mathbf{a}_z$$

- Then by *Biot-Savart's* law, we have

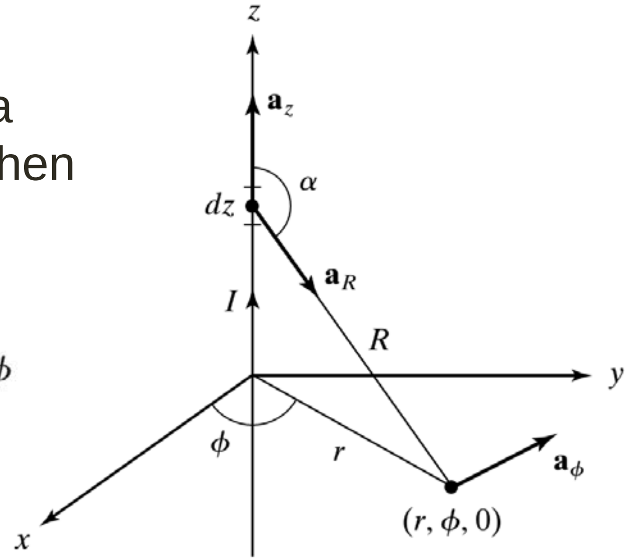
$$[\mathbf{B}]_{(0,2,2)} = \frac{\mu_0}{4\pi}\frac{I\,d\mathbf{l}\times\mathbf{a}_R}{R^2} = \frac{\mu_0}{4\pi}\frac{I\,d\mathbf{l}\times\mathbf{R}}{R^3}$$

$$= \frac{\mu_0}{4\pi}\frac{0.002\mathbf{a}_x \times (-\mathbf{a}_x + 2\mathbf{a}_y + 2\mathbf{a}_z)}{27} = \frac{0.001\mu_0}{27\pi}(-\mathbf{a}_y + \mathbf{a}_z)\ \mathrm{Wb/m^2}$$

VI. MAGNETIC FIELD

■ ***Example 1.11***: Magnetic field of an infinitely long wire

An infinitely long straight wire along z-axis with 1A in +z direction. Find the magnetic flux density everywhere.

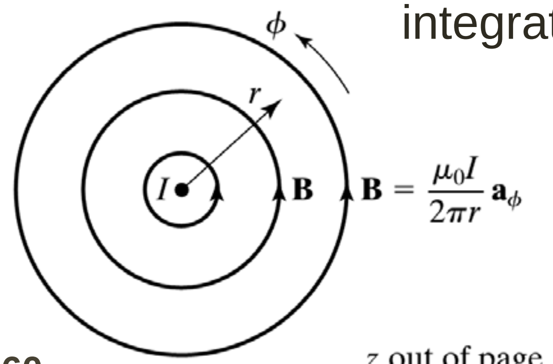


- At a point on x-y plane of $(r, \phi, 0)$, consider a differential length dz on the wire at (0,0,z). Then via *Biot-Savart's law*

$$[d\mathbf{B}]_{(r,\phi,0)} = \frac{\mu_0}{4\pi}\frac{I\,dz\,\mathbf{a}_z \times \mathbf{a}_R}{R^2} = \frac{\mu_0 I\,dz}{4\pi}\frac{\sin\alpha}{R^2}\mathbf{a}_\phi$$

$$= \frac{\mu_0 I\,dz}{4\pi}\frac{r}{R^3}\mathbf{a}_\phi = \frac{\mu_0 I r\,dz}{4\pi(z^2+r^2)^{3/2}}\mathbf{a}_\phi$$

- The contribution throughout the entire wire can be obtained by integration



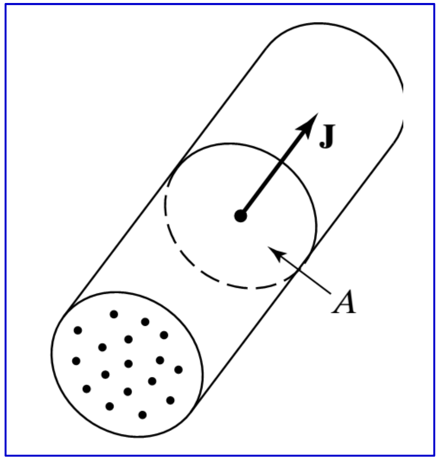
z out of page

$$[\mathbf{B}]_{(r,\phi,0)} = \int_{z=-\infty}^{\infty} d\mathbf{B} = \int_{z=-\infty}^{\infty}\frac{\mu_0 I r}{4\pi(z^2+r^2)^{3/2}}dz\,\mathbf{a}_\phi$$

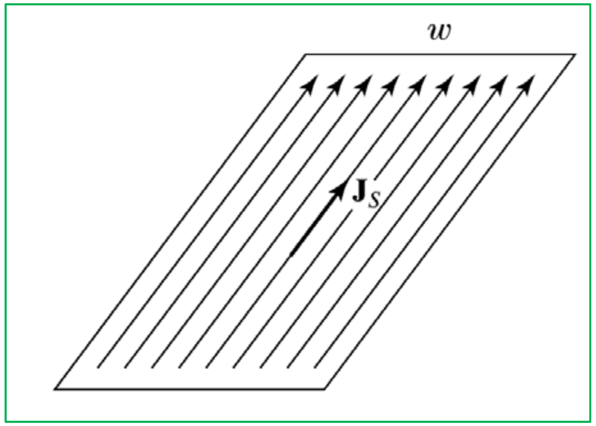
$$= \frac{\mu_0 I r}{4\pi}\left[\frac{z}{r^2\sqrt{z^2+r^2}}\right]_{z=-\infty}^{\infty}\mathbf{a}_\phi = \boxed{\frac{\mu_0 I}{2\pi r}\mathbf{a}_\phi}$$

VI. MAGNETIC FIELD

- Current Distributions:
  - Volume current density ($\mathbf{J}_{2D}$):
    - Current flows through a certain cross-section of area A
    - Unit: amperes/area such as A/cm^2 or A/m^2
    - Total current (**I**) = $\mathbf{J}_{2D}$ x Area if current flows uniformly; if not, $\boxed{I = \int \mathbf{J}_{2D} ds}$, where ds represents area integration



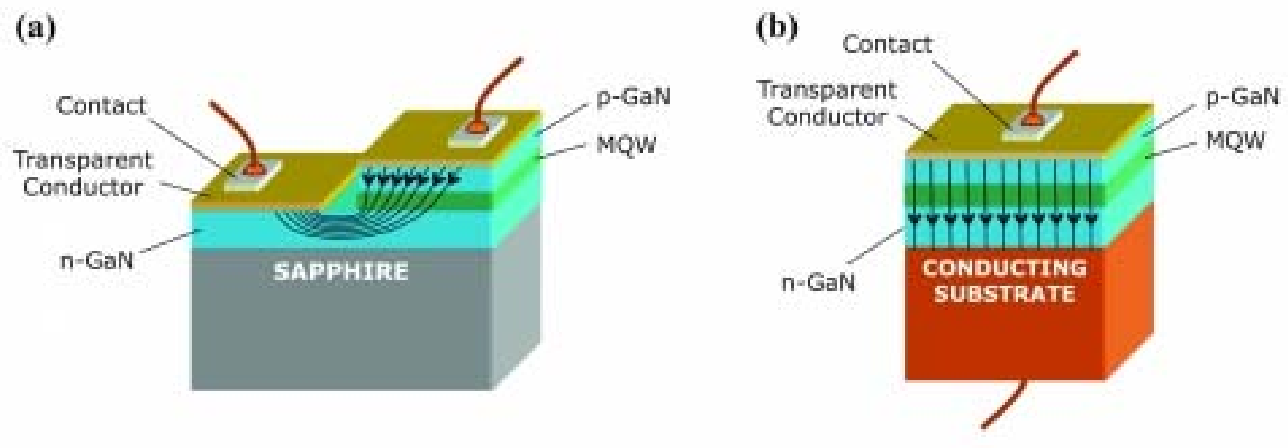
  - Surface current density ($\mathbf{J}_{1D}$):
    - Current flows through a certain width w
    - Unit: amperes/width such as A/cm or A/m
    - Total current (**I**) = $\mathbf{J}_{1D}$ x Area if current flows uniformly; if not, $\boxed{\mathbf{I} = \int \mathbf{J}_{1D} dl}$, where *dl* represents line integration



VI. MAGNETIC FIELD

Current Spreading (crowding):

- Since the resistance for the infinitesimal current components along their each flowing paths is NOT the same (path-dependent), current will not flow uniformly. Thus, current density is NOT total current divided by area.



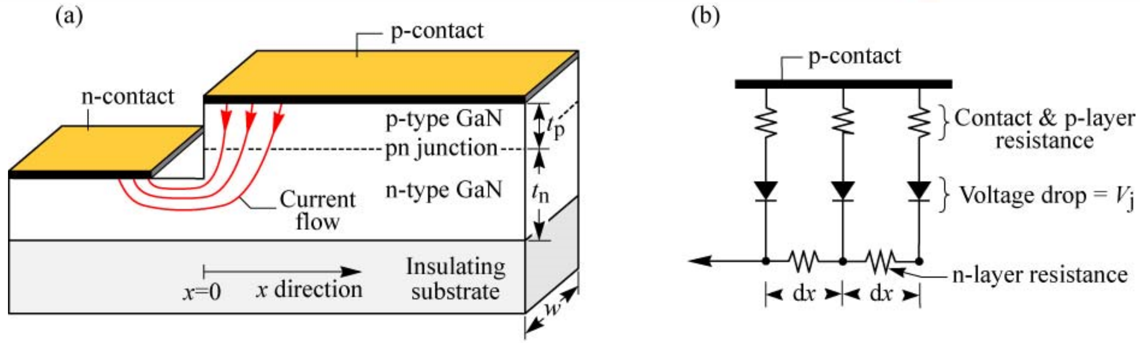
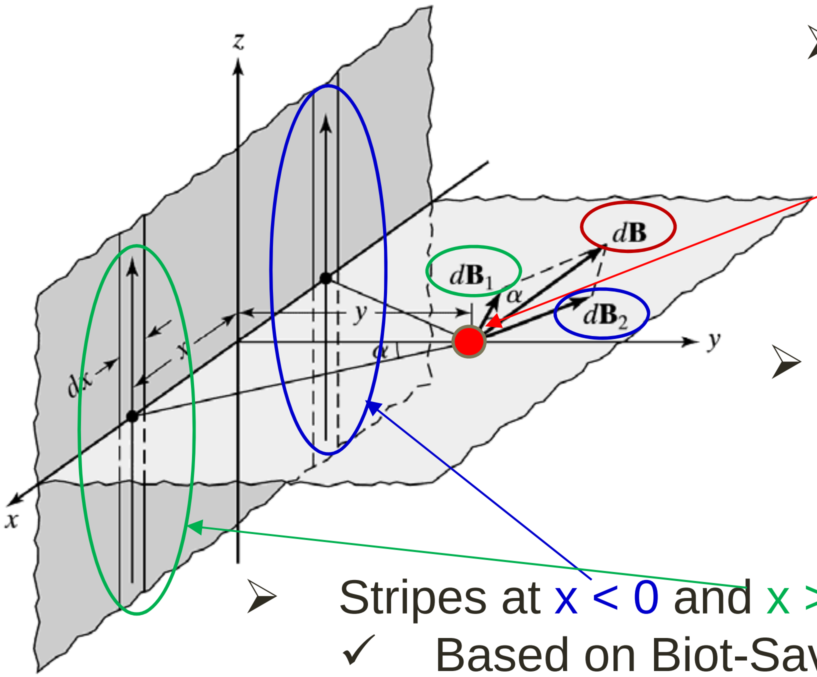


Fig. 8.7. (a) Current crowding in a mesa-structure GaN-based LED grown on an insulating substrate. (b) Equivalent circuit consisting of n-type and p-type layer resistances, p-type contact resistance, and ideal diodes representing the p-n junction.

E. F. Schubert
Light-Emitting Diodes (Cambridge Univ. Press)
www.LightEmittingDiodes.org

- ***Example 1.12***: Magnetic field of an infinite plane current sheet

An infinite current sheet in xz-plane of uniform current density $\mathbf{J}_S = \mathbf{J}_{S0}\mathbf{a}_z$ A/m, find the magnetic flux density (B) everywhere.

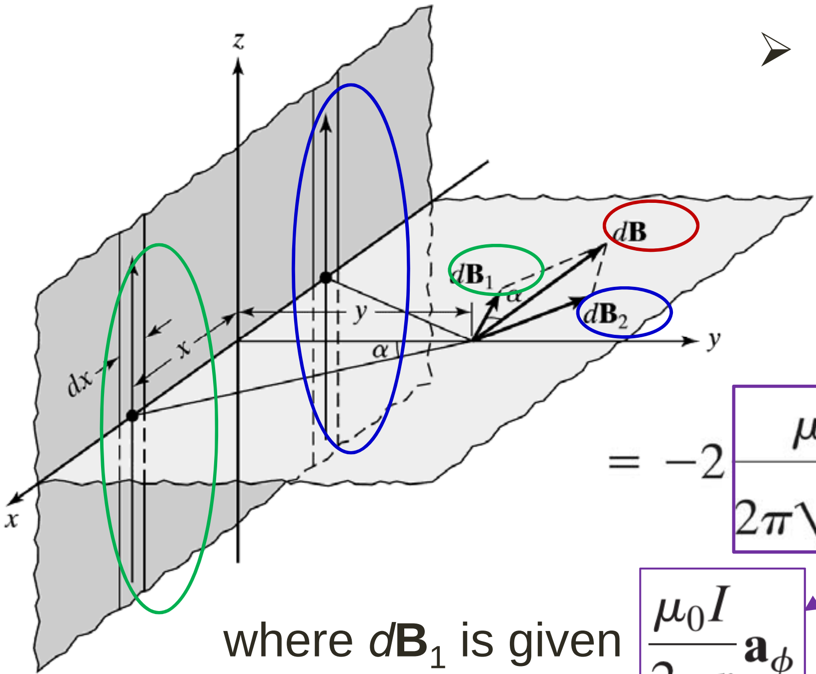


- Consider a point (0,y,0) at y-axis. Then we divide the current plane into infinite current vertical lines (along z-axis) with superposition used to integrate the overall magnetic fields by each strips (lines).

- There exists two vertical strips at x > 0 and x < 0 with the same distance x to the origin, which will induced the same magnitude of magnetic field, but some parts can be cancelled out.

- Stripes at x < 0 and x > 0:
  - ✓ Based on Biot-Savart's law, the resulting B's (⊥ **I** and $\mathbf{a}_R$) by x < 0 and x > 0 are $d\mathbf{B}_2$ and $d\mathbf{B}_1$, which can be divided into two components along x and y-axes.
  - ✓ Their components along y-axis can be cancelled out, so the only term left is along x-axis ($d\mathbf{B}$).

VI. MAGNETIC FIELD



➢ Then the resulting magnetic flux by applying Biot-Savart's law

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{I\,d\mathbf{l} \times \mathbf{a}_R}{R^2}$$

$$d\mathbf{B} = d\mathbf{B}_1 + d\mathbf{B}_2 = -2\,dB_1 \cos\alpha\,\mathbf{a}_x$$

$$= -2\boxed{\frac{\mu_0 J_{S0}\,dx}{2\pi\sqrt{x^2+y^2}}}\frac{y}{\sqrt{x^2+y^2}}\mathbf{a}_x = -\frac{\mu_0 J_{S0} y\,dx}{\pi(x^2+y^2)}\mathbf{a}_x$$

where $d\mathbf{B}_1$ is given $\boxed{\frac{\mu_0 I}{2\pi r}\mathbf{a}_\phi}$ (based on the results from Example 1.11).

➢ The total B flux from the entire current sheet is obtained by integration

$$[\mathbf{B}]_{(0,\,y,\,0)} = \int_{x=0}^{\infty} d\mathbf{B} = -\int_{x=0}^{\infty}\frac{\mu_0 J_{S0} y}{\pi(x^2+y^2)}dx\,\mathbf{a}_x = -\frac{\mu_0 J_{S0} y}{\pi}\left[\frac{1}{y}\tan^{-1}\frac{x}{y}\right]_{x=0}^{\infty}\mathbf{a}_x$$

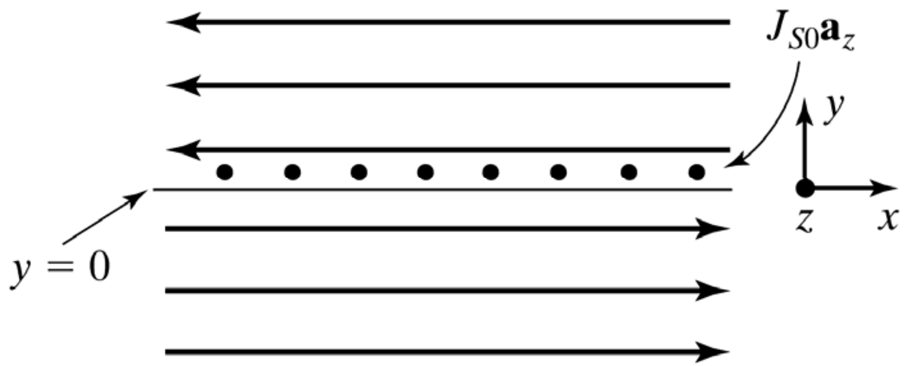
$$= -\frac{\mu_0 J_{S0}}{2}\mathbf{a}_x \quad \text{for} \quad y > 0 \qquad = \frac{\mu_0 J_{S0}}{2}\mathbf{a}_x \quad \text{for} \quad y < 0$$

$$\boxed{\mathbf{B} = \mp\frac{\mu_0 J_{S0}}{2}\mathbf{a}_x \quad \text{for} \quad y \gtrless 0}$$

VI. MAGNETIC FIELD

- The magnetic flux has the magnitude of $\mu_0 J_{S0}/2$ everywhere along the $\mp \mathbf{a}_x$ direction for $y \gtrless 0$

$$y > 0, \mathbf{B} = -\frac{\mu_0 J_{S0}}{2} \mathbf{a}_x$$



$$y < 0, \mathbf{B} = \frac{\mu_0 J_{S0}}{2} \mathbf{a}_x$$

- Define $\mathbf{a}_n$ to be unit normal vector away from the current sheet

$$\mathbf{a}_n = \pm \mathbf{a}_y \qquad \text{for} \quad y \gtrless 0$$

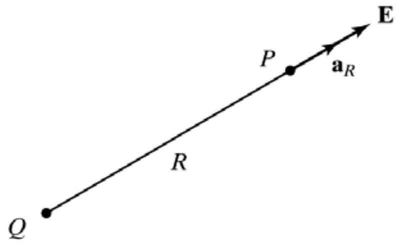
- And the magnetic flux B is

$$\mathbf{B} = \frac{\mu_0}{2}(J_{S0}\mathbf{a}_z) \times (\pm \mathbf{a}_y) \qquad \text{for} \quad y \gtrless 0 \qquad \boxed{\mathbf{B} = \frac{\mu_0}{2}\mathbf{J}_S \times \mathbf{a}_n}$$

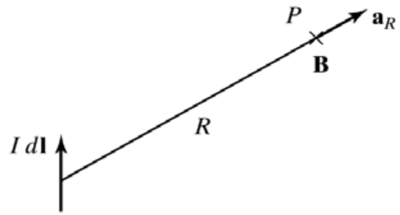
VI. MAGNETIC FIELD

- Analogy between magnetic field and electric field

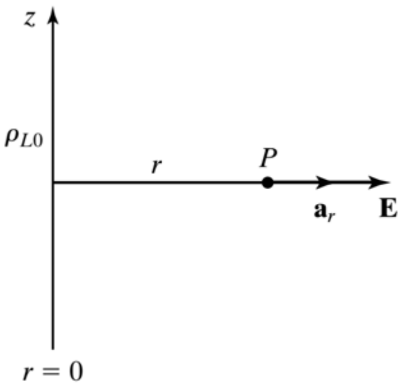
  - Charge distribution vs. current distribution



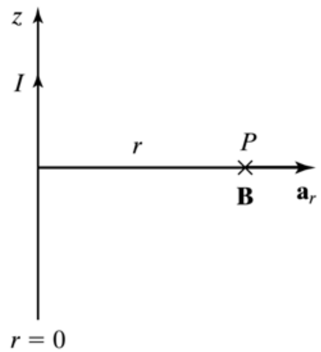
$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_0 R^2}\mathbf{a}_R \leftrightarrow \mathbf{B} = \frac{\mu_0 I}{4\pi R^2} d\mathbf{l} \times \mathbf{a}_R$$



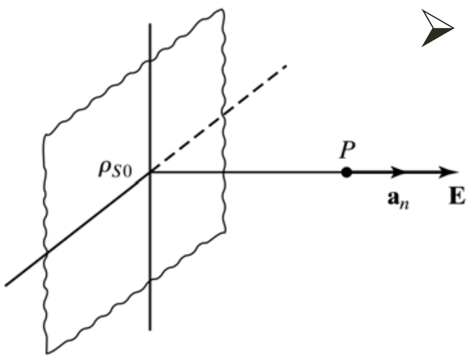
  - Infinite long line charge of uniform density vs. infinite long line current



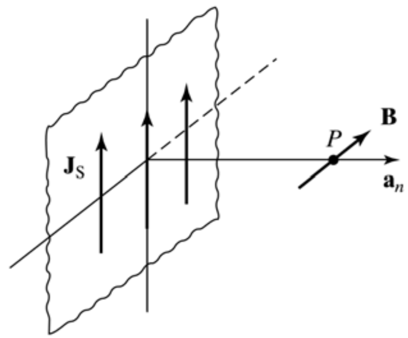
$$\mathbf{E} = \frac{\rho_{L0}}{2\pi\varepsilon_0 r}\mathbf{a}_r \leftrightarrow \mathbf{B} = \frac{\mu_0 I}{2\pi r}\mathbf{a}_\phi = \frac{\mu_0 I}{2\pi r}\mathbf{a}_z \times \mathbf{a}_r$$



  - Infinite plane of charge sheet vs. infinite plane of current sheet (both of uniform density)



$$\mathbf{E} = \frac{\rho_{L0}}{2\varepsilon_0}\mathbf{a}_n \leftrightarrow \mathbf{B} = \frac{\mu_0}{2}\mathbf{J}_S \times \mathbf{a}_n$$



VI. MAGNETIC FIELD

- Magnetic force
  - Magnetic flux is caused by current (i.e. charge movement), so the magnetic force (slide #1-57) can be expressed as

$$d\mathbf{F} = Id\mathbf{l} \times \mathbf{B} = \frac{dQ}{dt}\mathbf{v}dt \times \mathbf{B} = dQ\,\mathbf{v} \times \mathbf{B}$$

  - So

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} \quad \text{and} \quad \mathbf{B} = \frac{\mathbf{F}_m \times \mathbf{a}_m}{qv}$$

$$\Longrightarrow \boxed{\mathbf{B} = \lim_{qv \to 0} \frac{\mathbf{F}_m \times \mathbf{a}_m}{qv}}$$

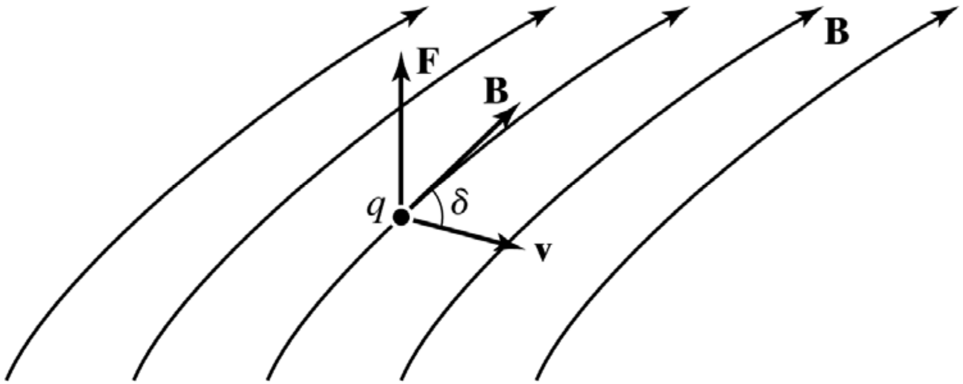


FIGURE 1.38

Force experienced by a test charge q moving with a velocity $\mathbf{v}$ in a magnetic field $\mathbf{B}$.

VII. LORENTZ FORCE EQUATION

- Force experienced by a charge in a magnetic field
  - A charge in an electric field (**E**) is experienced a force

$$\mathbf{F}_E = q\mathbf{E}$$

  - A "moving charge" in a magnetic field with a flux density (**B**) experiences a force (called Lorentz force)

$$\mathbf{F}_M = q\mathbf{v} \times \mathbf{B}$$

  - A combined force equation (Lorentz equation) is then given by

$$\boxed{\mathbf{F} = \mathbf{F}_E + \mathbf{F}_M = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})}$$

  - How to solve these two fields under a certain total force?
    - ✓ For a given **B** and one specific velocity, $\mathbf{F}_E$ can be found from the total force minus $\mathbf{F}_M$ since **E** is along the same direction of $\mathbf{F}_E$ (vice versa).
    - ✓ For a given **E**, **B** will be obtained from two "non-collinear velocities, since for $\mathbf{v} \times \mathbf{B}$, there could be two choices for v's

VII. LORENTZ FORCE EQUATION

- ***Example 1.13***: Finding the electric and magnetic fields from forces on a test charge

  A test charge *q* for 3 velocities experiences the forces by electric and magnetic fields

  - A charge in an electric field (**E**) is experienced a force

$$\begin{aligned} \mathbf{F}_1 &= qE_0\mathbf{a}_x && \text{for} \quad \mathbf{v}_1 = v_0\mathbf{a}_x \\ \mathbf{F}_2 &= qE_0(2\mathbf{a}_x + \mathbf{a}_y) && \text{for} \quad \mathbf{v}_2 = v_0\mathbf{a}_y \\ \mathbf{F}_3 &= qE_0(\mathbf{a}_x + \mathbf{a}_y) && \text{for} \quad \mathbf{v}_3 = v_0\mathbf{a}_z \end{aligned}$$

  - From Lorentz equation, we have

$$\begin{aligned} q\mathbf{E} + qv_0\mathbf{a}_x \times \mathbf{B} &= qE_0\mathbf{a}_x \\ q\mathbf{E} + qv_0\mathbf{a}_y \times \mathbf{B} &= q(2E_0\mathbf{a}_x + E_0\mathbf{a}_y) \\ q\mathbf{E} + qv_0\mathbf{a}_z \times \mathbf{B} &= q(E_0\mathbf{a}_x + E_0\mathbf{a}_y) \end{aligned}$$

  - By subtracting each other for the above three equations,

$$\begin{aligned} v_0(\mathbf{a}_y - \mathbf{a}_x) \times \mathbf{B} &= E_0(\mathbf{a}_x + \mathbf{a}_y) \\ v_0(\mathbf{a}_y - \mathbf{a}_z) \times \mathbf{B} &= E_0\mathbf{a}_x \end{aligned}$$

VII. LORENTZ FORCE EQUATION

$$v_0(\mathbf{a}_y - \mathbf{a}_x) \times \mathbf{B} = E_0(\mathbf{a}_x + \mathbf{a}_y)$$
$$v_0(\mathbf{a}_y - \mathbf{a}_z) \times \mathbf{B} = E_0\mathbf{a}_x$$

- For the first equation, let's try three direction for B:
  - ✓ x: $(\mathbf{a}_\mathrm{y} - \mathbf{a}_\mathrm{x}) \times \mathbf{a}_\mathrm{x} = -\mathbf{a}_\mathrm{z}$
  - ✓ y: $(\mathbf{a}_\mathrm{y} - \mathbf{a}_\mathrm{x}) \times \mathbf{a}_\mathrm{y} = -\mathbf{a}_\mathrm{z}$
  - ✓ z: $(\mathbf{a}_\mathrm{y} - \mathbf{a}_\mathrm{x}) \times \mathbf{a}_\mathrm{z} = \mathbf{a}_\mathrm{x} + \mathbf{a}_\mathrm{z}$
- The other formal way is:
  - ✓ B is perpendicular to $(\mathbf{a}_\mathrm{x} + \mathbf{a}_\mathrm{y})$ and $\mathbf{a}_\mathrm{x}$.
  - ✓ $(\mathbf{a}_\mathrm{x} + \mathbf{a}_\mathrm{y}) \times \mathbf{a}_\mathrm{x} = -\mathbf{a}_\mathrm{z}$, which is perpendicular to $(\mathbf{a}_\mathrm{x} + \mathbf{a}_\mathrm{y})$ and $\mathbf{a}_\mathrm{x}$.
  - ✓ Combine the above two, we know B is related to $\mathbf{a}_\mathrm{z}$

$$\mathbf{B} = C(\mathbf{a}_x + \mathbf{a}_y) \times \mathbf{a}_x = -C\mathbf{a}_z$$

- By replacing the above to second equation at top, we have

$$v_0(\mathbf{a}_y - \mathbf{a}_z) \times (-C\mathbf{a}_z) = E_0\mathbf{a}_x$$
$$-v_0 C\mathbf{a}_x = E_0\mathbf{a}_x \Rightarrow C = -E_0/v_0$$

VII. LORENTZ FORCE EQUATION

- Thus, $$\mathbf{B} = \frac{E_0}{v_0}\mathbf{a}_z$$

- Then

$$v_0\begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ -1 & 1 & 0 \\ B_x & B_y & B_z \end{vmatrix} = E_0(\mathbf{a}_x + \mathbf{a}_y)$$

$$v_0[B_z\mathbf{a}_x + B_z\mathbf{a}_y - (B_y + B_x)\mathbf{a}_z] = E_0\mathbf{a}_x + E_0\mathbf{a}_y$$

$$B_z = \frac{E_0}{v_0} \quad \text{and} \quad (B_y + B_x) = 0$$

$$v_0(\mathbf{a}_y - \mathbf{a}_x) \times \mathbf{B} = E_0(\mathbf{a}_x + \mathbf{a}_y)$$

$$v_0(\mathbf{a}_y - \mathbf{a}_z) \times \mathbf{B} = E_0\mathbf{a}_x$$

$$\boxed{\mathbf{B} = \frac{E_0}{v_0}\mathbf{a}_z} \qquad \boxed{\mathbf{E} = E_0(\mathbf{a}_x + \mathbf{a}_y)}$$

$$v_0\begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 0 & 1 & -1 \\ B_x & B_y & B_z \end{vmatrix} = E_0\mathbf{a}_x$$

$$v_0[(B_z + B_y)\mathbf{a}_x - B_x\mathbf{a}_y - B_x\mathbf{a}_z] = E_0\mathbf{a}_x$$

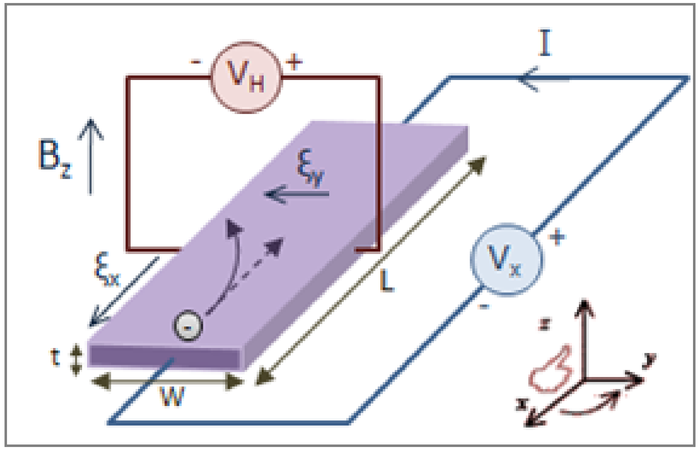
$$B_z + B_y = \frac{E_0}{v_0} \quad \text{and} \quad B_x = 0$$

- The alternative way is to assume

$$\mathbf{B} = B_x\mathbf{a}_x + B_y\mathbf{a}_y + B_z\mathbf{a}_z$$

VII. LORENTZ FORCE EQUATION

- Most important (common) applications: Hall Effect:
  - Equations of motion:



$$\frac{d\vec{p}}{dt} = -e\left(\vec{E} + \frac{\vec{p}}{m} \times \vec{B}\right) - \frac{\vec{p}}{\tau}$$

✓ Steady state:

x: $-eE_x - e\frac{B_z}{m}p_y - \frac{p_x}{\tau} = 0$

y: $-eE_y + e\frac{B_z}{m}p_x - \frac{p_y}{\tau} = 0$

where $\omega_c \equiv e\frac{B_z}{m}$ is defined as cyclotron frequency

- Hall coefficient: $\boxed{R_H \equiv \frac{E_y}{j_x B_z}}$

✓ px and py:

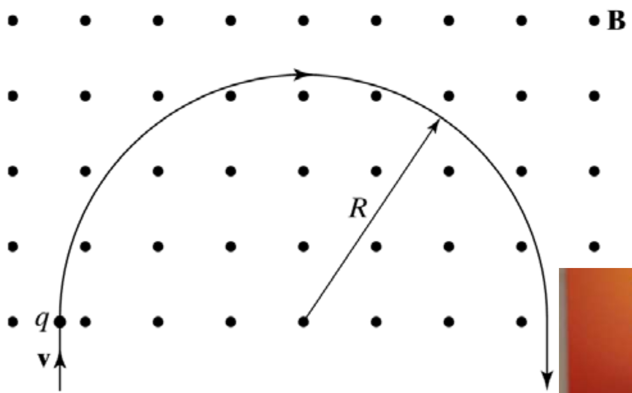
$$j_x = -nev_x = \frac{-nep_x}{m} \rightarrow p_x = \frac{mj_x}{-ne};\ j_y = -nev_y = \frac{-nep_y}{m} \rightarrow p_y = \frac{mj_y}{-ne}$$

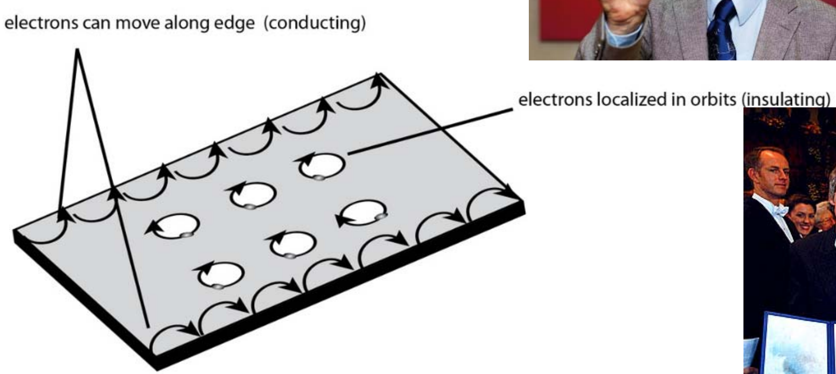
So $\boxed{R_H \equiv \frac{E_y}{j_x B_z} = \frac{-1}{ne}}$

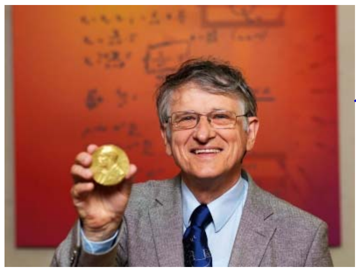
- Measurement sequence:
  - ✓ Measure V_y, then we get $E_y = V_y/w$
  - ✓ B and J_x (I_x/wt) are known, we then know R_H and n.

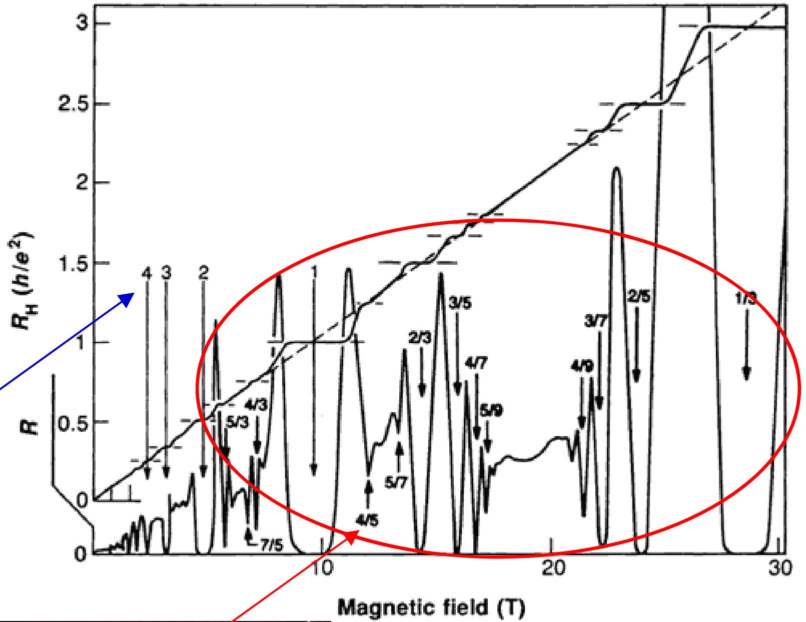
VII. LORENTZ FORCE EQUATION

- Circular motion for a charge in a magnetic field in free space
- Semi-circular motion for a charge in a real sample made of metal or semiconductor. This is the fundamental picture for Quantum Hall Effect, 1998 Nobel Physics Prize.









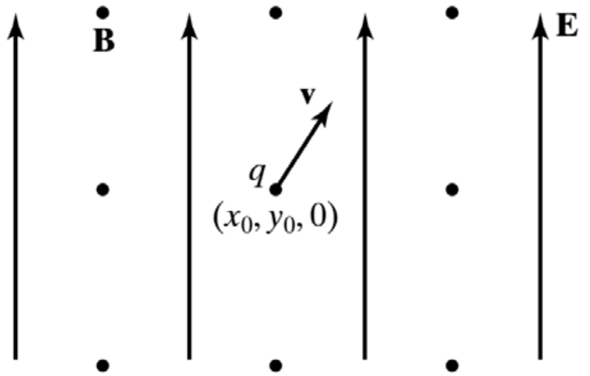
VII. LORENTZ FORCE EQUATION

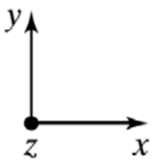
- Tracing the path of a charged particle
  - Requires the knowledge of mechanic force (i.e. F = ma) and the E (electric)/M (magnetic) forces via Lorentz equation

$$\mathbf{F} = m\mathbf{a} = \mathbf{F}_E + \mathbf{F}_M = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

  - For example, consider a two-dimensional motion (e.g. a charge moving in x-y plane) with $\mathbf{E} = E_0\mathbf{a}_y$ and $\mathbf{B} = B_0\mathbf{a}_z$
  - Assume at *t* = 0, the charge has an initial velocity

$$\mathbf{v} = v_{x0}\mathbf{a}_x + v_{y0}\mathbf{a}_y$$





  - The Lorentz force equation is

$$\begin{aligned}\mathbf{F} &= q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \\ &= qE_0\mathbf{a}_y + q(v_x\mathbf{a}_x + v_y\mathbf{a}_y + v_z\mathbf{a}_z) \times B_0\mathbf{a}_z \\ &= qB_0v_y\mathbf{a}_x + (qE_0 - qB_0v_x)\mathbf{a}_y\end{aligned}$$

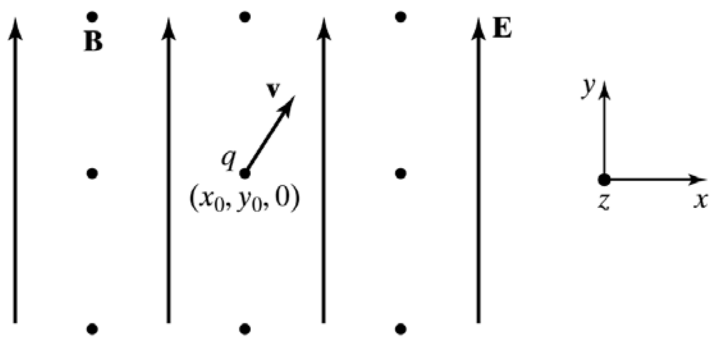
via motion equation ($F = m\frac{dv}{dt}$)

$$\frac{dv_x}{dt} = \frac{qB_0}{m}v_y \qquad \frac{dv_y}{dt} = \frac{qE_0}{m} - \frac{qB_0}{m}v_x \qquad \frac{dv_z}{dt} = 0$$

VII. LORENTZ FORCE EQUATION

- Eliminating v_y, we obtain

$$\frac{d^2 v_x}{dt^2} + \left(\frac{qB_0}{m}\right)^2 v_x = \left(\frac{q}{m}\right)^2 B_0 E_0$$



- Solution is

$$\boxed{v_x = \frac{E_0}{B_0} + C_1 \cos \omega_c t + C_2 \sin \omega_c t}$$

$$\boxed{v_y = -C_1 \sin \omega_c t + C_2 \cos \omega_c t}$$

- With the initial condition $\mathbf{v} = v_{x0}\mathbf{a}_x + v_{y0}\mathbf{a}_y$ and $\omega_c = qB_0/m$

$$v_x = \frac{E_0}{B_0} + \left(v_{x0} - \frac{E_0}{B_0}\right) \cos \omega_c t + v_{y0} \sin \omega_c t$$

$$v_y = -\left(v_{x0} - \frac{E_0}{B_0}\right) \sin \omega_c t + v_{y0} \cos \omega_c t$$

- By integration from $t = 0$ with $x = x_0$ and $y = y_0$, we have positions

$$x = x_0 + \frac{E_0}{B_0} t + \frac{1}{\omega_c}\left(v_{x0} - \frac{E_0}{B_0}\right) \sin \omega_c t + \frac{v_{y0}}{\omega_c}(1 - \cos \omega_c t)$$

$$y = y_0 - \frac{1}{\omega_c}\left(v_{x0} - \frac{E_0}{B_0}\right)(1 - \cos \omega_c t) + \frac{v_{y0}}{\omega_c} \sin \omega_c t$$

VII. LORENTZ FORCE EQUATION

- Special case: $B_0 = 0, \omega_c \to 0$
  - ✓ No magnetic field, and no circular motion so frequency is zero
  - ✓ Then the final solution becomes familiar (which can be solved by general Newton's law with only electric force present)

$$x = x_0 + v_{x0}t$$

$$y = y_0 + v_{y0}t + \frac{1}{2}\frac{qE_0}{m}t^2$$

$$v_x = v_{x0}$$

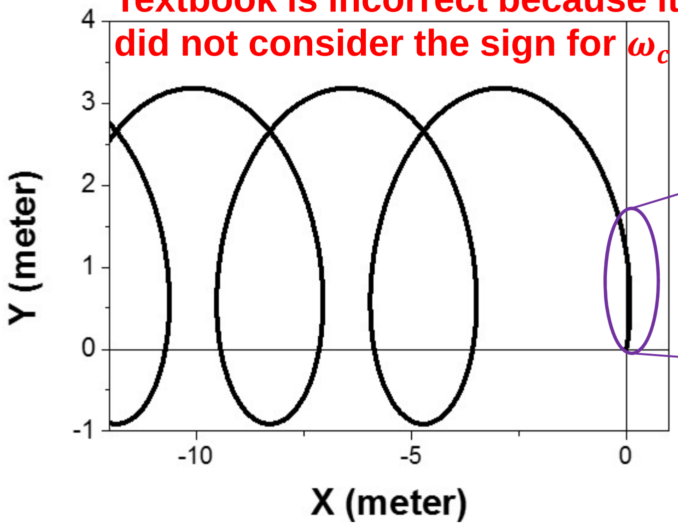
$$v_y = v_{y0} + \frac{qE_0}{m}t$$

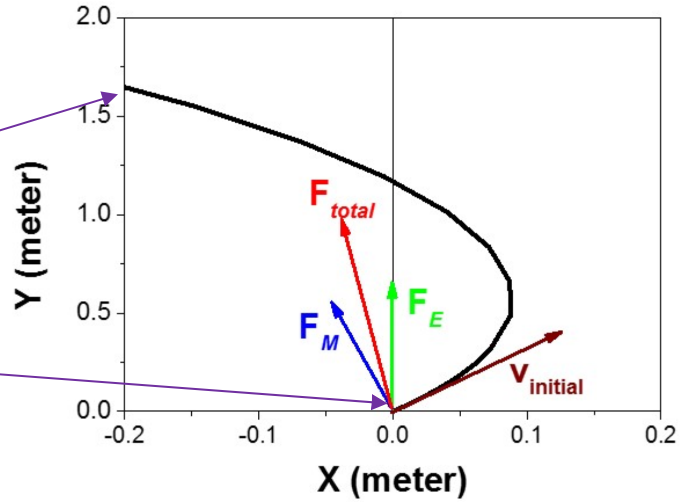
- Changed particle trace
  - ✓ Initial condition:

$$\boxed{x_0 = 0, \quad y_0 = 0, \quad E_0 = -10^3\,\text{V/m}, \quad B_0 = 10^{-4}\,\text{Wb/m}^2 \quad v_{x0} = 10^7\,\text{m/s, and } v_{y0} = 3 \times 10^7\,\text{m/s}}$$

$$\boxed{\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})}$$

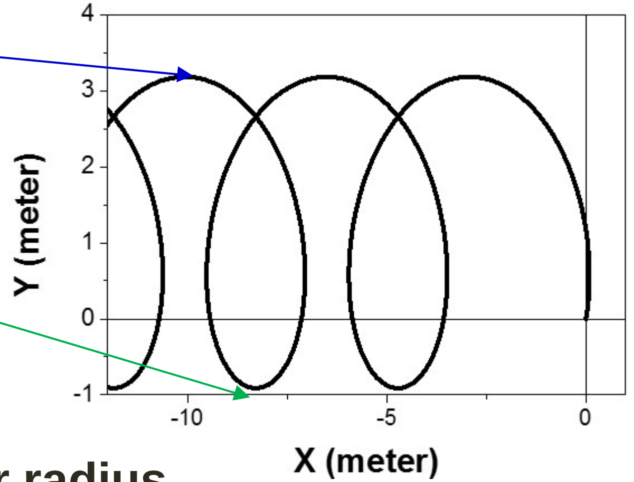
Textbook is incorrect because it did not consider the sign for ω_c





VII. LORENTZ FORCE EQUATION

- Why is the curvature at top and bottom different?



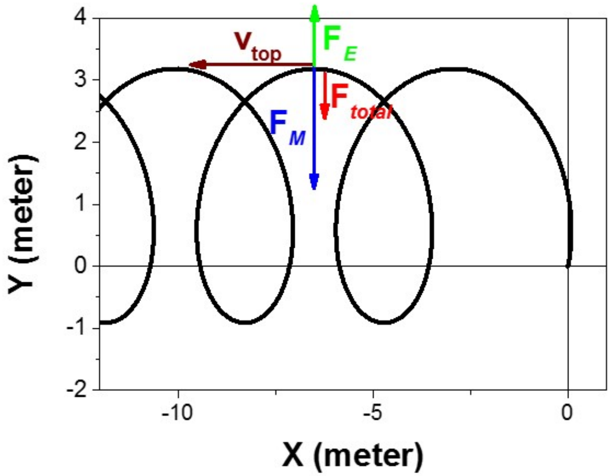
$$\boxed{\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})}$$

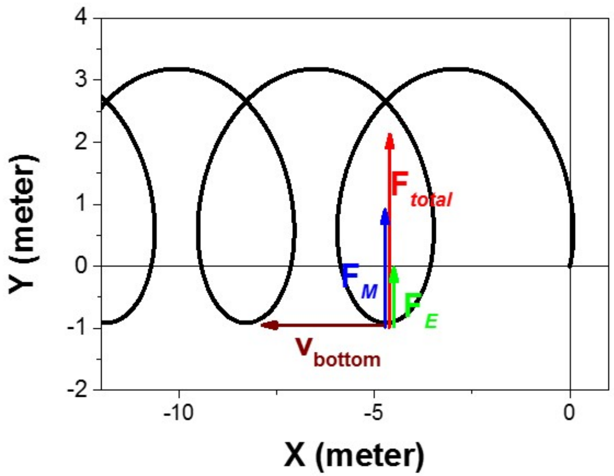
- ✓ It can be obtained from the magnitude of the centrifuge force

$$\boxed{F = \frac{mv^2}{r}}$$

Larger force, smaller radius, and more curved

- ✓ At top (bottom), total force is smaller (larger), so curvature is larger (smaller)





VIII. HOMEWORK DUE

- Due at March 30th (Wednesday) in class
- Chapter 1 problems: 1, 4, 7, 9, 11, 14, 16, 20, 22, 24, 26, 28, 30, 32, 34, 38, 40, 43, 44, 48, 51, 53, 54
- Sorry I know it’s a lot, but it’s the same no matter whose class you’re attending.