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CS 189 HW 3 #4

$\nabla J = \frac{\partial J}{\partial w_0} + \frac{\partial J}{\partial w}$ Find solutions to $\frac{\partial J}{\partial w_0} = 0$ and $\frac{\partial J}{\partial w} = 0$ to optimize J

$J(w, w_0) = (y - Xw - w_0 1)^T (y - Xw - w_0 1) + \lambda w^T w$

For w_0, rearrange as such:

$$J(w, w_0) = (y - Xw)^T (y - Xw) - (y - Xw)^T w_0 1 - w_0 1^T (y - Xw) + \underbrace{n w_0^2}_{w_0^2 1^T 1} + \lambda w^T w$$

$$\frac{\partial J}{\partial w_0} = 2 n w_0 - (y - Xw)^T 1 - 1^T (y - Xw)$$

$$0 = 2 n w_0 - (y - Xw)^T 1 - 1^T (y - Xw) \leftarrow (y - Xw)^T 1 = 1 \cdot (y - Xw) = 1^T (y - Xw) = \sum_i y_i - x_i^T w$$

$$2 n w_0 = 2 \cdot 1^T (y - Xw)$$

$$n w_0 = \sum_i y_i - x_i^T w \qquad x_i^T w = x_i \cdot w = w \cdot x_i$$

$$n w_0 = \left(\sum_i y_i\right) - w \cdot \sum_i x_i$$

$$w_0 = \frac{1}{n} \sum_i y_i - \frac{w}{n} \cdot \sum_i x_i$$

$$\boxed{\hat{w}_0 = \bar{y} - 0 = \bar{y}}$$

for w, rearrange J as such:

$$J(w, w_0) = (y - w_0 1)^T (y - w_0 1) - (y - w_0 1)^T Xw - \underbrace{(Xw)^T (y - w_0 1)}_{w^T X^T (y - w_0 1)} + \underbrace{(Xw)^T Xw}_{w^T X^T X w} + \underbrace{\lambda w^T w}_{\lambda w^T I w}$$

$$\frac{\partial J}{\partial w} = 0 - X^T (y - w_0 1) - X^T (y - w_0 1) + 2 X^T X w + 2 \lambda I w$$

$$\frac{\partial (w^T X^T X w)}{\partial w} = 2 X^T X w$$

since

$$\frac{\partial x^T A x}{\partial x} = 2 A x$$

for symmetric A

$$0 = -2 X^T (y - w_0 1) + 2 (X^T X + \lambda I) w$$

$$(X^T X + \lambda I) w = X^T (y - w_0 1)$$

$$(X^T X + \lambda I) w = X^T y - w_0 X^T 1 \qquad X^T 1 = \sum_i x_i = 0$$

$$(X^T X + \lambda I) w = X^T y$$

$$\boxed{\hat{w} = (X^T X + \lambda I)^{-1} X^T y}$$

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Hw 3 Problem 5

Show the MLE: $y_i | x_i \sim \mathcal{N}(\omega_0 + \omega_1 x_i, \sigma^2)$

$$\mathbb{P}(y_i) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y_i - \omega_0 - \omega_1 x_i)^2}{2\sigma^2}\right)$$

$$\mathbb{P}(Y|\Theta) = \mathcal{L}(\Theta|Y) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y_i - \omega_0 - \omega_1 x_i)^2}{2\sigma^2}\right)$$

$$l = \sum_{i=1}^{n} \ln\left(\frac{1}{\sqrt{2\pi}\sigma}\right) + \sum_{i=1}^{n}\left(-\frac{(y_i - \omega_0 - \omega_1 x_i)^2}{2\sigma^2}\right)$$

$$l = n \ln \frac{1}{\sqrt{2\pi}\sigma} - \frac{n}{2\sigma^2} \sum_{i=1}^{n} (y_i - \omega_0 - \omega_1 x_i)^2$$

$$\frac{\delta l}{\delta \omega_0} = -\frac{n}{\sigma^2} \sum_{i=1}^{n} (-y_i + \omega_0 + \omega_1 x_i)$$

$$0 = \sum_{i=1}^{n} (-y_i + \omega_0 + \omega_1 x_i)$$

$$\sum_{i=1}^{n} y_i - \omega_1 \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} \omega_0 = n\omega_0$$

$$\frac{1}{n} \sum_{i=1}^{n} y_i - \frac{\omega_1}{n} \sum_{i=1}^{n} x_i = \omega_0$$

$$\bar{y} - \omega_1 \bar{x} = \omega_0$$

$$\frac{\delta l}{\delta \omega_1} = \sum_{i=1}^{n} (-y_i x_i + \omega_0 x_i + \omega_1 x_i^2)$$

$$0 = -\sum_{i=1}^{n} y_i x_i + \sum_{i=1}^{n} \omega_0 x_i + \sum_{i=1}^{n} \omega_1 x_i^2$$

$$0 = -\sum_{i=1}^{n} y_i x_i + \sum_{i=1}^{n} (\bar{y} - \omega_1 \bar{x}) x_i + \sum_{i=1}^{n} \omega_1 x_i^2$$

$$\sum_{i=1}^{n} y_i x_i - \sum_{i=1}^{n} \bar{y} x_i = \omega_1 \left(-\sum_{i=1}^{n} \bar{x} x_i + \sum_{i=1}^{n} x_i^2\right)$$

$$\frac{\sum_{i=1}^{n} y_i x_i - \sum_{i=1}^{n} \bar{y} x_i \cdot \frac{n}{n}}{-\sum_{i=1}^{n} \bar{x} x_i \cdot \frac{n}{n} + \sum_{i=1}^{n} x_i^2} = \omega_1$$

$$\boxed{\frac{\sum_{i=1}^{n} y_i x_i - n\bar{y}\bar{x}}{\sum_{i=1}^{n} x_i^2 - n\bar{x}^2} = \omega_1} \approx \frac{\mathrm{cov}(X,Y)}{\mathrm{var}(X)}$$