High-order Lagrange elements in FreeFem++

A. Chabib, P-H. Tournier, F. Hecht
Laboratoire Jacques-Louis Lions
CNRS, Sorbonne Université UPMC , France
ALPINES team, Centre Inria de Paris

FreeFem Days
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Motivation

Finite element solution of **time-harmonic wave propagation problems**

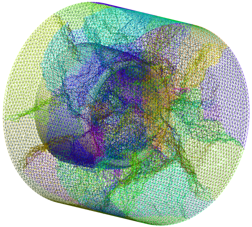
Helmholtz problem:

$$\begin{cases} -\Delta p - \omega^2 p = s & \text{in } \Omega \\ \text{Boundary conditions} & \text{on } \partial\Omega \end{cases}$$

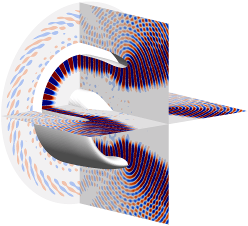
Finite element error:

$$\|p - p_h\|_{H^1} \leq C_1(\omega h)^k + C_2\omega^{(k+1)}h^k$$

Finite element mesh



Numerical field



High frequency (large ω)
Phenomena close to resonance

$\rightsquigarrow$ $\begin{cases} \text{Fine mesh (small } h) \\ \text{High-order basis functions (large } k) \end{cases}$

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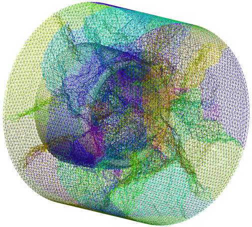

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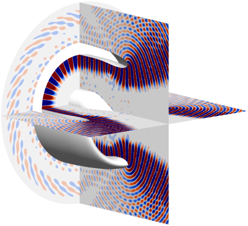

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Lagrange FE in FreeFem++ I

```
class TypeOfFE_P4Lagrange : public TypeOfFE {
 public:
  static const int k = 4;
  static const int ndf = (k + 2) * (k + 1) / 2;
  static int Data[];
  static double Pi_h_coef[];
  static const int nn[15][4];
  static const int aa[15][4];
  static const int ff[15];
  static const int il[15];
  static const int jl[15];
  static const int kl[15];

  TypeOfFE_P4Lagrange( ) : TypeOfFE(15, 1, Data, 4, 1, 15 + 6, 15, 0)
  {
    static const R2 Pt[15] = {R2(0 / 4., 0 / 4.), R2(4 / 4., 0 / 4.),
                              R2(0 / 4., 4 / 4.)...}

    // 3,4,5, 6,7,8, 9,10,11,
    int other[15] = {0, 1, 2, 5, 4, 3, 8, 7, 6, 11, 10, 9, 12, 13, 14};
    ...
  }
}
```

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  }
}
```

$\longrightarrow$ Interpolation order

$\longrightarrow$ # DOF =15

$\longrightarrow$ Topological information

Shape functions: $\Phi_i = \prod_{j=0}^{3} \frac{1}{\texttt{ff[i]}} \left(\lambda_{\texttt{nn[i][j]}} - \texttt{aa[i][j]}\right)$ `.hpp`

Nodes barycentric coordinates `.hpp`

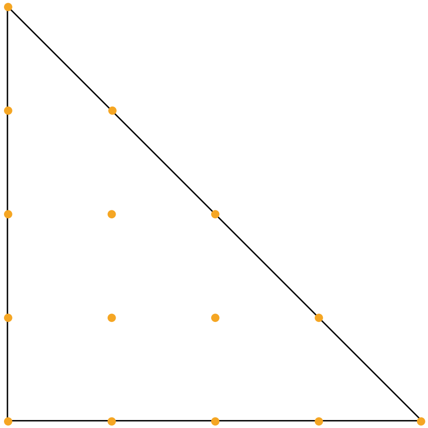
Lagrange FE in FreeFem++ II

```
TypeOfFE_P4Lagrange( ) : TypeOfFE(15, 1, Data, 4, 1, 15 + 6, 15, 0)
```

Argument	Value P_4	Value P_k
# dof	15	$\frac{(k+1)(k+2)}{2}$
Dimension of the function space	1	1
Array containing topological information	Data	Data
Number of subdivisions for plotting	4	k
Number of component sub-finite elements	1	1
nombre de dof + dof permutation non sym sur aretes	15+6	cell9
Number of interpolation points	15	$\frac{(k+1)(k+2)}{2}$
dof dont on a la connaissance avant	0	0

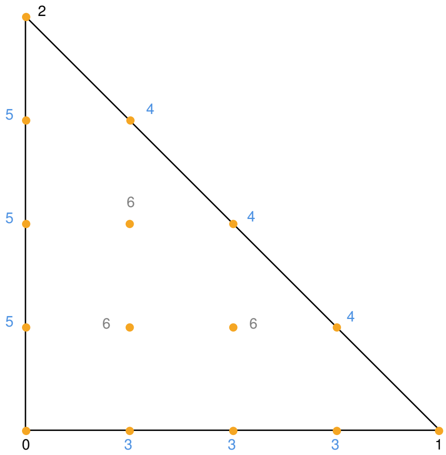
- 1: scalar function space(not vector field).
- Data: Numbering and relationship between DOFs and nodes.
- 4: To visualize the element, it's subdivided into k sub-elements.
- The 15 points where functions are evaluated to construct the interpolant.

Lagrange FE in FreeFem++ III



```
int TypeOfFE_P4Lagrange::Data[] = {
  // the support number  of the node of the dof
  0, 1, 2, 3, 3, 3, 4, 4, 4, 5, 5,  5,  6,  6,  6,
  // the number of the dof on the support
  0, 0, 0, 0, 1, 2, 0, 1, 2, 0, 1,  2,  0,  1,  2,
  // the node of the dof
  0, 1, 2, 3, 3, 3, 4, 4, 4, 5, 5,  5,  6,  6,  6,
  // the dof come from which FE (generaly 0)
  0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,  0,  0,  0,  0,
  // which are de dof on sub FE
  0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14,
  // for each compontant $j=0,N-1$ it give the sub FE ass
  0,
  // First dof
  0,
  // #dof
  15
  };
```

Lagrange FE in FreeFem++ III

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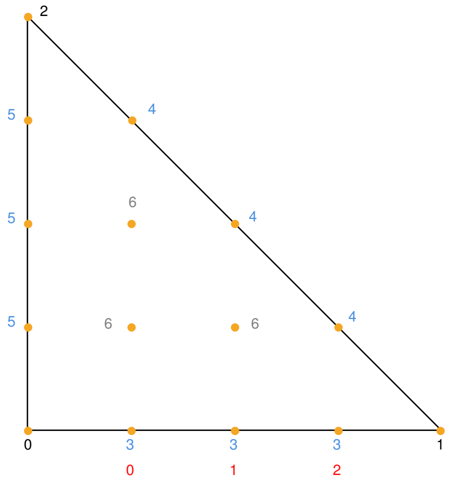


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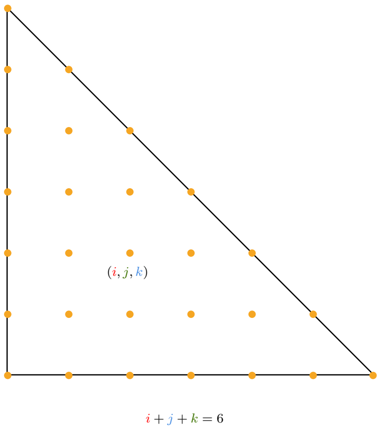
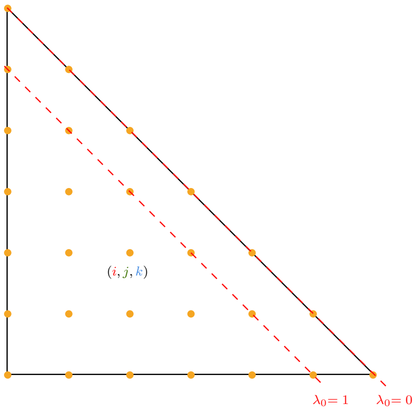
Shape functions



Let i, j, and k be the barycentric coordinates of a given node.

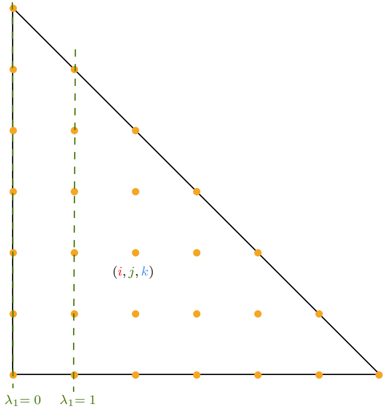
Shape functions



$i + j + k = 6$

$L_0 = \prod_{l<i} (\lambda_0 - l)$

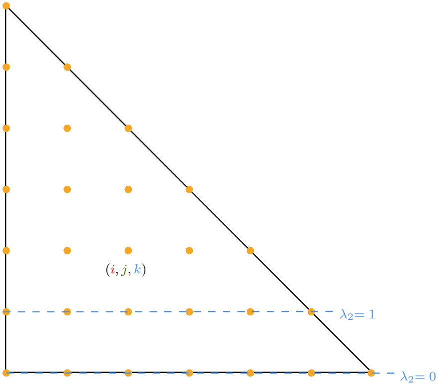
Shape functions



$i + j + k = 6$

$L_1 = \prod_{l<j} (\lambda_1 - l)$

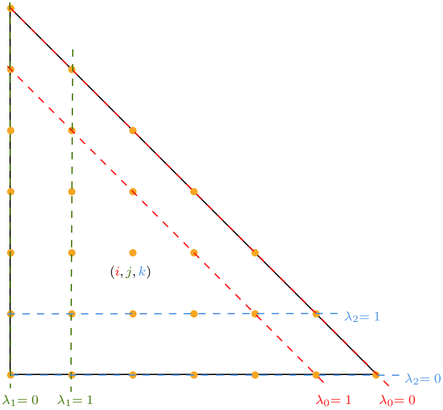
Shape functions



$i + j + k = 6$

$L_2 = \prod_{l<k} (\lambda_2 - l)$

Shape functions



$i + j + k = 6$

$L_0 = \prod_{l<i} (\lambda_0 - l)$ $\quad L_1 = \prod_{l<j} (\lambda_1 - l)$ $\quad L_2 = \prod_{l<k} (\lambda_2 - l)$

$$L = \frac{1}{i!j!k!} L_0 \times L_1 \times L_2$$

```
void BasisFctPK(int , vector<vector<long>> &nn,
        vector<vector<long>> &aa,
        vector<long> &ff) {
    int idx = 0;
    for (auto &coordinate : coordinate_list) {
        int i = coordinate[0];
        int j = coordinate[1];
        int k = coordinate[2];
        if (i + j + k == p) {
            int ID = 0;
            ff[idx] = factorial(i)*factorial(j)
                    *factorial(k);
            if (i > 0) {
                for (int ii = 0; ii < i ; ii++) {
                    nn[idx][ID] = 0;
                    aa[idx][ID] = ii;
                    ID++;
                }
            }
            // same for j and k
            idx++;
            }
        }
    }
```

Adding the FE in `FreeFem++`

- Declaration of a subclass `TypeOfFE_PkLagrange` of `TypeOfFE`
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- **Once the attributes are filled, the elements can be added to the DSL using:**

```
static TypeOfFE_PkLagrange PKLagrange(k); // Obj of the class TypeOfFE_PkLagrange
namespace Fem2D {
    ...
    static void init() {

        AddNewFE("PKLagrange", &PKLagrange);
        static ListOfTFE FE_PK("PKLagrange", &PKLagrange); //add Pk to the list of Common FE
    }
} // namespace Fem2D
LOADFUNC(Fem2D::init);
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Integrating polynomials of arbitrary order?

Quadrature Formula

Grundmann-Möller quadrature formula

Let $s \in \mathbb{N}$, $d = 2s + 1$, and $n \in \mathbb{N}$ a given dimension. Then:

$$I_n(f) = \sum_{i=0}^{s} (-1)^i 2^{-2s} \frac{(d+n-2i)^d}{i!(d+n-i)!} \sum_{\substack{|\boldsymbol{\beta}|=s-i \\ \beta_0 \geq \cdots \geq \beta_n}} \sum f\left(\frac{2\beta_1 + 1}{d+n-2i}, \ldots, \frac{2\beta_n + 1}{d+n-2i}\right)$$

is an invariant integration formula of degree d for the function f on an n-simplex,
where: $\boldsymbol{\beta} = (\beta_0, \beta_1, \ldots, \beta_n)$, and $|\boldsymbol{\beta}| = \beta_0 + \beta_1 + \cdots + \beta_n$

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GM formula:

- ☐ is capable of handling any arbitrary polynomial order ✓
- ☐ is exact for polynomials of degree d ✓
- ☐ is easily adaptable to any spatial dimension n ✓
- ☐ does not guarantee that all weights are positive X
- ☐ is prone to computational overflow issues X
- ☐ is prone to numerical stability issues X

Validation

Conclusion:

- ☐ Analysis of the existing 2D Lagrange finite elements
- ☐ Design of a generic 2D P_k element inspired by existing elements
- ☐ Development of associated quadrature formulas

Outlooks:

- ☐ More rigorous verification of convergence
- ☐ Addressing numerical instability in quadrature formulas
- ☐ Extension to the case of 3D Lagrange