



## High-order Lagrange elements in FreeFem++

A. CHABIB, P-H. TOURNIER, F. HECHT

Laboratoire Jacques-Louis Lions  
CNRS, Sorbonne Université UPMC , France  
ALPINES team, Centre Inria de Paris

FreeFem Days  
December 17, 2025

## Finite element solution of time-harmonic wave propagation problems

### Helmholtz problem:

$$\begin{cases} -\Delta \mathbf{p} - \omega^2 \mathbf{p} = \mathbf{s} & \text{in } \Omega \\ \text{Boundary conditions} & \text{on } \partial\Omega \end{cases}$$

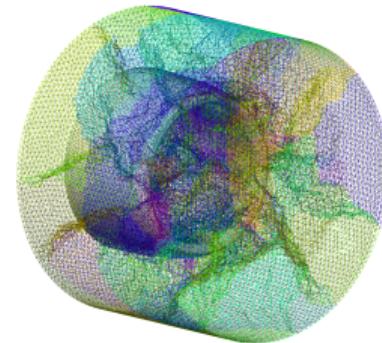
### Finite element error:

$$\|\mathbf{p} - \mathbf{p}_h\|_{H^1} \leq C_\omega h^k \|\mathbf{p}\|_{H^{k+1}}$$

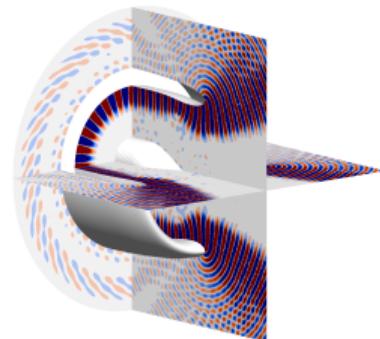
High frequency (large  $\omega$ )

Phenomena close to resonance

Finite element mesh



Numerical field



$\rightsquigarrow$   $\begin{cases} \text{Fine mesh (small } h\text{)} \\ \text{High-order basis functions (large } k\text{)} \end{cases}$

## Finite element solution of time-harmonic wave propagation problems

### Helmholtz problem:

$$\begin{cases} -\Delta \mathbf{p} - \omega^2 \mathbf{p} = \mathbf{s} & \text{in } \Omega \\ \text{Boundary conditions} & \text{on } \partial\Omega \end{cases}$$

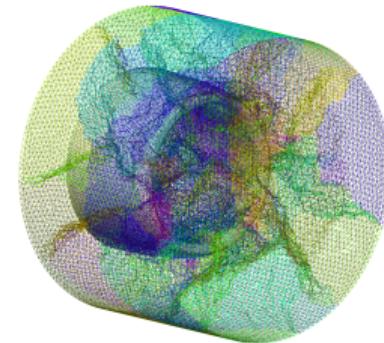
### Finite element error:

$$\|\mathbf{p} - \mathbf{p}_h\|_{H^1} \leq C_\omega h^k \|\mathbf{p}\|_{H^{k+1}}$$

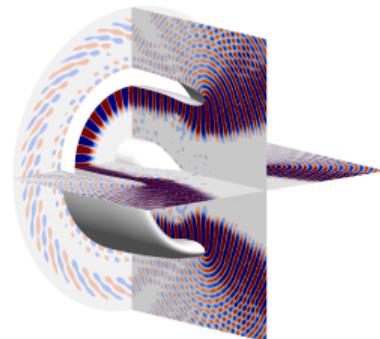
High frequency (large  $\omega$ )

Phenomena close to resonance

Finite element mesh



Numerical field



$\rightsquigarrow \begin{cases} \text{Fine mesh (small } h\text{)} & \checkmark \\ \text{High-order basis functions (large } k\text{)} & \text{X Limited to } P_4 \end{cases}$

# Lagrange FE in FreeFem++ |

```
class Type0fFE_P4Lagrange : public Type0fFE {
public:
    static const int k = 4;
    static const int ndf = (k + 2) * (k + 1) / 2;
    static int Data[];
    static double Pi_h_coef[];
    static const int nn[15][4];
    static const int aa[15][4];
    static const int ff[15];
    static const int il[15];
    static const int jl[15];
    static const int kl[15];

    Type0fFE_P4Lagrange( ) : Type0fFE(15, 1, Data, 4, 1, 15 + 6, 15, 0)
    {
        static const R2 Pt[15] = {R2(0 / 4., 0 / 4.), R2(4 / 4., 0 / 4.),
                                R2(0 / 4., 4 / 4.)...}

        // 3,4,5, 6,7,8, 9,10,11,
        int other[15] = {0, 1, 2, 5, 4, 3, 8, 7, 6, 11, 10, 9, 12, 13, 14};
        ...
    }
}
```

# Lagrange FE in FreeFem++ |

```
class Type0fFE_P4Lagrange : public Type0fFE {  
public:  
    static const int k = 4; → Interpolation order  
    static const int ndf = (k + 2) * (k + 1) / 2; → # DOF =15  
    static int Data[]; → Topological information  
    static double Pi_h_coef[]; → Used to construct normal Not needed for Lagrange  
    static const int nn[15][4];  
    static const int aa[15][4]; } Shape functions:  $\Phi_i = \prod_{j=0}^3 \frac{1}{ff[i]} (\lambda_{nn[i][j]} - aa[i][j]) .hpp$   
    static const int ff[15];  
    static const int il[15];  
    static const int jl[15]; } Nodes barycentric coordinates .hpp  
    static const int kl[15]; }
```

```
Type0fFE_P4Lagrange( ) : Type0fFE(15, 1, Data, 4, 1, 15 + 6, 15, 0)
```

Parent class constructor

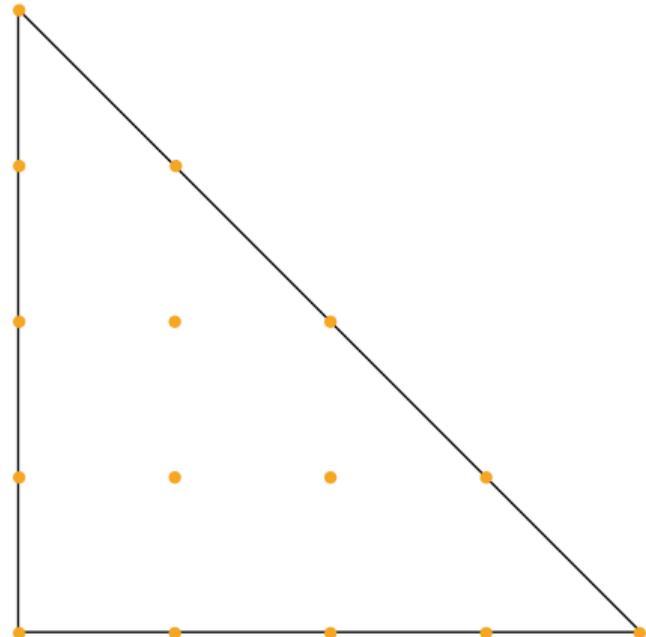
```
{  
    static const R2 Pt[15] = {R2(0 / 4., 0 / 4.), R2(4 / 4., 0 / 4.),  
                             R2(0 / 4., 4 / 4.)...}  
  
    // 3,4,5, 6,7,8, 9,10,11,  
    int other[15] = {0, 1, 2, 5, 4, 3, 8, 7, 6, 11, 10, 9, 12, 13, 14};  
    ...  
}
```

## Lagrange FE in FreeFem++ II

```
TypeOfFE_P4Lagrange( ) : TypeOfFE(15, 1, Data, 4, 1, 15 + 6, 15, 0)
```

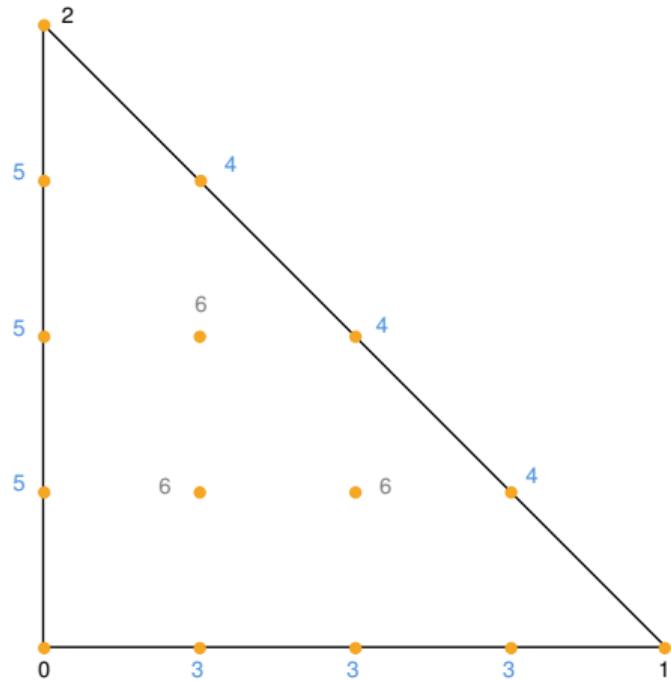
Argument	Value $P_4$	Value $P_k$
# dof	15	$\frac{(k+1)(k+2)}{2}$
Dimension FE	1	1
Array containing topological information	Data	Data
Number of subdivisions for plotting	4	$k$
Number of component sub-finite elements	1	1
# dof+ non-symmetric permutation of dofs on edges	15+6	$\# \text{dof} + 3 \times \frac{((k \% 2) ? (k - 1) : (k - 2))}{(k + 1)(k + 2)}$
Number of interpolation points	15	$\frac{2}{2}$
Normal components array	0	0

# Lagrange FE in FreeFem++ III



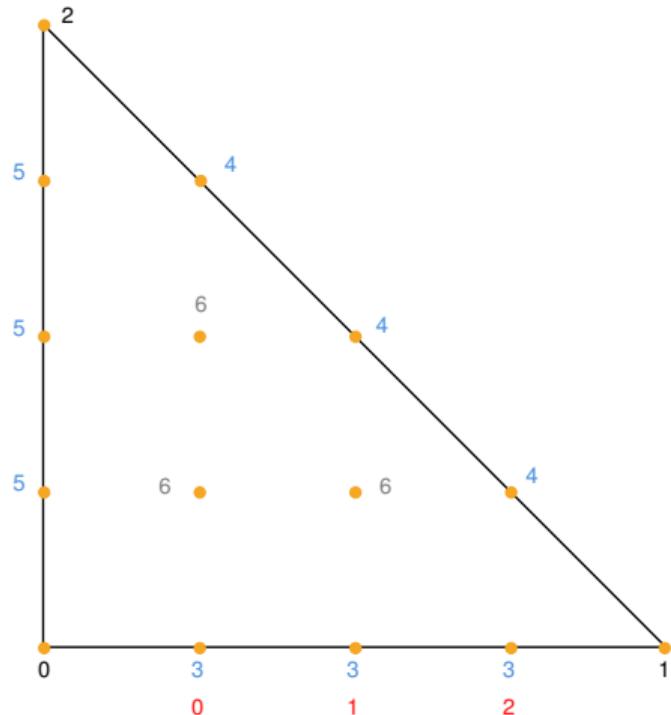
```
int Type0fFE_P4Lagrange::Data[] = {
    // the support number of the node of the dof
    0, 1, 2, 3, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 6,
    // the number of the dof on the support
    0, 0, 0, 0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2,
    // the node of the dof
    0, 1, 2, 3, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 6,
    // the dof come from which FE (generaly 0)
    0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
    // which are de dof on sub FE
    0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14,
    // for each compontant $j=0,N-1$ it give the sub FE ass
    0,
    // First dof
    0,
    // #dof
    15
};
```

# Lagrange FE in FreeFem++ III



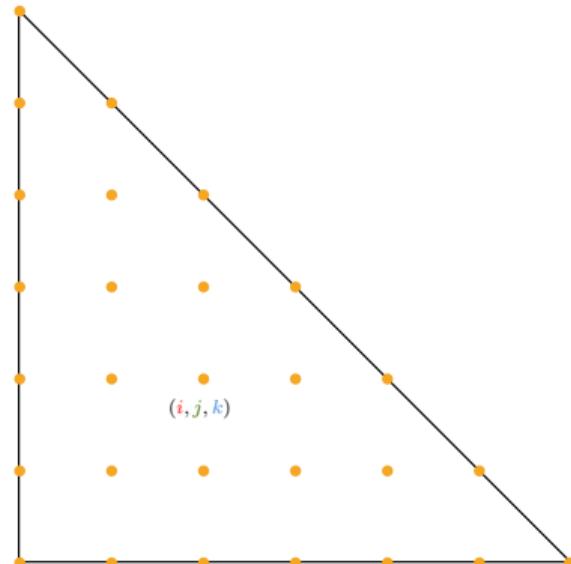
```
int Type0fFE_P4Lagrange::Data[] = {  
    // the support number of the node of the dof  
    0, 1, 2, 3, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 6,  
    // the number of the dof on the support  
    0, 0, 0, 0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2,  
    // the node of the dof  
    0, 1, 2, 3, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 6,  
    // the dof come from which FE (generaly 0)  
    0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,  
    // which are de dof on sub FE  
    0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14,  
    // for each compontant $j=0,N-1$ it give the sub FE ass  
    0,  
    // First dof  
    0,  
    // #dof  
    15  
};
```

# Lagrange FE in FreeFem++ III



```
int Type0fFE_P4Lagrange::Data[] = {  
    // the support number of the node of the dof  
    0, 1, 2, 3, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 6,  
    // the number of the dof on the support  
    0, 0, 0, 0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2,  
    // the node of the dof  
    0, 1, 2, 3, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 6,  
    // the dof come from which FE (generaly 0)  
    0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,  
    // which are de dof on sub FE  
    0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14,  
    // for each compontant $j=0,N-1$ it give the sub FE ass  
    0,  
    // First dof  
    0,  
    // #dof  
    15  
};
```

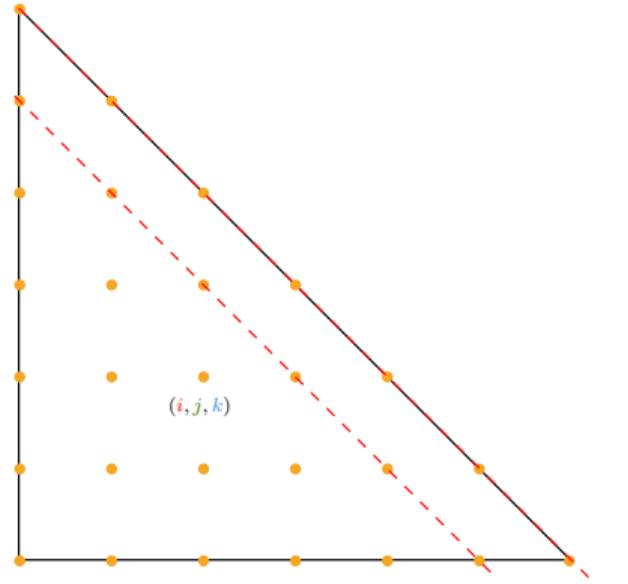
## Shape functions



$$i + j + k = 6$$

Let  $i$ ,  $j$ , and  $k$  be the barycentric coordinates of a given node.

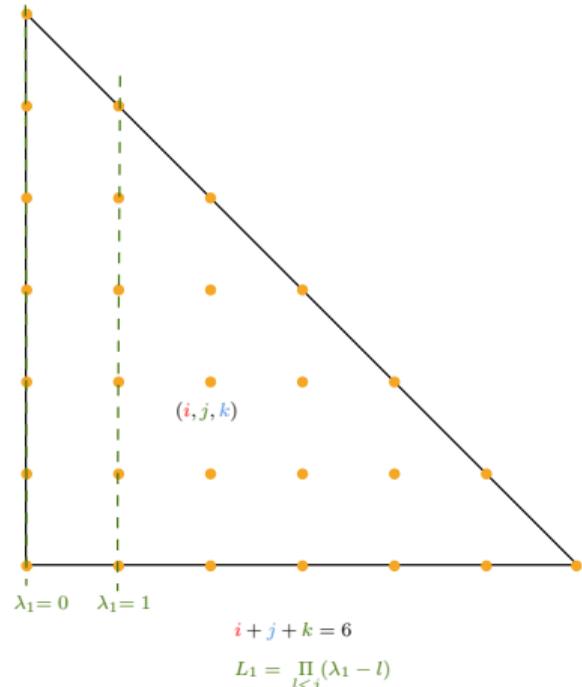
## Shape functions



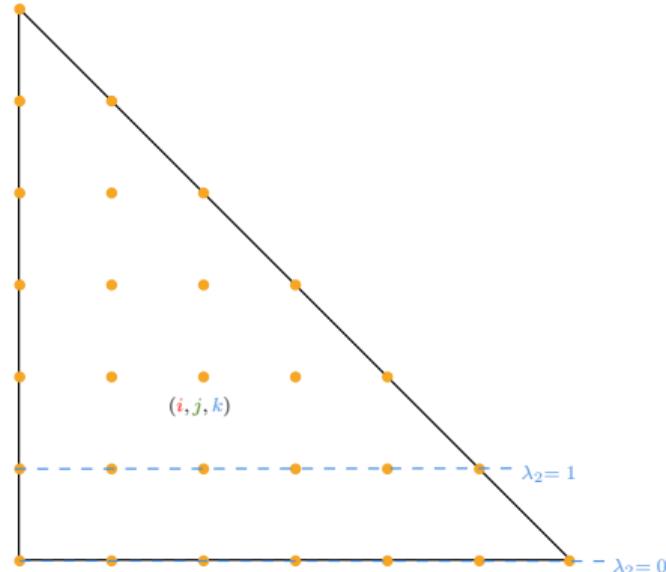
$$i + j + k = 6$$

$$L_0 = \prod_{l < i} (\lambda_0 - l)$$

# Shape functions



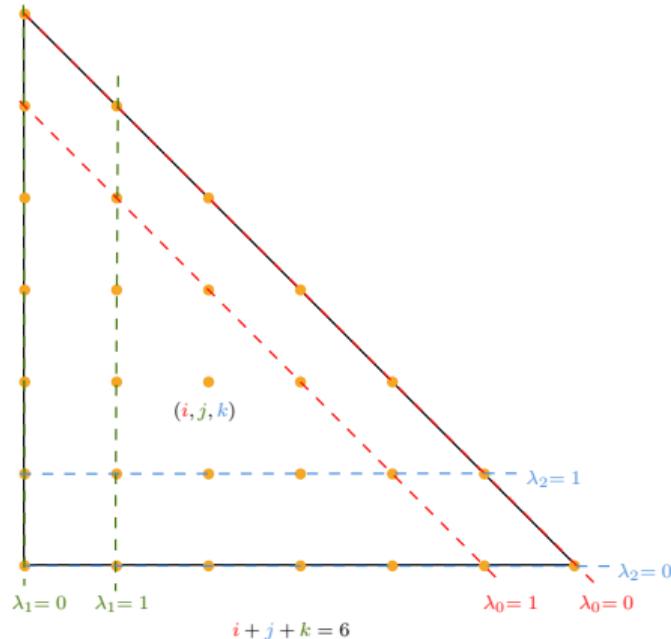
# Shape functions



$$i + j + k = 6$$

$$L_2 = \prod_{l < k} (\lambda_2 - l)$$

# Shape functions



$$L_0 = \prod_{l < i} (\lambda_0 - l) \quad L_1 = \prod_{l < j} (\lambda_1 - l) \quad L_2 = \prod_{l < k} (\lambda_2 - l)$$

$$L = \frac{1}{i!j!k!} L_0 \times L_1 \times L_2$$

```

void BasisFctPK(int , vector<vector<long>> &nn,
                 vector<vector<long>> &aa,
                 vector<long> &ff) {
    int idx = 0;
    for (auto &coordinate : coordinate_list) {
        int i = coordinate[0];
        int j = coordinate[1];
        int k = coordinate[2];
        if (i + j + k == p) {
            int ID = 0;
            ff[idx] = factorial(i)*factorial(j)
                      *factorial(k);
            if (i > 0) {
                for (int ii = 0; ii < i ; ii++) {
                    nn[idx][ID] = 0;
                    aa[idx][ID] = ii;
                    ID++;
                }
            }
            // same for j and k
            idx++;
        }
    }
}

```

## Adding the FE in FreeFem++

- Declaration of a subclass TypeOfFE\_PkLagrange of TypeOfFE
- Dynamic allocation of arrays depending on the interpolation order
- Arrays are filled on the fly during the creation of the object.

## Adding the FE in FreeFem++

- Declaration of a subclass TypeOfFE\_PkLagrange of TypeOfFE
- Dynamic allocation of arrays depending on the interpolation order
- Arrays are filled on the fly during the creation of the object.
- Once the attributes are filled, the elements can be added to the DSL using:

```
static TypeOfFE_PkLagrange PKLagrange(k); // Obj of the class TypeOfFE_PkLagrange
namespace Fem2D {
    ...
    static void init() {

        AddNewFE("PKLagrange", &PKLagrange);
        static ListOfTFE FE_PK("PKLagrange", &PKLagrange); //add Pk to the list of Common FE
    }
} // namespace Fem2D
LOADFUNC(Fem2D::init);
```

- Declaration of a subclass TypeOfFE\_PkLagrange of TypeOfFE
- Dynamic allocation of arrays depending on the interpolation order
- Arrays are filled on the fly during the creation of the object.
- Once the attributes are filled, the elements can be added to the DSL using:

```
static TypeOfFE_PkLagrange PKLagrange(k); // Obj of the class TypeOfFE_PkLagrange
namespace Fem2D {
    ...
    static void init() {

        AddNewFE("PKLagrange", &PKLagrange);
        static ListOfTFE FE_PK("PKLagrange", &PKLagrange); //add Pk to the list of Common FE
    }
} // namespace Fem2D
LOADFUNC(Fem2D::init);
```

Integrating polynomials of arbitrary order?

# Quadrature Formula

## Grundmann-Möller quadrature formula

Let  $s \in \mathbb{N}$ ,  $d = 2s + 1$ , and  $n \in \mathbb{N}$  a given dimension. Then:

$$I_n(f) = \sum_{i=0}^s (-1)^i 2^{-2s} \frac{(d+n-2i)^d}{i!(d+n-i)!} \sum_{\substack{|\beta|=s-i \\ \beta_0 \geq \dots \geq \beta_n}} f\left(\frac{2\beta_1+1}{d+n-2i}, \dots, \frac{2\beta_n+1}{d+n-2i}\right)$$

is an invariant integration formula of degree  $d$  for the function  $f$  on an  $n$ -simplex,  
where:  $\beta = (\beta_0, \beta_1, \dots, \beta_n)$ , and  $|\beta| = \beta_0 + \beta_1 + \dots + \beta_n$

# Quadrature Formula

## Grundmann-Möller quadrature formula

Let  $s \in \mathbb{N}$ ,  $d = 2s + 1$ , and  $n \in \mathbb{N}$  a given dimension. Then:

$$I_n(f) = \sum_{i=0}^s (-1)^i 2^{-2s} \frac{(d+n-2i)^d}{i!(d+n-i)!} \sum_{\substack{|\beta|=s-i \\ \beta_0 \geq \dots \geq \beta_n}} f\left(\frac{2\beta_1+1}{d+n-2i}, \dots, \frac{2\beta_n+1}{d+n-2i}\right)$$

is an invariant integration formula of degree  $d$  for the function  $f$  on an  $n$ -simplex,  
where:  $\beta = (\beta_0, \beta_1, \dots, \beta_n)$ , and  $|\beta| = \beta_0 + \beta_1 + \dots + \beta_n$

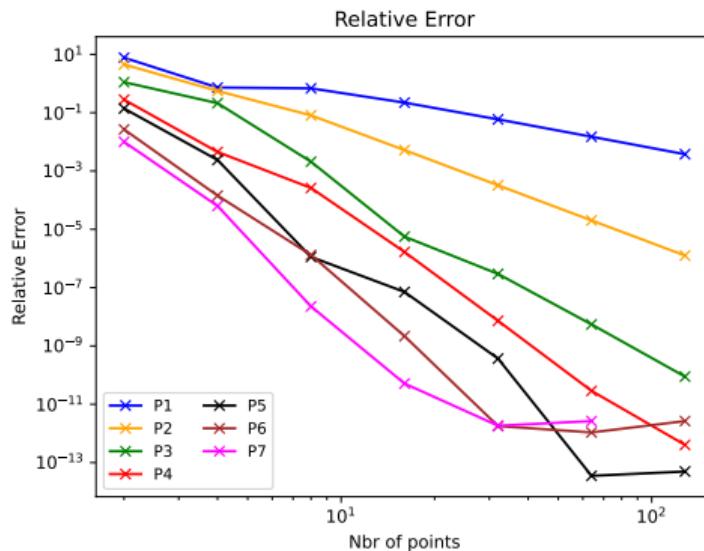
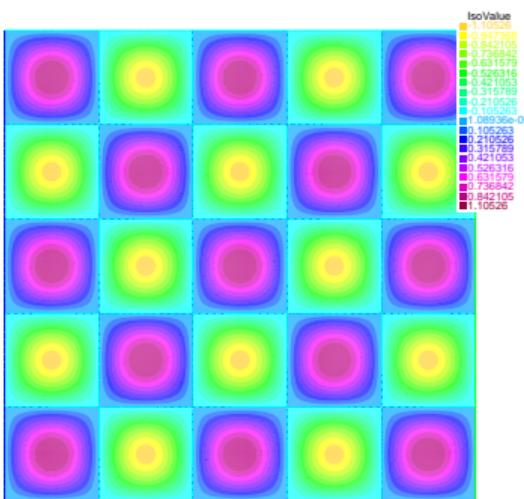
GM formula:

- |  |   |
|--|---|
| <input type="checkbox"/> is capable of handling any arbitrary polynomial order | ✓ |
| <input type="checkbox"/> is exact for polynomials of degree $d$                | ✓ |
| <input type="checkbox"/> is easily adaptable to any spatial dimension $n$      | ✓ |
| <input type="checkbox"/> does not guarantee that all weights are positive      | ✗ |
| <input type="checkbox"/> is prone to computational overflow issues             | ✗ |
| <input type="checkbox"/> is prone to numerical stability issues                | ✗ |

# Benchmark

Poisson problem

$$\begin{cases} -\Delta u = (5\pi)^2 \sin(5\pi x) \sin(5\pi y) & \text{in } \Omega = (0, 1)^2, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$



## Conclusion:

- Analysis of the existing 2D Lagrange finite elements
- Design of a generic 2D  $P_k$  element inspired by existing elements
- Development of associated quadrature formulas
- Online documentation on how to add a FE: <https://www.ljll.fr/hecht/ftp/old/freefem/old/manual-full.pdf>

## Outlooks:

- More rigorous verification of convergence
- Addressing numerical instability in quadrature formulas
- Extension to the case of 3D Lagrange