

High-order Lagrange elements in FreeFem++

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FreeFem Days
December 17, 2025

Motivation

Finite element solution of **time-harmonic wave propagation problems**

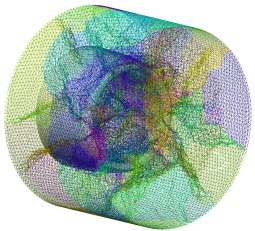
Helmholtz problem:

$$\begin{cases} -\Delta p - \kappa^2 p = s & \text{in } \Omega \\ \text{Boundary conditions} & \text{on } \partial\Omega \end{cases}$$

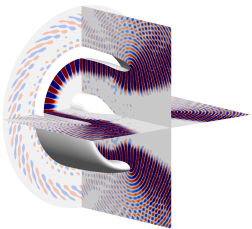
Finite element error:

$$\|p - p_h\|_{H^1} \leq C_1(\kappa h)^p + C_2 \kappa^{(p+1)} h^p$$

Finite element mesh



Numerical field

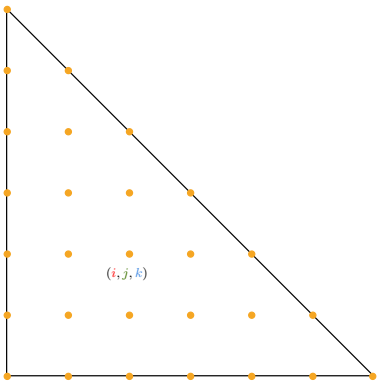


High frequency (large κ)
Phenomena close to resonance

$\rightsquigarrow$ Low approximation quality $\rightsquigarrow$

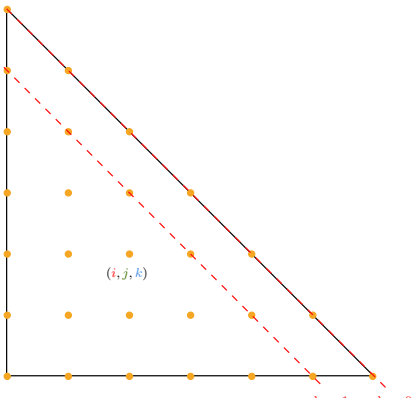
- Fine mesh (small h)
- High-order basis functions (large p)

Shape functions



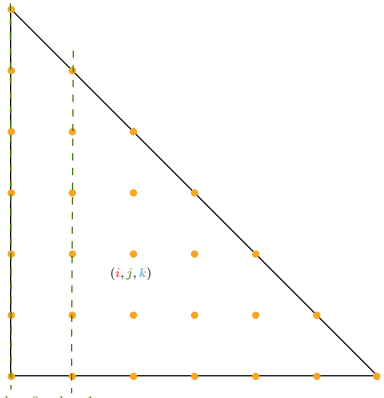
Let i, j, and k be the barycentric coordinates of a given node.

Shape functions



$$L_0 = \prod_{l<i} (\lambda_0 - l)$$

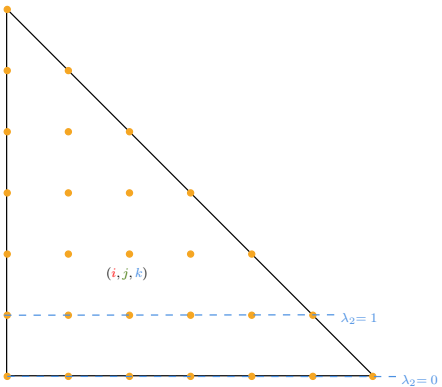
Shape functions



$i + j + k = 6$

$L_1 = \prod_{l<j} (\lambda_1 - l)$

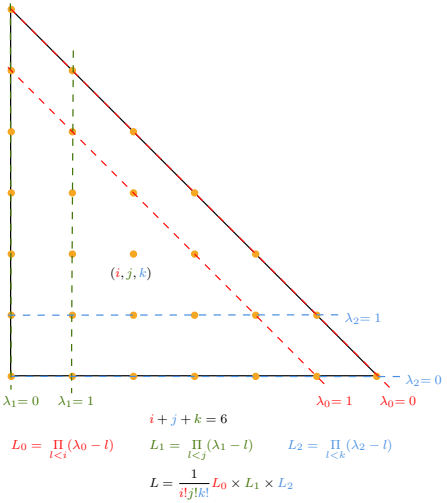
Shape functions



$i + j + k = 6$

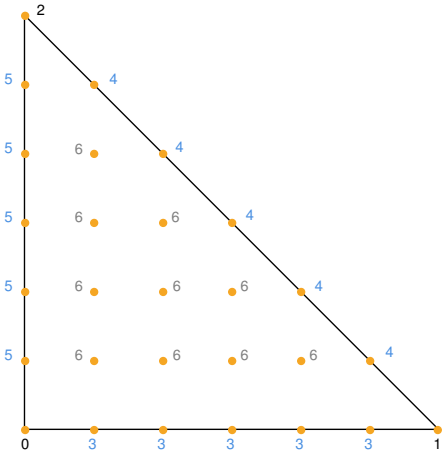
$$L_2 = \prod_{l<k} (\lambda_2 - l)$$

Shape functions



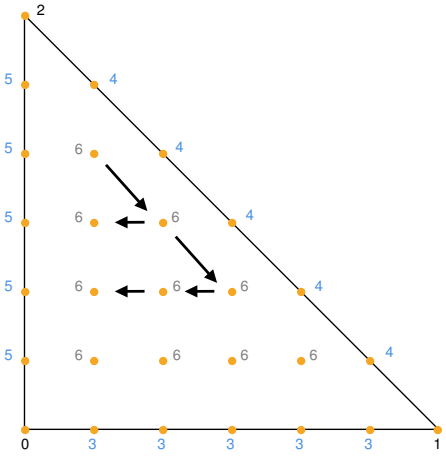

```
void BasisFctPK(int , vector<vector<long>> &lambdas,
        vector<vector<long>> &shift,
        vector<long> &ff) {
    int idx = 0;
    for (auto &coordinate : coordinate_list) {
        int i = coordinate[0];
        int j = coordinate[1];
        int k = coordinate[2];
        if (i + j + k == p) {
            int ID = 0;
            ff[idx] = factorial(i)*factorial(j)
                    *factorial(k);
            if (i > 0) {
                for (int ii = 0; ii < i ; ii++) {
                    lambdas[idx][ID] = 0;
                    shift[idx][ID] = ii;
                    ID++;
                }
            }
            // same for j and k
            idx++;
            }
        }
    }
```

Numerotation FreeFem++



- The vertices are numbered 0, 1, 2
- The nodes of the edges are numbered from 3 to 5
- The internal nodes keep the number 6

Numerotation FreeFem++



- The vertices are numbered 0, 1, 2
- The nodes of the edges are numbered from 3 to 5
- The internal nodes keep the number 6
- Internal points are ordered from top to bottom and from left to right

Conclusion