Lecture 3: Kalman filter

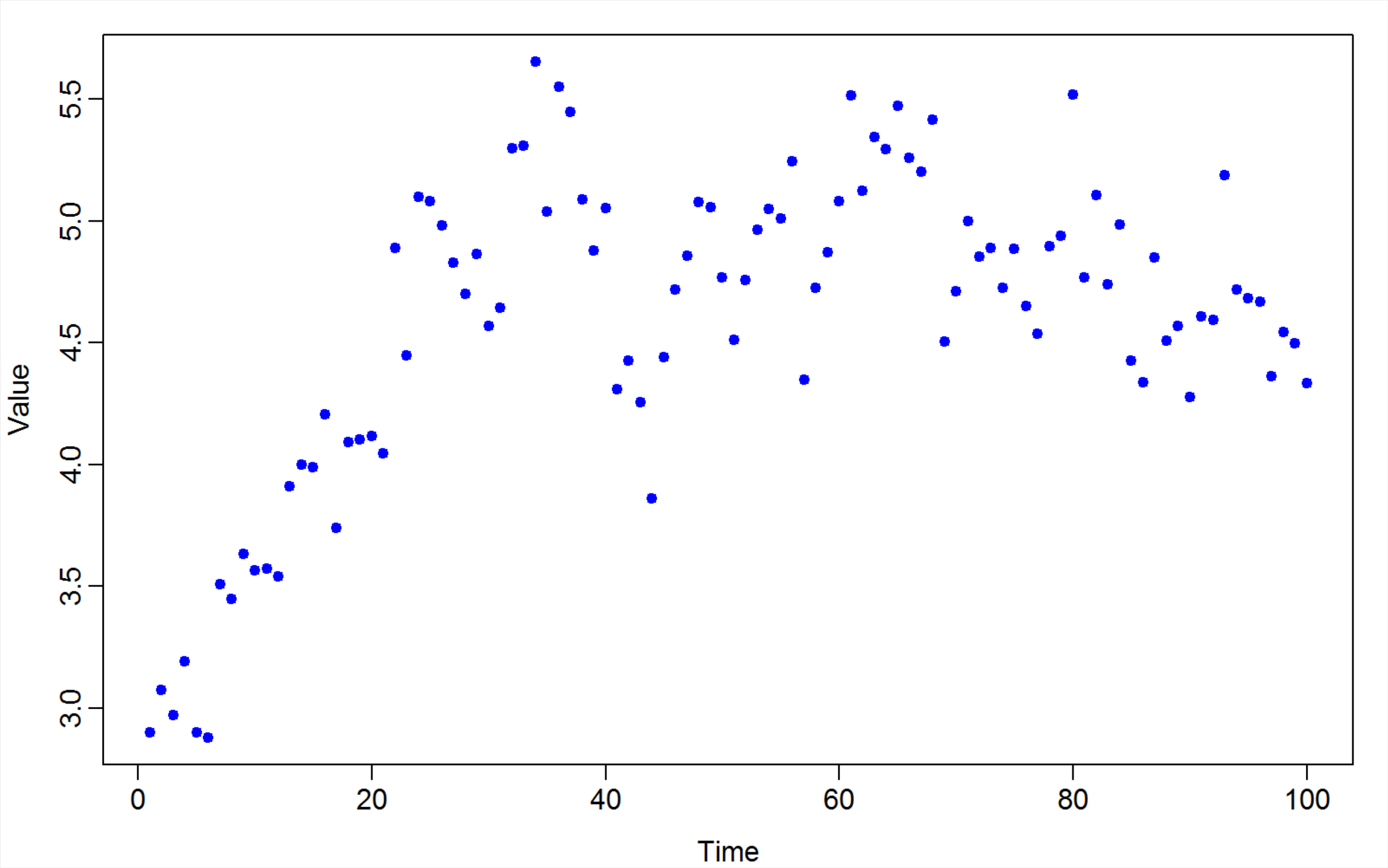
April 12, 2018

Directed random walk

$$x_{t+1} = x_t + \varepsilon_t$$
$$\varepsilon_t \sim Normal(\alpha, \sigma_x^2)$$
$$\log(y_t) \sim Normal(x_t, \sigma_y^2)$$

– Time-series follows a random-walk with a trend α

Data set



Questions

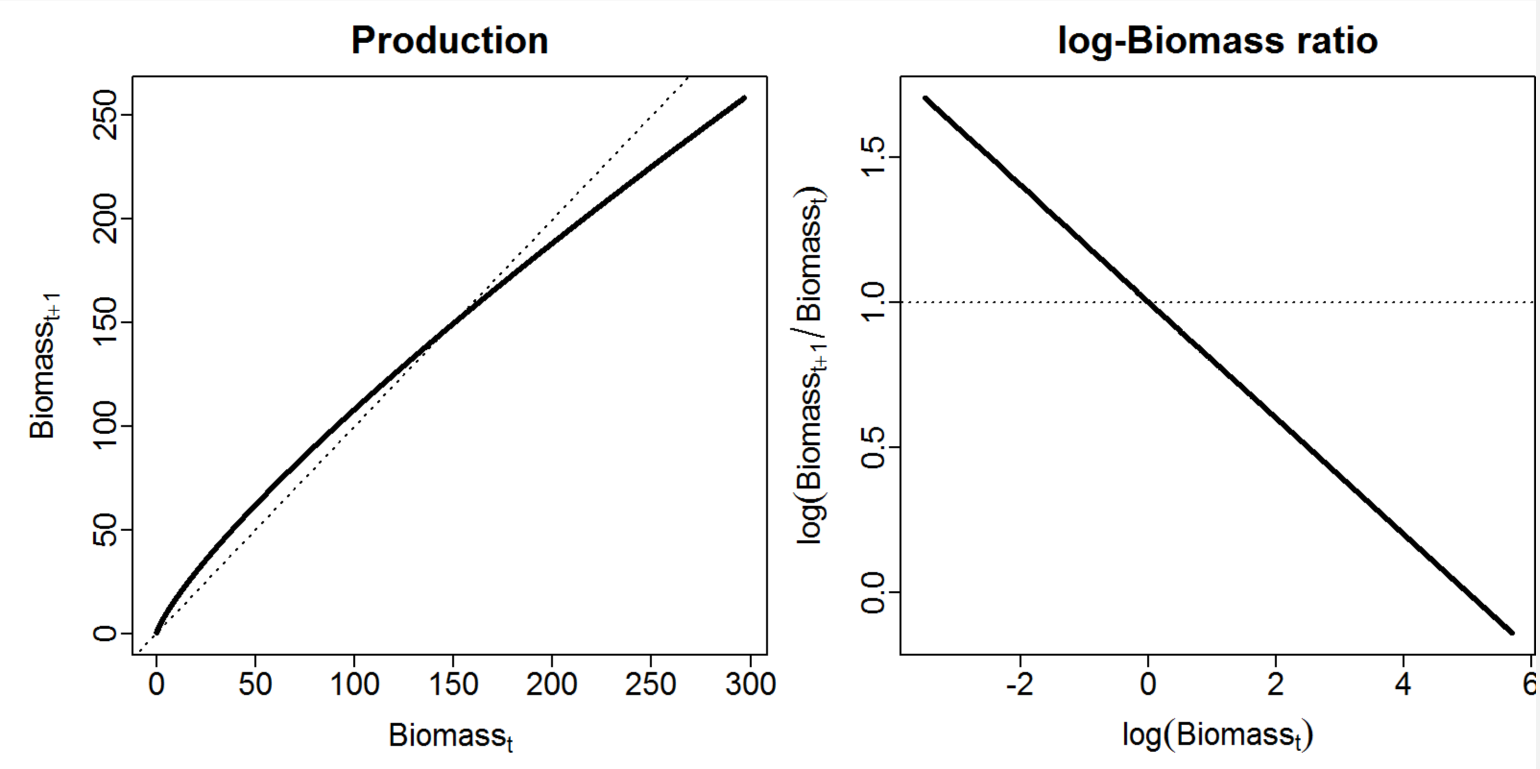
- Is there evidence of density dependence?
- If yes, what is the expected equilibrium?
- What is the stationary distribution?
  - Variance
- How “stable” is the model
  - Reactivitiy
  - Resilience
  - Resistence

Gompertz model

$$d_{t+1} = d_t \exp(\alpha - \beta \log(d_t) + \varepsilon_t)$$

$$\varepsilon_t \sim Normal(0, \sigma_d^2)$$

$$\log(b_t) \sim Normal\left(\log(d_t), \sigma_b^2\right)$$



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- Fits to an index of abundance, $\mathbf{b}$
- β is the strength of density dependence
  - Linear impact of $\log(d_t)$ on per-capita productivity
- ε_t is a lognormally distributed process error
- σ_ε^2 is the variance of log-process errors
- σ_b^2 is the variance of log-observation errors

Gompertz model (autoregressive parameterization)

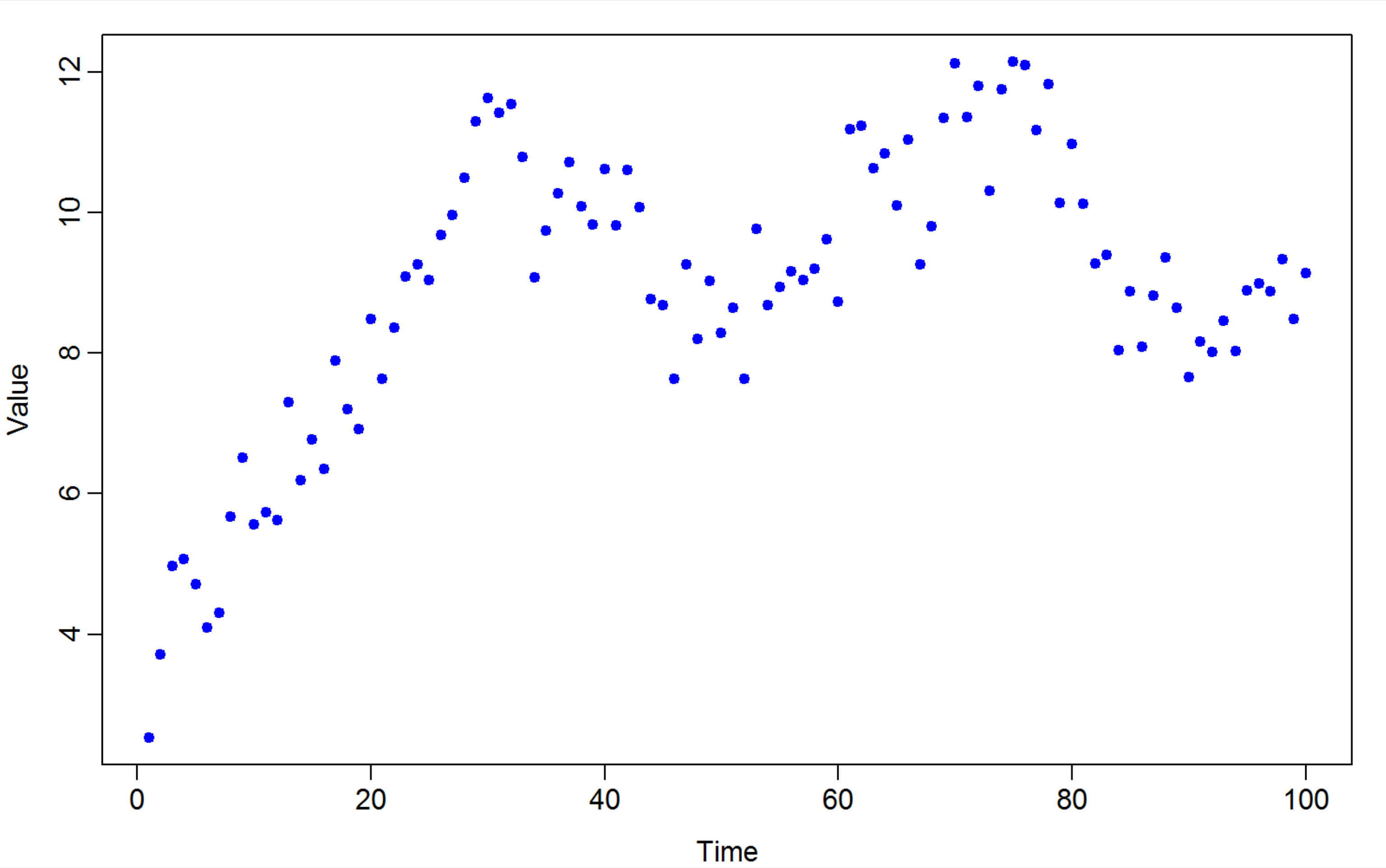
$$\log(d_{t+1}) = \alpha + (1-\beta)\log(d_t) + \varepsilon_t$$

$$\varepsilon_t \sim Normal(0, \sigma_\varepsilon^2)$$

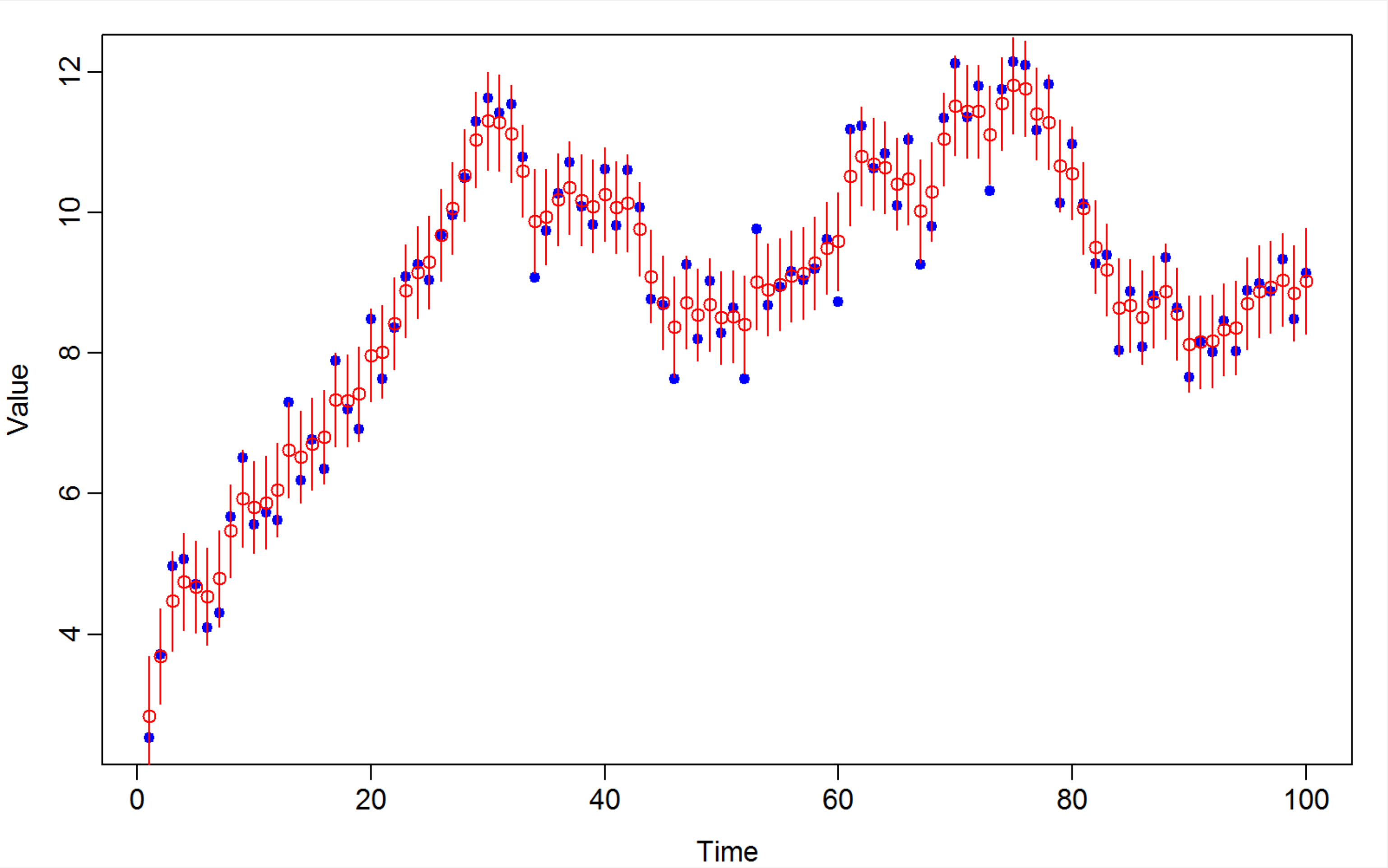
$$\log(b_t) \sim Normal(d_t, \sigma_b^2)$$

- Log-density follows an autoregressive process over time
- $\rho = 1 - \beta$ is "density dependence"
  - $\rho = 0$ means each year fluctuates independently
  - $\rho = 1$ means the population follows a random-walk with no equilibrium

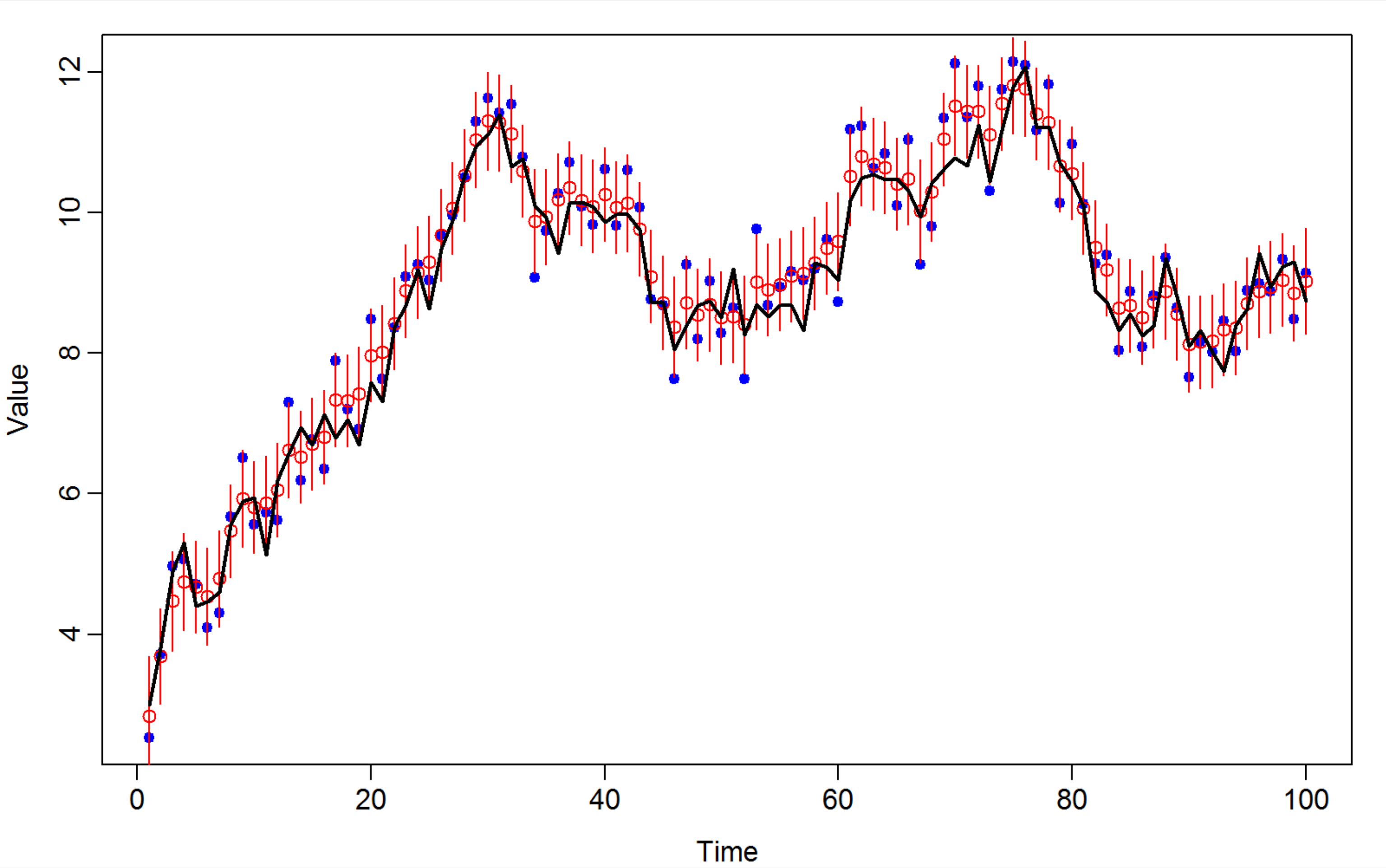
Gompertz model (autoregressive parameterization)



Gompertz model (autoregressive parameterization)



Gompertz model (autoregressive parameterization)



Gompertz model

Benefits

- Specifies an explicit model
  - Can select form of density dependence
- Can calculate stationary distribution

$$\lim_{t \to \infty} (\log(d_t)) = D$$

$$\mathbb{E}(D) = \frac{\alpha}{\beta}$$

$$\mathbb{V}(D) = \frac{\sigma_d^2}{\beta}$$

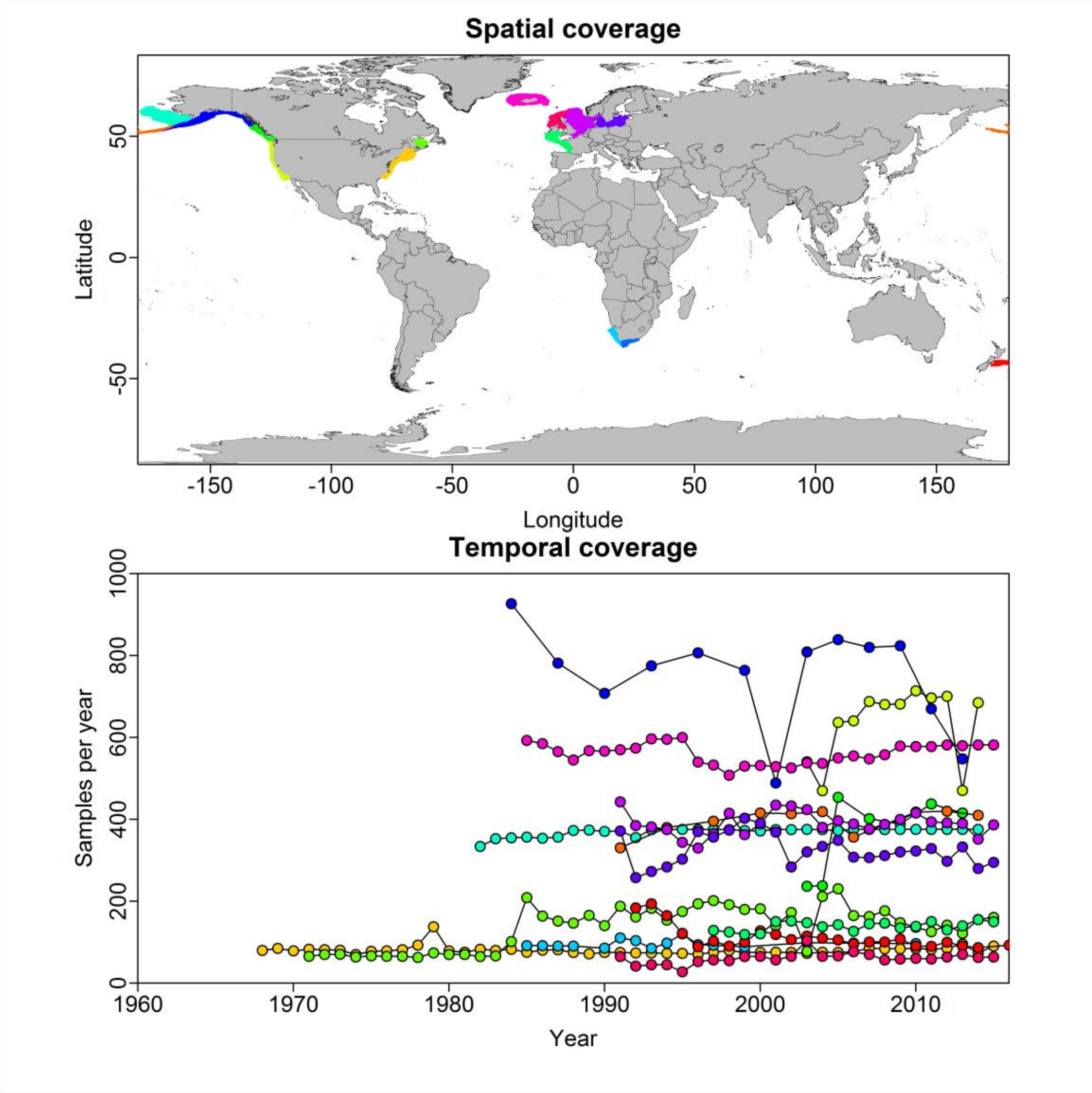
Map argument

[Look at map example in GitHub]

Try again with real data

Package *FishLife*

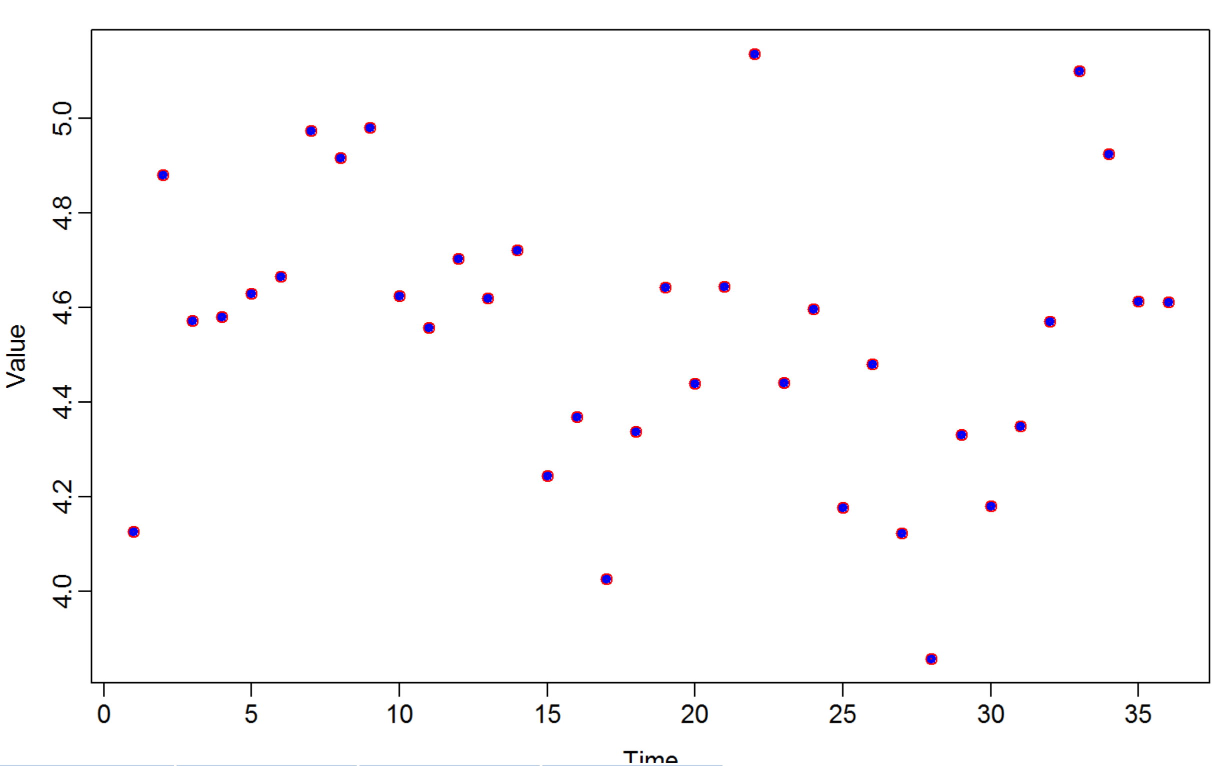
- Contains data for 100s of fish populations worldwide



Alaska pollock

Is this converged?

How to modify?



Param	Starting value	Lower	MLE	Upper	Final gradient	Std. Error
log_d0	0	-Inf	4.126071	Inf	3.75E-07	0.260514
log_sigmaP	1	-Inf	-1.3451	Inf	3.34E-06	0.117851
log_sigmaM	1	-Inf	-11.029	Inf	7.42E-10	26029.84
alpha	0	-Inf	2.651077	Inf	-1.42E-06	0.665236
rho	0	-Inf	0.419967	Inf	-2.64E-06	0.145993

Add sampling variance

Suppose you have many measurements x_i, what is the mean and standard error?

$$\hat{\mu} = \frac{1}{n_i} \sum_{i=1}^{n_i} x_i$$

$$\hat{\sigma} = \sqrt{\frac{1}{n_i - 1} \sum_{i=1}^{n_i} (x_i - \hat{\mu})^2}$$

$$\widehat{SE}(\hat{\mu}) = \frac{\hat{\sigma}}{\sqrt{n_i}}$$

Add sampling variance

If measurements x_i follow a lognormal distribution with mean $\hat{\mu}$ and standard deviation $\widehat{SE}(\hat{\mu})$, what is the log-standard deviation $\sigma_{lognormal}$ of the lognormal distribution?

$$CV = \frac{\widehat{SE}(\hat{\mu})}{\hat{\mu}} = \sqrt{e^{\sigma_{lognormal}^2} - 1}$$

Therefore:

$$\sigma_{lognormal}^2 = \log(CV^2 + 1)$$

Lab exercise

[Add sampling variance and re-run]

Alaska pollock

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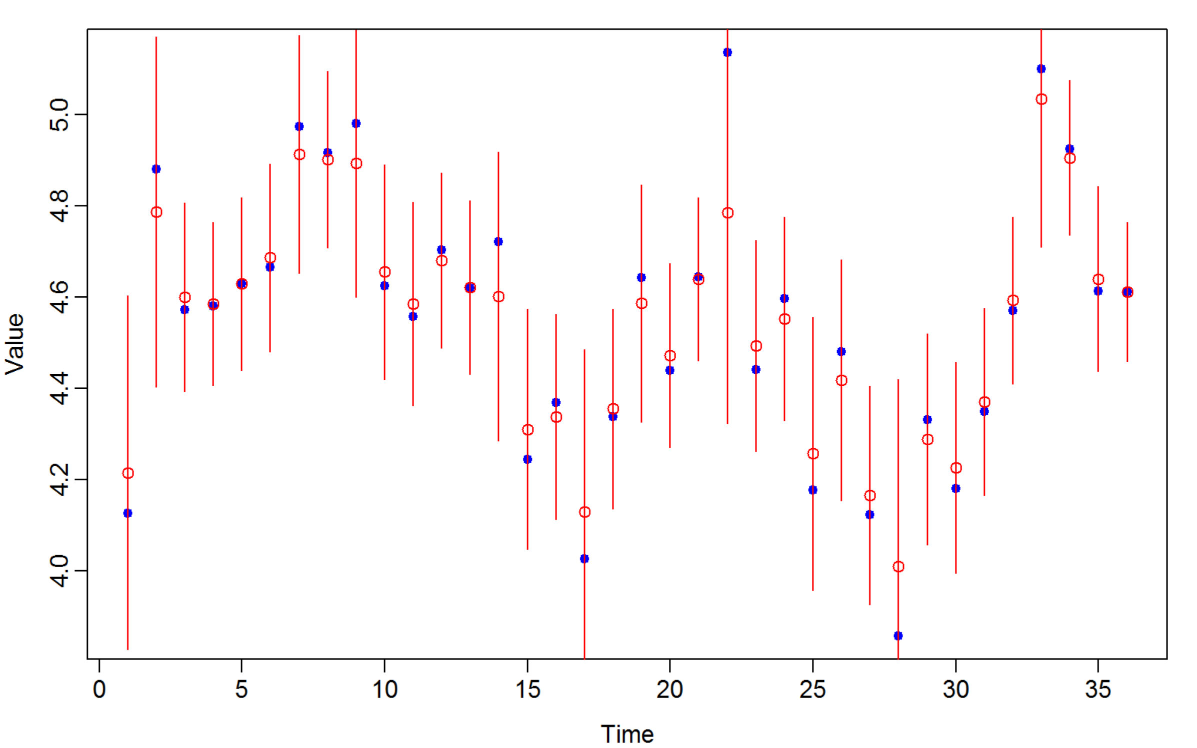
How to modify?



Param	Starting value	Lower	MLE	Upper	Final gradient	Std. Error
log_d0	0	-Inf	4.21494	Inf	2.35E-07	0.290923
log_sigmaP	1	-Inf	-1.54627	Inf	-1.26E-07	0.297971
log_sigma M	1	-Inf	-4.98037	Inf	-4.42E-09	13.00587
alpha	0	-Inf	2.071196	Inf	3.71E-06	1.02898
rho	0	-Inf	0.546332	Inf	1.67E-05	0.225704

Lab exercise

[Fix SigmaM at 0 and re-run]