IRREDUCIBLE REPRESENTATIONS OF FINITELY GENERATED NILPOTENT TORSION-FREE GROUPS

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An investigation of the structure of the quotient algebra, with respect to a prime ideal, of the group algebra of a finitely-generated nilpotent torsion-free group. Conditions are studied under which an irreducible representation of such a group is induced.

The object of this work is to derive results for irreducible representations of finitely-generated nilpotent and partly polycyclic torsion-free groups, analogous to those obtained by J. Dixmier [1] (see also [2] and [3]) concerning irreducible representations of nilpotent finite-dimensional Lie algebras.

Dixmier's principal result is the following [1].

Let $\mathfrak{G}$ be a nilpotent finite-dimensional Lie algebra over the field of complex numbers C (the passage to an arbitrary field P of zero characteristic is made in [2]), and let $U\mathfrak{G}$ be the universal enveloping algebra of $\mathfrak{G}$. Then the following conditions are equivalent:

1) I is a prime ideal of $U\mathfrak{G}$;

2) I is the maximal ideal of $U\mathfrak{G}$;

3) the center of the algebra $U\mathfrak{G}/I$ coincides with C (it is a finite extension of the field P, see [2]);

4) $U\mathfrak{G}/I$ is isomorphic to the algebra with generators p_i and q_i, with relations $p_iq_i - q_ip_i = 1$, $p_iq_j = q_jp_i$, $p_ip_j = p_jp_i$, $q_iq_j = q_jq_i$, $i \neq j$, $i, j = 1, 2, \ldots, k$.

Moreover under these conditions I is the kernel of some induced representation of the algebra $U\mathfrak{G}$.

Now let G be a finitely generated nilpotent torsion-free group, let PG be the group algebra of G over the field P, and let I be a two-sided ideal of PG, where the natural homomorphism $G \to PG/I$ has a trivial kernel.

THEOREM 1. The following conditions are equivalent:

1) I is a prime ideal of PG;

2) I is a maximal ideal;

3) the center of the algebra PG/I is a finite extension Q of the field P;

4) PG/I is the cross product of the field $Q = \langle Z \rangle$ and the group G/Z, where Z is the center of G. The cross product PG/I has no divisors of zero.

Under these conditions, I is the kernel of the induced irreducible representation of G.

The only divergence from Dixmier's theorem is in 4): if the algebra $U\mathfrak{G}/I$ for an algebraically closed field is completely determined by the number n, the algebra PG/I uniquely determines at least the group G/Z.

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P. Hall [5] was the first to study irreducible finitely generated representations of polycyclic groups. His basic result concerning the structure of an irreducible PG-module, considered as a PZ-module (Z is the center of G) establishes the implication 1) $\to$ 3) of Theorem 1. The implication 3) $\to$ 4) is obtained by applying the following theorem proved by the author in [4].

THEOREM 2. Let G be a nilpotent (not necessarily finitely generated) torsion-free subgroup of the group of invertible elements of some algebra U and let Z be the center of G. If $\langle Z \rangle$ is a ring without divisors of zero, then $\langle G \rangle$ is the cross product of $\langle Z \rangle$ and G/Z.

We note that Theorem 1 is meaningless if P is an algebraic extension of a finite field. Hall [5] showed that in this case all irreducible representations of a finitely generated nilpotent group are finite-dimensional,* and consequently the kernel of the mapping $G \to PG/I$ is nontrivial. If the field P is not an algebraic extension of a finite field, then every finitely generated nilpotent torsion-free group has an exact irreducible representation over P.

The implication 1) $\to$ 2) can be proved under less stringent conditions.

THEOREM 3. Let G be a finitely generated nilpotent group. Then every prime ideal of its group algebra PG over P is maximal.

In group-representation theory the question usually arises of the conditions under which a given irreducible representation is induced. We know of no example of an irreducible representation of a nilpotent group which is not induced, if the field P is algebraically closed; for Lie algebras there are such examples [1]. Theorems 5 and 6 give conditions under which a representation is induced.

The equivalence of 1)-4) of Theorem 1 does not hold for polycyclic groups. Thus 1) $\to$ 2) can be disproved for an appropriately chosen polycyclic group even if $I = 0$. With some distortion only the implication 1) $\to$ 4) can be retained.

THEOREM 4. Let M be an irreducible PG-module of the polycyclic group G. Then M is a finite direct sum of irreducible PG_k-modules M_k and $G\colon G_k < \infty$ and, if I_k is a prime ideal of PG_k corresponding to the module M_k, we have

1) PG_k/I_k is a ring without divisors of zero,

2) if Z_k is the center of the nilpotent radical of the group $\bar{G}_k$ (the image of G_k under the mapping $G_k \to PG_k/I_k$), then PG_k/I_k is the cross product of $\langle Z_k \rangle$ and $\bar{G}_k/Z_k$.

There is an analog of assertion 1) for solvable Lie algebras (see [3], Theorem 1.3).

Although a prime ideal is not necessarily a maximal ideal, there would be some regularity in the set of prime ideals if the following proposition were true.

PROPOSITION.† Let Z be a finitely generated torsion-free abelian group, let PZ be its group algebra over the field P, and let G be the group of automorphisms of Z considered as a group of automorphisms of the algebra PZ (G is isomorphic to the complete linear group GL over the ring of integers). Let I be an ideal of PZ with infinite index (i.e., $\dim PZ/I = \infty$) and let G(I) be the subgroup consisting of those automorphisms $g \in G$ of PZ, with respect to which I is invariant. Then G(I) has a subgroup G_0 of finite index such that, for all $g \in G_0$, $g(Z_0) = Z_0$ for some nontrivial subgroup Z_0 of Z of finite index. In other words G_0, as a subgroup of GL, is reducible to block-triangular form.

A problem of interest is the classification of the special D(PG) of the group algebra PG of a finitely generated nilpotent torsion-free group G, and the classification of the special D(PG/I) of the quotient algebra of the algebra PG with respect to its maximal ideals. I. M. Gel'fand and A. A. Kirillov [6] proved that the $D(U\mathfrak{G})$, for a nilpotent Lie algebra $\mathfrak{G}$ are classified by two natural parameters, while $D(U\mathfrak{G}/I)$, (I is a maximal ideal) are classified by one parameter.‡ For groups the position is more complicated. Even

*Note Added in Proof. P. Hall [5] has advanced the hypothesis that this result can be extended to apply to polycyclic groups. This hypothesis was confirmed by E. M. Levich [10]; another proof is given by the author in [11].

†Note Added in Proof. The author has learned that this proposition (stated also in [12]) has been proved by G. M. Bergman.

‡In the first case over a field of characteristic 0, in the second over an algebraically closed field of characteristic 0.

though we may appear to be optimistic, we risk the suggestion that D(PG) determines PG and D(PG/I) determines PG/I.

Notation and Definitions. We recall that a prime ideal I of a ring K is characterized by the following property: K/I has an exact irreducible module. If M is an irreducible K-module, then $I = \{k \in K, kM = 0\}$ is the prime ideal corresponding to the module M.

If G is a subgroup of the group K^* of invertible elements of an algebra K over a field P, then $\langle G\rangle$ denotes its enveloping algebra, i.e., the algebra of all elements K representable in the form $\Sigma p_i g_i$, $p_i \in P$, $g_i \in G$. If φ is a homomorphism of G into K^*, then we usually write $\langle G\rangle$ instead of $\langle \varphi(G)\rangle$.

The cross product is defined in [7]. For our purposes, the following special case is sufficient.

Let G be a subgroup of K^*, and let H be a normal divisor of G. Then $\langle G\rangle$ is called the cross product of G/H and $\langle H\rangle$, if the representations of different residue classes of G/H are linearly independent over $\langle H\rangle$.

An irreducible PG-module is called imprimitive, if $M = \oplus M_i$, where $gM_i = M_{i(g)}$ for all $g \in G$. The corresponding representation of the group G in the group of automorphisms of the module M is said to be induced with the representation $\varphi(G_i)$ of the stationary subgroup of the module M_i in the group of automorphisms of M_i; here $\varphi(G_i)$ is the restriction of the action of G_i to the module M_i. Conversely, if G_0 is a subgroup of G and $\varphi(G_0)$ is a representation of G_0 on M_0, then there is a prime PG-module $M = \oplus g_i M_0$, where g_i runs over the different right residue classes of G/G_0, on which the induced representation of G defined by

$$gM = g(\oplus g_i M_0) = \oplus g_j h_i M_0$$

acts, where h_i is calculated from the relation $gg_i = g_j h_i$, $h_i \in G_0$, and G_0 is a stationary subgroup of M_0.

PROPOSITION. Let G be a finitely generated nilpotent torsion-free group, let P be a field which is not absolutely algebraic, let $\overline{P}$ be the algebraic closure of P, and let $\tau\colon Z \to \overline{P}^*$ be an exact representation of the center Z of G. Let $Q(P \subset Q \subset \overline{P})$ be the minimal subfield of $\overline{P}$, containing the image of elements of Z and let $Q:P = n$. Let A be a maximal abelian normal divisor of G, let $A = Z \times A_1$, and let μ: $A \to Q^*$ be an exact representation of A coinciding with τ on Z; we retain the symbol μ for the composition representation $A \to Q^* \to GL(n, P)$. Let ρ be the representation of G induced by the representation μ of A. Then ρ is an exact irreducible representation of G.

A proof of this proposition is given by S. D. Berman and V. V. Sharaya in [9].

COROLLARY. Every finitely generated nilpotent torsion-free group has an exact irreducible representation over a field which is not an algebraic extension of a finite field.

This result is a consequence of the fact that the multiplicative group of such a field contains a torsion-free abelian group of arbitrarily large rank.

Proof of Theorem 1. We identify the elements of G with their images under the natural mapping $G \to PG/I$; let Z be the center of G.

1) $\to$ 3). If M is an exact irreducible PG/I-module, then $\langle Z\rangle$ must be contained in the centralizer of the module M; the latter is a field, and so $\langle Z\rangle$ is a ring without divisors of zero. Theorem 2 implies that

$$PG/I = \left\{\sum \zeta_i g_i\right\},\ \zeta_i \in \langle Z\rangle;$$

here the g_i are representations of distinct residue classes of G/Z and are linearly independent over $\langle Z\rangle$. Hence the center of the algebra PG/I coincides with $\langle Z\rangle$. By using Hall's reasoning (see [5], p. 610), we conclude that $\langle Z\rangle$ is a finite extension of the field P.

3) $\to$ 4). The first assertion of 4) follows from Theorem 2, the second from Theorem 1 in [7].

4) $\to$ 2). It suffices to show that, if $0 \neq J$ (a two-sided ideal of the algebra PG/I), then $J \cap \langle Z\rangle \neq 0$. This is proved by repeating the proof of Theorem 1 of [4] with $= 0$ in (1) of [4] replaced by $\in J$.

2) $\to$ 1) is obvious.

Now let Q be the center of the algebra PG/I and let $\tau: Z \to Q$ be the restriction to Z of the mapping $G \to PG/I$. Let ρ be the representation constructed under the assumption, and let J be the corresponding prime ideal. It follows from 4) that PG/J is the cross product of Q and G/Z, and the restriction to Z of the mappings $G \to PG/I$ and $G \to PG/J$ coincide. Hence $PQ/I \cong PG/J$ (see [4], Theorem 2). By virtue of 2), the ring PG/I is simple, and so the kernel of the composite mapping $PG \to PG/I \to PG/J$ coincides with I. This completes the proof of the theorem.

LEMMA 1. Let L be a ring whose left ideals satisfy the ascending chain condition, and let M be a finitely generated L-module. Then M contains a maximal L-submodule.

Proof. If $M = Lm_1 + \ldots + Lm_k$ and $M_1 = Lm_1 + \ldots + Lm_{k-1} \neq M$, then $N = M/M_1$ is an L-module with a single generator n. If N is reducible, then for some $n_1 \in N$ we have $Ln_1 \neq N$. But $n_1 = ln$, $l \in L$, $I_0 = Ll$ is a left ideal in L, and $I_0N \neq N$. Let I be the largest left ideal in L containing I_0, and such that $IN \neq N$. Then N/IN is an irreducible L-module and IN is the maximal L-module in N. This proves the lemma.

LEMMA 2. Suppose that M is an irreducible PG-module, G_0 is a subgroup of G of finite index, and M_0 is a PG_0-submodule of M. Then $\cap\, g_iM_0 = 0$, where $\{g_i\} = S$ is the set of representations of right residue classes G/G_0.

Proof. Let

$$M_1 = \cap\, g_iM_0.$$

Then $PGM_1 = \sum_k PG_0g_k\,(\bigcap_i g_iM_0) = \sum_k PG_0\,(\bigcap_i g_kg_iM_0) = \sum_k PG_0\,(\cap\, g_jM_0) = \sum_k\,(\cap\, g_jM_0) = M_1$, for any $g \in G$, the mapping $\alpha\,(g_i \to g_{\alpha(i)})$ of the set S into itself defined by $gg_i = g_{\alpha(i)}h$, $g_{\alpha(i)} \in S$, $h \in G_0$, is one-to-one.

Proof of Theorem 3. It is known that G has a normal torsion-free divisor N and a finite index in G. Our proof is by induction on $G{:}N$. Since every subgroup of G has a finite number of generators, the case $G{:}N = 1$ is contained in Theorem 1. Hence we assume that the theorem holds for the maximal subgroup G_0 of G containing N. Plainly $G : G_0 = p$ where p is a prime and G_0 is invariant in G; let g_i, $i = 1, \ldots, p$ be the representations of the residue classes in G/G_0.

Let M be an irreducible PG-module. Lemma 1 implies the existence of a maximal PG_0-module M_0, and it follows from Lemma 2 that $\cap\, g_iM_0 = 0$. Let I be the prime ideal of the algebra PG corresponding to the module M and let I_0 be the prime (and maximal) ideal of PG_0 corresponding to the module M/M_0. Then $I_0 \supset I \cap PG_0$. The condition $\cap\, g_iM_0 = 0$ implies that $\cap\, g_iI_0g_i^{-1} = I \cap PG_0$. Hence the ring $K = PG_0/I \cap PG_0$ is the direct sum of a finite number of simple rings K_i (if the L_i are maximal ideals of the ring K and $\cap\, L_i = 0$, then K is a finite direct sum of simple rings), uniquely determined up to their enumeration and isomorphic to PG_0/I_0. Hence $gK_ig^{-1} = K_{j(g)}$, $g \in G$, $M = \oplus K_iM = \oplus M_i$, and $gM_i = M_{j(g)}$. The irreducibility of M implies that G acts on K_i and M_i transitively. We may write

$$PG/I \cong \left\{\sum g_i\alpha_i\right\},\ \alpha_i \in K,\ g_i \in G/G_0.$$

Let J be an ideal of PG/I, $J \neq PG/I$, let the sum $\tau = \sum g_i\alpha_i \in J$ contain the minimum number n of terms, and let e_s be the unit of the ring K_s. Then $\tau e_s \neq 0$ for some S, and we may assume that $\alpha_i \in K_s$. If $n = 1$, then J contains K_s and also K, and since K contains the unit of the ring PG/I we have $J = PG/I$. Hence let $n \neq 1$. We may obviously assume that $g_1 = 1$, and since K_s is a simple ring we may assume that $\alpha_1 = e_s$ for some s and $\alpha_i \in K_s$ for this s. Calculating $\alpha\tau - \tau\alpha$, $\alpha \in K_s$, we obtain

$$\sum_{i=2}^{n} g_i\,(g_i^{-1}\alpha g_i\alpha_i - \alpha_i\alpha) \in J$$

and, since τ is minimal, we have $g_i^{-1}\alpha g_i\alpha_i = \alpha_i\alpha$. It follows that $K_s\alpha_i$ is a two-sided ideal in K_s, so that $K_s\alpha_i = K_s$, α_i is an invertible element in K_s, and $g_i^{-1}\alpha g_i \in K_s$ for all $\alpha \in K_s$. Since G_0 is the maximal subgroup of G, we have $gK_sg^{-1} = K_s$ for all $g \in G$, i.e., $K_s = K$. We have $\alpha g_i\alpha_i = g_i\alpha_i\alpha$, and we can choose new representations of the residue classes

$$GK^*/G_0K^*,\ g_i' = g_i\alpha_i,\ \tau = \sum g_i' \in J$$

$g_i'g_j' = g_k'\beta_{ij}$, and $\beta_{ij} \in C$ (the center of the ring $K = K_s$). Since K_s is simple, C is a field. Consider the subalgebra L of the algebra PG/J generated by $\{g_i'\} : L = \{\sum g_i'c_i'\}$, $c_i \in C$. The subalgebra L is finite-dimensional over C and $PG/J = KL$. If $J \cap L = L$, we have $J = PG/I$, because L contains the unit of the ring PG/I. If $J \cap L = J_L \neq J$, then $M_0 = J_L M \neq M$. Hence

$$(PG/I)\,M_0 = KLM_0 = KLJ_LM = J_LM = M_0,$$

which contradicts the irreducibility of M and proves the theorem.

LEMMA 3. Suppose that M is a PG-module, G is a polycyclic group, G_0 is an invariant subgroup of G, and $G: G_0 < \infty$. Then M is a finite direct sum of irreducible PG_0-modules.

Proof. Let g_i, $i \in S$, be representations of different residue classes of G/G_0. Lemmas 1 and 2 imply the existence of a maximal PG_0-submodule $M_1 \subset M$, such that $\bigcap_{i\in S} g_iM_1 = 0$. Let S_0 be the subset of those $i \in S$, such that $M_1 \cap (\bigcap_{i\in S_0} g_iM_1) = 0$, but $\bigcap_{i\in S_0} g_iM_1 = M_2 \neq 0$. Then $M_1 + M_2 = M$, $M_1 \cap M_2 = 0$, and so M_2 is an irreducible PG_0-submodule in M.

Let $S_1 = (i_1, \ldots, i_k)$ be a set of indices such that

$$g_{i_{l+1}}M_2 \cap \sum\nolimits_{\alpha=1}^{l} g_{i_\alpha}M_2 = 0$$

for all $1 \leq l \leq k-1$; we may assume that S_1 is the maximal set of indices with this property. If $M_3 = \sum_{\alpha\in S_1} g_{i_\alpha}M_2$, $M_3 \neq M$, then for some $i \in S$ $g_iM_2 \not\subset M_3$. Then $g_iM_2 \cap M_3 = 0$, since g_iM_2 is an irreducible PG_0-module, and this contradicts the definition of S_1. It follows from $\sum_{i\in S} g_iM_2 = M$, that $M = \bigoplus_{i\in S_1} g_iM_2$.

Proof of Theorem 4. Lemma 3 implies that we need only consider the case in which G possesses a normal series with infinite cyclic factors. The number s(G) of these factors is independent of the choice of the series, and we may make an inductive assumption concerning the validity of the theorem for groups with a smaller invariant s(G), in particular for the quotient group G/G_1, if G_1 is infinite. We may assume that each element $g \in G$ acts on M nontrivially.

Let C be a maximal abelian normal divisor of G. Assume first that $\langle C\rangle$ is a ring without divisors of 0. Let Z be the center of the nilpotent radical R of G. Then $Z \subset C$ and $\langle Z\rangle$ is a ring without divisors of 0. Theorem 2 implies that $\langle R\rangle$ is the cross product of $\langle Z\rangle$ and R/Z, and so $\langle R\rangle$ has no divisors of 0 (see [7]). It follows from Theorem 3 of [4] that $\langle G\rangle$ is the cross product of $\langle Z\rangle$ and G/Z. It follows from results in [7] that $\langle G\rangle$ has no divisors of 0.

The case in which $\langle C\rangle$ has divisors of 0 can be reduced to the above case. We first note that $\langle C\rangle$ contains no nilpotent elements: since the latter form a nilpotent ideal I of $\langle C\rangle$ (see [8], Chapter IV, Sec. 1) we have $gIg^{-1} = I$; then $IM \neq M$, IM is a G-module, and the irreducibility of M implies that $IM = 0$ and so $I = 0$. By virtue of the results in Secs. 4, 5, Chapter IV of [8], we have $0 = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \ldots \cap \mathfrak{p}_n$, where the $\mathfrak{p}_i$ are minimal simple ideals of $\langle C\rangle$, uniquely determined up to their enumeration. Let

$$G_0 = \{g \in G,\ g\,\mathfrak{p}_i g^{-1} = \mathfrak{p}_i,\ i = 1, 2, \ldots, n\}.$$

It follows from Lemma 3 the G_0-module M is the finite direct sum of irreducible G_0-modules L_j. Since $\mathfrak{p}_iL_j$ is a G_0-module we have $\mathfrak{p}_iL_j = 0$, since $\mathfrak{p}_iL_j \neq L_j$ and so $\mathfrak{p}_iL_j = 0$ for at least one i. Let φ_j be the restriction of the representation of the ring $\langle G_0\rangle$ to L_j; since $C \subset G_0$ we have $\varphi_j(\mathfrak{p}_i) = 0$. By virtue of the induction assumption, it suffices to consider the case in which $G_0 \to \varphi_j(G_0)$ is an isomorphism. Since G satisfies the maximal condition for subgroups, it follows from Lemma 3 that we may assume that C is a maximal abelian normal divisor of $G_0 \cong \varphi_j(G_0)$, and $\varphi_j(\mathfrak{p}_i) = 0$. Since $\langle C\rangle$ is a Noether ring, the application of the preceding reasoning to $\varphi(\langle G_0\rangle)$ leads, after a finite number of steps, to a situation in which $\langle C\rangle$ is a ring without divisors of 0.

THEOREM 5. Let G be a polycyclic group, let A be an abelian normal divisor of G, and let M be an irreducible PG-module. If for some ζ and m, $0 \neq \zeta \in \langle A\rangle$, $0 \neq m \in M$, we have $\zeta m = 0$, then M is an imprimitive module.

Proof. Let $S(m) = \{\zeta \in \langle A\rangle,\ \zeta m = 0,\ m \in M\}$. Clearly S(m) is an ideal in $\langle A\rangle$. Let

$$G(m) = \{g \in G,\ gS(m)g^{-1} = S(m)\}.$$

We have $G(m) \neq G$, since otherwise $S(m)M = 0$. There is $m_1 = t_1m$, $t_1 \in PG(m)$, such that

$$S(m_1) = S(tm_1) \text{ for all } t \in PG(m). \tag{1}$$

In fact $S(m)tm = 0$, so that $S(m) \subset S(tm)$, $t \in PG(m)$. Consider the sequence $S(m) \subset S(t_1m) \subset S(t_2t_1m) \subset \ldots$ of increasing ideals of the ring $\langle A \rangle$ such that $t_i \in PG(t_{i-1} \ldots t_1m)$. Our assertion can be inferred from the maximality condition holding in $\langle A \rangle$ for ideals.

2) Assume that m satisfies (1). Let $M_0 = \{tm\}$, $t \in PG(m)$. Then $M_0 \cap gM_0 = 0$ for $g \notin G(m)$. In fact if $t_1m = gt_2m$, $t_1, t_2 \in PG(m)$, we have

$$0 = S(m)\, gt_2m = g\,(g^{-1} S(m)\, g)\, t_2m, \quad g^{-1} S(m) gt_2m = 0$$

and, by virtue of 1), it follows that

$$g^{-1}S(m)\, g \subset S(m).$$

Moreover, $g^{-1}t_1m = t_2m$. Hence, as above, $gS(m)g^{-1} \subset S(m)$, so that

$$g^{-1}S(m)\, g \subset S(m) \subset g^{-1} S(m)\, g, \quad S(m) = g^{-1} S(m) g, \quad g \in G(m),$$

which yields a contradiction.

3) Let g_i be representations of different right residue classes of $G/G(m)$.

Then

$$M_0 \cap \left(\sum_{i=1}^{n} g_i M_0\right) = 0, \; g_i \notin G(m). \tag{2}$$

For $n = 1$ this is proved in 2). Assume that (2) holds for a number of terms in the parentheses smaller than n. Let

$$tm = \sum_{i=1}^{n} g_i t_i m. \tag{3}$$

Multiplication of (3) by $g_n S(m)\, g_n^{-1}$ yields

$$g_n S(m)\, g_n^{-1}\, tm = \sum_{i=1}^{n-1} g_i\, (g_i^{-1} g_n S(m)\, g_n^{-1} g_i)\, t_i m,$$

and this contradicts the induction hypothesis if both sides are different from 0. Hence

$$g_n S(m)\, g_n^{-1}\, tm = 0, \quad g_n S(m)\, g_n^{-1} \subset S(tm) \subset S(m).$$

It also follows from (3) that

$$g_n^{-1} tm - \sum_{i=1}^{n-1} g_n^{-1} g_i t_i m = t_n m. \tag{4}$$

Multiplying (4) by $g_n^{-1}S(m)g_n$ we obtain, as above,

$$g_n^{-1} S(m)\, g_n t_n m = 0, \; g_n^{-1} S(m)\, g_n \subset S(m).$$

Hence $g_n \in G(m)$, and this proves (3) and Theorem 5.

THEOREM 6. Let G be a group with the three generators a, b, and c and the relations $abc = cb$, $ab = ba$, and $ac = ca$. Let M be an irreducible PG-module, where P is an algebraically closed field. Suppose that $M = \{g^i m\}$ for some $g \in G$ and $m \in M$. Then M is imprimitive.

Proof. Theorem 5 implies that the $m_i = g^i m$ are linearly independent. Let $h \in G$, $hg = tgh$, $t = a^k$, $k \neq 0$, and let $hm_i = \sum \alpha_i^j m_j$, $h^{-1}m_i = \sum \gamma_i^j m_j$. Then $hgm_i = hm_{i+1} = \sum \alpha_{i+1}^j m_j = tghm_i = tg \sum \alpha_i^j m_j t \sum \alpha_i^j m_{j+1} = \sum \tau \alpha_i^j m_{j+1}$, $\tau \in P$, and so $\tau \alpha_i^j = \alpha_{i+1}^{j+1}$. Since $gh^{-1} = th^{-1}g$, we can prove similarly that $\gamma_i^j = \tau \gamma_{i+1}^{j+1}$. Finally $hh^{-1} = 1$, because $\sum \gamma_j^i \alpha_i^k = \sum \alpha_j^i \gamma_i^k = \delta_{jk}$. It now easily follows that the matrix (α_{ij}) cannot have more than one non-zero element in each row and in each column. This completes the proof of the theorem.

LITERATURE CITED

1. J. Dixmier, "Représentations irreducible des algèbres de Lie nilpotentes," Anais, Acad. Brasil, Ciancias, 35, 491-519 (1963).
2. Y. Nouaze and P. Gabriel, "Idéaux premier de l'algèbre envoloppante d'une algèbre de Lie nilpotentes," J. Algebra, 6, No. 1, 77-99 (1967).

3. J. Dixmier, "Représentations irreductible des algèbres de Lie resolubles," J. Math. Pures. et Appl., 45, 1-66 (1966).
4. A. E. Zalesskii, "Subgroups of rings without divisors of zero," Dokl. Akad. Nauk BSSR, 10, No. 10, 728-731 (1966).
5. P. Hall, "On the finiteness of certain soluble groups," Proc. London Math. Soc., 9, No. 3, 595-622 (1959).
6. I. M. Gel'fand and A. A. Kirillow, Sur les Corps Liés aux Algèbres Enveloppantes des Algèbres de Lie, Inst. Haut. et. Sci., Publ. Math., Paris, 31 (1966), pp. 509-524.
7. A. A. Bovdi, "Cross products of semigroups and rings," Dokl. Akad. Nauk SSSR, 137, No. 6, 1267-1269 (1961).
8. O. Zarisskii and P. Samyuél', Commutative Algebra [in Russian], Vol. 1, Moscow (1963).
9. S. D. Berman and V. V. Sharaya, Representations of one Class of Discrete Solvable Groups over an Arbitrary Algebraically Closed Field, Algebra and Mathematical Logic, Studies in Algebra, Kiev (1966), pp. 54-61.
10. E. M. Levich, "A problem suggested by P. Hall," Dokl. Akad. Nauk SSSR, 188, No. 6, 1241-1243 (1969).
11. A. E. Zalesskii, "The structure of an irreducible module over the group algebra of a polycyclic group," Dokl. Akad. Nauk BSSR, 14, No. 11, 977-980 (1970).
12. Dnestrovskaya Tetrad', Unsolved Problems in the Theory of Rings and Modules, Kishinev (1969).