A strong Dixmier–Moeglin equivalence for quantum Schubert cells

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A B S T R A C T

Dixmier and Moeglin gave an algebraic condition and a topological condition for recognising the primitive ideals among the prime ideals of the universal enveloping algebra of a finite-dimensional complex Lie algebra; they showed that the primitive, rational, and locally closed ideals coincide. In modern terminology, they showed that the universal enveloping algebra of a finite-dimensional complex Lie algebra satisfies the *Dixmier–Moeglin equivalence*.

We define quantities which measure how "close" an arbitrary prime ideal of a noetherian algebra is to being primitive, rational, and locally closed; if every prime ideal is equally "close" to satisfying each of these three properties, then we say that the algebra satisfies the *strong Dixmier–Moeglin equivalence*. Using the example of the universal enveloping algebra of $\mathfrak{sl}_2(\mathbb{C})$, we show that the strong Dixmier–Moeglin equivalence is strictly stronger than the Dixmier–Moeglin equivalence.

For a simple complex Lie algebra $\mathfrak{g}$, a non-root of unity $q \neq 0$ in an infinite field $\mathbb{K}$, and an element w of the Weyl group of $\mathfrak{g}$, De Concini, Kac, and Procesi have constructed a subalgebra

$U_q[w]$ of the quantised enveloping $\mathbb{K}$-algebra $U_q(\mathfrak{g})$. These *quantum Schubert cells* are known to satisfy the Dixmier–Moeglin equivalence and we show that they in fact satisfy the strong Dixmier–Moeglin equivalence. Along the way, we show that commutative affine domains, uniparameter quantum tori, and uniparameter quantum affine spaces satisfy the strong Dixmier–Moeglin equivalence.

1. Introduction

Throughout this paper, $\mathbb{K}$ denotes an infinite field of arbitrary characteristic and, unless otherwise stated, every algebra is a unital associative $\mathbb{K}$-algebra and every ideal is two-sided.

It is a difficult and often intractable problem to classify the irreducible representations of an algebra. Dixmier proposed that a good first step towards tackling this problem would be to find the kernels of the irreducible representations, that is the annihilators of the simple modules, namely the primitive ideals. In any ring, every primitive ideal is prime; Dixmier [9] and Moeglin [22] gave an algebraic condition and a topological condition for deciding whether or not a given prime ideal of the universal enveloping algebra of a finite-dimensional complex Lie algebra is primitive:

- A prime ideal P of a ring R is said to be *locally closed* if the singleton set $\{P\}$ is locally closed in the Zariski topology on Spec R. Equivalently, $\{P\}$ is the intersection of a Zariski-open subset of Spec R and a Zariski-closed subset of Spec R. (For a prime ideal P of a ring R, it is easily shown that P is locally closed if and only if P is strictly contained in the intersection of all prime ideals of R which strictly contain P.)
- A prime ideal P of a noetherian $\mathbb{K}$-algebra R is said to be *rational* if the field extension $\mathcal{Z}(\operatorname{Frac} R/P)$ of $\mathbb{K}$ is algebraic.

Dixmier and Moeglin proved that for a prime ideal of the universal enveloping algebra of a finite-dimensional complex Lie algebra, the properties of being primitive, locally closed, and rational are equivalent. In modern terminology, they proved that the universal enveloping algebra of a finite-dimensional complex Lie algebra satisfies the *Dixmier–Moeglin equivalence.*

Since the work of Dixmier and Moeglin on universal enveloping algebras of finite-dimensional complex Lie algebras, many more algebras have been shown to satisfy the Dixmier–Moeglin equivalence: [5, Corollary II.8.5] lists several quantised coordinate rings which satisfy the Dixmier–Moeglin equivalence; the first named author, Rogalski, and Sierra [1] have shown that twisted homogeneous coordinate rings of projective surfaces satisfy the Dixmier–Moeglin equivalence. However, Irving [15] and Lorenz [18] have shown that there exist noetherian algebras of infinite Gelfand–Kirillov dimension for

which the Dixmier–Moeglin equivalence fails. Moreover the first two named authors, León Sánchez, and Moosa [3] gave the first examples (to our knowledge) of noetherian algebras of finite Gelfand–Kirillov dimension which do not satisfy the Dixmier–Moeglin equivalence.

Our goal is to extend the notion of the Dixmier–Moeglin equivalence to all prime ideals, in a way which captures how "close" they are to being primitive. Of course, not all non-primitive prime ideals are created equal. For example, in the polynomial ring $\mathbb{C}[x, y]$, the primitive ideals are the maximal ideals $\langle x - \alpha, y - \beta \rangle$. For this reason, we think of the prime ideal $\langle x \rangle$ as being "closer" to being primitive than the prime ideal $\langle 0 \rangle$, in the same sense that it is "closer" to being maximal — that is, the height of $\langle x \rangle$ is greater than the height of $\langle 0 \rangle$.

In general, given a noetherian $\mathbb{K}$-algebra R and given a prime ideal P of R, we are interested in the *primitivity degree*, prim.deg P, of P, which we define as follows:

$$\text{prim.deg}\, P := \inf\{\text{ht}\, Q \mid Q \in \text{Prim}\, R/P\},$$

where $\text{Prim}\, R/P$ denotes the subspace of $\text{Spec}\, R/P$ consisting of the primitive ideals of R/P. This quantity gives a measure of how close the prime ideal P is to being primitive. Clearly, P is primitive if and only if prim.deg $P = 0$.

Remark 1.1. We would like to have a more representation-theoretic characterisation of primitivity degree, such as a way to realise the prime ideals of a given primitivity degree as the kernels of members of a family of representations. However we have not been able to find such a characterisation.

We use the notion of primitivity degree to extend the idea of the Dixmier–Moeglin equivalence to all prime ideals. To this end, we define generalisations of the notions of a locally closed ideal and a rational ideal.

It is easy to extend the notion of a rational ideal: for a prime ideal P of R, we define the *rationality degree*, rat.deg P, of P to be the transcendence degree of the field extension $\mathcal{Z}(\text{Frac}\, R/P)$ of $\mathbb{K}$. Clearly, P is rational if and only if rat.deg $P = 0$.

Remark 1.2. It seems reasonable to expect that, under some mild assumptions, the property that rat.deg $P = d$ should relate to the existence of a rational ideal of height d in R/P but it seems difficult to establish such a relationship.

In the same spirit of generalisation, we define the *local closure degree*, loc.deg P, of a prime ideal P of R to be the smallest nonnegative integer d such that $\bigcap_{Q \in \text{Spec}_{>d} R/P} Q \neq 0$, where $\text{Spec}_{>d} R/P$ denotes the subspace of $\text{Spec}\, R/P$ consisting of all prime ideals of R/P which are of height strictly greater than d. Clearly, P is locally closed if and only if loc.deg $P = 0$.

Remark 1.3. In the case that the noetherian $\mathbb{K}$-algebra R has finite Gelfand–Kirillov dimension, all prime ideals of R have finite height by [16, Corollary 3.16]. All of the algebras which will concern us in this paper have finite Gelfand–Kirillov dimension and so we shall always use the following equivalent characterisation of local closure degree: for a prime ideal P of R, loc.deg P is the smallest nonnegative integer d such that $\bigcap_{Q \in \mathrm{Spec}_{d+1} R/P} Q \neq 0$, where $\mathrm{Spec}_{d+1} R/P$ denotes the subspace of $\mathrm{Spec}\, R/P$ consisting of all prime ideals of R/P which are of height $d+1$. In this context, we shall prove (in the proof of Proposition 2.1) that if $P \in \mathrm{Spec}\, R$ is such that loc.deg $P = d$, then R/P has a locally closed ideal of height d.

Definition 1.4. A noetherian $\mathbb{K}$-algebra R is said to satisfy the *strong Dixmier–Moeglin equivalence* if every prime ideal P of R satisfies loc.deg $P = \mathrm{prim.deg}\, P = \mathrm{rat.deg}\, P$.

We remark that the strong Dixmier–Moeglin equivalence is strictly stronger than the Dixmier–Moeglin equivalence. Indeed the Dixmier–Moeglin equivalence simply says that if P is a prime ideal of a noetherian $\mathbb{K}$-algebra R, then

$$\mathrm{loc.deg}\, P = 0 \iff \mathrm{prim.deg}\, P = 0 \iff \mathrm{rat.deg}\, P = 0.$$

Even though the universal enveloping algebra, $U(\mathfrak{sl}_2(\mathbb{C}))$, of $\mathfrak{sl}_2(\mathbb{C})$ satisfies the Dixmier–Moeglin equivalence (as was shown in the original work of Dixmier and Moeglin), it fails to satisfy the strong Dixmier–Moeglin equivalence. Indeed, since $U(\mathfrak{sl}_2(\mathbb{C}))$ is a domain, $\langle 0 \rangle$ is a (completely) prime ideal of $U(\mathfrak{sl}_2(\mathbb{C}))$. By [6, Remark 4.6], all nonzero prime ideals of $U(\mathfrak{sl}_2(\mathbb{C}))$ are primitive, so that $\mathrm{prim.deg}\langle 0 \rangle = 1$. It is well known that the centre of $U(\mathfrak{sl}_2(\mathbb{C}))$ is given by the polynomials in the Casimir element; by [10, Corollary 4.2.3], $\mathcal{Z}(\mathrm{Frac}\, U(\mathfrak{sl}_2(\mathbb{C})))$ is given by the rational functions in the Casimir element, so that $\mathrm{rat.deg}\langle 0 \rangle = \mathrm{tr.deg}_{\mathbb{C}} \mathcal{Z}(\mathrm{Frac}\, U(\mathfrak{sl}_2(\mathbb{C}))) = 1$. By [6, Theorem 4.5 and Proposition 5.13], there are infinitely many height two prime ideals in $U(\mathfrak{sl}_2(\mathbb{C}))$ and their intersection is zero, so that $\mathrm{loc.deg}\langle 0 \rangle > 1$. Since, by [6, Theorem 4.5], there are no height three prime ideals in $U(\mathfrak{sl}_2(\mathbb{C}))$, the intersection of the height three prime ideals is nonzero (in fact it is the entirety of $U(\mathfrak{sl}_2(\mathbb{C}))$), so that $\mathrm{loc.deg}\langle 0 \rangle = 2$.

The goal of this paper is to prove that quantum Schubert cells, which we now briefly discuss (see Section 8 for more details), satisfy the strong Dixmier–Moeglin equivalence. Let $\mathfrak{g}$ be a simple complex Lie algebra of rank n and let $\pi = \{\alpha_1, \ldots, \alpha_n\}$ be the set of simple roots associated to a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ of $\mathfrak{g}$. Where $q \in \mathbb{K}^\times$ is not a root of unity and w is an element of the Weyl group of $\mathfrak{g}$, De Concini, Kac, and Procesi [8] defined a quantum analogue, $U_q[w]$, of the universal enveloping algebra of the nilpotent Lie algebra $\mathfrak{n}^+ \cap \mathrm{Ad}_w(\mathfrak{n}^-)$. These *quantum Schubert cells* $U_q[w]$ shall be our main objects of study.

It shall be useful to define a weaker version of the strong Dixmier–Moeglin equivalence which is often easy to prove and provides a useful stepping-stone to proving the strong Dixmier–Moeglin equivalence.

Definition 1.5. A noetherian $\mathbb{K}$-algebra R is said to satisfy the *quasi strong Dixmier–Moeglin equivalence* if every prime ideal P of R satisfies loc.deg P = rat.deg P.

With the quasi strong Dixmier–Moeglin equivalence in hand for a noetherian $\mathbb{K}$-algebra R, the problem is reduced to showing that every prime ideal P of R satisfies prim.deg P = rat.deg P. For a quantum Schubert cell $U_q[w]$, we prove this by exploiting the good behaviour of the poset of H-invariant prime ideals of $U_q[w]$, where H is a suitable algebraic $\mathbb{K}$-torus acting rationally on $U_q[w]$ by $\mathbb{K}$-algebra automorphisms.

This paper is organised as follows. First, we prove various general results about the (quasi) strong Dixmier–Moeglin equivalence (Section 2). Next, we consider various examples from the quantum world. Using Cauchon's theory of deleting derivations, one can relate the prime and primitive spectra of a quantum Schubert cell to those of an associated uniparameter quantum affine space, which can in turn be related via localisations to the prime and primitive spectra of a family of uniparameter quantum tori. Since there is a bi-increasing homeomorphism between the prime spectrum of a uniparameter quantum torus and the prime spectrum of its centre, which is a commutative affine domain, we are guided into a natural strategy: we shall prove the strong Dixmier–Moeglin equivalence first for commutative affine domains (Section 3), then for uniparameter quantum tori (Section 4), then for uniparameter quantum affine spaces (Section 6), and finally for quantum Schubert cells (Section 8). Partial results are also obtained for a larger class of algebras — we prove in Section 7 that every uniparameter Cauchon–Goodearl–Letzter extension satisfies the quasi strong Dixmier–Moeglin equivalence.

We have partial results for quantised coordinate rings and quantum Grassmannians and we have reason to believe that they satisfy the strong Dixmier–Moeglin equivalence; we will return to these algebras in a later paper.

2. General results on the (quasi) strong Dixmier–Moeglin equivalence

In this section we prove that, under some mild assumptions, the primitivity degree of a prime ideal is bounded above by its local closure degree, and then we prove transfer results for the quasi strong Dixmier–Moeglin equivalence for an algebra and its localisations.

2.1. An upper bound for the primitivity degree

Some of the implications needed to prove the Dixmier–Moeglin equivalence hold in a very general setting. Recall that a noetherian $\mathbb{K}$-algebra R is said to satisfy the *noncommutative Nullstellensatz* over $\mathbb{K}$ if R is a Jacobson ring and the endomorphism ring of every irreducible R-module is algebraic over $\mathbb{K}$. By [5, Lemma II.7.15], for any noetherian $\mathbb{K}$-algebra R which satisfies the noncommutative Nullstellensatz over $\mathbb{K}$ and for any prime ideal P of R, we have

$$P \text{ is locally closed} \implies P \text{ is primitive} \implies P \text{ is rational}. \tag{1}$$

We have generalised the first implication above to a large class of algebras:

Proposition 2.1. *Let R be a noetherian $\mathbb{K}$-algebra of finite Gelfand–Kirillov dimension which has the property that every locally closed ideal is primitive (this is the case if, for example, R satisfies the noncommutative Nullstellensatz over $\mathbb{K}$). Then for any prime ideal P of R, we have $\mathrm{loc.deg}\, P \geq \mathrm{prim.deg}\, P$.*

Proof. Let $P \in \mathrm{Spec}\, R$ be such that $\mathrm{loc.deg}\, P = d$. We claim that the algebra $B := R/P$ has a locally closed ideal of height d. Indeed if not, then every prime ideal Q of height d in B is such that $\bigcap_{Q \subsetneq T \in \mathrm{Spec}\, B} T = Q$. It follows that $\bigcap_{Q \in \mathrm{Spec}_d B} \left(\bigcap_{Q \subsetneq T \in \mathrm{Spec}\, B} T \right) = \bigcap_{Q \in \mathrm{Spec}_d B} Q$, from which we immediately get $\bigcap_{T \in \mathrm{Spec}_{>d} B} T = \bigcap_{Q \in \mathrm{Spec}_d B} Q$ i.e. $\bigcap_{T \in \mathrm{Spec}_{d+1} B} T = \bigcap_{Q \in \mathrm{Spec}_d B} Q$. This is a contradiction because $\mathrm{loc.deg}\, P = d$ implies that the intersection $\bigcap_{Q \in \mathrm{Spec}_d B} Q$ is trivial while the intersection $\bigcap_{T \in \mathrm{Spec}_{d+1} B} T$ is not. This establishes the claim that the algebra $B = R/P$ has a locally closed ideal of height d; since this ideal is also primitive, the proof is complete. $\square$

We do not know whether the second implication in (1) can be similarly generalised but we will prove, on a case-by-case basis, that for a prime ideal P of a commutative affine domain, a uniparameter quantum torus, a uniparameter quantum affine space, or a quantum Schubert cell, we have

$$\mathrm{prim.deg}\, P = \mathrm{rat.deg}\, P.$$

2.2. Transferring the quasi strong Dixmier–Moeglin equivalence

Recall that a noetherian $\mathbb{K}$-algebra R is said to satisfy the quasi strong Dixmier–Moeglin equivalence if, for every prime ideal P of R, we have $\mathrm{loc.deg}\, P = \mathrm{rat.deg}\, P$.

Lemma 2.2. *Let R be a noetherian $\mathbb{K}$-algebra of finite Gelfand–Kirillov dimension which is a domain and in which every prime ideal is completely prime. Let $\mathcal{E}$ be a right Ore set of regular elements of R which is finitely generated as a multiplicative system. Then for any $d \in \mathbb{N} \backslash \{0\}$, we have*

$$\bigcap_{P \in \mathrm{Spec}_d R} P \neq 0 \iff \bigcap_{Q \in \mathrm{Spec}_d R\mathcal{E}^{-1}} Q \neq 0.$$

It follows immediately that $\mathrm{loc.deg}\langle 0 \rangle_R = \mathrm{loc.deg}\langle 0 \rangle_{R\mathcal{E}^{-1}}$, where $\langle 0 \rangle_R$ and $\langle 0 \rangle_{R\mathcal{E}^{-1}}$ denote the zero ideals of R and $R\mathcal{E}^{-1}$ respectively.

Proof. Let $\mathcal{E}$ be generated as a multiplicative system by $x_1, \ldots, x_n$. Since all prime ideals of R are completely prime, the conditions $P \cap \mathcal{E} = \emptyset$ and $x_1, \ldots, x_n \notin P$ are equivalent for every prime ideal P of R.

By [13, Theorem 10.20], extension $(P \mapsto P\mathcal{E}^{-1})$ and contraction $(Q \mapsto Q \cap R)$ are mutually inverse increasing homeomorphisms between $\{P \in \mathrm{Spec}\, R \mid P \cap \mathcal{E} = \emptyset\} = \{P \in$

$\operatorname{Spec} R \mid x_1, \ldots, x_n \notin P\}$ and $\operatorname{Spec} R\mathcal{E}^{-1}$, so that since both extension and contraction send the zero ideal to the zero ideal, we get

$$\bigcap_{P \in \operatorname{Spec}_d R, \ x_1, \ldots, x_n \notin P} P \neq 0 \iff \bigcap_{Q \in \operatorname{Spec}_d R\mathcal{E}^{-1}} Q \neq 0. \qquad (2)$$

We claim that

$$\bigcap_{P \in \operatorname{Spec}_d R, \ x_1, \ldots, x_n \notin P} P \neq 0 \iff \bigcap_{P \in \operatorname{Spec}_d R} P \neq 0. \qquad (3)$$

One implication is trivial. For the other, suppose that $\bigcap_{P \in \operatorname{Spec}_d R, \ x_1, \ldots, x_n \notin P} P \neq 0$ and choose any $0 \neq r$ which belongs to this intersection. Then $0 \neq rx_1 \cdots x_n \in \bigcap_{P \in \operatorname{Spec}_d R} P$, verifying (3). Now (2) and (3) immediately give the result. $\square$

Lemma 2.3. *Let R be a noetherian $\mathbb{K}$-algebra of finite Gelfand–Kirillov dimension in which every prime ideal is completely prime. Then R satisfies the quasi strong Dixmier–Moeglin equivalence if and only if for every right Ore set $\mathcal{E}$ of regular elements of R which is finitely generated as a multiplicative system, the algebra $R\mathcal{E}^{-1}$ satisfies the quasi strong Dixmier–Moeglin equivalence.*

Proof. Suppose that R satisfies the quasi strong Dixmier–Moeglin equivalence and that $\mathcal{E}$ is a right Ore set of regular elements of R which is finitely generated as a multiplicative system. Every prime ideal of $R\mathcal{E}^{-1}$ takes the form $P\mathcal{E}^{-1}$ for some $P \in \operatorname{Spec} R$ with $P \cap \mathcal{E} = \emptyset$. Denoting by $\overline{\mathcal{E}}$ the image of $\mathcal{E}$ in R/P, we have

$$
\begin{aligned}
\operatorname{loc.deg} P\mathcal{E}^{-1} &= \operatorname{loc.deg} \langle 0 \rangle_{R\mathcal{E}^{-1}/P\mathcal{E}^{-1}} \\
&= \operatorname{loc.deg} \langle 0 \rangle_{(R/P)\overline{\mathcal{E}}^{-1}} \\
&= \operatorname{loc.deg} \langle 0 \rangle_{R/P} \qquad \text{(Lemma 2.2)} \\
&= \operatorname{loc.deg} P \\
&= \operatorname{rat.deg} P.
\end{aligned}
$$

Since it is clear that $\operatorname{rat.deg} P = \operatorname{rat.deg} P\mathcal{E}^{-1}$, we are done.

The converse follows simply by taking $\mathcal{E} = \emptyset$. $\square$

Proposition 2.4. *Let R be a noetherian $\mathbb{K}$-algebra of finite Gelfand–Kirillov dimension in which every prime ideal is completely prime. Suppose that for every $P \in \operatorname{Spec} R$, there exists a right Ore set $\mathcal{E}$ of regular elements of R/P which is finitely generated as a multiplicative system, such that $(R/P)\mathcal{E}^{-1}$ satisfies the quasi strong Dixmier–Moeglin equivalence. Then R itself satisfies the quasi strong Dixmier–Moeglin equivalence.*

Proof. Choose any $P \in \operatorname{Spec} R$. We have

$$\operatorname{loc.deg} P = \operatorname{loc.deg}\langle 0 \rangle_{R/P}$$
$$= \operatorname{loc.deg}\langle 0 \rangle_{(R/P)\mathcal{E}^{-1}} \qquad \left(\text{Lemma 2.2}\right)$$
$$= \operatorname{rat.deg}\langle 0 \rangle_{(R/P)\mathcal{E}^{-1}}.$$

Since it is clear that $\operatorname{rat.deg}\langle 0 \rangle_{(R/P)\mathcal{E}^{-1}} = \operatorname{rat.deg} P$, we are done. $\square$

Remark 2.5. The result of Proposition 2.4 holds if, rather than assuming that $(R/P)\mathcal{E}^{-1}$ satisfies the quasi strong Dixmier–Moeglin equivalence, we simply assume that, in $(R/P)\mathcal{E}^{-1}$, we have $\operatorname{loc.deg}\langle 0 \rangle = \operatorname{rat.deg}\langle 0 \rangle$.

3. The strong Dixmier–Moeglin equivalence in the commutative case

If there is to be any hope that the strong Dixmier–Moeglin equivalence will hold for any quantum algebras, one should first check that it holds for commutative affine domains. Before checking this, let us introduce the useful notion of Tauvel's height formula:

Definition 3.1. *Tauvel's height formula* is said to hold in a $\mathbb{K}$-algebra R if for every prime ideal P of R, the following equality holds:

$$\operatorname{GKdim} R/P = \operatorname{GKdim} R - \operatorname{ht} P.$$

It is well known that Tauvel's height formula holds in commutative affine domains; as we shall remark later, it has also been shown to hold in several interesting quantum algebras, including all of those which interest us in this paper.

In commutative affine domains, the notions of primitive, locally closed, and rational ideals all agree with the notion of a maximal ideal, so the following result is not surprising.

Proposition 3.2. *Every commutative affine domain over $\mathbb{K}$ satisfies the strong Dixmier–Moeglin equivalence.*

Proof. Let R be a commutative affine domain over $\mathbb{K}$ and let $P \in \operatorname{Spec} R$. We claim that

$$\operatorname{prim.deg} P = \operatorname{K.dim} R/P = \operatorname{rat.deg} P. \qquad (4)$$

Indeed, R/P is itself a commutative affine domain, so that every primitive (i.e. maximal) ideal of R/P has height $\operatorname{K.dim} R/P$. It follows that $\operatorname{prim.deg} P = \operatorname{K.dim} R/P$. Moreover, by standard results of commutative algebra, we have

$$\operatorname{rat.deg} P = \operatorname{tr.deg}_{\mathbb{K}} \mathcal{Z}(\operatorname{Frac} R/P)$$

$$= \text{tr.deg}_{\mathbb{K}} \text{Frac}(R/P)$$
$$= \text{K.dim} \, R/P,$$

so that (4) is proved.

If we set $d = \text{prim.deg} \, P = \text{K.dim} \, R/P = \text{rat.deg} \, P$, then all maximal ideals of R/P have height d, so that $\text{Spec}_{d+1} R/P$ is empty and hence $\bigcap_{Q \in \text{Spec}_{d+1} R/P} Q = R/P \neq 0$. Since R is a Jacobson ring, we get $\bigcap_{Q \in \text{Spec}_d R/P} Q = 0$, so that $\text{loc.deg} \, P = d$. This completes the proof. $\square$

Remark 3.3. Affine prime noetherian polynomial identity algebras over $\mathbb{K}$ can be shown to satisfy the strong Dixmier–Moeglin equivalence by a proof essentially the same as the proof above.

Remark 3.4. Let P be a prime ideal of a commutative affine domain R over $\mathbb{K}$. Since Gelfand–Kirillov dimension and Krull dimension agree in commutative affine domains, Tauvel's height formula gives $\text{K.dim} \, R/P = \text{K.dim} \, R - \text{ht} \, P$. Now we conclude from Proposition 3.2 and equation (4) that

$$\text{loc.deg} \, P = \text{prim.deg} \, P = \text{rat.deg} \, P = \text{K.dim} \, R - \text{ht} \, P.$$

4. The strong Dixmier–Moeglin equivalence for uniparameter quantum tori

Let N be a positive integer and let $\Lambda = (\lambda_{i,j}) \in \mathcal{M}_N(\mathbb{K}^\times)$ be a multiplicatively skew-symmetric matrix. The *quantum torus* associated to Λ is denoted by $\mathcal{O}_\Lambda((\mathbb{K}^\times)^N)$ or $\mathbb{K}_\Lambda[T_1^{\pm 1}, \ldots, T_N^{\pm 1}]$ and is presented as the $\mathbb{K}$-algebra generated by $T_1^{\pm 1}, \ldots, T_N^{\pm 1}$ with relations

$$T_i T_i^{-1} = T_i^{-1} T_i = 1 \text{ for all } i, \ T_j T_i = \lambda_{j,i} T_i T_j \text{ for all } i, j.$$

The algebra $\mathcal{O}_\Lambda((\mathbb{K}^\times)^N)$ can be written as the iterated skew-Laurent extension

$$\mathbb{K}[T_1^{\pm 1}][T_2^{\pm 1}; \sigma_2] \cdots [T_N^{\pm 1}; \sigma_N],$$

where for each $j \in [\![2, N]\!]$, σ_j is the automorphism of $\mathbb{K}[T_1^{\pm 1}][T_2^{\pm 1}; \sigma_2] \cdots [T_{j-1}^{\pm 1}; \sigma_{j-1}]$ defined by $\sigma_j(T_i) = \lambda_{j,i} T_i$ for all $i \in [\![1, j-1]\!]$. As such, $\mathcal{O}_\Lambda((\mathbb{K}^\times)^N)$ is a noetherian domain and there is a monomial $\mathbb{K}$-basis for $\mathcal{O}_\Lambda((\mathbb{K}^\times)^N)$ given by $\{T_1^{i_1} \cdots T_N^{i_N} \mid (i_1, \ldots, i_N) \in \mathbb{Z}^N\}$. By [5, Corollary II.7.18], $\mathcal{O}_\Lambda((\mathbb{K}^\times)^N)$ satisfies the noncommutative Nullstellensatz over $\mathbb{K}$ and by [5, Theorem II.9.14], $\mathcal{O}_\Lambda((\mathbb{K}^\times)^N)$ is catenary and satisfies Tauvel's height formula.

We recall from [12, Section 1] some useful facts about quantum tori. For $\underline{i} = (i_1, \ldots, i_N) \in \mathbb{Z}^N$, we set $\underline{T}^{\underline{i}} := T_1^{i_1} \cdots T_N^{i_N}$. For any $\underline{s}, \underline{t} \in \mathbb{Z}^N$, we have $\underline{T}^{\underline{s}} \underline{T}^{\underline{t}} = \sigma(\underline{s}, \underline{t}) \underline{T}^{\underline{t}} \underline{T}^{\underline{s}}$, where $\sigma: \mathbb{Z}^N \times \mathbb{Z}^N \to \mathbb{K}^\times$ is the alternating bicharacter which sends any $((s_1, \ldots, s_N), (t_1, \ldots, t_N))$ to $\prod_{i,j=1}^N \lambda_{i,j}^{s_i t_j}$.

When S is the subgroup $\{\underline{s} \in \mathbb{Z}^N \mid \sigma(\underline{s}, -) \equiv 1\}$ of $\mathbb{Z}^N$, the centre of $\mathcal{O}_\Lambda((\mathbb{K}^\times)^N)$ is spanned over $\mathbb{K}$ by those $\underline{T}^{\underline{s}}$ with $\underline{s} \in S$. Where $\underline{b_1}, \ldots, \underline{b_r}$ is a basis for S, the centre of $\mathcal{O}_\Lambda((\mathbb{K}^\times)^N)$ is a commutative Laurent polynomial ring in $(\underline{T}^{\underline{b_1}})^{\pm 1}, \ldots, (\underline{T}^{\underline{b_r}})^{\pm 1}$. Moreover, $\mathcal{O}_\Lambda((\mathbb{K}^\times)^N)$ is a free module over its centre with basis $\underline{T}^{\underline{t}}$, where $\underline{t}$ runs over any transversal for S in $\mathbb{Z}^N$.

There is a bi-increasing homeomorphism, known as *extension*, from $\operatorname{Spec} \mathcal{Z}(\mathcal{O}_\Lambda((\mathbb{K}^\times)^N))$ to $\operatorname{Spec} \mathcal{O}_\Lambda((\mathbb{K}^\times)^N)$ given by $I \mapsto \langle I \rangle$ (where $\langle I \rangle$ denotes the ideal of $\mathcal{O}_\Lambda((\mathbb{K}^\times)^N)$ generated by I). The inverse of this map is given by $J \mapsto J \cap \mathcal{Z}(\mathcal{O}_\Lambda((\mathbb{K}^\times)^N))$ and is known as *contraction* from $\operatorname{Spec} \mathcal{O}_\Lambda((\mathbb{K}^\times)^N)$ to $\operatorname{Spec} \mathcal{Z}(\mathcal{O}_\Lambda((\mathbb{K}^\times)^N))$. In fact, contraction and extension define mutually inverse increasing bijections between the set of all ideals of $\mathcal{O}_\Lambda((\mathbb{K}^\times)^N)$ and the set of all ideals of its centre.

Computing the rationality degree of a prime ideal P of $\mathcal{O}_\Lambda((\mathbb{K}^\times)^N)$ requires study of the centre of the algebra $\operatorname{Frac}(\mathcal{O}_\Lambda((\mathbb{K}^\times)^N)/P)$. The following general lemma is folklore, but we haven't been able to locate it in the literature.[4]

Lemma 4.1. *Let R be a noetherian domain and suppose that every nonzero ideal of R intersects $\mathcal{Z}(R)$ nontrivially. Then*

$$\mathcal{Z}(\operatorname{Frac} R) \cong \operatorname{Frac} \mathcal{Z}(R).$$

Proof. $\operatorname{Frac} \mathcal{Z}(R)$ embeds naturally into $\mathcal{Z}(\operatorname{Frac} R)$. Let $z \in \mathcal{Z}(\operatorname{Frac} R)$ and set $I = \{a \in R \mid za \in R\}$. Then I is a nonzero ideal of R and thus contains a nonzero element c of $\mathcal{Z}(R)$. Now $z = (zc)c^{-1} \in \operatorname{Frac} \mathcal{Z}(R)$. $\square$

Proposition 4.2. *For a completely prime ideal P of $\mathcal{O}_\Lambda((\mathbb{K}^\times)^N)$, we have*

$$\mathcal{Z}\left(\operatorname{Frac} \frac{\mathcal{O}_\Lambda((\mathbb{K}^\times)^N)}{P}\right) \cong \operatorname{Frac} \mathcal{Z}\left(\frac{\mathcal{O}_\Lambda((\mathbb{K}^\times)^N)}{P}\right).$$

Proof. Set $R = \mathcal{O}_\Lambda((\mathbb{K}^\times)^N)$ and let P be a completely prime ideal of R. By Lemma 4.1, it will suffice to show that every nonzero ideal of R/P intersects $\mathcal{Z}(R/P)$ nontrivially. This follows easily from fact that every ideal of R is generated by its intersection with $\mathcal{Z}(R)$. $\square$

Proposition 4.3. *For any ideal I of $\mathcal{O}_\Lambda((\mathbb{K}^\times)^N)$, we have*

$$\mathcal{Z}\left(\frac{\mathcal{O}_\Lambda((\mathbb{K}^\times)^N)}{I}\right) \cong \frac{\mathcal{Z}(\mathcal{O}_\Lambda((\mathbb{K}^\times)^N))}{I \cap \mathcal{Z}(\mathcal{O}_\Lambda((\mathbb{K}^\times)^N))}.$$

Proof. Set $R = \mathcal{O}_\Lambda((\mathbb{K}^\times)^N)$. We may clearly assume that I is a proper ideal and that R is noncommutative. We claim that $\mathcal{Z}(R/I) = (\mathcal{Z}(R)+I)/I$. Indeed the inclusion $\mathcal{Z}(R/I) \supseteq$

$(\mathcal{Z}(R)+I)/I$ is obvious. Suppose that $x \in R$ is central modulo I. We may choose elements $\underline{0}, \underline{i_1}, \dots, \underline{i_n}$ of a transversal for S in $\mathbb{Z}^N$ and central elements $z_0, z_1, \dots, z_n$ of R such that

$$x = z_0 + \sum_{a=1}^{n} z_a \underline{T}^{i_a}.$$

Fixing any $b \in [\![1, n]\!]$, there exists j_b belonging to the chosen transversal for S in $\mathbb{Z}^N$ such that $\sigma(j_b, i_b) \neq 1$. Since $\underline{T}^{j_b} x (\underline{T}^{j_b})^{-1} = x$ modulo I, we have

$$\sum_{a=1}^{n} (1 - \sigma(\underline{j_b}, \underline{i_a})) z_a \underline{T}^{i_a} \in I$$

and hence, by [12, Proposition 1.4], each $(1 - \sigma(\underline{j_b}, \underline{i_a})) z_a$ must belong to I. Since $\sigma(\underline{j_b}, \underline{i_b}) \neq 1$, we must have $z_b \in I$. Because $b \in [\![1, n]\!]$ was chosen arbitrarily, we get $z_1, \dots, z_n \in I$ and hence $x = z_0$ modulo I, completing the proof. $\quad\square$

The quantum torus $\mathcal{O}_\Lambda((\mathbb{K}^\times)^N)$ is called a *uniparameter* quantum torus if there exists a non-root of unity $q \in \mathbb{K}^\times$ and an additively skew-symmetric matrix $A = (a_{i,j}) \in \mathcal{M}_N(\mathbb{Z})$ such that $\Lambda = (q^{a_{i,j}})$; in this case, we write $\mathcal{O}_{q,A}((\mathbb{K}^\times)^N)$ for $\mathcal{O}_\Lambda((\mathbb{K}^\times)^N)$. By [5, Corollary II.6.10], all prime ideals of $\mathcal{O}_{q,A}((\mathbb{K}^\times)^N)$ are completely prime so that Proposition 4.2 applies. We are now ready to prove the main result of this section.

Theorem 4.4. *The uniparameter quantum tori $\mathcal{O}_{q,A}((\mathbb{K}^\times)^N)$ satisfy the strong Dixmier–Moeglin equivalence.*

Proof. Set $R = \mathcal{O}_{q,A}((\mathbb{K}^\times)^N)$ and choose any $P \in \operatorname{Spec} R$. As we have just noted, P is guaranteed to be completely prime. Recall that $\mathcal{Z}(R)$ is a commutative Laurent polynomial ring; in particular, $\mathcal{Z}(R)$ is a commutative affine domain, so that it satisfies the strong Dixmier–Moeglin equivalence by Proposition 3.2. By Propositions 4.2 and 4.3, we have

$$\mathcal{Z}(\operatorname{Frac} R/P) \cong \operatorname{Frac} \mathcal{Z}(R/P) \cong \operatorname{Frac} \frac{\mathcal{Z}(R)}{\mathcal{Z}(R) \cap P}.$$

It follows that rat.deg P = rat.deg($\mathcal{Z}(R) \cap P$). Since $\mathcal{Z}(R)/(\mathcal{Z}(R) \cap P)$ is a commutative affine domain, Remark 3.4 gives rat.deg $P = \operatorname{K.dim} \mathcal{Z}(R) - \operatorname{ht}(\mathcal{Z}(R) \cap P)$. Since extension and contraction are mutually inverse increasing homeomorphisms between $\operatorname{Spec} \mathcal{Z}(R)$ and $\operatorname{Spec} R$, we have $\operatorname{ht}(\mathcal{Z}(R) \cap P) = \operatorname{ht} P$, so that

$$\text{rat.deg } P = \operatorname{K.dim} \mathcal{Z}(R) - \operatorname{ht} P.$$

Every maximal ideal of $\mathcal{Z}(R)$ has height $\operatorname{K.dim} \mathcal{Z}(R)$ and hence so does every maximal ideal of R. By [12, Corollary 1.5], the primitive ideals of R are exactly its maximal ideals,

so that every primitive ideal of R has height K.dim $\mathcal{Z}(R)$. Now the catenarity of R gives prim.deg $P = \text{K.dim } \mathcal{Z}(R) - \text{ht } P$ and, in particular, prim.deg $P = \text{rat.deg } P$.

Let us set $d = \text{prim.deg } P = \text{rat.deg } P = \text{K.dim } \mathcal{Z}(R) - \text{ht } P$. Since all maximal (i.e. primitive) ideals of R have height K.dim $\mathcal{Z}(R)$, all maximal (i.e. primitive) ideals of R/P have height d. Now $\text{Spec}_{d+1} R/P$ is empty so that $\bigcap_{Q \in \text{Spec}_{d+1} R/P} Q = R/P \neq 0$. Since R is a Jacobson ring, we get $\bigcap_{Q \in \text{Spec}_d R/P} Q = 0$ and hence loc.deg $P = d$, completing the proof. $\quad\square$

5. Primer on H-stratification

Our next aim is to show that uniparameter quantum affine spaces (which we shall later define) satisfy the strong Dixmier–Moeglin equivalence. For this, we will make use of the H-stratification theory of Goodearl and Letzter (for details on this theory, see [5, II.2]). Indeed, an examination of the H-stratification (a notion which we define in this section) of a uniparameter quantum affine space reveals that every (prime homomorphic image of a) uniparameter quantum affine space localises to a (prime homomorphic image of a) uniparameter quantum torus. This allows us to transfer the quasi strong Dixmier–Moeglin equivalence from uniparameter quantum tori to uniparameter quantum affine spaces in Section 6. Further examination of the H-stratification of a uniparameter quantum affine space allows us to calculate the primitivity degrees of the prime ideals and hence, in the next section, complete the proof that uniparameter quantum affine spaces satisfy the strong Dixmier–Moeglin equivalence.

The material in this section shall be useful beyond quantum affine spaces, so we work in a more general setting. Let us suppose that R is a noetherian $\mathbb{K}$-algebra and that $H = (\mathbb{K}^{\times})^r$ is an algebraic $\mathbb{K}$-torus acting rationally on R by $\mathbb{K}$-algebra automorphisms. We refer to H-invariant prime ideals as H-*prime* ideals. We denote by H-$\text{Spec } R$ the H-*spectrum* of R, namely the subspace of $\text{Spec } R$ consisting of all H-prime ideals. Let us assume further that every H-prime ideal J of R is *strongly H-rational* in the sense that the fixed field $\mathcal{Z}(\text{Frac}(R/J))^H$ is $\mathbb{K}$ (in all of the algebras which will concern us in this paper, [5, Theorem II.6.4] guarantees that every H-prime ideal is strongly H-rational).

For an ideal I of R, $(I : H) := \bigcap_{h \in H} h \cdot I$ is the largest H-invariant ideal of R contained in I. It is well known that if P is a prime ideal of R, then $(P : H)$ is an H-prime ideal of R. For an H-prime ideal J of R, the H-*stratum* of $\text{Spec } R$ associated to J is denoted by $\text{Spec}_J R$ and is defined by $\text{Spec}_J R = \{P \in \text{Spec } R \mid (P : H) = J\}$. The H-strata form a partition of $\text{Spec } R$, usually referred to as the H-*stratification*. This stratification plays a crucial role in understanding the prime ideals of R and, as we shall see later in this section, the primitive ideals of R. By [5, Theorem II.2.13], for each H-prime ideal J of R, there is a bi-increasing homeomorphism from $\text{Spec}_J R$ to the prime spectrum of an appropriate commutative Laurent polynomial algebra over $\mathbb{K}$; the *Krull dimension* of the H-stratum $\text{Spec}_J R$ is defined to be the Krull dimension of this commutative Laurent polynomial algebra.

Let us make a useful observation on the Krull dimension of H-strata under localisation. Let $\mathcal{E}$ be a right Ore set in R consisting of regular H-eigenvectors with rational H-eigenvalues. There is a natural induced rational action of H on $R\mathcal{E}^{-1}$ by $\mathbb{K}$-algebra automorphisms. Extension and contraction restrict to mutually inverse increasing homeomorphisms between the set of H-prime ideals of R which do not intersect $\mathcal{E}$ and the set of H-prime ideals of $R\mathcal{E}^{-1}$. Moreover, for any H-prime ideal J of R which does not intersect $\mathcal{E}$, extension and contraction restrict to mutually inverse increasing homeomorphisms between $\mathrm{Spec}_J R$ and $\mathrm{Spec}_{J\mathcal{E}^{-1}} R\mathcal{E}^{-1}$. We deduce:

Lemma 5.1. *Let an algebraic $\mathbb{K}$-torus H act rationally on a noetherian $\mathbb{K}$-algebra R by $\mathbb{K}$-algebra automorphisms and suppose that all H-prime ideals of R are strongly H-rational. Let $\mathcal{E}$ be a right Ore set in R consisting of regular H-eigenvectors with rational H-eigenvalues. Then for any H-prime ideal J of R which does not intersect $\mathcal{E}$, we have*

$$\mathrm{K.dim}\,\mathrm{Spec}_J R = \mathrm{K.dim}\,\mathrm{Spec}_{J\mathcal{E}^{-1}} R\mathcal{E}^{-1}.$$

Under the further assumptions that R has finitely many H-prime ideals and that R satisfies the noncommutative Nullstellensatz over $\mathbb{K}$, [5, Theorem II.8.4] says that R satisfies the Dixmier–Moeglin equivalence and that the primitive ideals of R are exactly those prime ideals which are maximal in their H-strata. Assuming further that R is catenary and that the H-strata of R satisfy a technical condition (given in inequality (5)), we now show that if P is a prime ideal of R belonging to $\mathrm{Spec}_J R$ for an H-prime ideal J of R and if $M \supseteq P$ is a primitive (i.e. maximal) element of $\mathrm{Spec}_J R$, then $\mathrm{ht}\, M/P = \mathrm{prim.deg}\, P$ (and we compute these quantities in terms of the Krull dimension of $\mathrm{Spec}_J R$). Crucially, this allows us to look only at a single H-stratum of R in order to compute $\mathrm{prim.deg}\, P$.

Proposition 5.2. *Let R be a catenary noetherian $\mathbb{K}$-algebra satisfying the noncommutative Nullstellensatz over $\mathbb{K}$ and let H be an algebraic $\mathbb{K}$-torus acting rationally on R by $\mathbb{K}$-algebra automorphisms. Suppose that H-$\mathrm{Spec}\, R$ is finite, that all H-prime ideals of R are strongly H-rational, and that for any pair of H-prime ideals $J \subseteq J'$ of R, we have*

$$\mathrm{K.dim}\,\mathrm{Spec}_J R + \mathrm{ht}\, J \leq \mathrm{K.dim}\,\mathrm{Spec}_{J'} R + \mathrm{ht}\, J'. \tag{5}$$

Then for any H-prime ideal J of R, any $P \in \mathrm{Spec}_J R$, and any primitive element $M \supseteq P$ of $\mathrm{Spec}_J R$, we have

$$\mathrm{prim.deg}\, P = \mathrm{ht}\, M/P = \mathrm{K.dim}\,\mathrm{Spec}_J R + \mathrm{ht}\, J - \mathrm{ht}\, P. \tag{6}$$

Proof. Let M be a primitive element of $\mathrm{Spec}_J R$ which contains P. Then M is maximal in $\mathrm{Spec}_J R$, so that $\mathrm{ht}\, M/J = \mathrm{K.dim}\,\mathrm{Spec}_J R$. It follows from the catenarity of R that

$$\mathrm{ht}\, M/P = \mathrm{K.dim}\,\mathrm{Spec}_J R + \mathrm{ht}\, J - \mathrm{ht}\, P. \qquad (7)$$

Every primitive ideal of R/P corresponds to a primitive ideal of R which contains P. Choose any such primitive ideal N of R and say N belongs to $\mathrm{Spec}_{J'} R$ for an H-prime ideal J' of R. It is clear that $J \subseteq J'$.

Since N is maximal in $\mathrm{Spec}_{J'} R$, we have $\mathrm{ht}\, N/J' = \mathrm{K.dim}\,\mathrm{Spec}_{J'} R$. It follows from the catenarity of R that

$$\mathrm{ht}\, N/P = \mathrm{K.dim}\,\mathrm{Spec}_{J'} R + \mathrm{ht}\, J' - \mathrm{ht}\, P. \qquad (8)$$

Equations (7) and (8), along with the assumption (5), show that the height of an arbitrary primitive ideal of R/P is at least $\mathrm{ht}\, M/P$. Since M/P is itself primitive, we get $\mathrm{ht}\, M/P = \mathrm{prim.deg}\, P$; combining this with equation (7) gives the result. $\quad\square$

Remark 5.3. Except for the inequality (5), the conditions of Proposition 5.2 are known to hold for many interesting algebras. Much of the rest of this paper is concerned with verifying inequality (5) for uniparameter quantum affine spaces (Section 6) and quantum Schubert cells (Section 8). Our proofs rely on knowledge of the dimensions of the H-strata [2,4] and on knowledge of the posets of H-prime ideals [11,12,21].

6. The strong Dixmier–Moeglin equivalence for uniparameter quantum affine spaces

In a further step towards proving the strong Dixmier–Moeglin equivalence for quantum Schubert cells, we prove it in this section for uniparameter quantum affine spaces.

6.1. Quantum affine spaces

Let N be a positive integer and let $\Lambda = (\lambda_{i,j}) \in \mathcal{M}_N(\mathbb{K}^\times)$ be a multiplicatively skew-symmetric matrix. The *quantum affine space* associated to Λ is denoted by $\mathcal{O}_\Lambda(\mathbb{K}^N)$ or $\mathbb{K}_\Lambda[T_1, \ldots, T_N]$ and is presented as the $\mathbb{K}$-algebra with generators $T_1, \ldots, T_N$ and relations

$$T_j T_i = \lambda_{j,i} T_i T_j \text{ for all } i, j \in [\![1, N]\!].$$

The algebra $\mathcal{O}_\Lambda(\mathbb{K}^N)$ can be written as the iterated skew-polynomial extension

$$\mathbb{K}[T_1][T_2; \sigma_2] \cdots [T_N; \sigma_N],$$

where, for each $j \in [\![2, N]\!]$, σ_j is the automorphism of $\mathbb{K}[T_1][T_2; \sigma_2] \cdots [T_{j-1}; \sigma_{j-1}]$ defined by $\sigma_j(T_i) = \lambda_{j,i} T_i$ for all $i \in [\![1, j-1]\!]$. As such, $\mathcal{O}_\Lambda(\mathbb{K}^N)$ is a noetherian domain. By [5, Corollary II.7.18], $\mathcal{O}_\Lambda(\mathbb{K}^N)$ satisfies the noncommutative Nullstellensatz over $\mathbb{K}$ and by [5, Theorem II.9.14], $\mathcal{O}_\Lambda(\mathbb{K}^N)$ is catenary and satisfies Tauvel's height formula.

6.2. H-stratification of $\operatorname{Spec}\mathcal{O}_\Lambda(\mathbb{K}^N)$

The algebraic $\mathbb{K}$-torus $H = (\mathbb{K}^\times)^N$ acts rationally on $\mathcal{O}_\Lambda(\mathbb{K}^N)$ by $\mathbb{K}$-algebra automorphisms as follows:

$$(a_1, \ldots, a_N) \cdot T_i = a_i T_i \text{ for all } i \in [\![1, N]\!] \text{ and all } (a_1, \ldots, a_N) \in H.$$

For a subset Δ of $\{1, \ldots, N\}$, let K_Δ be the ideal of $\mathcal{O}_\Lambda(\mathbb{K}^N)$ generated by those T_i with $i \in \Delta$. The ideal K_Δ is clearly an H-invariant completely prime ideal of $\mathcal{O}_\Lambda(\mathbb{K}^N)$. Goodearl and Letzter have shown [12, Proposition 2.11] that all H-prime ideals of $\mathcal{O}_\Lambda(\mathbb{K}^N)$ take this form, namely that $H\text{-}\operatorname{Spec}\mathcal{O}_\Lambda(\mathbb{K}^N) = \{K_\Delta \mid \Delta \subseteq \{1, \ldots, N\}\}$. For any $\Delta \subseteq \{1, \ldots, N\}$, the H-stratum of $\mathcal{O}_\Lambda(\mathbb{K}^N)$ associated to K_Δ (which shall be denoted by $\operatorname{Spec}_\Delta(\mathcal{O}_\Lambda(\mathbb{K}^N))$) is given by

$$\operatorname{Spec}_\Delta(\mathcal{O}_\Lambda(\mathbb{K}^N)) = \left\{ P \in \operatorname{Spec}\mathcal{O}_\Lambda(\mathbb{K}^N) \mid P \cap \{T_i \mid i \in [\![1, N]\!]\} = \{T_i \mid i \in \Delta\} \right\}.$$

6.3. Uniparameter quantum affine spaces

The quantum affine space $\mathcal{O}_\Lambda(\mathbb{K}^N)$ is called a *uniparameter* quantum affine space if there exists a non-root of unity $q \in \mathbb{K}^\times$ and an additively skew-symmetric matrix $A = (a_{i,j}) \in \mathcal{M}_N(\mathbb{Z})$ such that $\Lambda = (q^{a_{i,j}})$. In this case, we denote $\mathcal{O}_\Lambda(\mathbb{K}^N) = \mathbb{K}_\Lambda[T_1, \ldots, T_N]$ by $\mathcal{O}_{q,A}(\mathbb{K}^N) = \mathbb{K}_{q,A}[T_1, \ldots, T_N]$. By [5, Corollary II.6.10], every prime ideal of $\mathcal{O}_{q,A}(\mathbb{K}^N)$ is completely prime.

We use a transfer result from Section 2 to show that $\mathcal{O}_{q,A}(\mathbb{K}^N)$ satisfies the quasi strong Dixmier–Moeglin equivalence.

Proposition 6.1. *The uniparameter quantum affine spaces $\mathcal{O}_{q,A}(\mathbb{K}^N)$ satisfy the quasi strong Dixmier–Moeglin equivalence.*

Proof. Set $R = \mathcal{O}_{q,A}(\mathbb{K}^N) = \mathbb{K}_{q,A}[T_1, \ldots, T_N]$. Choose any $P \in \operatorname{Spec} R$ and say $P \in \operatorname{Spec}_\Delta R$ for a subset Δ of $\{1, \ldots, N\}$. Let $\mathcal{E}$ be the multiplicative system in R generated by those T_i for which $i \notin \Delta$. Then $\mathcal{E}$ satisfies the Ore condition on both sides in R and, denoting by $\overline{\mathcal{E}}$ and $\hat{\mathcal{E}}$ its images in R/P and R/K_Δ respectively, we have

$$(R/P)\overline{\mathcal{E}}^{-1} \cong ((R/K_\Delta)\hat{\mathcal{E}}^{-1})/((P/K_\Delta)\hat{\mathcal{E}}^{-1}).$$

The uniparameter quantum torus $(R/K_\Delta)\hat{\mathcal{E}}^{-1}$ satisfies the strong Dixmier–Moeglin equivalence by Theorem 4.4 and hence so does its homomorphic image $(R/P)\overline{\mathcal{E}}^{-1}$. The result now follows from Proposition 2.4. $\square$

6.4. The strong Dixmier–Moeglin equivalence for uniparameter quantum affine spaces

Since we have proven that $\mathcal{O}_{q,A}(\mathbb{K}^N)$ satisfies the quasi strong Dixmier–Moeglin equivalence, proving that prim.deg P = rat.deg P holds for all prime ideals P of $\mathcal{O}_{q,A}(\mathbb{K}^N)$ will establish the strong Dixmier–Moeglin equivalence for $\mathcal{O}_{q,A}(\mathbb{K}^N)$.

In order to invoke Proposition 5.2, which gives us an expression for the primitivity degree of any prime ideal P of $\mathcal{O}_{q,A}(\mathbb{K}^N)$ in terms of the dimension of the H-stratum to which P belongs, we must prove an inequality relating the dimensions of H-strata of $\mathcal{O}_{q,A}(\mathbb{K}^N)$. First we introduce some new notation:

Notation 6.2. *Let Δ be a subset of $\{1, \ldots, N\}$ and set $\{\ell_1 < \ldots < \ell_d\} = \{1, \ldots, N\} \backslash \Delta$. We define the* skew-adjacency matrix, *$A(\Delta)$, of Δ to be the $d \times d$ additively skew-symmetric submatrix of $A = (a_{i,j}) \in \mathcal{M}_N(\mathbb{Z})$ whose (s,t) entry $(s < t)$ is a_{ℓ_s, ℓ_t}.*

For any $\Delta \subseteq \{1, \ldots, N\}$, it follows from [4, Theorem 3.1] that the dimension of the H-stratum $\mathrm{Spec}_\Delta(\mathcal{O}_{q,A}(\mathbb{K}^N))$ corresponding to the H-prime ideal $K_\Delta = \langle T_i \mid i \in \Delta \rangle$ is exactly $\dim_\mathbb{Q}(\ker A(\Delta))$. In fact, [4, Theorem 3.1] applies to a more general class of algebras called *uniparameter Cauchon–Goodearl–Letzter extensions* (see Section 7).

Proposition 6.3. *For any pair of H-prime ideals $K_\Delta \subseteq K_{\Delta'}$ of $\mathcal{O}_{q,A}(\mathbb{K}^N)$, we have*

$$\mathrm{K.dim}\,\mathrm{Spec}_\Delta(\mathcal{O}_{q,A}(\mathbb{K}^N)) + \mathrm{ht}\,K_\Delta \leq \mathrm{K.dim}\,\mathrm{Spec}_{\Delta'}(\mathcal{O}_{q,A}(\mathbb{K}^N)) + \mathrm{ht}\,K_{\Delta'}.$$

Proof. Since $K_\Delta \subseteq K_{\Delta'}$, we clearly have $\Delta \subseteq \Delta'$. The matrix $A(\Delta')$ is an $(N - |\Delta'|)$-square submatrix of the $(N - |\Delta|)$-square matrix $A(\Delta)$, so that $\mathrm{rk}\,A(\Delta') \leq \mathrm{rk}\,A(\Delta)$ and

$$(N - |\Delta'|) - \dim_\mathbb{Q}(\ker A(\Delta')) \leq (N - |\Delta|) - \dim_\mathbb{Q}(\ker A(\Delta)).$$

Hence, we have

$$\dim_\mathbb{Q}(\ker A(\Delta)) + |\Delta| \leq \dim_\mathbb{Q}(\ker A(\Delta')) + |\Delta'|. \tag{9}$$

Tauvel's height formula holds in $\mathcal{O}_{q,A}(\mathbb{K}^N)$, so that

$$\mathrm{ht}\,K_\Delta = \mathrm{GKdim}\,\mathcal{O}_{q,A}(\mathbb{K}^N) - \mathrm{GKdim}(\mathcal{O}_{q,A}(\mathbb{K}^N)/K_\Delta) = N - (N - |\Delta|) = |\Delta|$$

and similarly $\mathrm{ht}\,K_{\Delta'} = |\Delta'|$. Now (9) and [4, Theorem 3.1] give

$$\mathrm{K.dim}\,\mathrm{Spec}_\Delta(\mathcal{O}_{q,A}(\mathbb{K}^N)) + \mathrm{ht}\,K_\Delta \leq \mathrm{K.dim}\,\mathrm{Spec}_{\Delta'}(\mathcal{O}_{q,A}(\mathbb{K}^N)) + \mathrm{ht}\,K_{\Delta'}. \quad \square$$

With Proposition 6.3 in hand, we can apply Proposition 5.2 to $\mathcal{O}_{q,A}(\mathbb{K}^N)$ in our proof of the main result of this section:

Theorem 6.4. *The uniparameter quantum affine spaces $\mathcal{O}_{q,A}(\mathbb{K}^N)$ satisfy the strong Dixmier–Moeglin equivalence.*

Proof. Set $R = \mathcal{O}_{q,A}(\mathbb{K}^N) = \mathbb{K}_{q,A}[T_1, \ldots, T_N]$. We showed in Proposition 6.1 that R satisfies the quasi strong Dixmier–Moeglin equivalence, so what remains is to prove that $\text{prim.deg}\, P = \text{rat.deg}\, P$ for all prime ideals P of R.

Let P be any prime ideal of R and say $P \in \text{Spec}_\Delta R$ for a subset Δ of $\{1, \ldots, N\}$. Proposition 5.2 gives

$$\text{prim.deg}\, P = \text{K.dim}\, \text{Spec}_\Delta R + \text{ht}\, K_\Delta - \text{ht}\, P.$$

Let $\mathcal{E}$ be the multiplicative system in R generated by those T_i for which $i \notin \Delta$. Then $\mathcal{E}$ satisfies the Ore condition on both sides in R and, denoting by $\hat{\mathcal{E}}$ its image in R/K_Δ, we have $R\mathcal{E}^{-1}/P\mathcal{E}^{-1} \cong ((R/K_\Delta)\hat{\mathcal{E}}^{-1})/((P/K_\Delta)\hat{\mathcal{E}}^{-1})$. Notice that $(R/K_\Delta)\hat{\mathcal{E}}^{-1}$ is a uniparameter quantum torus and that $P\mathcal{E}^{-1} \in \text{Spec}_{K_\Delta \mathcal{E}^{-1}} R\mathcal{E}^{-1}$.

Since R is catenary and noetherian, so is $R\mathcal{E}^{-1}$. Moreover, $R\mathcal{E}^{-1}$ can be obtained from $\mathbb{K}$ by a finite number of skew-polynomial and skew-Laurent extensions; in particular, $R\mathcal{E}^{-1}$ is a constructible $\mathbb{K}$-algebra in the sense of [20, 9.4.12], so that $R\mathcal{E}^{-1}$ satisfies the noncommutative Nullstellensatz over $\mathbb{K}$ by [20, Theorem 9.4.21]. From the discussion of the effect of localisation on H-stratification (Section 5), we deduce that $R\mathcal{E}^{-1}$ satisfies the conditions of Proposition 5.2 and that

$$\text{prim.deg}\, P\mathcal{E}^{-1} = \text{K.dim}\, \text{Spec}_{K_\Delta \mathcal{E}^{-1}} R\mathcal{E}^{-1} + \text{ht}\, K_\Delta \mathcal{E}^{-1} - \text{ht}\, P\mathcal{E}^{-1}$$

$$= \text{K.dim}\, \text{Spec}_\Delta R + \text{ht}\, K_\Delta - \text{ht}\, P,$$

so that

$$\text{prim.deg}\, P\mathcal{E}^{-1} = \text{prim.deg}\, P.$$

Since the uniparameter quantum torus $(R/K_\Delta)\hat{\mathcal{E}}^{-1}$ satisfies the strong Dixmier–Moeglin equivalence (Theorem 4.4), so does its homomorphic image $R\mathcal{E}^{-1}/P\mathcal{E}^{-1}$. So $\text{prim.deg}\langle 0 \rangle = \text{rat.deg}\langle 0 \rangle$ holds in $R\mathcal{E}^{-1}/P\mathcal{E}^{-1}$, which can be rephrased by saying that in $R\mathcal{E}^{-1}$, we have $\text{prim.deg}\, P\mathcal{E}^{-1} = \text{rat.deg}\, P\mathcal{E}^{-1}$. Since we have shown that $\text{prim.deg}\, P = \text{prim.deg}\, P\mathcal{E}^{-1}$ and it is clear that $\text{rat.deg}\, P\mathcal{E}^{-1} = \text{rat.deg}\, P$, we have $\text{prim.deg}\, P = \text{rat.deg}\, P$, as required. $\square$

7. CGL extensions and the deleting derivations algorithm

In the terminology introduced in [17, Definition 3.1], let $R = \mathbb{K}[X_1][X_2; \sigma_2, \delta_2] \cdots [X_N; \sigma_N; \delta_N]$ be a uniparameter *Cauchon–Goodearl–Letzter (CGL)* extension. This class

of algebras contains many quantum algebras such as quantum matrices and, more generally, quantum Schubert cells. In particular, there exists an algebraic $\mathbb{K}$-torus $H = (\mathbb{K}^\times)^d$ acting rationally on R by $\mathbb{K}$-algebra automorphisms, there exists $q \in \mathbb{K}^\times$ not a root of unity, and there exists an additively skew-symmetric matrix $A = (a_{i,j}) \in \mathcal{M}_N(\mathbb{Z})$ such that

 (i) For all $j \in [\![2, N]\!]$, δ_j is locally nilpotent;
 (ii) For all $j \in [\![2, N]\!]$, there exists $q_j \in \mathbb{K}^\times$ not a root of unity such that $\sigma_j \circ \delta_j = q_j \delta_j \circ \sigma_j$;
 (iii) For all $j \in [\![2, N]\!]$ and all $i \in [\![1, j-1]\!]$, we have $\sigma_j(X_i) = q^{a_{j,i}} X_i$;
 (iv) $X_1, \ldots, X_N$ are H-eigenvectors;
 (v) The set $\{\lambda \in \mathbb{K}^\times \mid \text{there exists } h \in H \text{ such that } h \cdot X_1 = \lambda X_1\}$ is infinite;
 (vi) For all $j \in [\![2, N]\!]$, there exists $h_j \in H$ such that $h_j \cdot X_j = q_j X_j$ and, for $i \in [\![1, j-1]\!]$, $h_j \cdot X_i = q^{a_{j,i}} X_i$.

R is a noetherian domain and it satisfies the noncommutative Nullstellensatz over $\mathbb{K}$ by [5, Theorem II.7.17]. By [5, Theorem II.6.9], all prime ideals of R are completely prime. The Gelfand–Kirillov dimension of R is N by [5, Lemma II.9.7]. The algebra R has finitely many H-prime ideals by a result of Cauchon (we shall discuss this in more detail later in this section), so that R satisfies the Dixmier–Moeglin equivalence by [5, Theorem II.8.4].

Cauchon [7] introduced an algorithm (now known as the *deleting derivations algorithm*) which relates the prime spectrum and the H-stratification of R to those of the quantum affine space $\overline{R}$ which results from "deleting" the derivations δ_i (for a survey of this algorithm, see [4, Section 2C]). Following the notation of [7], $\overline{R}$ is, more precisely, a uniparameter quantum affine space in indeterminates $T_1, \ldots, T_N$ with commutation relations given by q and the matrix A, i.e.

$$\overline{R} = \mathbb{K}_{q,A}[T_1, \ldots, T_N] = \mathcal{O}_{q,A}(\mathbb{K}^N).$$

There is a *canonical injection* φ of $\operatorname{Spec} R$ into $\operatorname{Spec} \overline{R}$ (see [7, Section 4]), which Cauchon used to construct a partition of $\operatorname{Spec} R$ which we now describe.

Let W be the power set of $\{1, \ldots, N\}$. For any $\Delta \in W$, set $\operatorname{Spec}_\Delta R = \varphi^{-1}(\operatorname{Spec}_\Delta \overline{R})$, where $\operatorname{Spec}_\Delta \overline{R}$ denotes the stratum in $\operatorname{Spec} \overline{R}$ associated to the H-prime ideal $K_\Delta = \langle T_i \mid i \in \Delta \rangle$ (see Subsection 6.2). Denote by W' the set of those $\Delta \in W$ with $\operatorname{Spec}_\Delta R \neq \emptyset$. The elements of W are called the *diagrams* of the CGL extension R and the elements of W' are called the *Cauchon diagrams* of R. By [7, Proposition 4.4.1], we have

$$\operatorname{Spec} R = \bigsqcup_{\Delta \in W'} \operatorname{Spec}_\Delta R.$$

This is called the *canonical partition* of $\operatorname{Spec} R$ and, by [7, Théorème 5.5.2], it coincides with the partition of $\operatorname{Spec} R$ into H-strata. Let us make this more precise.

For any Cauchon diagram Δ of R, the canonical injection φ restricts to a bi-increasing homeomorphism from $\mathrm{Spec}_\Delta\, R$ to $\mathrm{Spec}_\Delta\, \overline{R}$ ([7, Théorèmes 5.1.1 and 5.5.1]). Moreover, by [7, Lemme 5.5.8 and Théorème 5.5.2], we have the following description of the H-prime ideals of R:

(i) For any $\Delta \in W'$, there is a (unique) H-invariant (completely) prime ideal J_Δ of R such that $\varphi(J_\Delta) = K_\Delta$;
(ii) $H\text{-}\mathrm{Spec}\, R = \{J_\Delta \mid \Delta \in W'\}$;
(iii) $\mathrm{Spec}_{J_\Delta}\, R = \mathrm{Spec}_\Delta\, R$ for all $\Delta \in W'$.

The invertible map $\Delta \mapsto J_\Delta$ from W' to $H\text{-}\mathrm{Spec}\, R$ is increasing but, in general, its inverse is not.

We are now in position to establish the quasi strong Dixmier–Moeglin equivalence for uniparamater CGL extensions.

Theorem 7.1. *Every uniparameter CGL extension satisfies the quasi strong Dixmier–Moeglin equivalence.*

Proof. Let R be a uniparameter CGL extension. Recall that both in R and in the uniparameter quantum affine space $\overline{R}$, all prime ideals are completely prime.

Choose any $P \in \mathrm{Spec}\, R$ and say $P \in \mathrm{Spec}_\Delta\, R$ for a Cauchon diagram Δ of R. Let $\mathcal{E}$ be the image in $\overline{R}/\varphi(P)$ of the multiplicative system in $\overline{R}$ generated by those T_i for which $i \in \{1, \ldots, N\}\backslash\Delta$. By [7, Théoremè 5.4.1], $\mathcal{E}$ satisfies the Ore condition on both sides in $\overline{R}/\varphi(P)$ and there exists a finitely generated multiplicative system $\mathcal{F}$ in R/P satisfying the Ore condition on both sides such that

$$(R/P)\mathcal{F}^{-1} \cong (\overline{R}/\varphi(P))\mathcal{E}^{-1}. \tag{10}$$

Since $\overline{R}$ is a uniparameter quantum affine space, it satisfies the strong Dixmier–Moeglin equivalence (Theorem 6.4) and hence so does every homomorphic image of $\overline{R}$. In particular, $\overline{R}/\varphi(P)$ satisfies the strong Dixmier–Moeglin equivalence. Hence, by Lemma 2.3, $(\overline{R}/\varphi(P))\mathcal{E}^{-1}$ satisfies the quasi strong Dixmier–Moeglin equivalence. The result now follows from (10) and Proposition 2.4. $\square$

Regarding the strong Dixmier–Moeglin equivalence, we can prove the following partial result.

Theorem 7.2. *If R is a catenary uniparameter CGL extension such that for any pair of H-prime ideals $J \subseteq J'$ of R, the following inequality holds:*

$$\mathrm{K.dim}\, \mathrm{Spec}_J\, R + \mathrm{ht}\, J \leq \mathrm{K.dim}\, \mathrm{Spec}_{J'}\, R + \mathrm{ht}\, J', \tag{11}$$

then R satisfies the strong Dixmier–Moeglin equivalence.

Proof. Since R satisfies the quasi strong Dixmier–Moeglin equivalence (Theorem 7.1), we need only show that for every prime ideal P of R, we have prim.deg $P = $ rat.deg P. By [5, Theorem II.8.4], R and $\overline{R}$ satisfy the Dixmier–Moeglin equivalence and, in each of these two algebras, the primitive ideals are exactly the prime ideals which are maximal in their H-strata.

Suppose that P is a prime ideal of R with $P \in \mathrm{Spec}_\Delta R$ for a Cauchon diagram Δ of R. Choose any primitive (i.e. maximal) element $M \supseteq P$ of $\mathrm{Spec}_\Delta R$. Since φ restricts to a bi-increasing homeomorphism from $\mathrm{Spec}_\Delta R$ to $\mathrm{Spec}_\Delta \overline{R}$, we get that $\varphi(M)$ is a maximal (i.e. primitive) element of $\mathrm{Spec}_\Delta \overline{R}$ and that $\varphi(M)$ contains $\varphi(P)$. Proposition 6.3 and the assumption (11) allow us to invoke Proposition 5.2 to get

$$\mathrm{ht}\, M/P = \mathrm{prim.deg}\, P \ \text{ and } \ \mathrm{ht}\, \varphi(M)/\varphi(P) = \mathrm{prim.deg}\, \varphi(P). \tag{12}$$

Moreover, since φ restricts to a bi-increasing homeomorphism from $\mathrm{Spec}_\Delta R$ to $\mathrm{Spec}_\Delta \overline{R}$, it induces a length-preserving one-to-one correspondence between the chains of prime ideals from P to M and the chains of prime ideals from $\varphi(P)$ to $\varphi(M)$. It follows that

$$\mathrm{ht}\, M/P = \mathrm{ht}\, \varphi(M)/\varphi(P). \tag{13}$$

We deduce from (12) and (13) that prim.deg $P = $ prim.deg $\varphi(P)$. Now, recalling that the uniparameter quantum affine space $\overline{R}$ satisfies the strong Dixmier–Moeglin equivalence (by Theorem 6.4) and that, by [7, Théoremè 5.4.1], $\mathrm{Frac}(R/P) \cong \mathrm{Frac}(\overline{R}/\varphi(P))$, we have

$$\mathrm{prim.deg}\, P = \mathrm{prim.deg}\, \varphi(P)$$
$$= \mathrm{rat.deg}\, \varphi(P)$$
$$= \mathrm{rat.deg}\, P,$$

as required. $\quad\square$

Remark 7.3. We do not know whether or not there are any CGL extensions in which the inequality (11) fails.

8. Quantum Schubert cells

We discuss quantum Schubert cells and their uniparameter CGL extension structure. Yakimov [23, Theorem 5.7] has shown that these algebras are catenary and satisfy Tauvel's height formula. We show that they satisfy inequality (11) so that, by Theorem 7.2, they satisfy the strong Dixmier–Moeglin equivalence.

8.1. The algebras $U_q[w]$ and their uniparameter CGL extension structure

Let $\mathfrak{g}$ be a simple complex Lie algebra of rank n and let $\pi := \{\alpha_1, \ldots, \alpha_n\}$ be the set of simple roots associated to a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ of $\mathfrak{g}$. The set π is a basis of a real Euclidean vector space E, whose inner product we denote by $(-, -)$. Recall that the Weyl group of $\mathfrak{g}$, which we denote by $\mathcal{W}$, is the subgroup of the orthogonal group of E generated by the reflections $s_i := s_{\alpha_i}$, for $i = 1, \ldots, n$, with reflecting hyperplanes $H_i := \{\beta \in E \mid (\beta, \alpha_i) = 0\}$, $i = 1, \ldots, n$.

Where $q \in \mathbb{K}^\times$ is not a root of unity and w is any element of $\mathcal{W}$, De Concini, Kac, and Procesi [8] defined a quantum analogue, $U_q[w]$, of the universal enveloping algebra of the nilpotent Lie algebra $\mathfrak{n}^+ \cap \mathrm{Ad}_w(\mathfrak{n}^-)$, where Ad denotes the adjoint action. We refer the reader to [4, Subsection 3C] for a description of the *quantum Schubert cell $U_q[w]$* as a certain subalgebra of $U_q^+(\mathfrak{g})$, where $U_q(\mathfrak{g})$ is the quantised enveloping algebra of $\mathfrak{g}$ over $\mathbb{K}$ associated to the above data.

$\mathcal{W}$ is a Coxeter group with respect to the generators $s_1, \ldots, s_n$ and we define the length, $\ell(w)$, of w to be the smallest N such that there exist $i_j \in \{1, \ldots, n\}$ satisfying $w = s_{i_1} \cdots s_{i_N}$. Let us fix this *reduced* expression for w. It is well known that $\beta_1 = \alpha_{i_1}$, $\beta_2 = s_{i_1}(\alpha_{i_2}), \ldots, \beta_N = s_{i_1} \cdots s_{i_{N-1}}(\alpha_{i_N})$ are distinct positive roots and that the set $\{\beta_1, \ldots, \beta_N\}$ does not depend on the chosen reduced expression for w.

Cauchon proved [7, Proposition 6.1.2 and Lemme 6.2.1] that $U_q[w]$ is a uniparameter CGL extension in N indeterminates with the following associated additively skew-symmetric matrix:

$$
A := \begin{pmatrix}
0 & (\beta_1, \beta_2) & \cdots & & \cdots & (\beta_1, \beta_N) \\
-(\beta_1, \beta_2) & 0 & (\beta_2, \beta_3) & & & (\beta_2, \beta_N) \\
\vdots & \ddots & \ddots & \ddots & & \vdots \\
\vdots & & \ddots & & 0 & (\beta_{N-1}, \beta_N) \\
-(\beta_1, \beta_N) & \cdots & & \cdots & -(\beta_{N-1}, \beta_N) & 0
\end{pmatrix}. \tag{14}
$$

Theorem 7.1 immediately gives:

Proposition 8.1. *The quantum Schubert cells $U_q[w]$ satisfy the quasi strong Dixmier–Moeglin equivalence.*

8.2. The strong Dixmier–Moeglin equivalence for $U_q[w]$

Considering $U_q[w]$ as a uniparameter CGL extension in N indeterminates with associated additively skew-symmetric matrix A (see (14)), recall that J_Δ denotes the H-prime ideal of $U_q[w]$ associated to a Cauchon diagram Δ of $U_q[w]$. The remaining work lies in proving that for any pair of H-prime ideals $J_\Delta \subseteq J_{\Delta'}$ of $U_q[w]$, the following inequality holds:

$$\text{K.dim}\,\text{Spec}_\Delta\,U_q[w] + \text{ht}\,J_\Delta \le \text{K.dim}\,\text{Spec}_{\Delta'}\,U_q[w] + \text{ht}\,J_{\Delta'}.$$

This will allow us to invoke Theorem 7.2 to show that $U_q[w]$ satisfies the strong Dixmier–Moeglin equivalence.

In contrast to that of most algebras supporting an H-action, the poset structure of the H-spectrum of $U_q[w]$ is known. Let us denote by $\le$ the Bruhat order on $\mathcal{W}$ and let us set $\mathcal{W}^{\le w} := \{u \in \mathcal{W} \mid u \le w\}$. The posets H-$\text{Spec}\,U_q[w]$ and $\mathcal{W}^{\le w}$ are isomorphic. In order to describe an isomorphism due to Cauchon–Mériaux and Geiger–Yakimov, we introduce some notation:

Notation 8.2. *Recall that we have fixed a reduced expression* $w = s_{i_1}\cdots s_{i_N}$ *for* w. *Let* $\Delta \subseteq \{1,\ldots,N\}$ *be any (not necessarily Cauchon) diagram.*

(i) *For all* $k = 1,\ldots,N$, *we set*

$$s_{i_k}^\Delta := \begin{cases} s_{i_k} & \text{if } k \in \Delta \\ id & \text{otherwise.} \end{cases}$$

(ii) *We set* $\{l_1 < \cdots < l_d\} := \{1,\ldots,N\}\backslash\Delta$ *and* $j_r = i_{l_r}$ *for all* $r = 1,\ldots,d$.

(iii) *We set* $w^\Delta := s_{i_1}^\Delta \cdots s_{i_N}^\Delta \in \mathcal{W}$.

(iv) *We set* $A(w^\Delta)$ *to be the* $d \times d$ *additively skew-symmetric submatrix of* A *whose* (s,t)-*entry* $(s < t)$ *is* $(\beta_{j_s}, \beta_{j_t})$.

Cauchon and Mériaux [21, Corollary 5.3.1] showed that the map

$$H\text{-}\text{Spec}\,U_q[w] \to \mathcal{W}^{\le w};\ J_\Delta \mapsto w^\Delta, \tag{15}$$

where Δ runs over the set of Cauchon diagrams of $U_q[w]$, is a bijection; they asked whether or not this bijection is an isomorphism of posets and this question was answered affirmatively by Geiger and Yakimov [11, Theorem 4.4].

Lemma 8.3. *For any Cauchon diagram* Δ *of* $U_q[w]$, *we have* $\text{ht}\,J_\Delta = |\Delta|$.

Proof. Set $R = U_q[w]$ and recall that $\overline{R}$ denotes the uniparameter quantum affine space (in indeterminates $T_1,\ldots,T_N$ say) which results from "deleting" the derivations in the expression of R as a uniparameter CGL extension in N indeterminates. Recall that $K_\Delta = \langle T_i \mid i \in \Delta \rangle$ is the image of J_Δ under the canonical injection $\varphi\colon \text{Spec}\,R \to \text{Spec}\,\overline{R}$.

Let $\mathcal{E}$ be the image in $\overline{R}/K_\Delta$ of the multiplicative system in $\overline{R}$ generated by those T_i for which $i \notin \Delta$. Then $\mathcal{E}$ satisfies the Ore condition on both sides in $\overline{R}/K_\Delta$ and it follows from [7, Théoremè 5.4.1] both that R/J_Δ embeds in the uniparameter quantum torus $(\overline{R}/K_\Delta)\mathcal{E}^{-1}$ and that $\text{Frac}(R/J_\Delta) \cong \text{Frac}((\overline{R}/K_\Delta)\mathcal{E}^{-1})$. By [19, Corollary 2.2], the uniparameter quantum torus $(\overline{R}/K_\Delta)\mathcal{E}^{-1}$ is Tdeg-stable (in the sense of

[24, Section 1]). Therefore, we can apply [24, Proposition 3.5(4)] to get GKdim $R/J_\Delta =$ GKdim$(\overline{R}/K_\Delta)\mathcal{E}^{-1} = N - |\Delta|$.

Since R satisfies Tauvel's height formula, we conclude that

$$N - |\Delta| = \text{GKdim } R/J_\Delta = \text{GKdim } R - \text{ht } J_\Delta = N - \text{ht } J_\Delta,$$

and so ht $J_\Delta = |\Delta|$, as desired. $\square$

We are now in position to establish the crucial inequality required to prove that quantum Schubert cells satisfy the strong Dixmier–Moeglin equivalence.

Proposition 8.4. *For any pair of H-prime ideals $J_\Delta \subseteq J_{\Delta'}$ of $U_q[w]$, we have*

$$\text{K.dim Spec}_\Delta\, U_q[w] + \text{ht } J_\Delta \leq \text{K.dim Spec}_{\Delta'}\, U_q[w] + \text{ht } J_{\Delta'}.$$

Proof. As we have noted, $U_q[w]$ is a uniparameter CGL extension in N indeterminates with associated additively skew-symmetric matrix A. By [2, Theorems 2.3 and 3.1], we have

$$\text{K.dim Spec}_\Delta U_q[w] = \dim_\mathbb{Q} \ker(w^\Delta + w) \text{ and } \text{K.dim Spec}_{\Delta'} U_q[w] = \dim_\mathbb{Q} \ker A(w^{\Delta'}).$$

From the poset isomorphism (H-Spec $U_q[w] \to \mathcal{W}^{\leq w}$; $J_\Delta \mapsto w^\Delta$), we deduce that $w^\Delta \leq w^{\Delta'}$. Since the diagrams Δ and Δ' are Cauchon, the subexpressions w^Δ and $w^{\Delta'}$ of $w = s_{i_1} \cdots s_{i_N}$ are reduced by [21, Corollary 5.3.1(2)]. Since $w^\Delta \leq w^{\Delta'}$, [14, Corollary 5.8] allows us to choose a diagram (not necessarily Cauchon) $\widetilde{\Delta} \subseteq \Delta'$ such that $w^{\widetilde{\Delta}} = w^\Delta$ and the subexpression $w^{\widetilde{\Delta}}$ of $w = s_{i_1} \cdots s_{i_N}$ is reduced. Now K.dim Spec$_\Delta U_q[w] = \dim_\mathbb{Q} \ker(w^{\widetilde{\Delta}} + w)$, so that [2, Theorem 3.1] gives K.dim Spec$_\Delta U_q[w] = \dim_\mathbb{Q} \ker A(w^{\widetilde{\Delta}})$.

$A(w^{\Delta'})$ is an $(N - |\Delta'|)$-square submatrix of the $(N - |\widetilde{\Delta}|)$-square matrix $A(w^{\widetilde{\Delta}})$, so that rk $A(w^{\Delta'}) \leq$ rk $A(w^{\widetilde{\Delta}})$ and hence $\dim_\mathbb{Q} \ker A(w^{\widetilde{\Delta}}) + |\widetilde{\Delta}| \leq \dim_\mathbb{Q} \ker A(w^{\Delta'}) + |\Delta'|$ and

$$\text{K.dim Spec}_\Delta U_q[w] + |\widetilde{\Delta}| \leq \text{K.dim Spec}_{\Delta'} U_q[w] + |\Delta'|. \tag{16}$$

By Lemma 8.3, we have ht $J_\Delta = |\Delta|$ and ht $J_{\Delta'} = |\Delta'|$. Since w^Δ and $w^{\widetilde{\Delta}}$ are equal as elements of $\mathcal{W}$, we have $\ell(w^\Delta) = \ell(w^{\widetilde{\Delta}})$. But since the subexpressions w^Δ and $w^{\widetilde{\Delta}}$ of $w = s_{i_1} \cdots s_{i_N}$ are reduced, we have $\ell(w^\Delta) = |\Delta|$ and $\ell(w^{\widetilde{\Delta}}) = |\widetilde{\Delta}|$; hence $|\Delta| = |\widetilde{\Delta}|$.

Now we have $|\widetilde{\Delta}| = \text{ht } J_\Delta$ and $|\Delta'| = \text{ht } J_{\Delta'}$, so that the result now follows from (16). $\square$

Yakimov has shown [23, Theorem 5.7] that $U_q[w]$ is catenary. We have discussed the uniparameter CGL extension structure of $U_q[w]$. Proposition 8.4 provides the final condition required for us to apply Theorem 7.2 to $U_q[w]$, giving our main result:

Theorem 8.5. *The quantum Schubert cells $U_q[w]$ satisfy the strong Dixmier–Moeglin equivalence.*

Acknowledgments

References

[1] J. Bell, D. Rogalski, S.J. Sierra, The Dixmier–Moeglin equivalence for twisted homogeneous coordinate rings, Israel J. Math. 180 (2010) 461–507.

[2] Jason Bell, Karel Casteels, Stéphane Launois, Primitive ideals in quantum Schubert cells: dimension of the strata, Forum Math. 26 (3) (2014) 703–721.

[3] Jason Bell, Stéphane Launois, Omar León Sánchez, Rahim Moosa, Poisson algebras via model theory and differential-algebraic geometry, arXiv:1406.0867v3, 2015.

[4] Jason P. Bell, Stéphane Launois, On the dimension of H-strata in quantum algebras, Algebra Number Theory 4 (2) (2010) 175–200.

[5] Ken A. Brown, Ken R. Goodearl, Lectures on Algebraic Quantum Groups, Adv. Courses Math. CRM Barcelona, Birkhäuser Verlag, Basel, 2002.

[6] Stefan Catoiu, Ideals of the enveloping algebra $U(\mathrm{sl}_2)$, J. Algebra 202 (1) (1998) 142–177.

[7] Gérard Cauchon, Effacement des dérivations et spectres premiers des algèbres quantiques, J. Algebra 260 (2) (2003) 476–518.

[8] C. De Concini, V.G. Kac, C. Procesi, Some quantum analogues of solvable Lie groups, in: Geometry and Analysis, Bombay, 1992, Tata Inst. Fund. Res., Bombay, 1995, pp. 41–65.

[9] J. Dixmier, Idéaux primitifs dans les algèbres enveloppantes, J. Algebra 48 (1) (1977) 96–112.

[10] Jacques Dixmier, Enveloping Algebras, Grad. Stud. Math., vol. 11, American Mathematical Society, Providence, RI, 1996, revised reprint of the 1977 translation.

[11] Joel Geiger, Milen Yakimov, Quantum Schubert cells via representation theory and ring theory, Michigan Math. J. 63 (1) (2014) 125–157.

[12] K.R. Goodearl, E.S. Letzter, Prime and primitive spectra of multiparameter quantum affine spaces, in: Trends in Ring Theory, Miskolc, 1996, in: Conf. Proc., Can. Math. Soc., vol. 22, Amer. Math. Soc., Providence, RI, 1998, pp. 39–58.

[13] K.R. Goodearl, R.B. Warfield Jr., An Introduction to Noncommutative Noetherian Rings, second ed., London Math. Soc. Stud. Texts, vol. 61, Cambridge University Press, Cambridge, 2004.

[14] James E. Humphreys, Reflection Groups and Coxeter Groups, Cambridge Stud. Adv. Math., vol. 29, Cambridge University Press, Cambridge, 1990.

[15] Ronald S. Irving, Noetherian algebras and nullstellensatz, in: Séminaire d'Algèbre Paul Dubreil 31ème année, Paris, 1977–1978, in: Lecture Notes in Math., vol. 740, Springer, Berlin, 1979, pp. 80–87.

[16] Günter R. Krause, Thomas H. Lenagan, Growth of Algebras and Gelfand–Kirillov Dimension, revised ed., Grad. Stud. Math., vol. 22, American Mathematical Society, Providence, RI, 2000.

[17] S. Launois, T.H. Lenagan, L. Rigal, Quantum unique factorisation domains, J. Lond. Math. Soc. (2) 74 (2) (2006) 321–340.

[18] Martin Lorenz, Primitive ideals of group algebras of supersoluble groups, Math. Ann. 225 (2) (1977) 115–122.

[19] Martin Lorenz, On the transcendence degree of group algebras of nilpotent groups, Glasg. Math. J. 25 (2) (1984) 167–174.

[20] J.C. McConnell, J.C. Robson, Noncommutative Noetherian Rings, revised ed., Grad. Stud. Math., vol. 30, American Mathematical Society, Providence, RI, 2001, with the cooperation of L.W. Small.

[21] Antoine Mériaux, Gérard Cauchon, Admissible diagrams in quantum nilpotent algebras and combinatoric properties of Weyl groups, Represent. Theory 14 (2010) 645–687.

[22] C. Moeglin, Idéaux primitifs des algèbres enveloppantes, J. Math. Pures Appl. (9) 59 (3) (1980) 265–336.
[23] Milen Yakimov, A proof of the Goodearl–Lenagan polynormality conjecture, Int. Math. Res. Not. IMRN (9) (2013) 2097–2132.
[24] James J. Zhang, On Gelfand–Kirillov transcendence degree, Trans. Amer. Math. Soc. 348 (7) (1996) 2867–2899.