# A RING PRIMITIVE ON THE RIGHT BUT NOT ON THE LEFT

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Jacobson [1, p. 4] writes, "It is not known whether [right] primitivity implies left primitivity. It seems unlikely that it does, but no examples of [right] primitive rings which are not left primitive are known." Such an example is here constructed.

Let $D$ be a division ring, and $\alpha: D \to D$ an endomorphism. Let $A$ be the ring of formal polynomials $\sum_{i \geq 0} d_i Y^i$ ($d_i \in D$, nonzero for only finitely many $i$) with multiplication determined by the rule $Yd = \alpha(d)Y$. Such rings are without zero divisors, and every left ideal of one is principal. The proofs are exactly as for the ordinary commutative rings of polynomials, cf. [2, p. 483].

Now let $D = \mathbf{Q}(X)$, where $\mathbf{Q}$ denotes the field of rationals, and let $\alpha$ be the map $r(X) \to r(X^2)$. In this case, we have the following result.

PROPOSITION 1. *Any subring $B \subset A$ containing $X$ and $Y$ is right primitive.*

PROOF. We observe that for any $r \in \mathbf{Q}(X)$ there is a unique $r^* \in \mathbf{Q}(X)$ such that $(r(X) + r(-X))/2 = r^*(X^2)$. We use this in defining the structure of a right $A$-module on $\mathbf{Q}(X)$ such that, if $r$ and $s$ are elements of $\mathbf{Q}(X)$, $r \cdot s = rs$ and $r \cdot Y = r^*$. To see that this is actually a right $A$-module structure, it suffices to verify that $(r \cdot Y) \cdot s = (r \cdot \alpha(s)) \cdot Y$, i.e., that $r^* s = (r\alpha(s))^*$. Now we have $(r^*s)(X^2) = \frac{1}{2}(r(X) + r(-X))s(X^2) = \frac{1}{2}(r(X)\alpha(s)(X) + r(-X)\alpha(s)(-X)) = (r\alpha(s))^*(X^2)$, whence $r^*s = (r\alpha(s))^*$.

We observe:

$$X^n \cdot Y^m = \begin{cases} X^{n/2^m} & \text{if } 2^m \mid n, \\ 0 & \text{otherwise.} \end{cases}$$

Let $M$ be the $B$-submodule of $\mathbf{Q}(X)$ that is generated by $X$. We claim that $M$ is irreducible. For let us be given a nonzero element $P(X)/Q(X) \in M$, where $P(X)$ and $Q(X)$ are polynomials. Take $2^a > \deg P$. Let $c = 1/(\text{leading coefficient of } P)$. Then $[P(X)/Q(X)] \cdot cQ(X)X^{2^a - \deg P}\, Y^a = [cP(X)X^{2^a - \deg P}] \cdot Y^a$. The element in square brackets is a polynomial with leading term $X^{2^a}$, and constant term zero. Hence $Y^a$, applied to it, gives $X$, which in turn generates $M$.

Next we show that $M$ is faithful: Let $0 \neq b = \sum r_i(X) Y^i \in B$. Choose

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a polynomial $P(X)$ such that the $R_i(X)=P(X)r_i(X)$ are all polynomials. We shall find $m\in M$ such that $m\cdot\sum R_i(X)Y^i\neq 0$, which will give us $mP(X)\cdot b\neq 0$.

Let $j$ be the least integer such that $R_j\neq 0$. Let

$$d=\max_i(\deg R_i-\deg R_j).$$

Choose $n>0$ so that $2^n\geqq\deg R_j$, $2^{n-j-1}\geqq d$. Consider $X^{2^n-\deg R_j}\cdot\sum R_i(X)Y^i$. The exponent of $X$ in the highest-power term of $X^{2^n-\deg R_j}\cdot R_i(X)Y^i$ is $(2^n-\deg R_j+\deg R_i)/2^i$, or less if this is not integral. For $i=j$ this comes to $2^{n-j}$. For $i>j$, it is $\leqq(2^n+d)/2^i<2^{n-j-1}+d\leqq 2^{n-j-1}+2^{n-j-1}=2^{n-j-1}$. So only the $j$th term contributes to the coefficient of $X^{2^{n-j}}$, which is thus nonzero. Q.E.D.

Now, for every odd integer $n>0$, let $v_n$ be the valuation on $\boldsymbol{Q}(X)$ induced by the $n$th cyclotomic polynomial. For $r\in\boldsymbol{Q}(X)$, we have $v_n(r(X^2))=v_n(r(X))$.[1]

Suppose, in the more general context of the second paragraph, that a valuation $v$ on $D$ satisfies $v(\alpha(d))=v(d)$ for all $d\in D$. Then it follows immediately that $v$ is extended to a valuation on $A$ by the definition $v(\sum d_iY^i)=\min_i v(d_i)$.

Suppose we have an infinite set $V$ of valuations of this sort, with the properties that at any $d\in D-\{0\}$, only finitely many of them are nonzero, and that for each $v\in V$, there is an $x_v\in D$ on which all valuations are nonnegative, and $v$ is positive. Let us designate by $B$ the intersection of the valuation subrings of the valuations in $V$, i.e., the subring consisting of those elements on which all the valuations are nonnegative.

In the specific case in question, $B$ contains $X$ and $Y$, and so, by Proposition 1, is right primitive. But we shall now show that, in general, it cannot be left primitive.

PROPOSITION 2. *$B$ is not left primitive.*

PROOF. Let $I$ be any left ideal of $B$. We shall show that either it is not maximal or the annihilator in $B$ of $B/I$ is not zero.

$AI$, being a left ideal of $A$, is principal. Let $g$ be a generator. We can assume its leading coefficient is 1.

*Case* 1. $g$ has $Y$-degree $d>0$.

General observation: given any $v\in V$ and nonzero $a=\sum d_iY^i\in A$,

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[1] One of many ways to see this is as follows: Consider $r(X)$ as a meromorphic function on the complex plane. It is clear that the order of $r(X^2)$ at $z_0$ is exactly the order of $r$ at $z_0^2$ if $z_0\neq 0$. But if $z_0$ is a primitive $n$th root of unity for odd $n$, so is $z_0^2$, and the orders of $r$ at $z_0$ and $z_0^2$ are the same. Hence the orders of $r(X)$ and $r(X^2)$ at $z_0$ are the same.

there must be some $i$ such that $v(a)=v(d_i)$. If $i_0$ is the greatest such, we shall say that $a$ is of relativized $v$-degree $i_0$. It is clear that the relativized $v$-degree of $aa'$ is the sum of the relativized $v$-degrees of $a$ and $a'$.

Choose $v$ such that $v(g)=0$. Then $g$ has relativized $v$-degree $d$. Hence any nonzero element of $AI$ has relativized $v$-degree $>0$.

Now $x_v \notin I$. Hence, if $I$ were maximal, we could write $bx_v+i=1$, $b\in B$, $i\in I$. But then we would have $i=1-bx_v$, which has relativized $v$-degree 0; contradiction. Hence $I$ is not maximal.

*Case* 2. $g=1$.

General observation: given nonzero $d\in D$, an element of $A$ can be written in the form $bd$ with $b\in B$ if and only if it can be written $db'$ with $b'=d^{-1}bd\in B$, since all valuations of $b$ and $d^{-1}bd$ are the same. Hence $Bd=dB$.

Now let $1=\sum a_k i_k$, $a_k\in A$, $i_k\in I$. For a sufficiently large (but finite!) product $x$ of the $x_v$'s, $xa_k\in B$ for all $k$. Hence $x=\sum(xa_k)i_k\in I$.

Hence $I\supset Bx=xB$, and so the module $B/I$ is not faithful. Q.E.D.

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## References

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