PRIMITIVE IDEAL

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Baire Space

Definition 0.1. A topology space is called Baire if $\cap_{i\in I}O_i$ is dense, where O_i is dense open set, and $|I| < |k|$.

Remark 0.2. If R is countable generated, we may repalce all $< |k|$ by countable.

Lemma 0.3. *Let R be a prime semiprimitive ring. Then every non-empty open set is dense.*

Proof. Suppose U is a non-empty open set, V is open and $U \cap V = \emptyset$. Let $I = \cap_{P\notin U}P$, $J = \cap_{P\notin V}P$, then $IJ \subseteq I \cap J = 0$, but $U \neq \emptyset$, so $I \neq 0$, so $J = 0$. So $V = \emptyset$. $\square$

Kaplansky Ring

Definition 0.4. R is said to be Kaplansky ring provided the primtive ideal space of every homomorphic image of R is a Baire space.

Lemma 0.5. *Suppose R is Jacobson ring which is (one-side) noetherian, Then R is Kaplansky if and only if Prim R/P is Baire for every prime ideal P in R.*

Note that Prim $R/I = (\text{Prim } R/P_1) \cup \cdots \cup (\text{Prim } R/P_n)$, where $\{P_1, \cdots, P_n\}$ is the set of prime ideal mininal over I. and the proof is clear.

Primitive Spectrum

Prim R is sub-topological space of Spec(R) consist of (left) primitive ideal.

Lemma 0.6. *Let R be a prime semiprimitive ring. Prim R is Baire if and only if $\cap_{i\in I}U_i \neq \emptyset$ for every $|I| < |k|$ and $U_i, i \in I$ are non-empty open set in Prim R.*

Proof. $U \doteq \cap_{i\in I}U_i$, for every non-empty open set V, $U \cap V = \cap_{i\in I}U_i \cap V$ is non-empty by assumption. So U is dense. $\square$

We may write above lemma as:

Lemma 0.7. *Let R be a prime semiprimitive ring. Prim R is Baire if and only if $\cup_{i\in I}W_i$ is proper for every $|I| < |k|$ and $W_i, i \in I$ are proper closed set in Prim R.*

thus we may sate:

Lemma 0.8. *Let R be a prime semiprimitive ring. Prim R is Baire if and only if for every set of ideal $J_{i i\in I}$ with $|I| < |k|$, there exist an (left) primitive ideal don't contain any J_i.*

If R is (left) noetherian, R has only finitely many prime ideal minimal over J_i. So we have

Lemma 0.9. *Let R be a noetherian prime semiprimitive ring. Prim R is Baire if and only if for every set of prime ideal $P_{i i\in I}$ with $|I| < |k|$, there exist an (left) primitive ideal don't contain any P_i.*

Assume that k is a field, R is a Noetherian k-algebra, with $\dim_k(R) < |k|$, so R is a Jacobson ring, satisfies Nullstellensatz, thus if P is a primitive, then it is rational, if P is locally closed, then P is primitive.

In this case, J.P.Bell have 0 is Rational if and only if there is a set X of cardinality less than $|k|$ and a set of nonzero prime ideals $\{P_x : x \in X\}$ such that every nonzero prime ideal P of R contains P_x for some $x \in X$.

So If R is prime ring satisfies above assumption. Then 0 is primitive if and only if 0 is rational and Prim R is Baire.

If P is a Prime ideal in R, Then P is primitive if and only if P is rational and Prim R/P is Baire.

Remark 0.10. Prim R is Baire in above case has easy version.

References

[1] The Dixmier-Moeglin equivalence, Morita equivalence, and homeomorphism of spectra. (English summary) J. Algebra 534 (2019), 228–244.

[2] Baire category and Laurent extensions. Canadian J. Math. 31 (1979), no. 4, 824–830.