Thus, Zorn's Lemma applies, giving us a prime ideal $P^* \in \mathcal{X}$ that is minimal among the ideals in $\mathcal{X}$. Since any prime ideal contained in P^* is in $\mathcal{X}$, we conclude that P^* is a minimal prime ideal of R. $\square$

Given an ideal I in a ring R and a prime ideal P containing I, we may apply Proposition 3.3 in the ring R/I to see that the prime ideal P/I contains a minimal prime Q/I of R/I. Then Q is a prime ideal of R which contains I and is minimal among the primes containing I. By way of abbreviation, we say that Q is a *prime minimal over* I.

Theorem 3.4. *In a right or left noetherian ring R, there exist only finitely many minimal prime ideals, and there is a finite product of minimal prime ideals (repetitions allowed) that equals zero.*

Proof. Note that the following proof does not require the full force of the right or left noetherian hypothesis, but only the ACC on two-sided ideals.

It suffices to prove that there exist prime ideals $P_1, \ldots, P_n$ in R such that $P_1P_2 \cdots P_n = 0$. To see this, note that after replacing each P_i by a minimal prime ideal contained in it, we may assume that each P_i is minimal. Since any minimal prime P contains $P_1P_2 \cdots P_n$, it must contain some P_j, whence $P = P_j$ by minimality. Thus the minimal prime ideals of R are contained in the finite set $\{P_1, \ldots, P_n\}$.

Suppose that no finite product of prime ideals in R is zero. Let $\mathcal{X}$ be the set of those ideals K in R that do not contain a finite product of prime ideals. Since $\mathcal{X}$ contains 0, it is nonempty. By the noetherian hypothesis (not Zorn's Lemma!), there exists a maximal element $K \in \mathcal{X}$.

As R/K is a counterexample to the theorem, we may replace R by R/K. Thus we may assume, without loss of generality, that no finite product of prime ideals in R is zero, while all nonzero ideals of R contain finite products of prime ideals.

In particular, 0 cannot be a prime ideal. Hence, there exist nonzero ideals I, J in R such that $IJ = 0$. Then there exist prime ideals $P_1, \ldots, P_m, Q_1, \ldots, Q_n$ in R with $P_1P_2 \cdots P_m \subseteq I$ and $Q_1Q_2 \cdots Q_n \subseteq J$. But then

$$P_1P_2 \cdots P_m Q_1 Q_2 \cdots Q_n = 0,$$

contradicting our supposition.

Therefore some finite product of prime ideals in R is zero. $\square$

The use of the noetherian condition in the proof of Theorem 3.4 to pass from R to R/K is known as *noetherian induction*. Since R/K is as small as possible among factor rings of R violating the theorem, it is known as a *minimal criminal*. (For this terminology we are indebted to Reinhold Baer, who remarked that, as in the larger world, it is the minimal criminal who is apprehended.)

In general, a ring may have infinitely many minimal prime ideals, as the following example shows.

A such that the cosets $a_i + AJ(R)$ generate $A/AJ(R)$, then $a_1, \dots, a_n$ generate A. (To see this, consider the module $A/(a_1R + \cdots + a_nR)$.)

• PRIME IDEALS IN DIFFERENTIAL OPERATOR RINGS •

In this section, we will in some sense write down all of the prime ideals of a special class of noetherian rings – the differential operator rings $R[x;\delta]$, where R is assumed to be a commutative noetherian $\mathbb{Q}$-algebra. This will illustrate some of the phenomena that occur in more general settings and will also give us more examples of primitive noetherian rings.

Lemma 3.18. *Let R be a ring, δ a derivation on R, and $S = R[x;\delta]$.*

(a) *If I is a right ideal of R, then IS is a right ideal of S and $IS \cap R = I$.*

(b) *If I is a δ-ideal of R, then IS is an ideal of S and $IS = SI$.*

(c) *If J is an ideal of S, then $J \cap R$ is a δ-ideal of R.*

Proof. Most of this is an easy computation. In (a), $IS \cap R = I$ because R is a direct summand of S as a left R-module. For (b), note that, if I is a δ-ideal and $a \in I$, then, because $xa = ax + \delta(a)$ and $\delta(a) \in I$, we have $xa \in IS$. For (c), note that if $a \in J \cap R$, then $\delta(a) = xa - ax \in J \cap R$. $\square$

Lemma 3.19. *Let R be a commutative integral domain of characteristic zero with a nonzero derivation δ, and let $S = R[x;\delta]$. If I is a nonzero ideal of S, then $I \cap R \neq 0$.*

Proof. Pick a nonzero element $s = s_n x^n + \cdots$ from I with degree n and leading coefficient s_n, and assume that $n \geq 1$. Choose $r \in R$ such that $\delta(r) \neq 0$, and look at the element $sr - rs$. An immediate calculation shows that

$$sr - rs = ns_n\delta(r)x^{n-1} + [\text{terms of degree less than } n-1].$$

Since under our hypotheses $ns_n\delta(r) \neq 0$, we see that I contains a nonzero element of degree $n-1$. Hence, iterating this argument, we conclude that I contains a nonzero element of degree 0. $\square$

Lemma 3.20. *If R is a ring, δ a derivation on R, and P a minimal prime ideal of R such that R/P has characteristic zero, then P is a δ-ideal.*

Proof. Let $Q = \{r \in R \mid \delta^n(r) \in P \text{ for all } n \geq 0\}$. Using Leibniz's Rule (Exercise 2K), it is clear that Q is an ideal of R and is contained in P. We show that Q is prime as follows. Consider any $a, b \in R \setminus Q$. Choose nonnegative integers r and s as small as possible so that $\delta^r(a)$ and $\delta^s(b)$ are not in P, and then choose $c \in R$ such that $\delta^r(a)c\delta^s(b) \notin P$. Now use Leibniz's Rule to expand $\delta^{r+s}(acb)$, as follows:

$$\begin{aligned}\delta^{r+s}(acb) &= \sum_{i=0}^{r+s} \binom{r+s}{i} \delta^{r+s-i}(a)\delta^i(cb) \\ &= \sum_{i=0}^{r+s}\sum_{j=0}^{i} \binom{r+s}{i}\binom{i}{j} \delta^{r+s-i}(a)\delta^{i-j}(c)\delta^j(b).\end{aligned}$$

Since $\delta^{r+s-i}(a) \in P$ whenever $i > s$ and $\delta^j(b) \in P$ whenever $j < s$, all of the terms in the last summation are in P except for $\binom{r+s}{s}\binom{s}{s}\delta^r(a)c\delta^s(b)$, which is not in P because $\delta^r(a)c\delta^s(b)$ is not and R/P has characteristic zero. Thus, $\delta^{r+s}(acb) \notin P$, and so $acb \notin Q$, which shows that Q is prime. Since P is a minimal prime, we must have $P = Q$, and then, since Q is clearly a δ-ideal, the result follows. $\square$

In the next two proofs, we shall make use of Exercise 2ZA. For the case of a differential operator ring, it may be phrased as follows. Let R be a ring, δ a derivation on R, and $S = R[x; \delta]$. If I is a δ-ideal of R and $\hat{\delta}$ the derivation on R/I induced by δ, then $S/IS \cong (R/I)[\hat{x}; \hat{\delta}]$.

Lemma 3.21. *Let R be a noetherian $\mathbb{Q}$-algebra with a derivation δ. Let $S = R[x; \delta]$, and let P be a prime ideal of S. Then $P \cap R$ is a prime ideal of R.*

Proof. Since $P \cap R$ is a δ-ideal of R (Lemma 3.18), we can use Exercise 2ZA to reduce to a differential operator ring over $R/(P \cap R)$. Hence, we may assume that $P \cap R = 0$. If Q is any minimal prime of R, then R/Q has characteristic zero (since $R \supseteq \mathbb{Q}$), and so, by Lemma 3.20, Q is a δ-ideal. According to Theorem 3.4, there are minimal primes $Q_1, \dots, Q_m$ in R such that $Q_1 Q_2 \cdots Q_m = 0$. From Lemma 3.18, we infer that each $Q_i S$ is an ideal of S, and that

$$(Q_1 S)(Q_2 S) \cdots (Q_m S) = Q_1 Q_2 \cdots Q_m S = 0.$$

Since P is prime, we have $Q_i S \subseteq P$ for some index i. Hence, $Q_i \subseteq P \cap R = 0$, and so $P \cap R = Q_i$ is a prime ideal, as claimed. $\square$

Theorem 3.22. *Let R be a commutative noetherian $\mathbb{Q}$-algebra and $S = R[x; \delta]$ a differential operator ring.*

(a) *If P is any prime ideal of S, then $P \cap R$ is a prime δ-ideal of R.*

(b) *If Q is a prime δ-ideal of R, then QS is a prime ideal of S such that $QS \cap R = Q$. Furthermore, if P is any prime ideal of S such that $P \cap R = Q$, then either $P = QS$ or $\delta(R) \subseteq Q$, and in the latter case S/QS and S/P are commutative rings.*

(c) *All prime factor rings of S are domains.*

Proof. (a) This is contained in Lemmas 3.18 and 3.21.

(b) By Lemma 3.18, QS is an ideal of S such that $QS \cap R = Q$. From Exercise 2ZA we have that $S/QS \cong (R/Q)[\hat{x}; \hat{\delta}]$, where $\hat{\delta}$ is the derivation on R/Q induced by δ. Since R/Q is a domain, S/QS is a domain (Exercise 2O), and hence QS is a prime ideal of S.

If P is a prime ideal of S such that $P \cap R = Q$ but $P \neq QS$, then the image of P/QS in $(R/Q)[\hat{x}; \hat{\delta}]$ is a nonzero ideal I such that $I \cap (R/Q) = 0$. It follows from Lemma 3.19 that $\hat{\delta} = 0$, whence $\delta(R) \subseteq Q$. Moreover, $(R/Q)[\hat{x}; \hat{\delta}]$ is then an ordinary polynomial ring over the commutative ring R/Q. Thus in this case S/QS is commutative, as is S/P (since $P \supseteq QS$).

(c) In the notation of part (b), if $P = QS$, we have already seen that S/P is a domain. Otherwise, S/P is a commutative prime ring, and again it is a domain. $\square$

One way to summarize Theorem 3.22 is to say that the prime ideals of S are parametrized by the prime δ-ideals of R. If Q is a prime δ-ideal of R and $\delta(R) \nsubseteq Q$, there is a unique prime ideal of S that contracts to Q (that is, whose intersection with R equals Q), namely, QS. If Q is a prime δ-ideal of R and $\delta(R) \subseteq Q$, then S/QS is a commutative ring isomorphic to an ordinary polynomial ring $(R/Q)[\hat{x}]$. In this case, the primes of S that contract to Q correspond to the primes of $(R/Q)[\hat{x}]$ that contract to zero in R/Q; these in turn correspond precisely to the primes in $K[\hat{x}]$, where K is the quotient field of R/Q.

Exercise 3ZE below shows that Lemmas 3.20 and 3.21 and Theorem 3.22 are all false in characteristic p.

Exercise 3W. Let R be a polynomial ring $k[x]$ where k is an algebraically closed field of characteristic zero, and let $S = R[y; \delta]$ where $\delta = x\frac{d}{dx}$. Show that the only δ-ideals of R are 0 and the ideals $x^n R$ (for $n = 0, 1, \dots$). Show that the only prime ideals of S are 0 and xS together with $xS + (y - \alpha)S$ for all $\alpha \in k$. Then show (without the computations used in Exercises 3O,P) that S is right primitive. [Hint: If $\alpha \in k$ is nonzero, then $xS + (x - \alpha)S = S$. Hence, no proper right ideal containing $(x - \alpha)S$ can contain a nonzero prime ideal.] $\square$

Exercise 3X. Let R be a polynomial ring $k[x]$ where k is a field of characteristic zero, and let δ be any nonzero k-linear derivation on R. Show that there is a nonzero polynomial $g \in R$ such that $\delta = g\frac{d}{dx}$. If $S = R[y; \delta]$, show that S is right and left primitive. $\square$

We end the section with an example showing that Theorem 3.22 does not carry over to general skew polynomial rings $R[x; \alpha, \delta]$, even in characteristic zero.

Exercise 3Y. Let k be a field of characteristic 0, and let $U(\mathfrak{sl}_2(k)) = R[f; \alpha, \delta]$ as in Exercise 2S, where R is the k-subalgebra generated by e and h. Since $\mathfrak{sl}_2(k)$ is, by definition, a Lie subalgebra of $M_2(k)$, the vector space $V = k^2$ becomes a left $U(\mathfrak{sl}_2(k))$-module, such that the module multiplication of any element of $\mathfrak{sl}_2(k)$ with any column vector from V is given by matrix multiplication.

Show that V is a simple $U(\mathfrak{sl}_2(k))$-module, and conclude that its annihilator, call it P, is a left primitive ideal of $U(\mathfrak{sl}_2(k))$. (In fact, $U(\mathfrak{sl}_2(k))/P \cong M_2(k)$, and so P is a maximal ideal.) Now show that $P \cap R = \langle e^2, h^2 - 1 \rangle$, and conclude that $P \cap R$ is neither an α-ideal nor a δ-ideal of R. Finally, show that $P \cap R$ is not a prime (or even semiprime) ideal of R. $\square$