Journal of Algebra 400 (2014) 56–71

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Poisson spectra in polynomial algebras

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ARTICLE INFO

Article history:
Received 9 July 2013
Available online 20 December 2013
Communicated by J.T. Stafford

MSC:
17B63
16S36

Keywords:
Poisson algebra
Poisson prime ideal
Polynomial algebra

ABSTRACT

A significant class of Poisson brackets on the polynomial algebra $\mathbb{C}[x_1,\dots,x_n]$ is studied and, for this class of Poisson brackets, the Poisson prime ideals, Poisson primitive ideals and symplectic cores are determined. Moreover it is established that these Poisson algebras satisfy the Poisson Dixmier–Moeglin equivalence.

1. Introduction

To understand a Poisson bracket on the polynomial algebra $\mathbb{C}[x_1,\dots,x_n]$, one should identify the Poisson prime ideals, which correspond to the varieties in $\mathbb{C}^n$ that inherit the Poisson structure, and the *symplectic cores* [2] which are the algebraic analogues of symplectic leaves. For each point $p\in\mathbb{C}^n$, with corresponding maximal ideal M_p, there is a unique largest Poisson ideal $\mathcal{P}(M_p)$ contained in M_p. The ideal $\mathcal{P}(M_p)$, which is necessarily Poisson prime, is said to be *Poisson primitive* and is called the *Poisson core* of M_p. Two points p and q are in the same symplectic core when $\mathcal{P}(M_p)=\mathcal{P}(M_q)$. See [5, Sections 6, 7] for a full discussion of symplectic cores and their relationship with symplectic leaves.

In [10], the authors analyzed Poisson brackets on the polynomial algebra $\mathbb{C}[x,y,z]$ in three indeterminates x,y,z, including a class of Poisson brackets determined by Jacobians. In particular, for an

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1 The second author is supported by National Research Foundation of Korea, Grant 2009-0071102, and thanks the Korea Institute for Advanced Study for the warm hospitality during part of the period of preparation of this paper.

0021-8693/$ – see front matter
http://dx.doi.org/10.1016/j.jalgebra.2013.11.004

arbitrary rational function $s/t \in \mathbb{C}(x, y, z)$, they analyzed the prime and primitive Poisson ideals for the Poisson bracket such that, for $f, g \in \mathbb{C}[x, y, z]$,

$$\{f, g\} = t^2 \operatorname{Jac}(f, g, s/t), \tag{1.1}$$

where Jac denotes the Jacobian determinant.

The main purpose of this paper is to generalize the results in [10] to the general polynomial algebra $A := \mathbb{C}[x_1, x_2, \dots, x_n]$, $n \geqslant 3$, equipped with a Poisson bracket which is determined by $n-2$ rational functions and which generalizes (1.1). As in [10], the results will be illustrated using particular examples.

Fix $s_1, t_1, \dots, s_{n-2}, t_{n-2} \in A$ such that s_i and $t_i \neq 0$ are coprime for each $i = 1, 2, \dots, n-2$. In Section 2 it is shown that there is a Poisson bracket on the quotient field B of A such that, for all $f, g \in B$,

$$\{f, g\} = (t_1 \dots t_{n-2})^2 \operatorname{Jac}(f, g, s_1/t_1, s_2/t_2, \dots, s_{n-2}/t_{n-2}).$$

The purpose of the factor $(t_1 \dots t_{n-2})^2$ is to ensure that this restricts to a Poisson bracket on A.

The Poisson prime ideals of A for the above bracket are determined in Section 3, where Definition 3.2 uses the terminology *residually null*, respectively *proper*, for Poisson prime ideals P where the induced Poisson bracket on A/P is zero, respectively non-zero. The residually null Poisson prime ideals of A form a Zariski closed set of the prime spectrum of A and can often be found explicitly using elementary commutative algebra. We shall determine the proper Poisson prime ideals of A in terms of a finite set of localizations A_γ of A, each of which has a subalgebra C_γ that is a polynomial ring in $n-2$ variables and is contained in the Poisson centre of A_γ. As the Poisson bracket on C_γ is trivial, any prime ideal Q of C_γ is Poisson. Although QA_γ need not be prime, it is a Poisson ideal and the finitely many minimal prime ideals of A_γ over QA_γ are Poisson prime ideals of A_γ. Taking the intersection of each of these with A, we obtain finitely many Poisson prime ideals of A. The main result is that every proper Poisson prime ideal P of A_γ occurs in this way with $Q = PA_\gamma \cap C_\gamma$. The passage between Poisson prime ideals of A_γ and those of A can then be handled by standard localization techniques. This will be illustrated using examples with $n = 4$ in which case the algebras C_γ are polynomial algebras in two indeterminates. The main example is the Poisson bracket associated with 2×2 quantum matrices with which the reader may be familiar. We also consider actions on A, as Poisson automorphisms, of subgroups of the multiplicative group $(\mathbb{C}^*)^n$.

In Section 4, we determine the Poisson primitive ideals and symplectic cores of A and show that A satisfies the Poisson Dixmier–Moeglin equivalence discussed in [12, 2.4] and [6]. Here, as indeed is the case with the Poisson prime ideals, the varieties determined by $n-2$ polynomials of the form $\lambda_i s_i - \mu_i t_i$, $i = 1, 2, \dots, n-2$, where $(\lambda_i, \mu_i) \in \mathbb{C}^2 \backslash \{(0, 0)\}$ for all i, play an important role.

2. Poisson brackets

Definition 2.1. A *Poisson algebra* is a $\mathbb{C}$-algebra A with a Poisson bracket, that is a bilinear product $\{-,-\} : A \times A \to A$ such that A is a Lie algebra under $\{-,-\}$ and, for all $a \in A$, the *hamiltonian* $\operatorname{ham}(a) := \{a, -\}$ is a $\mathbb{C}$-derivation of A.

Notation 2.2. Let A denote the polynomial algebra $\mathbb{C}[x_1, \dots, x_n]$ in n indeterminates and let B denote the quotient field $\mathbb{C}(x_1, \dots, x_n)$ of A. For $1 \leqslant i \leqslant n$, let ∂_i be the derivation $\frac{\partial}{\partial x_i}$ of B. For $b_1, \dots, b_n \in B$, let $\operatorname{Jac}_M(b_1, \dots, b_n)$ denote the Jacobian matrix $(\partial_j(b_i))$ and let $\operatorname{Jac}(b_1, \dots, b_n)$ denote the Jacobian determinant $|\operatorname{Jac}_M(b_1, \dots, b_n)|$. Thus the ith row of $\operatorname{Jac}_M(b_1, \dots, b_n)$ is $\nabla(b_i)$ where $\nabla = (\partial_1, \partial_2, \dots, \partial_n)$ is the gradient.

Let $a, f_1, f_2, f_3, \dots, f_{n-2} \in B$ and, for $f, g \in B$, let

$$\{f, g\} = a \operatorname{Jac}(f, g, f_1, f_2, \dots, f_{n-2}). \tag{2.1}$$

Poisson brackets of this form, with $a = 1$, appear in the literature of mathematical physics, for example see [9,17]. Our aim in this section is to give an algebraic proof that (2.1) defines a Poisson bracket on the rational function field B.

For an $(n-2) \times n$ matrix M over B and $1 \leqslant i < j \leqslant n$, let M_{ij} be the $(n-2) \times (n-2)$ minor obtained by deleting columns i and j of M and taking the determinant. Let D be the $(n-2) \times n$ matrix with ith row $\nabla(f_i)$. Then

$$\{x_i, x_j\} = (-1)^{i+j-1} a D_{ij}.$$

Also, if $u_1, u_2, \dots, u_{n-2} \in B$ are such that $a = u_1 u_2 \dots u_{n-2}$ then

$$\{x_i, x_j\} = (-1)^{i+j-1} E_{ij},$$

where E is the $(n-2) \times n$ matrix with ith row $u_i \nabla(f_i)$.

Lemma 2.3. *Let $1 \leqslant i \leqslant n$ and let $a, \phi_1, \phi_2, \dots, \phi_{i-1}, \phi_{i+1}, \dots \phi_n \in B$. The map $\delta : B \to B$ given by*

$$\delta(b) = a \operatorname{Jac}(\phi_1, \phi_2, \dots, \phi_{i-1}, b, \phi_{i+1}, \dots, \phi_n)$$

is a derivation of B.

Proof. Denoting the ij-minor of the Jacobian matrix $\operatorname{Jac}_M(\phi_1, \phi_2, \dots, \phi_n)$ by m_{ij},

$$\delta = a\big((-1)^{i+1} m_{i1} \partial_1 + (-1)^{i+2} m_{i2} \partial_2 + \dots + (-1)^{i+n} m_{in} \partial_n\big).$$

Hence δ is a derivation of B. $\square$

Theorem 2.4. *Let $a, f_1, f_2, f_3, \dots, f_{n-2} \in B$ and let $\{-,-\}$ be as in (2.1). Then $\{-,-\}$ is a Poisson bracket on B.*

Proof. Applying Lemma 2.3 with $(\phi_1, \phi_2, \dots, \phi_n) = (f, g, f_1, f_2, \dots, f_{n-2})$ and $i = 1, 2$, we see that, if $\{-,-\}$ is defined as in (2.1), then $\{f, -\}$ and $\{-, g\}$ are derivations. Also $\{-,-\}$ is clearly antisymmetric so it remains to show that it satisfies the Jacobi identity.

We begin with the case $a = 1$ where we can exploit the n-Jacobi identity for the Jacobian [16]. Given an ordered set $F = \{f_0, f_1, f_2, f_3, \dots, f_{n-2}\}$ of $n-1$ elements of B, define $\partial_F : B \to B$

$$\partial_F(h) = \operatorname{Jac}(h, f_0, f_1, f_2, \dots, f_{n-2}).$$

There is a minor difference here to [16] where h appears in the rightmost argument. Note that $\partial_F(f_i) = 0$ for $0 \leqslant i \leqslant n-2$. When $a = 1$, $\{g, f\} = \partial_F(g)$, where $F = \{f, f_1, f_2, f_3, \dots, f_{n-2}\}$. The n-Jacobi identity for the Jacobian says that, for $h_1, h_2, \dots, h_n \in B$,

$$\begin{aligned}\partial_F\big(\operatorname{Jac}(h_1, h_2, \dots, h_{n-1}, h_n)\big) &= \operatorname{Jac}\big(\partial_F(h_1), h_2, \dots, h_{n-1}, h_n\big) \\ &\quad + \operatorname{Jac}\big(h_1, \partial_F(h_2), \dots, h_{n-1}, h_n\big) + \dots + \operatorname{Jac}\big(h_1, h_2, \dots, \partial_F(h_n)\big).\end{aligned}$$

The proof of this in [16] is presented for the algebra of C^∞-functions on a real manifold but it is valid for the rational function field B. It is first checked when each $h_i = x_i$ and then, using derivation properties as in the proof of [10, Proposition 1.14], extended first to the polynomial algebra and then to the rational function field.

Let $f, g, h \in B$. Then

$$\begin{aligned}\big\{\{g,h\}, f\big\} &= \partial_F\big(\{g,h\}\big) = \partial_F\big(\mathrm{Jac}(g,h,f_1,f_2,\dots,f_{n-2})\big)\\ &= \mathrm{Jac}\big(\partial_F(g),h,f_1,f_2,\dots,f_{n-2}\big) + \mathrm{Jac}\big(g,\partial_F(h),f_1,f_2,\dots,f_{n-2}\big)\\ &\quad \text{(by the n-Jacobi identity, the other summands being 0)}\\ &= \big\{\{g,f\},h\big\} + \big\{g,\{h,f\}\big\} = -\big\{\{f,g\},h\big\} - \big\{\{h,f\},g\big\}.\end{aligned}$$

Thus $\{-,-\}$ satisfies the Jacobi identity and is a Poisson bracket on B.

Now let $a \in B$. We need to show that the bracket $a\{-,-\}$ satisfies the Jacobi identity. As

$$a\big\{f, a\{g,h\}\big\} = a^2\big\{f,\{g,h\}\big\} + a\{g,h\}\{f,a\}$$

and $\{-,-\}$ satisfies the Jacobi identity, it suffices to show that

$$\{g,h\}\{f,a\} + \{f,g\}\{h,a\} + \{h,f\}\{g,a\} = 0$$

for all $a, f, g, h \in B$. As $\{g,h\}\{f,-\} + \{f,g\}\{h,-\} + \{h,f\}\{g,-\}$ and the similar maps, where three of a, f, g, h are fixed, are derivations, it suffices to show that

$$\{x_i,x_j\}\{x_k,x_\ell\} + \{x_k,x_i\}\{x_j,x_\ell\} + \{x_j,x_k\}\{x_i,x_\ell\} = 0 \tag{2.2}$$

for $1 \leqslant i \leqslant j \leqslant k \leqslant \ell \leqslant n$. Clearly (2.2) holds when any two of i, j, k, ℓ are equal so we may assume that $i < j < k < \ell$. In this case (2.2) is, using Notation 2.2,

$$D_{ij}D_{k\ell} - D_{ik}D_{j\ell} + D_{jk}D_{i\ell} = 0.$$

This is a Plücker relation for the $(n-2)\times n$ matrix D, see [4, Theorem 1.3], or [8, Chapter VII §6], where Plücker relations are called p-relations. Indeed it is one of the three-term Plücker relations stated explicitly in [8, footnote on p. 311]. In the notation of [8], where subscripts indicate included rather than excluded rows, it is

$$p_{i_1\dots i_{n-4}k\ell}\,p_{i_1\dots i_{n-4}ij} + p_{i_1\dots i_{n-4}\ell j}\,p_{i_1\dots i_{n-4}ki} + p_{i_1\dots i_{n-4}\ell i}\,p_{i_1\dots i_{n-4}kj} = 0,$$

where $\{i_1,\dots,i_{n-4}\} = \{1,2,3,\dots,n\}\backslash\{i,j,k,\ell\}$. □

Theorem 2.5. *If $f_1, f_2, \dots, f_{n-2}$ are algebraically dependent over $\mathbb{C}$ then the Poisson bracket $\{-,-\} = 0$.*

Proof. Let $0 \neq G = G(y_1,\dots,y_{n-2}) \in \mathbb{C}[y_1,\dots,y_{n-2}]$ be of minimal total degree such that $G(f_1, f_2, \dots, f_{n-2}) = 0$. Without loss of generality, we may assume that the degree in y_1 of G is at least one. Let

$$G = \sum_{\mathbf{r}=(r_1,\dots,r_{n-2})} \alpha_{\mathbf{r}} y_1^{r_1} y_2^{r_2} \dots y_{n-2}^{r_{n-2}}.$$

Let $f, g \in B$ and let δ be the derivation in Lemma 2.3, in the case where $i = 3$, $\phi_1 = f$, $\phi_2 = g$ and, for $4 \leqslant j \leqslant n$, $\phi_j = f_{j-2}$. Then $\delta(f_k) = 0$ for $2 \leqslant k \leqslant n-2$, whereas $\delta(f_1) = \{f,g\}$. Then

$$\begin{aligned}0 &= \delta\big(G(f_1, f_2, \dots, f_{n-2})\big)\\ &= \delta\Bigg(\sum_{\mathbf{r}=(r_1,\dots,r_{n-2})} \alpha_{\mathbf{r}} f_1^{r_1} f_2^{r_2} \dots f_{n-2}^{r_{n-2}}\Bigg)\\ &= \Bigg(\sum_{\mathbf{r}} r_1\alpha_{\mathbf{r}} f_1^{r_1-1} f_2^{r_2} \dots f_{n-2}^{r_{n-2}}\Bigg)\{f, g\}.\end{aligned}$$

By the minimality of G, $\sum_{\mathbf{r}} r_1\alpha_{\mathbf{r}} f_1^{r_1-1} f_2^{r_2} \dots f_{n-2}^{r_{n-2}} \neq 0$ so $\{f, g\} = 0$. $\square$

3. Poisson spectra

The following definitions and the claims made for them are well-known. Appropriate references include [5,6,10,14].

Definitions 3.1. Let A be a Poisson algebra with bracket $\{-,-\}$. The *Poisson centre* of A, denoted $\mathrm{PZ}(A)$, of A is $\{a \in A\colon\ \{a, r\} = 0$ for all $r \in A\}$.

An ideal I of A is a *Poisson ideal* if $\{i, r\} \in I$ for all $i \in I$ and $r \in A$. If I is a Poisson ideal of A then A/I is a Poisson algebra with $\{a + I, b + I\} = \{a, b\} + I$ for all $a, b \in A$. A Poisson ideal P of A is *Poisson prime* if, for all Poisson ideals I and J of A, $IJ \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$. If A is Noetherian then this is equivalent to saying that P is both a prime ideal and a Poisson ideal. The *Poisson spectrum* of A, written $\mathrm{P.spec}\, A$, is the set of all Poisson prime ideals of A. A maximal ideal M of A is said to be a *Poisson maximal ideal* if it is also a Poisson ideal. This is not equivalent to saying that M is maximal as a Poisson ideal.

The *Poisson core* of an ideal I of A, denoted $\mathcal{P}(I)$, is the largest Poisson ideal of A contained in I. If I is a prime ideal of A then $\mathcal{P}(I)$ is Poisson prime. Consequently if A is Noetherian and P is a minimal prime ideal over a Poisson ideal I then P is Poisson. A Poisson ideal P of A is *Poisson primitive* if $P = \mathcal{P}(M)$ for some maximal ideal M of A. Every Poisson primitive ideal is Poisson prime. We can view $\mathcal{P}$ as a surjective map from the maximal spectrum $\mathrm{maxspec}(A)$ to the Poisson primitive spectrum $\mathrm{P.prim}(A)$, the latter being the set of Poisson primitive ideals of A. With the topologies induced from the Zariski topology on $\mathrm{spec}(A)$, $\mathcal{P}$ is continuous. The fibres of $\mathcal{P}$ are the *symplectic cores* introduced in [2]. These partition $\mathrm{maxspec}(A)$ and are the algebraic counterparts of symplectic leaves. In the case where A is the polynomial algebra $\mathbb{C}[x_1, \dots, x_n]$ the natural identification between $\mathbb{C}^n$ and $\mathrm{maxspec}\, A$ allows us to regard symplectic cores as giving a partition of $\mathbb{C}^n$. Thus if $p \in \mathbb{C}^n$ has corresponding maximal ideal

$$M_p = \big\{f \in A\colon\ f(p) = 0\big\}$$

then the symplectic core of p is

$$\big\{q \in \mathbb{C}^n\colon\ \mathcal{P}(M_q) = \mathcal{P}(M_p)\big\}.$$

If $\mathcal{S}$ is a multiplicatively closed subset of a Poisson algebra A then the localization $A_{\mathcal{S}}$ is also a Poisson algebra, with $\{as^{-1}, bt^{-1}\}$ computed using the quotient rule for derivations. If P is a Poisson prime ideal of A then the quotient field $Q(A/P)$ is a Poisson algebra and P is said to be *rational* if $\mathrm{PZ}(Q(A/P)) = \mathbb{C}$. For a Poisson prime ideal P of an affine Poisson algebra A, there is, by [12, 1.7(i) and 1.10], a sequence of implications:

$$P \text{ is locally closed} \quad\Rightarrow\quad P \text{ is Poisson primitive} \quad\Rightarrow\quad P \text{ is rational}.$$

To establish the *Poisson Dixmier–Moeglin equivalence*, it is enough to show that if P is a rational Poisson prime ideal of A then P is locally closed. For further discussion of this, see [5,6].

A $\mathbb{C}$-algebra automorphism θ of a Poisson algebra A is a *Poisson automorphism* of A if $\theta(\{a,b\}) = \{\theta(a), \theta(b)\}$ for all $a, b \in A$ and is a *Poisson anti-automorphism* of A if $\theta(\{a,b\}) = \{\theta(b), \theta(a)\}$ for all $a, b \in A$. Under composition, the set of all Poisson automorphisms and Poisson anti-automorphisms of A is a group in which the Poisson automorphisms form a normal subgroup of index 2.

The height of a prime ideal P of A will be denoted $\operatorname{ht} P$.

Definition 3.2. Let A be a Poisson algebra and I be a Poisson ideal of A. Following [10, Definition 1.8], we say that I is *residually null* if the induced Poisson bracket on A/I is zero. This is equivalent to saying that I contains all elements of the form $\{a,b\}$ where $a, b \in A$, or that I contains all such elements where $a, b \in G$ for some generating set G for A. We shall also say that a Poisson ideal is a *proper Poisson ideal* if it is not residually null.

Lemma 3.3. *Let A be a Poisson algebra.*

(i) *Every residually null Poisson primitive ideal P is a Poisson maximal ideal.*
(ii) *A Poisson algebra A is Poisson simple if and only if there does not exist a non-zero Poisson primitive ideal of A.*
(iii) *Let $A = \mathbb{C}[x_1, \dots, x_n]$ be a polynomial algebra with a Poisson bracket. Let P be a proper Poisson prime ideal in A of height $\geqslant n-2$. Then P is locally closed and Poisson primitive.*

Proof. (i) Let M be a maximal ideal of A such that $P = \mathcal{P}(M)$. Suppose that P is residually null. Then M is Poisson and $P = \mathcal{P}(M) = M$ is maximal.

(ii) The 'only if' part is clear. For the converse, suppose that A is not simple, let I be a proper ideal of A that is Poisson and let M be a maximal ideal of A containing I. Then $\mathcal{P}(M)$ is Poisson primitive and $0 \neq I \subseteq \mathcal{P}(M)$.

(iii) Since P is proper Poisson, $\{x_k, x_j\} \notin P$ for some pair k, j. Let Q be a Poisson prime ideal containing P properly. Then $\operatorname{ht} Q > n-2$ and hence Q is residually null by [10, Proposition 3.2]. It follows that $\{x_k, x_j\} \in Q$. Thus P is locally closed and hence, by [12, 1.7(i)], P is Poisson primitive. $\square$

Notation 3.4. For the remainder of the paper, let $A = \mathbb{C}[x_1, \dots, x_n]$ and $B = \mathbb{C}(x_1, \dots, x_n)$, where $n \geqslant 3$. Let $s_1, t_1, \dots, s_{n-2}, t_{n-2} \in A$ be such that s_i and $t_i \neq 0$ are coprime for $1 \leqslant i \leqslant n-2$. Let $f_i = s_i t_i^{-1} \in B$, $1 \leqslant i \leqslant n-2$, and let $a = t_1^2 t_2^2 \dots t_{n-2}^2$. By Theorem 2.4, there is a Poisson bracket on B such that $\{f,g\} = a\operatorname{Jac}(f, g, f_1, f_2, \dots, f_{n-2})$ for all $f, g \in B$. Thus $\{f,g\} = \det J$, where J is the $n \times n$ matrix with first row ∇f, second row ∇g and, for $3 \leqslant i \leqslant n$, ith row $t_{i-2}^2 \nabla(s_{i-2} t_{i-2}^{-1})$. In the notation of 2.2, with each $u_i = t_i^2$,

$$\{x_i, x_j\} = (-1)^{i+j-1} a D_{ij} = (-1)^{i+j-1} E_{ij}.$$

Note that $t_i^2 \partial_j(s_i t_i^{-1}) \in A$ for $1 \leqslant i \leqslant n-2$ and $1 \leqslant j \leqslant n$. It follows that $\{f,g\} \in A$ for all $f, g \in A$ and hence that A is a Poisson subalgebra of B.

If $f_1, \dots, f_{n-2}$ are algebraically dependent over $\mathbb{C}$ then the Poisson bracket $\{-,-\} = 0$, by Theorem 2.5, so, henceforth, we assume that $f_1, \dots, f_{n-2}$ are algebraically independent over $\mathbb{C}$.

Example 3.5. Let $n = 4$ and let

$$s_1 = x_1x_4 - x_2x_3, \qquad t_1 = 1, \qquad s_2 = x_2, \qquad t_2 = x_3.$$

Then, in the notation of 2.2,

$$E = \begin{pmatrix} t_1^2 \nabla(f_1) \\ t_2^2 \nabla(f_2) \end{pmatrix} = \begin{pmatrix} x_4 & -x_3 & -x_2 & x_1 \\ 0 & x_3 & -x_2 & 0 \end{pmatrix}$$

and the resulting Poisson bracket on $\mathbb{C}[x_1, x_2, x_3, x_4]$ is such that:

$$\begin{aligned} \{x_1, x_2\} &= x_1x_2, & \{x_1, x_3\} &= x_1x_3, & \{x_1, x_4\} &= 2x_2x_3, \\ \{x_2, x_3\} &= 0, & \{x_2, x_4\} &= x_2x_4, & \{x_3, x_4\} &= x_3x_4. \end{aligned}$$

This is the well-known Poisson bracket associated with 2×2 quantum matrices, see [12, 2.9]. This example will be used to illustrate our methods and results.

Example 3.6. Let $n = 4$ and let $s_1 = x_1 + x_2 + x_3 + x_4$, $s_2 = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4$ and $t_1 = t_2 = 1$. In the notation of 2.2,

$$E = \begin{pmatrix} 1 & 1 & 1 & 1 \\ x_2 + x_3 + x_4 & x_1 + x_3 + x_4 & x_1 + x_2 + x_4 & x_1 + x_2 + x_3 \end{pmatrix}$$

and, for the resulting Poisson bracket on $\mathbb{C}[x_1, x_2, x_3, x_4]$,

$$\begin{aligned} \{x_1, x_2\} &= x_3 - x_4, & \{x_1, x_3\} &= x_4 - x_2, & \{x_1, x_4\} &= x_2 - x_3, \\ \{x_2, x_3\} &= x_1 - x_4, & \{x_2, x_4\} &= x_3 - x_1, & \{x_3, x_4\} &= x_1 - x_2. \end{aligned}$$

Here the elementary symmetric polynomials s_1 and s_2 are Poisson central. The prime ideal generated by $x_1 - x_2$, $x_1 - x_3$ and $x_1 - x_4$ is residually null Poisson as are all the maximal ideals of the form $(x_1 - \lambda, x_2 - \lambda, x_3 - \lambda, x_4 - \lambda)$.

As $\{x_i, x_j\}$ is homogeneous of degree one, the Poisson bracket here is the Kirillov–Kostant–Souriau bracket [1, III.5.5] for a 4-dimensional Lie algebra $\mathfrak{g}$ in which $z := x_1 + x_2 + x_3 + x_4$ is central. If $\mathfrak{s} = \mathfrak{g}/\mathbb{C}z$ then it is a routine calculation to check that $[\mathfrak{s}, \mathfrak{s}] = \mathfrak{s}$ whence, by [3, 3.2.4], $\mathfrak{s} \simeq \mathfrak{sl}_2$.

Examples 3.7. The examples in 3.5 and 3.6 exhibit very different symmetry properties. In Example 3.6, there is alternating symmetry in the following sense. For each $\alpha \in S_4$, there are $\mathbb{C}$-automorphisms ϕ_α and θ_α of A such that, for $1 \leqslant i \leqslant 4$, $\phi_\alpha(x_i) = x_{\alpha(i)}$ and $\theta_\alpha(x_i) = \operatorname{sgn}\alpha x_{\alpha(i)}$. Then θ_α is a Poisson automorphism. It is enough to check this for the generators (1 2) and (1 2 3 4) of S_4, for which

$$\theta_{(1\ 2)} : x_1 \mapsto -x_2,\ x_2 \mapsto -x_1,\ x_3 \mapsto -x_3,\ x_4 \mapsto -x_4$$

and

$$\theta_{(1\ 2\ 3\ 4)} : x_1 \mapsto -x_2,\ x_2 \mapsto -x_3,\ x_3 \mapsto -x_4,\ x_4 \mapsto -x_1.$$

Note that $\theta_\alpha(s_2) = s_2$ for all $\alpha \in S_4$, whereas $\theta_\alpha(s_1) = \operatorname{sgn}\alpha s_1$. If α is even then $\phi_\alpha = \theta_\alpha$ and if α is odd then ϕ_α is a Poisson anti-automorphism.

In Example 3.5, there is a well-known action of the group

$$H := \left\{(h_1, h_2, h_3, h_4) \in \left(\mathbb{C}^*\right)^4 \colon h_1h_4 = h_2h_3\right\}$$

on $A = \mathbb{C}[x_1, x_2, x_3, x_4]$, acting as Poisson automorphisms with $x_i \mapsto h_ix_i$ for $1 \leqslant i \leqslant n$. If, in [15, 1.2], we take $A_{2,\Gamma}^{P,Q} = \mathbb{C}[y_1, x_1, y_2, x_2]$ with $q_1 = q_2 = 0$, $p_1 = p_2 = -2$ and $\gamma_{12} = -1$ then $A_{2,\Gamma}^{P,Q}$ is Poisson isomorphic to A via the map $y_1 \mapsto x_2$, $x_1 \mapsto x_3$, $y_2 \mapsto x_1$, $x_2 \mapsto x_4$. The above action of H on A then corresponds to the action of H on $A_{2,\Gamma}^{P,Q}$ specified in [15] and, by [15, 2.6], the H-prime Poisson ideals of A are as follows:

$$\begin{gathered} 0, \\ x_2A, \quad x_3A, \quad DA, \\ x_1A+x_2A, \quad x_2A+x_4A, \quad x_2A+x_3A, \quad x_1A+x_3A, \quad x_3A+x_4A, \\ x_1A+x_2A+x_3A, \quad x_1A+x_2A+x_4A, \\ x_1A+x_3A+x_4A, \quad x_2A+x_3A+x_4A, \\ x_1A+x_2A+x_3A+x_4A, \end{gathered}$$

where $D = x_1x_4 - x_2x_3$, the determinant.

This is a special case of a general situation. The multiplicative group $(\mathbb{C}^*)^n$ acts, as algebra automorphisms, on A by the rule

$$(h_1, \dots, h_n).f = f(h_1x_1, \dots, h_nx_n).$$

With E_{ij} as in Notation 2.2, let H be the subgroup

$$\{(h_1, \dots, h_n)\colon h.E_{ij} = h_ih_jE_{ij} \text{ for } 1 \leqslant i, j \leqslant n\}.$$

If $h = (h_1, \dots, h_n) \in H$ then, for $1 \leqslant i, j \leqslant n$,

$$\begin{aligned} \{h.x_i, h.x_j\} &= h_ih_j\{x_i, x_j\} \\ &= h_ih_j(-1)^{i+j-1}E_{ij} \\ &= (-1)^{i+j-1}h.E_{ij} \\ &= h.\{x_i, x_j\}. \end{aligned}$$

Thus H acts on A by Poisson automorphisms. In Example 3.5, $H \simeq (\mathbb{C}^*)^3$, the 3-torus. Note that H might be trivial.

Notation 3.8. Consider the group

$$H' := \{h \in (\mathbb{C}^*)^n\colon h.s_i \in \mathbb{C}s_i \text{ and } h.t_i \in \mathbb{C}t_i \text{ for all } i\}.$$

This group is readily calculated from the data and its elements sometimes, but not always, act on A as Poisson automorphisms. In Example 3.5, all elements of H' act as Poisson automorphisms. However, in Example 3.6, where $H' = \{(h_1, h_1, h_1, h_1)\}$, only $(1, 1, 1, 1)$ acts as a Poisson automorphism. For $1 \leqslant i \leqslant n-2$, let $\sigma_i : H' \to \mathbb{C}$ and $\tau_i : H' \to \mathbb{C}$ be such that $h.s_i = \sigma_i(h)s_i$ and $h.t_i = \tau_i(h)t_i$. Let $\rho : H' \to \mathbb{C}$ be such that, for $h \in H'$, $\rho(h) = \sigma_1(h)\tau_1(h)\dots\sigma_{n-2}(h)\tau_{n-2}(h)$. The next result gives a criterion for an element of H' to act as a Poisson automorphism of A.

Proposition 3.9. *Let $h = (h_1, h_2, \dots, h_n) \in H'$. Then h acts as a Poisson automorphism if and only if $\rho(h) = h_1 \dots h_n$.*

Proof. For all $g \in A$ and $1 \leqslant i \leqslant n-2$, $h.\partial_j(g) = h_j^{-1}\partial_j(h.g)$ so, for $1 \leqslant j \leqslant n$, $h.t_i^2\partial_j(s_i/t_i) = h_j^{-1}\sigma_i(h)\tau_i(h)t_i^2\partial_j(s_i/t_i)$. In other words, when h acts on the $(n-2) \times n$ matrix E whose rows are $t_i^2\nabla(s_i/t_i)$, the jth column gets multiplied by h_j^{-1} and the ith row by $\sigma_i(h)\tau_i(h)$. It follows that, for $1 \leqslant k, \ell \leqslant n$,

$$h.\{x_k, x_\ell\} = (h_1 \dots h_n)^{-1}h_kh_\ell\rho(h)\{x_k, x_\ell\}.$$

As $\{h.x_k, h.x_\ell\} = h_kh_\ell\{x_k, x_\ell\}$, the result follows. □

Examples 3.10. Proposition 3.9 is nicely illustrated in Examples 3.5 and 3.6. In Example 3.5, for $h = (h_1, h_2, h_3, h_1^{-1}h_2h_3) \in H'$, $\sigma_1(h) = h_1h_4 = h_2h_3$, $\sigma_2(h) = h_2$, $\tau_1(h) = 1$ and $\tau_2(h) = h_3$ so $\rho(h) = h_1h_2h_3h_4$ for all $h \in H'$. Here $H' = H$. In Example 3.6, for $h = (h_1, h_1, h_1, h_1) \in H'$, $\sigma_1(h) = h_1$, $\sigma_2(h) = h_1^2$, $\tau_1(h) = 1$ and $\tau_2(h) = 1$ so, unless $h_1 = 1$, $\rho(h) = h_1^3 \neq h_1h_2h_3h_4 = h_1^4$.

Lemma 3.11. *Let R be a commutative Noetherian $\mathbb{C}$-algebra that is a domain and let δ be a $\mathbb{C}$-derivation of R. Let K denote the subring of constants, that is $K = \{r \in R\colon \delta(r) = 0\}$. Then K is algebraically closed in R.*

Proof. The proof is essentially the same as that of [10, Lemma 3.1] but with the word 'algebraic' replacing the word 'integral' and with the insertion of a leading coefficient k_n that need not be 1. □

Lemma 3.12. *Let P be a proper Poisson prime ideal of A. Then $s_i \notin P$ or $t_i \notin P$ for each $i = 1, \dots, n-2$.*

Proof. If $s_i \in P$ and $t_i \in P$ for some i then P is residually null since $t_i^2\partial_j(\frac{s_i}{t_i}) = t_i\partial_j(s_i) - s_i\partial_j(t_i) \in P$ for $1 \leqslant j \leqslant n$. □

The proof of our main result, Theorem 3.19, involves the relationship between transcendence degree and heights of prime ideals.

Notation 3.13. Let K be a field, R be an integral domain which is also an affine K-algebra, $Q(R)$ be the field of quotients of R and L be a field extension of K. We shall denote the Krull dimension of an affine K-algebra R by $\dim(R)$ and the transcendence degree of L by $\mathrm{tr.deg}_K(L)$. Following [18, Chapter 6], we extend the latter notation to R by taking $\mathrm{tr.deg}_K(R)$ to be the number of elements in any maximal algebraically independent set of elements in R. By [19, Corollary 14.29] and [18, Theorem 6.35], $\mathrm{tr.deg}_K(Q(R)) = \dim(R) = \mathrm{tr.deg}_K(R)$. Note also that any algebraically independent set of elements in R can be extended to a maximally algebraically independent set in R, see [18, Example 6.4 and Remark 6.6]. We shall simply write $\mathrm{tr.deg}(R)$ for $\mathrm{tr.deg}_K(R)$ if no confusion arises. By [19, Corollary 14.32],

$$\mathrm{ht}(P) + \dim(R/P) = \dim(R).$$

Notation 3.14. Let Γ be the set of all sequences $((\gamma_1, \delta_1), \dots, (\gamma_{n-2}, \delta_{n-2}))$ of length $n-2$ in $\{0,1\} \times \{0,1\}$. Call an element γ of Γ *dense* if, for each $i = 1, \dots, n-2$, $(\gamma_i, \delta_i) \neq (0,0)$. To each $\gamma \in \Gamma$, we associate a finite subset S_γ of $\{s_1, \dots, s_{n-2}, t_1, \dots, t_{n-2}\}$, a finite subset $V_\gamma = \{v_1, \dots, v_{n-2}\}$ of B, a multiplicatively closed subset M_γ of A and a localization A_γ of A as follows:

$$s_i \in S_\gamma \quad \Leftrightarrow \quad \gamma_i = 1 \quad \text{and} \quad t_i \in S_\gamma \quad \Leftrightarrow \quad \delta_i = 1,$$
$$v_i = \begin{cases} s_i/t_i & \text{if } \delta_i = 1, \\ t_i/s_i & \text{otherwise,} \end{cases}$$

M_γ is the multiplicatively closed subset of A generated by the elements of S_γ, and A_γ is the localization $M_\gamma^{-1}A$. Note that if γ is dense then $v_i \in A_\gamma$ for each $i = 1, \dots, n-2$. In this case, denote by C_γ the subalgebra of A_γ generated by $v_1, \dots, v_{n-2}$. Since $s_1/t_1, \dots, s_{n-2}/t_{n-2}$ are algebraically independent over $\mathbb{C}$, the transcendence degree of C_γ is $n-2$.

For example, in Example 3.5, if $\gamma = \{(0,1),(1,0)\}$ then the sequences S_γ and V_γ are respectively $(t_1, s_2) = (1, x_2)$ and $(s_1/t_1, t_2/s_2) = (x_1x_4 - x_2x_3, x_3/x_2)$.

Notation 3.15. For $P \in \mathrm{P.spec}(A)$, let $\gamma(P) = ((\gamma_1, \delta_1), \dots, (\gamma_{n-2}, \delta_{n-2}))$ be the sequence such that, for each i, $\gamma_i = 0 \Leftrightarrow s_i \in P$ and $\delta_i = 0 \Leftrightarrow t_i \in P$.

For example, in Example 3.5, if $P = x_1A + x_3A$ then the sets $\gamma(P)$, $S_{\gamma(P)}$ and $V_{\gamma(P)}$ are, respectively $\{(0,1),(1,0)\}$, $\{x_2, 1\}$ and $\{x_1x_4 - x_2x_3, x_3/x_2\}$.

The next lemma amounts to observing some restrictions on $\gamma(P)$.

Lemma 3.16. *Let P be a Poisson prime ideal of A.*

(i) *If P is proper Poisson then $\gamma(P)$ is dense.*
(ii) *If $t_i = 1$ for some i then, in $\gamma(P)$, $\delta_i = 1$ and, in $V_{\gamma(P)}$, $v_i = s_i$.*

Proof. (i) holds because, by Lemma 3.12, we cannot have $s_i \in P$ and $t_i \in P$ for any i and (ii) holds because $t_i \notin P$. □

Remark 3.17. The converse to (i) is false as can be seen from Example 3.5 where, for the residually null Poisson prime ideal $P = x_1A + x_2A + x_4A$, $\gamma(P) = \{(0,1),(0,1)\}$ is dense.

Notation 3.18. The Poisson spectrum $\mathrm{P.spec}\,A$ can be partitioned using Γ. For $\gamma \in \Gamma$, let

$$\mathrm{P.spec}_\gamma A = \{P \in \mathrm{P.spec}\,A \mid S_\gamma = S_{\gamma(P)}\}.$$

The set $\mathrm{P.spec}_\gamma A$ may be empty. Indeed, by Lemma 3.16(ii), if $t_i = 1$ for some i then $\mathrm{P.spec}_\gamma A = \emptyset$ whenever $\delta_i = 0$.

Our strategy in attempting to understand $\mathrm{P.spec}\,A$ is based on the following:

(i) $\mathrm{P.spec}\,A$ is the disjoint union of the subsets $\mathrm{P.spec}_\gamma A$ taken over $\gamma \in \Gamma$.
(ii) By standard localization theory, if $P \in \mathrm{P.spec}_\gamma A$ then PA_γ is a Poisson prime ideal of A_γ and $P = A \cap PA_\gamma$.
(iii) When γ is dense, Theorem 3.19 below determines the Poisson prime ideals of A_γ in terms of prime ideals of the polynomial algebra C_γ.
(iv) When γ is not dense, every Poisson prime ideal in $\mathrm{P.spec}_\gamma A$ is residually null.

The next result determines the proper Poisson prime ideals in A_γ when γ is dense.

Theorem 3.19. *Let $\gamma \in \Gamma$ be dense.*

(i) *Let I be an ideal of C_γ and let Q be a prime ideal of A_γ that is minimal over IA_γ. Then Q is a Poisson prime ideal of A_γ.*
(ii) *If Q is a non-zero proper Poisson prime ideal of A_γ then $\mathrm{ht}(Q) = \mathrm{ht}(Q \cap C_\gamma)$.*
(iii) *If Q is a non-zero proper Poisson prime ideal of A_γ then Q is a minimal prime ideal over $(Q \cap C_\gamma)A_\gamma$.*

Proof. (i) Since $v_i = s_i/t_i$ or $v_i = t_i/s_i$ and $\nabla \frac{t_i}{s_i} = -s_i^{-2}(t_i^2 \nabla \frac{s_i}{t_i})$, the subalgebra C_γ is contained in the Poisson centre of A_γ. Hence IA_γ is a Poisson ideal of A_γ and, by [14, 1.4], every prime ideal of A_γ minimal over IA_γ is Poisson.

(ii) By Noether's Normalization Theorem, as stated in [19, 14.14], there exist non-negative integers m, d, with $d \leqslant m$, and $y_1, \dots, y_m \in C_\gamma$ such that $y_1, \dots, y_m$ are algebraically independent over $\mathbb{C}$, C_γ is integral over $\mathbb{C}[y_1, \dots, y_m]$ and $(Q \cap C_\gamma) \cap \mathbb{C}[y_1, \dots, y_m] = \sum_{i=d+1}^m \mathbb{C}[y_1, \dots, y_m]y_i$. So $m = \mathrm{tr.deg}_{\mathbb{C}}(C_\gamma) = n-2$ and $d = \mathrm{tr.deg}_{\mathbb{C}}(C_\gamma/Q \cap C_\gamma)$. The algebraically independent subset $\{y_1, \dots, y_{n-2}\}$ can be extended to a maximal algebraically independent subset $\{y_1, \dots, y_{n-2}, z_1, z_2\}$ of A_γ. Thus A_γ is algebraic over $\mathbb{C}[y_1, \dots, y_{n-2}, z_1, z_2]$. As $y_{d+1}, \dots, y_{n-2} \in Q$, A_γ/Q is algebraic over $\mathbb{C}[y_1 + Q, \dots, y_d + Q, z_1 + Q, z_2 + Q]$. It follows that

$$\mathrm{tr.deg}(A_\gamma/Q) \leqslant d + 2 = \mathrm{tr.deg}(C_\gamma/Q \cap C_\gamma) + 2,$$

and, from Notation 3.13, that

$$\mathrm{ht}(Q) \geqslant \mathrm{ht}(Q \cap C_\gamma). \tag{3.1}$$

Now suppose that $\mathrm{ht}(Q) > \mathrm{ht}(Q \cap C_\gamma)$. Then

$$\mathrm{tr.deg}(A_\gamma/Q) \leqslant \mathrm{tr.deg}(C_\gamma/Q \cap C_\gamma) + 1.$$

Let T be the set of all non-zero elements of the integral domain $C_\gamma/Q \cap C_\gamma$ and let $K = T^{-1}(C_\gamma/Q \cap C_\gamma)$ be the quotient field of $C_\gamma/Q \cap C_\gamma$. Thus $T^{-1}(A_\gamma/Q)$ is an affine K-algebra. Let L be the quotient field of A_γ/Q which is also the quotient field of $T^{-1}(A_\gamma/Q)$. Then $\mathrm{tr.deg}_{\mathbb{C}}(L) \leqslant \mathrm{tr.deg}_{\mathbb{C}}(K) + 1$ and, by [19, 12.56], $\mathrm{tr.deg}_K L \leqslant 1$. Hence there exists $w \in T^{-1}(A_\gamma/Q)$ such that $T^{-1}(A_\gamma/Q)$ is algebraic over $K[w]$. Moreover C_γ is contained in the Poisson centre of A_γ, whence the Poisson bracket in $K[w]$ is trivial. The constant subring of the hamiltonian $\mathrm{ham}\, w$ contains $K[w]$ and, by Lemma 3.11, it contains $T^{-1}(A_\gamma/Q)$. Hence, for any $b \in T^{-1}(A_\gamma/Q)$, the constant subring of $\mathrm{ham}\, b$ contains $K[w]$ and, again by Lemma 3.11, $\mathrm{ham}\, b = 0$. Thus the Poisson bracket in $T^{-1}(A_\gamma/Q)$ is trivial, which is impossible since Q is proper Poisson. Therefore we must have equality in (3.1), that is, $\mathrm{ht}(Q) = \mathrm{ht}(Q \cap C_\gamma)$.

(iii) Let Q' be a minimal prime ideal over $(Q \cap C_\gamma)A_\gamma$ such that $Q' \subseteq Q$. Then Q' is Poisson prime, by (i), and is proper Poisson. Suppose that $Q' \neq Q$. Then $\mathrm{ht}(Q') < \mathrm{ht}(Q)$ and

$$\mathrm{ht}\big(Q'\big) = \mathrm{ht}\big(Q' \cap C_\gamma\big) = \mathrm{ht}(Q \cap C_\gamma) = \mathrm{ht}(Q),$$

a clear contradiction. □

Remark 3.20. Let P be a proper Poisson prime ideal of A and let $\gamma = \gamma(P)$. By Theorem 3.19, PA_γ is a minimal prime ideal over $PA_\gamma \cap C_\gamma$ and $\mathrm{ht}(P) = \mathrm{ht}(PA_\gamma) = \mathrm{ht}(PA_\gamma \cap C_\gamma)$. Denote by $\mathrm{Pht}(P)$ the maximal length ℓ of a chain of distinct Poisson prime ideals

$$0 = P_0 \subset P_1 \subset P_2 \subset \cdots \subset P_\ell = P$$

of A. Clearly $\mathrm{Pht}(P) \leqslant \mathrm{ht}(P)$. But, if $j = \mathrm{ht}\, P = \mathrm{ht}\, PA_\gamma = \mathrm{ht}(PA_\gamma \cap C_\gamma)$ then a chain of prime ideals of C_γ of length j inside $PA_\gamma \cap C_\gamma$ gives rise to chain of Poisson prime ideals of A of length j inside P so $\mathrm{Pht}(P) \geqslant \mathrm{ht}(PA_\gamma \cap C_\gamma) = \mathrm{ht}(P)$, whence $\mathrm{Pht}(P) = \mathrm{ht}(P)$. There are many examples of residually null Poisson prime ideals P for which $\mathrm{Pht}(P) < \mathrm{ht}(P)$, for example the symmetric algebra of $\mathfrak{sl}_2$ with the Kirillov–Kostant–Souriau bracket, where the unique Poisson maximal ideal M has $\mathrm{Pht}(M) = 2$ and $\mathrm{ht}(M) = 3$, see [10, Example 4.1]. For an example where $\mathrm{Pht}(P) < \mathrm{ht}(P)$ and P is not residually null, see [11, Remark 5.13].

Example 3.21. To illustrate Theorem 3.19 and the strategy outlined in Notation 3.18, we return to Example 3.5 and describe all Poisson prime ideals. It is easy to see that the residually null Poisson prime ideals in this case are the height two prime ideal $x_2A + x_3A$, the height three prime ideals $x_1A + x_2A + x_4A$ and $x_1A + x_3A + x_4A$ and all prime ideals containing one, or more, of these.

Let $D = x_1x_4 - x_2x_3 = s_1$, the determinant. The dense subsets of Γ for which $\mathrm{P.spec}_\gamma A$ can be non-empty and the corresponding sets S_γ and V_γ are:

$$\begin{aligned}
\gamma_1 &:= \big\{(0,1),(0,1)\big\}, & S_{\gamma_1} &= \{1, x_3\}, & V_{\gamma_1} &= \{D, x_2/x_3\},\\
\gamma_2 &:= \big\{(0,1),(1,1)\big\}, & S_{\gamma_2} &= \{x_2, 1, x_3\}, & V_{\gamma_2} &= \{D, x_2/x_3\},\\
\gamma_3 &:= \big\{(1,1),(0,1)\big\}, & S_{\gamma_3} &= \{D, 1, x_3\}, & V_{\gamma_3} &= \{D, x_2/x_3\},\\
\gamma_4 &:= \big\{(1,1),(1,1)\big\}, & S_{\gamma_4} &= \{D, x_2, 1, x_3\}, & V_{\gamma_4} &= \{D, x_2/x_3\},\\
\gamma_5 &:= \big\{(1,1),(1,0)\big\}, & S_{\gamma_5} &= \{D, x_2, 1\}, & V_{\gamma_5} &= \{D, x_3/x_2\},\\
\gamma_6 &:= \big\{(0,1),(1,0)\big\}, & S_{\gamma_6} &= \{x_2, 1\}, & V_{\gamma_6} &= \{D, x_3/x_2\}.
\end{aligned}$$

Consequently $C_\gamma = C_1 := \mathbb{C}[D, x_2/x_3]$ or $C_\gamma = C_2 := \mathbb{C}[D, x_3/x_2]$.

Let P be a proper Poisson prime ideal of A and let $\gamma = \gamma(P)$. Then PA_γ is minimal over $PA_\gamma \cap C_\gamma$. Suppose that $C_\gamma = C_1$ so that $\gamma = \gamma_i$ for some i with $1 \leqslant i \leqslant 4$. The prime ideals of C_1 are 0, the principal ideals fC_1, where f is irreducible in C_1 and the maximal ideals $(D-\lambda)C_1 + (x-\mu)C_1$, where $x = x_2/x_3$.

If $i=1$ then $D, x_2 \in P$ and thus $PA_\gamma \cap C_1 = xC_1 + DC_1$ so P is minimal over $x_2A + DA = x_2A + x_1x_4A$. It follows that $P = x_2A + x_1A$ or $P = x_2A + x_4A$. In both cases $A/P \simeq \mathbb{C}[y,z]$ with $\{y,z\} = yz$.

If $i=2$ then $D \in P$, and $PA_\gamma \cap C_1 = DC_1$ or $PA_\gamma \cap C_1 = DC_1 + (x-\mu)C_1$ for some non-zero $\mu \in \mathbb{C}$. In this case either $P = DA$ or $P = DA + (x_2 - \mu x_3)A$. In the latter case, A/P is isomorphic to $\mathbb{C}[x_1,x_3,x_4]/(x_1x_4 - \mu x_3^2)$ with the bracket induced by the Poisson bracket on $\mathbb{C}[x_1,x_3,x_4]$ such that $\{f,g\} = x_3 \operatorname{Jac}(f,g,x_1x_4 - \mu x_3^2)$ for all $f,g \in \mathbb{C}[x_1,x_3,x_4]$. It is easy to see, using [10, Theorem 3.8], that the non-zero Poisson prime ideals of A/P are residually null.

If $i=3$ then $PA_\gamma \cap C_1 = xC_1$ or $PA_\gamma \cap C_1 = xC_1 + (D-\lambda)C_1$ for some non-zero $\lambda \in \mathbb{C}$ so either $P = x_2A$ or $P = x_2A + (D-\lambda)A$ and, in the latter case, A/P is isomorphic to $\mathbb{C}[x_1^{\pm1}, x_3]$ with $\{x_1, x_3\} = x_1x_3$.

If $i=4$ then $PA_\gamma \cap C_1 = 0$ or $PA_\gamma \cap C_1 = fC_1$, for some irreducible $f \in C_1$ that is not an associate of D or x, or $PA_\gamma \cap C_1 = (x-\mu)C_1 + (D-\lambda)C_1$ for some non-zero $\mu, \lambda \in \mathbb{C}$. In the third of these cases, $P = (x_2 - \mu x_3)A + (D-\lambda)A$ and $A/P \simeq \mathbb{C}[x_1,x_3,x_4]/(x_1x_4 - \mu x_3^2 - \lambda)$. In the second case, f remains irreducible in the polynomial extensions $\mathbb{C}[x, D, x_3] = \mathbb{C}[x, x_1x_4, x_3]$ and $\mathbb{C}[x, x_1x_4, x_3, x_1]$ and in the localization T of the latter at the multiplicatively closed subset generated by x_1 and S_γ. It follows that f is irreducible in A_γ since A_γ is a subalgebra of T. Hence if j is the minimal non-negative integer such that $fx_3^j \in A$ and $g = fx_3^j$ then g is irreducible in A and $P = fA_\gamma \cap A = gA$. Examples of Poisson prime ideals arising in this way include the principal ideals generated by $g_0 = D - \lambda$, $g_1 = (x_2 - \lambda x_3) = x_3(x-\lambda)$, where $\lambda \in \mathbb{C}\backslash\{0\}$, $g_2 = x_1x_3x_4 - x_2x_3^2 + x_2 = Dx_3 + x_2 = (D+x)x_3$, $g_3 = x_1x_2x_4 - x_2^2x_3 + x_3 = Dx_2 + x_3 = (xD+1)x_3$, $g_4 = (D^2 - x^3)x_3^3 = D^2x_3^3 - x_2^3$ and $g_5 = D^2x_2^3 - x_3^3 = x_3^3(D^2x^3 - 1)$. Here the pairs g_2, g_3 and g_4, g_5 show how the choice of v_2, which is not symmetric between x_2 and x_3, takes account of the inherent symmetry between x_2 and x_3. In general, if $f(D, x^{-1})$ is irreducible in C_2, where $x^{-1} = x_3x_2^{-1}$, then there is an irreducible polynomial $g(D,x)$ such that $g(D,x) = x^k f(D, x^{-1})$ for some $k \geqslant 0$.

The symmetry between x_2 and x_3 is more explicit in the analysis for γ_5 and γ_6, which are analogous to γ_3 and γ_1 respectively. Here the Poisson prime ideals are $P = x_3A$ or $P = x_3A + (D-\lambda)A$, $\lambda \in \mathbb{C}\backslash\{0\}$, for γ_5, and $P = x_3A + x_1A$ or $P = x_3A + x_4A$, for γ_6.

In the case where $t_1 = t_2 = \cdots = t_{n-2} = 1$, if γ is such that $\operatorname{P.spec} A_\gamma$ is non-empty, then, by Lemma 3.16, each $\delta_i = 1$, each $v_i = s_i$ and $C_\gamma = \mathbb{C}[s_1, s_2, \ldots, s_{n-2}]$ which, under our working assumption, is a polynomial subalgebra of A.

Corollary 3.22. *Suppose that $t_1 = \cdots = t_{n-2} = 1$, let $C = \mathbb{C}[s_1, \ldots, s_{n-2}]$ and let P be a proper Poisson prime ideal of A. Then there exists a prime ideal Q of C such that P is a minimal prime ideal over QA.*

Proof. Let $\gamma = \gamma(P)$. In this case, $C \subseteq A \subseteq A_\gamma$. By Theorem 3.19, PA_γ is a minimal prime of A_γ over $PA_\gamma \cap C$ and it follows easily that P is a minimal prime of A over $(P \cap C)A$. $\square$

Example 3.23. In Example 3.21, each irreducible polynomial f in C_1 leads to a single principal Poisson prime ideal. This is not always the case. For example, consider the case where $n=4$, $s_1 = x_1x_4 - x_2x_3$, $s_2 = x_2x_3$ and $t_1 = t_2 = 1$. Thus Corollary 3.22 applies. The Poisson bracket on A is such that

$$\begin{aligned} \{x_1,x_2\} &= -x_1x_2, & \{x_1,x_3\} &= x_1x_3, & \{x_1,x_4\} &= 0, \\ \{x_2,x_3\} &= 0, & \{x_2,x_4\} &= -x_2x_4, & \{x_3,x_4\} &= x_3x_4. \end{aligned}$$

Note that if $f = s_2$ or $f = s_1 + s_2$ then f is irreducible in $C := \mathbb{C}[s_1, s_2]$ but factorises as the product of two irreducible factors in A, each generating a principal Poisson prime ideal. This gives rise to four height one Poisson prime ideals of A, x_iA for $1 \leqslant i \leqslant 4$.

Example 3.24. This example illustrates the situation where A has an element of the form $\lambda_i s_i - \mu_i t_i$ for two different values of i. This gives rise to residually null Poisson prime ideals. Let $n = 4$ and let $s_1 = x_1 + x_2 + x_3 + x_4$, $t_1 = 1$, $s_2 = x_1 + x_4$ and $t_2 = x_2 + x_3$. Then $s_1 = \lambda_1 s_1 - \mu_1 t_1 = \lambda_2 s_2 - \mu_2 t_2$, where $\lambda_1 = \lambda_2 = 1$, $\mu_1 = 0$ and $\mu_2 = -1$. The Poisson bracket on A in this example is given by

$$\begin{aligned} \{x_1, x_2\} &= s_1, & \{x_1, x_3\} &= -s_1, & \{x_1, x_4\} &= 0, \\ \{x_2, x_3\} &= 0, & \{x_2, x_4\} &= s_1, & \{x_3, x_4\} &= -s_1. \end{aligned}$$

Here the height one prime ideal $P = s_1 A$ is residually null Poisson and $\gamma(P) = ((0,1),(1,1))$ is dense. Notice that $PA_\gamma \cap C_\gamma$ contains both $v_1 = s_1$ and $v_2 + 1 = (x_2 + x_3)^{-1} s_1$ and that $\mathrm{ht}(PA_\gamma \cap C_\gamma) = 2$ whereas $\mathrm{ht}(PA_\gamma) = 1$. This shows that the condition in Theorem 3.19(ii) that P is proper Poisson is necessary.

4. Poisson primitive spectra

Notation 4.1. For each $i = 1, \dots, n-2$ and for each $(\lambda_i, \mu_i) \in \mathbb{C}^2 \setminus \{(0,0)\}$, set

$$f^i_{\lambda_i,\mu_i} = \lambda_i s_i - \mu_i t_i.$$

Observe that

$$t_i^2 \nabla \frac{s_i}{t_i} = \begin{cases} \lambda_i^{-1}(t_i \nabla f^i_{\lambda_i,\mu_i} - f^i_{\lambda_i,\mu_i} \nabla t_i) & \text{if } \lambda_i \neq 0, \\ \mu_i^{-1}(s_i \nabla f^i_{\lambda_i,\mu_i} - f^i_{\lambda_i,\mu_i} \nabla s_i) & \text{if } \mu_i \neq 0. \end{cases} \tag{4.1}$$

Lemma 4.2. *For $p = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{C}^n$, let*

$$M_p = (x_1 - \alpha_1)A + (x_2 - \alpha_2)A + \cdots + (x_n - \alpha_n)A$$

and, for $1 \leqslant i \leqslant n-2$, let $g_i = f^i_{t_i(p), s_i(p)}$. Then M_p is a Poisson ideal if and only if the rank of the Jacobian matrix $(\frac{\partial g_i}{\partial x_j}(p))$ is less than $n-2$.

Proof. Since

$$\left(t_i^2 \nabla \frac{s_i}{t_i}\right)(p) = (t_i \nabla s_i - s_i \nabla t_i)(p) = \nabla g_i(p),$$

the rank of the Jacobian matrix $(\frac{\partial g_i}{\partial x_j}(p))$ is less than $n-2$ if and only if

$$\{x_k - \alpha_k, x_\ell\}(p) = \begin{vmatrix} e_k \\ e_\ell \\ (t_1^2 \nabla \frac{s_1}{t_1})(p) \\ \vdots \\ (t_{n-2}^2 \nabla \frac{s_{n-2}}{t_{n-2}})(p) \end{vmatrix} = 0$$

for all $1 \leqslant k, \ell \leqslant n$. Hence the result follows. $\square$

Lemma 4.3. *Suppose that the parameters λ_i and μ_i are such that the ideal $I := f^1_{\lambda_1,\mu_1} A + \cdots + f^{n-2}_{\lambda_{n-2},\mu_{n-2}} A$ is a proper ideal of A and let P be a minimal prime ideal over I.*

(i) *P is a Poisson prime ideal with height less than or equal to $n-2$.*
(ii) *If P is residually null then it is not Poisson primitive.*
(iii) *If P is proper Poisson then* $\operatorname{ht} P = n-2$ *and P is locally closed and Poisson primitive.*

Proof. (i) It follows from (4.1) that I is a Poisson ideal so P is a Poisson prime ideal. The height of P is less than or equal to $n-2$ by [19, 15.4].

(ii) By Lemma 3.3(i), any residually null Poisson primitive ideal is maximal and hence has height n. By (i), P is not Poisson primitive.

(iii) Let $\gamma = \gamma(P)$ which is dense by Lemma 3.16(i). Let $1 \leqslant i \leqslant n-2$. Suppose that $t_i \notin P$. Then $t_i \in S_\gamma$, $v_i = s_i/t_i$, and $\lambda_i v_i - \mu_i = t_i^{-1} f^i_{\lambda_i,\mu_i} \in PA_\gamma$. Note that $\lambda_i \neq 0$, otherwise $0 \neq \mu_i t_i = -f^i_{\lambda_i,\mu_i} \in P$. Similarly if $t_i \in P$ then $s_i \in S_\gamma$, $v_i = t_i/s_i$, $\lambda_i - \mu_i v_i = s_i^{-1} f^i_{\lambda_i,\mu_i} \in PA_\gamma$ and $\mu_i \neq 0$. Therefore $PA_\gamma \cap C_\gamma$ must be the maximal ideal of C_γ generated by the elements m_i, where $m_i = v_i - \frac{\mu_i}{\lambda_i}$ if $t_i \notin P$ and $m_i = v_i - \frac{\lambda_i}{\mu_i}$ if $t_i \in P$. By Theorem 3.19(ii), $n-2 = \operatorname{ht} PA_\gamma = \operatorname{ht} P$, and hence P is locally closed and Poisson primitive by Lemma 3.3(iii). $\square$

The following result computes the primitive core of a non-Poisson maximal ideal and the symplectic core of the corresponding point p of $\mathbb{C}^n$.

Theorem 4.4. *Let $p \in \mathbb{C}^n$ be a point, with corresponding maximal ideal M_p of A, such that M_p is not Poisson. For $1 \leqslant i \leqslant n-2$, let $g_i = t_i(p)s_i - s_i(p)t_i$, as in Lemma 4.2, and let $I_p = g_1A + g_2A + \cdots + g_{n-2}A$. Then the Poisson core $\mathcal{P}(M_p)$ is the unique minimal prime ideal P over I_p such that $P \subset M_p$. The symplectic core of p is*

$$\{q \in \mathbb{C}^n\colon\ M_q \text{ is not Poisson and } P \subset M_q\}.$$

Proof. As $I_p \subseteq M_p$, the ideal I_p is a proper ideal. Let P be a minimal prime ideal over I_p contained in M_p. By Lemma 4.3(i), P is Poisson so $P \subseteq \mathcal{P}(M_p) \subset M_p$. Both P and $\mathcal{P}(M_p)$ must be proper Poisson, otherwise M_p would be Poisson. By Lemma 4.3(iii) and [10, Proposition 3.2], $\operatorname{ht} P = n-2 \geqslant \operatorname{ht} \mathcal{P}(M_p)$. Hence $P = \mathcal{P}(M_p)$ and P is unique. The rest follows. $\square$

Remark 4.5. The computation of the Poisson core based on Theorem 4.4 is effective because the minimal primes over the ideal I_p in the statement can be computed, for example using the function minimalPrimes in Macaulay2 [7], in which the function isSubset(-,-) can be used to check whether $P \subset M_p$.

We next determine the Poisson primitive ideals of A and establish the Poisson Dixmier–Moeglin equivalence.

Corollary 4.6. *The Poisson primitive ideals of A are the Poisson maximal ideals, as specified in Lemma 4.2, and the proper Poisson ideals that are minimal prime ideals over a proper ideal $f^1_{\lambda_1,\mu_1}A + \cdots + f^{n-2}_{\lambda_{n-2},\mu_{n-2}}A$, as specified in Lemma 4.3.*

Proof. Poisson maximal ideals are always Poisson primitive, so it follows from Lemma 4.3(iii) that the listed ideals are Poisson primitive. The converse is immediate from Theorem 4.4. $\square$

Theorem 4.7. *The Poisson algebra A satisfies the Poisson Dixmier–Moeglin equivalence.*

Proof. Let P be a rational Poisson prime ideal and let $1 \leqslant i \leqslant n-2$. As $s_i/t_i \in \mathrm{PZ}(Q(A))$, P contains $f^i_{\lambda_i,\mu_i}$ for some $(\lambda_i,\mu_i) \in \mathbb{C}^2 \backslash \{(0,0)\}$, and therefore P contains a proper ideal of the form

$$I = f^1_{\lambda_1,\mu_1}A + \cdots + f^{n-2}_{\lambda_{n-2},\mu_{n-2}}A.$$

If P is residually null then $\mathbb{C} = \mathrm{PZ}(Q(A/P)) = Q(A/P)$ so P is a Poisson maximal ideal and hence is locally closed. So we can assume that P is proper Poisson. Let Q be a minimal prime ideal over I such that $Q \subseteq P$. By Lemma 4.3(i), Q is Poisson and, as P is proper Poisson, so also is Q. It follows from Lemma 4.3(iii) that $\operatorname{ht} Q = n-2$ and Q is locally closed. But, by [10, Proposition 3.2], $\operatorname{ht} P \leqslant n-2$ so $P = Q$. Thus P is locally closed. $\square$

Remark 4.8. It follows from Theorem 4.7 that $\mathrm{P.prim}\, A$ is homeomorphic to the space of symplectic cores. See [5, Remark 9.2(d)]. This can also be seen directly from Theorem 4.4.

Examples 4.9. In Example 3.5, for a point $p = (\alpha, \beta, \gamma, \delta)$, $g_1 = D - \mu$, where $\mu = \alpha\delta - \beta\gamma$, and $g_2 = \gamma x_2 - \beta x_3$. So the primitive core of the maximal ideal M_p corresponding to p is determined by μ and the pair (β, γ). By Lemma 4.2, the points corresponding to the Poisson maximal ideals, and so belonging to singleton symplectic cores, are those of the forms $(\alpha, 0, 0, \delta)$, $\alpha, \delta \in \mathbb{C}$, $(0, 0, \gamma, 0)$, $\gamma \in \mathbb{C}$ and $(0, \beta, 0, 0)$, $\beta \in \mathbb{C}$.

For $\mu, \rho \in \mathbb{C}$, not both 0, there is a symplectic core

$$\left\{(\alpha, \beta, \rho\beta, \delta)\colon \alpha, \beta, \delta \in \mathbb{C}, \alpha\delta = \rho\beta^2 + \mu\right\},$$

corresponding to the primitive core $(D - \mu)A + (x_3 - \rho x_2)A$. This includes the case $\rho = 0 \neq \mu$, where the symplectic core is

$$\left\{\left(\alpha, \beta, 0, \mu\alpha^{-1}\right)\colon \alpha, \beta \in \mathbb{C}^*\right\}$$

and the primitive core is $(D - \mu)A + x_3 A$. For $\mu \neq 0$, there is also a symplectic core

$$\left\{\left(\alpha, 0, \gamma, \mu\alpha^{-1}\right)\colon \alpha, \gamma \in \mathbb{C}^*\right\},$$

corresponding to the primitive core $(D - \mu)A + x_2 A$. The remaining symplectic cores arise in the case where $\mu = 0$ and $\gamma = 0$ or $\beta = 0$. They, and their corresponding primitive cores, are

$$\begin{gathered}
\left\{(\alpha, \beta, 0, 0)\colon \alpha, \beta \in \mathbb{C}^*\right\}, \quad x_3 A + x_4 A; \\
\left\{(\alpha, 0, \gamma, 0)\colon \alpha, \gamma \in \mathbb{C}^*\right\}, \quad x_2 A + x_4 A; \\
\left\{(0, \beta, 0, \delta)\colon \delta, \beta \in \mathbb{C}^*\right\}, \quad x_1 A + x_3 A; \\
\left\{(0, 0, \gamma, \delta)\colon \gamma, \delta \in \mathbb{C}^*\right\}, \quad x_1 A + x_2 A.
\end{gathered}$$

The symplectic cores are the symplectic leaves as given in [13, Example 2.4].

In Example 3.24, let $p = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathbb{C}^4$ and let $\lambda := \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$. It follows easily from Lemma 4.2 that M_p is a Poisson maximal ideal if and only if $\lambda = 0$. Suppose that $\lambda \neq 0$. In the notation of Theorem 4.4, $g_1 = x_1 + x_2 + x_3 + x_4 - \lambda$ and $g_2 = (\lambda_2 + \lambda_3)(x_1 + x_4) - (\lambda_1 + \lambda_4)(x_2 + x_3)$. As these degree one polynomials have linearly independent leading terms, they generate a prime ideal I_p of A and so $\mathcal{P}(M_p) = I_p$. Let $q = (\mu_1, \mu_2, \mu_3, \mu_4) \in \mathbb{C}^4$ and let $\mu = \mu_1 + \mu_2 + \mu_3 + \mu_4$. Then

$$\begin{aligned}
I_p \subset M_q \quad &\Leftrightarrow \quad \mu = \lambda \quad \text{and} \quad (\lambda_2 + \lambda_3)(\mu_1 + \mu_4) = (\lambda_1 + \lambda_4)(\mu_2 + \mu_3) \\
&\Leftrightarrow \quad \mu = \lambda \quad \text{and} \quad \bigl(\lambda - (\lambda_1 + \lambda_4)\bigr)(\mu_1 + \mu_4) = (\lambda_1 + \lambda_4)\bigl(\lambda - (\mu_1 + \mu_4)\bigr) \\
&\Leftrightarrow \quad \mu = \lambda \quad \text{and} \quad \mu_1 + \mu_4 = \lambda_1 + \lambda_4.
\end{aligned}$$

It follows from Theorem 4.4 that the symplectic core of p is

$$\left\{(\mu_1, \mu_2, \mu_3, \mu_4) \in \mathbb{C}^4\colon \mu_1 + \mu_4 = \lambda_1 + \lambda_4 \text{ and } \mu_2 + \mu_3 = \lambda_2 + \lambda_3\right\}.$$

We end with a result identifying some simple Poisson algebras that occur as quotients of A in the case where each $t_i = 1$.

Corollary 4.10. *Suppose that $t_i = 1$ for each i and that $s_1, \ldots, s_{n-2}$ are algebraically independent. Let $(\mu_1, \ldots, \mu_{n-2}) \in \mathbb{C}^{n-2}$ be such that $P := (s_1 - \mu_1)A + \cdots + (s_{n-2} - \mu_{n-2})A$ is a prime ideal of A. Let $X \subset \mathbb{C}^n$ be the variety determined by P. Then P is Poisson prime. Moreover X is nonsingular if and only if A/P is Poisson simple.*

Proof. In Notation 4.1, let $\lambda_i = 1$ so that $f^i_{\lambda_i,\mu_i} = s_i - \mu_i$. By Lemma 4.3(i), P is Poisson. By Lemma 3.16(ii), $C_{\gamma(P)} = \mathbb{C}[s_1, \ldots, s_{n-2}]$ so it follows, by Theorem 3.19(ii), that $\operatorname{ht} P = n - 2$. Hence $\dim X = 2$. Let Q be a Poisson primitive ideal of A such that $P \subseteq Q$. By Theorem 4.6 and Lemma 4.3, either $\operatorname{ht} Q = n - 2$, in which case $Q = P$, or Q is the maximal ideal corresponding to a singularity of X. Hence A/P has no non-zero Poisson primitive ideal if and only if X is nonsingular. The result now follows from Lemma 3.3(ii). $\square$

References

[1] K.A. Brown, K.R. Goodearl, Lectures on Algebraic Quantum Groups, Adv. Courses Math. CRM Barcelona, Birkhäuser, Basel, Boston, Berlin, 2002.
[2] K.A. Brown, I. Gordon, Poisson orders, symplectic reflection algebras and representation theory, J. Reine Angew. Math. 559 (2003) 193–216.
[3] K. Erdmann, M.J. Wildon, Introduction to Lie Algebras, Springer, London, 2006.
[4] I.M. Gelfand, M. Kapranov, A. Zelevinsky, Discriminants, Resultants, and Multidimensional Determinants, Birkhäuser, Boston, 1994.
[5] K.R. Goodearl, Semiclassical limits of quantized coordinate rings, in: D.V. Huynh, S. Lopez-Permouth (Eds.), Advances in Ring Theory, Birkhäuser, Basel, 2009, pp. 165–204, arXiv:0812.1612v1 [math.QA].
[6] K.R. Goodearl, A Dixmier–Moeglin equivalence for Poisson algebras with torus actions, in: D.V. Huynh, S.K. Jain, S.R. Lopez-Permouth (Eds.), Algebra and Its Applications, in: Contemp. Math., vol. 419, 2006, pp. 131–154.
[7] D.R. Grayson, M.E. Stillman, Macaulay2, a software system for research in algebraic geometry, available at http://www.math.uiuc.edu/Macaulay2/.
[8] W.V.D. Hodge, D. Pedoe, Methods of Algebraic Geometry, vol. 1, Cambridge Math. Lib., Cambridge University Press, Cambridge, 1994 (reissue).
[9] J. Grabowski, G. Marmo, A.M. Perelomov, Poisson structures: towards a classification, Modern Phys. Lett. A 8 (18) (1993) 1719–1733.
[10] D.A. Jordan, Sei-Qwon Oh, Poisson brackets and Poisson spectra in polynomial algebras, Contemp. Math. 562 (2012) 169–187.
[11] D.A. Jordan, Ore extensions and Poisson algebras, Glasg. Math. J. (13 August 2013), http://dx.doi.org/10.1017/S0017089513000293, Firstview Articles, published online.
[12] Sei-Qwon Oh, Symplectic ideals of Poisson algebras and the Poisson structure associated to quantum matrices, Comm. Algebra 27 (1999) 2163–2180.
[13] Sei-Qwon Oh, Symplectic submanifolds and symplectic ideals, J. Lie Theory 16 (1) (2006) 131–138.
[14] Sei-Qwon Oh, Poisson prime ideals of Poisson polynomial rings, Comm. Algebra 35 (2007) 3007–3012.
[15] Sei-Qwon Oh, Quantum and Poisson structures of multi-parameter symplectic and Euclidean spaces, J. Algebra 319 (2008) 4485–4535.
[16] A.N. Panov, n-Poisson and n-Sklyanin brackets, J. Math. Sci. 110 (2002) 2322–2329.
[17] R. Przybysz, On one class of exact Poisson structures, J. Math. Phys. 42 (2001) 1913–1920.
[18] L.H. Rowen, Graduate Algebra: Commutative View, American Mathematical Society, Providence RI, 2006.
[19] R.Y. Sharp, Steps in Commutative Algebra, second edition, London Math. Soc. Stud. Texts, vol. 51, Cambridge University Press, 2000.