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Convergence properties in the nonhyperbolic case $x_{n+1} = \frac{x_{n-1}}{1+f(x_n)}$

Steven Kalikow [a], Peter M. Knopf [b,*], Ying Sue Huang [b], Gabor Nyerges [c]

[a] Department of Mathematics, University of Memphis, Memphis, TN, USA
[b] Department of Mathematics, Pace University, Pleasantville, NY, USA
[c] Department of Electrical, Computer and Communications Engineering, London South Bank University, London, UK

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Abstract

Consider the difference equation $x_{n+1} = \frac{x_{n-1}}{1+f(x_n)}$ where f is in a certain class of increasing continuous functions. In particular, the class includes all functions of the form $f(x) = \alpha x^{\beta}$ with $\alpha > 0$ and $\beta > 0$. The set of initial conditions (x_0, x_1) in the first quadrant that converge to any given boundary point of the first quadrant forms a unique increasing continuous function. Furthermore, all of the positive solutions x_n are stable under small perturbations of the initial point (x_0, x_1).

Keywords: Difference equations; Nonhyperbolic; Convergence; Uniqueness; Stability

1. Introduction

In their book [2], Kulenović and Ladas consider the positive solutions for the class of difference equations of the form $x_{n+1} = \frac{x_{n-1}}{1+Ax_n}$ with $A > 0$. This is a second-order difference equation. We use (x_0, x_1) to represent the initial condition (initial point). In this paper, all of our initial conditions (x_0, x_1) satisfy $x_0 \geqslant 0$ and $x_1 \geqslant 0$. We will often refer to (x_{2n}, x_{2n+1}) as a solution to the second-order difference equation under consideration. These equations are nonhyperbolic at their fixed point $(0, 0)$. Kulenović and Ladas give some partial results on the convergence of

* Corresponding author.
E-mail address: pknopf@pace.edu (P.M. Knopf).

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these equations. In particular, they show that either $x_{2n+1} \downarrow 0$ or $x_{2n+2} \downarrow 0$. They ask if there exist initial conditions under which the sequence x_n converges to 0. Kent shows the existence of such initial conditions in her paper [1]. She generalizes this result to a much wider class of second-order difference equations that exhibits this similar behavior. There still remained open the question as to the precise nature of the set of positive initial conditions under which the sequence will converge to 0. Janssen and Tjaden [3] showed that if $x_0 = 1$, then there exists a unique initial value x_1 such that the difference equation converges to 0 for the case $A = 1$.

In this paper, we show that the set of initial conditions (x_0, x_1) in the first quadrant that converge to any given point on the boundary is a unique increasing continuous curve. Furthermore, we generalize this result to difference equations of the form $x_{n+1} = \frac{x_{n-1}}{1+f(x_n)}$ where f is in a certain class of functions that include $f(x) = Ax$ with $A > 0$. We prove that the positive solutions are stable under small perturbations of the initial conditions.

In Section 3, we prove that it is not sufficient for the function f to be an increasing continuous function with $f(0) = 0$ in order to obtain our results. We construct a function f with these properties so that some of the solutions to the difference equation (1) are not stable. Furthermore, for initial conditions of the form (x_0, x_1) with $0 < x_0 \leqslant 1/2$ and $x_1 > 0$ and sufficiently small, the solutions to the difference equation all converge to the same limit point $(0, 0)$.

2. The main results

Our first theorem concerns the stability of the solutions to the difference equation under perturbations of the initial conditions.

Theorem 1. *Consider the difference equation*

$$x_{n+1} = \frac{x_{n-1}}{1 + f(x_n)}, \tag{1}$$

where f is a continuous nonnegative increasing function on $[0, \infty)$. *We also assume that there exists $\xi > 0$ such that $f(x) \leqslant cx^L$ for all $0 \leqslant x < \xi$ for some positive constants c and L. Let* (x_0, x_1) *be any initial condition. Then for every $\epsilon > 0$, there exists $\delta > 0$ such that if $|x_0 - y_0| < \delta$ and $|x_1 - y_1| < \delta$, we have $|x_n - y_n| < \epsilon$ for all $n \geqslant 0$ where the sequences x_n and y_n satisfy the difference equation* (1).

Proof. Clearly x_{2n} and x_{2n+1} are decreasing sequences and hence each converge. As $x_{2n+1} f(x_{2n}) = x_{2n-1} - x_{2n+1}$, then $x_{2n+1} f(x_{2n}) \to 0$. Thus either $x_{2n+1} \to 0$ or $f(x_{2n}) \to 0$. But clearly $f(0) = 0$ and since f is an increasing function, then as $f(x) \to 0$ we must have $x \to 0$. So either $x_{2n+1} \to 0$ or $x_{2n} \to 0$. Consider the case $x_{2n} \to 0$ and $x_{2n+1} \to a$ for some $a > 0$. Given $\epsilon > 0$ with $\epsilon < \min(1, \frac{a}{4})$, choose N sufficiently large so that $x_{2n} < \epsilon$ and $|x_{2n+1} - a| < \epsilon$ for all $n \geqslant N$. Next choose $\delta > 0$ sufficiently small so that if $|x_0 - y_0| < \delta$ and $|x_1 - y_1| < \delta$, then $|x_{2N} - y_{2N}| < \epsilon$ and $|x_{2N+1} - y_{2N+1}| < \epsilon$. This clearly can be done because of the continuity of f. Since y_{2n} and y_{2n+1} are decreasing sequences, in order to prove the theorem, it is sufficient to show that $y_{2n} < 2\epsilon$ and $y_{2n+1} - a > -\epsilon_1$ for all $n \geqslant N$ where $\epsilon_1 \to 0$ as $\epsilon \to 0$. Since $x_{2N} < \epsilon$, $|x_{2N} - y_{2N}| < \epsilon$, and y_{2n} is decreasing, then $y_{2n} < 2\epsilon$ for $n \geqslant N$. We show by induction that

$$y_{2N+2k} \leqslant \frac{y_{2N}}{[1 + f(\frac{a}{4})]^k} \tag{2}$$

and

$$y_{2N+2k+1} \geqslant y_{2N+1} \prod_{i=1}^{k} \left\{ 1 + \frac{c y_{2N}^{L}}{[1 + f(\frac{a}{4})]^{Li}} \right\}^{-1} \tag{3}$$

for $k \geqslant 1$. It is easy to check that (2) and (3) hold for $k \geqslant 1$. From the difference equation and the induction hypothesis we have

$$y_{2N+2k+2} \leqslant \frac{y_{2N}}{[1 + f(\frac{a}{4})]^{k}[1 + f(y_{2N+2k+1})]}. \tag{4}$$

It is easy to check that the product factor in (3) satisfies the following estimate:

$$\prod_{i=1}^{k} \left\{ 1 + \frac{c y_{2N}^{L}}{[1 + f(\frac{a}{4})]^{Li}} \right\}^{-1} \geqslant e^{-c_1 \epsilon^{L}} \tag{5}$$

for ϵ sufficiently small for some positive constant c_1 since $y_{2N} < 2\epsilon$. Inserting the obvious bound $e^{-c_1 \epsilon^{L}} \geqslant 1/2$ into (3) gives $y_{2N+2k+1} \geqslant y_{2N+1}/2 > a/4$ and this result inserted into (4) establishes (2). Similarly, from the difference equation and the induction hypothesis, we have:

$$y_{2N+2k+3} \geqslant \frac{y_{2N+1}}{1 + f(y_{2N+2k+2})} \prod_{i=1}^{k} \left\{ 1 + \frac{c y_{2N}^{L}}{[1 + f(\frac{a}{4})]^{Li}} \right\}^{-1}.$$

Since

$$f(y_{2N+2k+2}) \leqslant f\left(\frac{y_{2N}}{[1 + f(\frac{a}{4})]^{k+1}} \right) \leqslant \frac{c y_{2N}^{L}}{[1 + f(\frac{a}{4})]^{L(k+1)}}$$

then

$$y_{2N+2k+3} \geqslant y_{2N+1} \prod_{i=1}^{k+1} \left\{ 1 + \frac{c y_{2N}^{L}}{[1 + f(\frac{a}{4})]^{Li}} \right\}^{-1}$$

which proves (3). By (3) and (5) we finally obtain $y_{2N+2k+1} \geqslant y_{2N+1} e^{-c_1 \epsilon^{L}} \geqslant (a - 2\epsilon)(1 - c_1 \epsilon^{L})$ for ϵ sufficiently small which proves the theorem for the case $x_{2n} \to 0$ and $x_{2n+1} \to a$ for $a > 0$. The proof for the case $x_{2n} \to a$ and $x_{2n+1} \to 0$ for $a > 0$ is similar. The case $x_{2n} \to 0$ and $x_{2n+1} \to 0$ is trivial, since by arguing as before we can arrange $x_{2N} < 2\epsilon$ and $x_{2N+1} < 2\epsilon$ for some large N. The fact that y_{2n} and y_{2n+1} are decreasing sequences implies $y_{2n} < 2\epsilon$ and $y_{2n+1} < 2\epsilon$ for all $n \geqslant N$. $\square$

The following definition and lemma will be helpful to prove some of the results.

Definition. We define an ordering, $\succcurlyeq$, on the limit of a sequence (x_{2n}, x_{2n+1}) in the following way:

(i) $(0, b) \succcurlyeq (0, b')$ if $b \geqslant b' \geqslant 0$;
(ii) $(0, b) \succcurlyeq (b', 0)$ for all $b \geqslant 0$ and $b' \geqslant 0$;
(iii) $(b, 0) \succcurlyeq (b', 0)$ if $0 \leqslant b \leqslant b'$.

With slight abuse of notation, we will sometimes denote $\lim_{n\to\infty}(x_{2n}, x_{2n+1}) = (a, b)$ by $\lim_{n\to\infty}(x_0, x_1) = (a, b)$ or by $(x_0, x_1) \to (a, b)$ where (x_0, x_1) is the initial condition of the sequence (x_{2n}, x_{2n+1}).

Lemma 2. *Suppose* $\{x_n\}$ *and* $\{y_n\}$ *satisfy the difference equation* $x_{n+1} = x_{n-1}/[1 + f(x_n)]$ *such that* f *is an increasing continuous function on* $[0, \infty)$ *with* $f(0) = 0$. *If the initial conditions satisfy* $x_0 \leqslant y_0$ *and* $x_1 \geqslant y_1$, *then* $x_{2n} \leqslant y_{2n}$ *and* $x_{2n+1} \geqslant y_{2n+1}$ *for all* $n \geqslant 0$, *and* $\lim_{n\to\infty}(x_{2n}, x_{2n+1}) \succcurlyeq \lim_{n\to\infty}(y_{2n}, y_{2n+1})$.

Proof. By the beginning of the proof of Theorem 1, either $x_{2n} \to 0$ or $x_{2n+1} \to 0$ as $n \to \infty$. Since x_{2n} and x_{2n+1} are both decreasing in n, then we must have $\lim_{n\to\infty}(x_{2n}, x_{2n+1}) = (b, 0)$ or $(0, b)$ for some $b \geqslant 0$. By induction, we have

$$x_{2n+2} = \frac{x_{2n}}{1 + f(x_{2n+1})} \leqslant \frac{y_{2n}}{1 + f(y_{2n+1})} = y_{2n+2}$$

and

$$x_{2n+3} = \frac{x_{2n+1}}{1 + f(x_{2n+2})} \geqslant \frac{y_{2n+1}}{1 + f(y_{2n+2})} = y_{2n+3}$$

since f is an increasing function. The result $\lim_{n\to\infty}(x_{2n}, x_{2n+1}) \succcurlyeq \lim_{n\to\infty}(y_{2n}, y_{2n+1})$ now easily follows. $\square$

Theorem 3. *Consider the difference equation defined by*

$$x_{n+1} = \frac{x_{n-1}}{1 + f(x_n)},$$

where f *is a continuous and increasing function on* $[0, \infty)$ *such that for all* $a \geqslant 0$ *we have*:

$$\lim_{(t,x)\to(1+f(a),0)} \frac{f(tx) - f(x)}{(t-1)f(x)} = L_1(a) \tag{6}$$

and

$$\lim_{(t,x)\to(1,a)} \frac{f(tx) - f(x)}{(t-1)f(x)} = L_2(a), \tag{7}$$

where $L_1(a)$ *and* $L_2(a)$ *are positive numbers. Then the set of all initial points* (x_0, x_1) *in the first quadrant that converge to the point* $(a, 0)$ *forms a unique continuous increasing function on* $[a, \infty)$ *for each* $a \geqslant 0$. *Furthermore, the set of all initial points* (x_0, x_1) *in the first quadrant that converge to the point* $(0, a)$ *forms a unique continuous increasing function on* $[0, \infty)$ *for each* $a \geqslant 0$.

We will show that $f(0) = 0$, and so conditions (6) and (7) are equivalent for the case $a = 0$.

Special Case. If $f \in C^1([0, \infty))$, f is increasing on $[0, \infty)$, $\lim_{x\to 0} \frac{xf'(x)}{f(x)} > 0$, and $f'(a) > 0$ for $a > 0$, then conditions (6) and (7) will be satisfied for all $a \geqslant 0$.

Examples of functions that satisfy the conditions of Theorem 3 include all functions of the form $f(x) = \alpha x^\beta$ for any $\alpha > 0$ and $\beta > 0$. One can also construct examples of the form $f(x) = \alpha x^\beta g(x)$ where $g(x)$ behaves like $[\log(1/x)]^\gamma$ for any $-\infty < \gamma < \infty$ and x sufficiently small.

The following theorem is a generalization of Theorem 3.

Theorem 4. *Consider the difference equation*:

$$x_{n+1} = \frac{x_{n-1}}{1 + f(x_n)g(x_n)} \tag{8}$$

such that f and g are continuous on $[0,\infty)$, $g(0)>0$, fg is increasing on $[0,\infty)$, f satisfies condition (7) in Theorem 3, and in place of condition (6) we require

$$\lim_{(t,x)\to(1+f(a)g(a),0)} \frac{f(tx)-f(x)}{(t-1)f(x)} = L_1(a) \tag{6$*$}$$

for all $a\geqslant 0$. Then the set of initial points (x_0,x_1) in the first quadrant that converge to the point $(a,0)$ forms a unique continuous increasing function on $[a,\infty)$ for each $a\geqslant 0$. Furthermore, the set of initial points (x_0,x_1) in the first quadrant that converge to the point $(0,a)$ forms a unique continuous increasing function on $[0,\infty)$ for each $a\geqslant 0$.

In order to prove Theorem 4, we establish a couple of lemmas concerning the behavior of f. In Lemma 5, for brevity, we will denote the constant $L_2=L_2(0)$ from condition (7).

Lemma 5. *If f is a nonnegative continuous increasing function on $[0,\infty)$ and satisfies condition (7) for $a=0$, then for every $\zeta>0$, there exists $\xi>0$ such that*

$$cx^{L_2+\zeta}\leqslant f(x)\leqslant cx^{L_2-\zeta}$$

whenever $0\leqslant x<\xi$ for some constant $c>0$.

Proof. By condition (7) for the case $a=0$, given $\zeta_0>0$, there exists $\xi_0>0$ such that for $0\leqslant x\leqslant\xi_0$ and $1-\xi_0<t<1$, we have

$$(L_2+\zeta_0)(t-1)\leqslant\frac{f(tx)-f(x)}{f(x)}\leqslant(L_2-\zeta_0)(t-1).$$

Fix x such that $0<x<\xi_0$. Define $0<x_{i+1}<x_i$ for $i=0,1,\dots,n-1$, so that $x_0=\xi_0$, $x=x_n$, $\Delta x_i=x_{i+1}-x_i$, and $t_i=1+\frac{\Delta x_i}{x_i}$, with Δx_i chosen small enough to insure $-\xi_0<\frac{\Delta x_i}{x_i}<0$. This allows us to obtain the following estimate:

$$(L_2+\zeta_0)\sum_{i=0}^{n-1}\frac{\Delta x_i}{x_i}\leqslant\sum_{i=0}^{n-1}\frac{f(x_{i+1})-f(x_i)}{f(x_i)}\leqslant(L_2-\zeta_0)\sum_{i=0}^{n-1}\frac{\Delta x_i}{x_i}.$$

Since $f:[0,\infty)\to[0,\infty)$ is a continuous increasing function and by arranging $\sup_i\Delta x_i\to 0$ as $n\to\infty$, we get:

$$(L_2+\zeta_0)\int_{\xi_0}^{x}\frac{du}{u}\leqslant\int_{\xi_0}^{x}\frac{df(u)}{f(u)}\leqslant(L_2-\zeta_0)\int_{\xi_0}^{x}\frac{du}{u}.$$

After integrating and exponentiating we obtain the estimate:

$$\frac{f(\xi_0)}{\xi_0^{L_2+\zeta_0}}x^{L_2+\zeta_0}\leqslant f(x)\leqslant\frac{f(\xi_0)}{\xi_0^{L_2-\zeta_0}}x^{L_2-\zeta_0} \tag{9}$$

for $0<x<\xi_0$. Next we replace ζ_0 with an arbitrarily small ζ and let ξ replace ξ_0 so that Eq. (9) becomes:

$$\frac{f(\xi)}{\xi^{L_2+\zeta}}x^{L_2+\zeta}\leqslant f(x)\leqslant\frac{f(\xi)}{\xi^{L_2-\zeta}}x^{L_2-\zeta} \tag{10}$$

for $0<x<\xi$. Choosing $x=\xi$ in Eq. (9) and referring to (10) gives:

$$\frac{f(\xi_0)}{\xi_0^{L_2+\zeta_0}}\xi^{\zeta_0-\zeta}x^{L_2+\zeta}\leqslant f(x)\leqslant\frac{f(\xi_0)}{\xi_0^{L_2-\zeta_0}}\xi^{\zeta-\zeta_0}x^{L_2-\zeta}$$

for $0 < x < \xi$. So for all $0 < x < \xi^{\frac{\zeta_0}{\zeta}-1}$, we obtain

$$\frac{f(\xi_0)}{\xi_0^{L_2+\zeta_0}} x^{L_2+2\zeta} \leqslant f(x) \leqslant \frac{f(\xi_0)}{\xi_0^{L_2-\zeta_0}} x^{L_2-2\zeta}.$$

Note that $f(0) = 0$ since f is continuous. $\square$

Lemma 6. *Suppose the sequence x_n satisfies the conditions of Theorem* 4. *If* $\lim_{n\to\infty} x_{2n} = 0$ *and* $\lim_{n\to\infty} x_{2n+1} = a$, *then*

$$\lim_{n\to\infty} \frac{f(x_{2n+1}) - f(a)}{f(x_{2n})} = \frac{g(0)L_2(a)}{g(a)L_1(a)}$$

for all $a \geqslant 0$.

Proof. By (8), we have $f([1 + f(x_{2n+1})g(x_{2n+1})]x_{2n+2}) = f(x_{2n})$. For sufficiently large n, by condition (6∗) we obtain

$$g(a)f(x_{2n+1})f(x_{2n+2})\big[L_1(a) + \varepsilon_1(n)\big] = f(x_{2n}) - f(x_{2n+2}), \tag{11}$$

where $\varepsilon_1(n) \to 0$ as $n \to \infty$. Also

$$f\big(\big[1 + f(x_{2n+2})g(x_{2n+2})\big]x_{2n+3}\big) = f(x_{2n+1})$$

by (8). For sufficiently large n, by condition (7) we get:

$$g(0)f(x_{2n+2})f(x_{2n+3})\big[L_2(a) + \varepsilon_2(n)\big] = f(x_{2n+1}) - f(x_{2n+3}), \tag{12}$$

where $\varepsilon_2(n) \to 0$ as $n \to \infty$. Since

$$f(x_{2n+3}) = f\left(\frac{x_{2n+1}}{1 + f(x_{2n+2})g(x_{2n+2})}\right),$$

and $f(x_{2n+2}) \to 0$ as $n \to \infty$ by Lemma 5, then by condition (7), for sufficiently large n, we get

$$f(x_{2n+3}) = f(x_{2n+1}) - \varepsilon_3(n), \tag{13}$$

where $\varepsilon_3(n) \to 0$ as $n \to \infty$. From (12) and (13), we obtain

$$g(0)f(x_{2n+1})f(x_{2n+2})\big[L_2(a) + \varepsilon_4(n)\big] = f(x_{2n+1}) - f(x_{2n+3}), \tag{14}$$

where $\varepsilon_4(n) \to 0$ as $n \to \infty$. From (11) and (14) we see that

$$f(x_{2n+1}) - f(x_{2n+3}) = \left[\frac{g(0)L_2(a)}{g(a)L_1(a)} + \varepsilon_5(n)\right]\big[f(x_{2n}) - f(x_{2n+2})\big], \tag{15}$$

where $\varepsilon_5(n) \to 0$ as $n \to \infty$. We now sum both sides of (15):

$$\begin{aligned}
&\sum_{k=0}^{\infty}\left[\frac{g(0)L_2(a)}{g(a)L_1(a)} - \varepsilon_6(n)\right]\big[f(x_{2n+2k}) - f(x_{2n+2k+2})\big] \\
&\quad\leqslant \sum_{k=0}^{\infty}\big[f(x_{2n+2k+1}) - f(x_{2n+2k+3})\big] \\
&\quad\leqslant \sum_{k=0}^{\infty}\left[\frac{g(0)L_2(a)}{g(a)L_1(a)} + \varepsilon_6(n)\right]\big[f(x_{2n+2k}) - f(x_{2n+2k+2})\big],
\end{aligned} \tag{16}$$

where $\varepsilon_6(n) = \sup_{k \geqslant n} \varepsilon_5(k)$ or

$$\left[\frac{g(0)L_2(a)}{g(a)L_1(a)} - \varepsilon_6(n)\right] f(x_{2n}) \leqslant f(x_{2n+1}) - f(a) \leqslant \left[\frac{g(0)L_2(a)}{g(a)L_1(a)} + \varepsilon_6(n)\right] f(x_{2n})$$

which proves the lemma. □

Lemma 7. *Suppose f satisfies the conditions of Theorem 4, $y_0 = x_0 > 0$, and $y_1 = rx_1 > 0$ with $r > 1$. If $\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} y_{2n} = 0$, and $\lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} y_{2n+1} = a$ with $a \geqslant 0$, then*

$$\frac{f(y_{2n})}{f(x_{2n})} \leqslant 1 \quad \textit{and} \quad \frac{f(y_{2n+1}) - f(a)}{f(x_{2n+1}) - f(a)} \geqslant c_1$$

for some constant $c_1 > 1$ for all n sufficiently large. If $\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} y_{2n} = a$, and $\lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} y_{2n+1} = 0$ with $a \geqslant 0$, then

$$\frac{f(y_{2n}) - a}{f(x_{2n}) - a} \leqslant 1 \quad \textit{and} \quad \frac{f(y_{2n+1})}{f(x_{2n+1})} \geqslant c_2$$

for some constant $c_2 > 1$ for all n sufficiently large.

Proof. We show by induction that $y_{2n} \leqslant x_{2n}$ and $y_{2n+1} \geqslant rx_{2n+1}$ for all integers $n \geqslant 0$. Since fg is an increasing function, we have

$$y_{2n+2} = \frac{y_{2n}}{1 + f(y_{2n+1})g(y_{2n+1})} \leqslant \frac{x_{2n}}{1 + f(x_{2n+1})g(x_{2n+1})} = x_{2n+2}$$

and

$$y_{2n+3} = \frac{y_{2n+1}}{1 + f(y_{2n+2})g(y_{2n+2})} \geqslant \frac{rx_{2n+1}}{1 + f(x_{2n+2})g(x_{2n+2})} = rx_{2n+3}$$

and thus we have proved the induction hypothesis. Since f is an increasing function, we immediately see that $f(y_{2n}) \leqslant f(x_{2n})$ and $f(y_{2n+1}) \geqslant f(rx_{2n+1})$. By condition (7), for N sufficiently large and some $t > 1$ sufficiently close to 1, we have

$$f(rx_{2n+1}) \geqslant f(tx_{2n+1}) \geqslant \left[1 + \frac{L_2(a)}{2}(t-1)\right] f(x_{2n+1})$$

for all $n \geqslant N$. The conclusions of Lemma 7 now easily follow. □

Lemma 8. *Suppose f satisfies the conditions of Theorem 4, $y_0 = rx_0 > 0$ with $r > 1$, and $0 < y_1 \leqslant x_1$. If $\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} y_{2n} = 0$, and $\lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} y_{2n+1} = a$ with $a \geqslant 0$, then*

$$\frac{f(y_{2n})}{f(x_{2n})} \geqslant c_3 \quad \textit{and} \quad \frac{f(y_{2n+1}) - f(a)}{f(x_{2n+1}) - f(a)} \leqslant 1$$

for some constant $c_3 > 1$ for all n sufficiently large. If $\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} y_{2n} = a$, and $\lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} y_{2n+1} = 0$ with $a \geqslant 0$, then

$$\frac{f(y_{2n}) - a}{f(x_{2n}) - a} \geqslant c_4 \quad \textit{and} \quad \frac{f(y_{2n+1})}{f(x_{2n+1})} \leqslant 1$$

for some constant $c_4 > 1$ for all n sufficiently large.

Proof. The proof is similar to the proof of Lemma 7. □

Proof of Theorem 4. Lemma 5 and the properties that g is continuous and $g(0) > 0$ imply that fg satisfies the hypotheses for f in Theorem 1. By the beginning of the proof of Theorem 1, (x_{2n}, x_{2n+1}) converges to either $(0, a)$ or $(a, 0)$ for some $a \geqslant 0$. First consider $(0, a)$ as the point of convergence of the sequence with $a > 0$. We first show existence; namely, for every $x_0 \geqslant 0$ there exists $x_1 \geqslant a$ such that $(x_{2n}, x_{2n+1}) \to (0, a)$. Clearly $(x_{2n}, 0)$ converges to $(x_0, 0)$. We claim that for any given $\delta > 0$ we can pick $x_1^{(\delta)}$ sufficiently large so that starting with the initial condition $(x_0, x_1^{(\delta)})$, after N steps we will have $x_{2N} < \delta$ and $x_{2N+1}^{(\delta)} > 2a$ for some N. To justify this claim, observe that

$$x_{2N+1}^{(\delta)} = \frac{x_1}{\prod_{k=1}^{N}[1 + f(x_{2k})g(x_{2k})]} \geqslant \frac{x_1}{[1 + f(x_0)g(x_0)]^N}.$$

Now pick $x_1 \geqslant \max(2a, 1)[1 + f(x_0)g(x_0)]^N$ to insure that $x_{2N+1}^{(\delta)} \geqslant \max(2a, 1)$. We pick N sufficiently large so that

$$x_{2N} = \frac{x_0}{\prod_{k=1}^{N}[1 + f(x_{2k-1}^{(\delta)})g(x_{2k-1}^{(\delta)})]} \leqslant \frac{x_0}{[1 + f(1)g(1)]^N} < \delta$$

as claimed. For brevity, we will refer to the convergence of the sequence (x_{2n}, x_{2n+1}) with initial condition (x_0, x_1) as the convergence of the initial condition (x_0, x_1). Let us consider the convergence of the initial conditions $(x_{2N}, x_{2N+1}^{(\delta)})$ and $(0, x_{2N+1}^{(\delta)})$. Observe that $(x_0, x_1^{(\delta)})$ converges to the same point as $(x_{2N}, x_{2N+1}^{(\delta)})$. Since $(0, x_{2N+1}^{(\delta)})$ is a fixed point with $x_{2N+1}^{(\delta)} \geqslant 2a$, then by Theorem 1, by picking δ sufficiently small, $(x_{2N}, x_{2N+1}^{(\delta)})$ and hence $(x_0, x_1^{(\delta)})$ must converge to a point say $(0, b)$ such that $b > a$. Fix $x_0 \geqslant 0$. Let $x_1' = \inf\{x_1 \geqslant 0\colon\ (x_0, x_1) \to (0, b) \text{ with } b \geqslant a\}$. Since $(x_0, 0) \to (x_0, 0)$ and $(x_0, x_1^{(\delta)}) \to (0, b)$ with $b > a$, then by Lemma 2, x_1' must exist. There are two cases to consider. For the first case, suppose that $(x_0, x_1') \to (0, b')$ with $b' \geqslant a$. If $b' = a$, then we are done. So assume $b' > a$. By Theorem 1, we can choose $\delta > 0$ so small that $(x_0, x_1' - \delta) \to (0, b'')$ and $b'' > a$. But then $\inf\{x_1 \geqslant 0\colon\ (x_0, x_1) \to (0, b) \text{ with } b \geqslant a\} \leqslant x_1' - \delta$ which is a contradiction. For the second case, suppose that $(x_0, x_1') \to (0, b')$ with $b' < a$ or $(x_0, x_1') \to (b', 0)$ with $b' \geqslant 0$. By Theorem 1, we can choose $\delta > 0$ sufficiently small so that either $(x_0, x_1' + \delta) \to (0, b'')$ and $b'' < a$ or $(x_0, x_1' + \delta) \to (b'', 0)$ with $b'' \geqslant 0$. This implies that $\inf\{x_1 \geqslant 0\colon\ (x_0, x_1) \to (0, b) \text{ with } b \geqslant a\} \geqslant x_1' + \delta$ which is a contradiction. If $(a, 0)$ is the point of convergence with $a \geqslant 0$, the proof is similar.

Our second step is to show that the set of initial points that converge to $(0, a)$ for $a \geqslant 0$ is a function. We may assume that $x_0 > 0$, since if $x_0 = 0$ then it is obvious that we must have $x_1 = a$ in order for $(x_0, x_1) \to (0, a)$. Furthermore, we may also assume $x_1 > 0$ since if $x_0 > 0$ then it is obvious that we must have $x_1 > a$ in order for $(x_0, x_1) \to (0, a)$. It is sufficient to show that given initial conditions (x_0, x_1) and (y_0, y_1), if $y_0 = x_0$, $y_1 \geqslant x_1$, $\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} y_{2n} = 0$, and $\lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} y_{2n+1} = a$, then $y_1 = x_1$. Suppose $y_1 \geqslant r x_1$ with $r > 1$. By Lemma 6 we can conclude

$$\lim_{n\to\infty} \frac{[f(y_{2n+1}) - f(a)]f(x_{2n})}{[f(x_{2n+1}) - f(a)]f(y_{2n})} = 1.$$

But Lemma 7 implies

$$\frac{[f(y_{2n+1}) - f(a)]f(x_{2n})}{[f(x_{2n+1}) - f(a)]f(y_{2n})} \geqslant c_1 > 1$$

for all $n \geqslant N$ for N sufficiently large which is a contradiction. The proof for the case $(a,0)$ with $a > 0$ is similar. We point out that this case requires a modification of Lemma 6 with the conclusion that

$$\lim_{n\to\infty} \frac{f(x_{2n}) - f(a)}{f(x_{2n-1})} = \frac{g(0)L_2(a)}{g(a)L_1(a)}$$

when $\lim_{n\to\infty} x_{2n} = a$ and $\lim_{n\to\infty} x_{2n+1} = 0$.

Next, we show that this function, which we will denote by $h(x)$, is increasing. First consider the case that the convergence point is $(0,a)$ for $a \geqslant 0$. From the difference equation (8), it is obvious that $h(0) = a$ and $h(x_0) > a$ for $x_0 > 0$. Thus we only need to consider initial points (x_0, x_1) such that $x_0 > 0$ and $x_1 > 0$. We will assume that (x_0, x_1) and (y_0, y_1) are points on the graph of h with $0 < x_0 < y_0$ and $0 < y_1 \leqslant x_1$, and then show a contradiction. So $y_0 \geqslant rx_0$ with $r > 1$. By Lemma 6 we can conclude that

$$\lim_{n\to\infty} \frac{[f(y_{2n+1}) - f(a)]f(x_{2n})}{[f(x_{2n+1}) - f(a)]f(y_{2n})} = 1.$$

But Lemma 8 implies

$$\frac{[f(y_{2n+1}) - f(a)]f(x_{2n})}{[f(x_{2n+1}) - f(a)]f(y_{2n})} \leqslant \frac{1}{c_3} < 1$$

for all $n \geqslant N$ for N sufficiently large which is a contradiction. The proof for the case $(a,0)$ is similar.

Finally, we show that the function is continuous. Fix $a \geqslant 0$. Let $(x, h_1(x))$ be the set of initial conditions that converge to $(a,0)$, and let $(x, h_2(x))$ be the set of initial conditions that converge to $(0,a)$. Since h_1 and h_2 are increasing, in order to show that h_1 and h_2 are continuous, it is sufficient to show that $h_1 : [a,\infty) \to [0,\infty)$ and $h_2 : [0,\infty) \to [a,\infty)$ are each surjective. Choose any $c \in [0,\infty)$. Define $b = h_2(c)[1 + f(c)g(c)]$. Since the initial condition $(c, h_2(c))$ converges to $(0,a)$, then the initial condition (b,c) converges to $(0,a)$. Thus $h_1(b) = c$, which proves that h_1 is surjective. A similar proof shows that h_2 is surjective. $\square$

3. Instability and nonuniqueness results

We will construct an increasing continuous function f with $f(0) = 0$ such that there exist solutions to the difference equation (1) that are not stable. Furthermore, with the f we construct, the uniqueness result in Theorem 4 does not follow.

We use the following lemma to establish our counterexample.

Lemma 9. *Suppose F is a nonnegative nondecreasing function, $F(x) \leqslant 1$ for all $0 \leqslant x \leqslant c$, and $x_{n+1} = x_{n-1}/[1 + F(x_n)]$. If $0 \leqslant x_1 \leqslant x_0/2 \leqslant c$, then $x_{2n+1} \leqslant x_{2n}/2$ for all $n \geqslant 0$.*

Proof. By induction, we have

$$\begin{aligned} x_{2n+3} &= \frac{x_{2n+1}}{1 + F(x_{2n+2})} \leqslant \frac{x_{2n}}{2[1 + F(x_{2n+2})]} = \frac{x_{2n}}{2[1 + F(\frac{x_{2n}}{1+F(x_{2n+1})})]} \\ &\leqslant \frac{x_{2n}}{2[1 + F(\frac{2x_{2n+1}}{1+F(x_{2n+1})})]} \leqslant \frac{x_{2n}}{2[1 + F(x_{2n+1})]} = \frac{1}{2}x_{2n+2} \end{aligned}$$

as we wished to show. $\square$

Counterexample. We will construct an increasing continuous function f with $f(0)=0$ such that there are solutions to the difference equation (1) with the following property. For every x_0 with $0 < x_0 \leqslant 1/2$, there exists $\varepsilon(x_0) > 0$ such that whenever $0 < x_1 \leqslant \varepsilon(x_0)$, the solutions to the difference equation with these initial conditions (x_0, x_1) will all converge to the same limit point $(0,0)$. Thus the fixed point solution $(x_0, 0)$ is not stable for all $0 < x_0 \leqslant 1/2$. Furthermore, this also shows nonuniqueness of the solutions, since for a given x_0 with $0 < x_0 \leqslant 1/2$, there are an infinite number of initial conditions whose solutions converge to the same limit point $(0,0)$.

Proof. We define f as a limit of a sequence of functions f_k which we construct inductively. First let

$$f_1(x) = \begin{cases} x, & x > \frac{1}{4}, \\ \frac{1}{8}, & x \leqslant \frac{1}{4}. \end{cases}$$

Given $f_j(x)$ and initial condition $(x_0^{(k)}, x_1^{(k)})$, if we iterate with $x_{n+1} = x_{n-1}/[1+f_j(x_n)]$, we will let $(x_{j,2n}^{(k)}, x_{j,2n+1}^{(k)})$ denote the $2n$th and $(2n+1)$st terms of the sequence. However our actual construction will be accomplished in the following manner. Given $f_k(x)$ and initial condition $(x_0^{(k)}, x_1^{(k)}) = (1/2, x_{k-1,2n(k-1)+1}^{(k-1)})$, we will iterate with $x_{n+1} = x_{n-1}/[1+f_k(x_n)]$ thereby producing the $2n$th and $(2n+1)$st terms $(x_{k,2n}^{(k)}, x_{k,2n+1}^{(k)})$. We define $x_{0,2n(0)}^{(0)} = 1/2$ and $x_{0,2n(0)+1}^{(0)} = 1/4$. For $k \geqslant 1$, define $n(k)$ to be the smallest integer such that $x_{k,2n(k)}^{(k)} \leqslant 3x_{k-1,2n(k-1)}^{(k-1)}/4$. For $k \geqslant 1$, define

$$f_{k+1}(x) = \begin{cases} f_k(x), & x > x_1^{(k)}, \\ \frac{x - x_1^{(k)}}{2^{k+2}[x_1^{(k)} - x_{k,2n(k)+1}^{(k)}]} + \frac{1}{2^{k+1}}, & x_{k,2n(k)+1}^{(k)} < x \leqslant x_1^{(k)}, \\ \frac{1}{2^{k+3}}, & x \leqslant x_{k,2n(k)+1}^{(k)}, \end{cases}$$

and

$$f(x) = \lim_{k\to\infty} f_k(x) \tag{17}$$

for $x > 0$ and $f(0) = 0$. We will prove that the difference equation $x_{n+1} = x_{n-1}/[1 + f(x_n)]$, where f is defined by (17), has the following property. For every initial condition (x_0, x_1) such that $0 < x_0 \leqslant 1/2$ and $0 < x_1 \leqslant \varepsilon(x_0)$ for $\varepsilon(x_0) > 0$ sufficiently small, we have $\lim_{n\to\infty}(x_{2n}, x_{2n+1}) = (0,0)$.

We will show by induction that

$$x_{k,2n(k)+1}^{(k)} \leqslant x_{k+1,2n(k)+1}^{(k)} \leqslant x_{k+1,2n(k)}^{(k)} \leqslant x_{k,2n(k)}^{(k)}, \tag{18}$$

for all $k \geqslant 1$ and that

$$x_{k,2n(k)}^{(k)} \leqslant \frac{1}{2}\left(\frac{3}{4}\right)^k \tag{19}$$

for all $k \geqslant 0$. First note that $x_{k+1,2n(k)+1}^{(k)} \leqslant x_{k+1,2n(k)}^{(k)}$ by Lemma 9, as $x_{k+1,0}^{(k)} = x_0^{(k)} = 1/2$ and $x_{k+1,1}^{(k)} = x_1^{(k)} \leqslant 1/4$. Since $x_0^{(k)} = 1/2$ and $f_k(x) \geqslant 1/2^{k+2}$, there must exist a smallest integer $n(k) \geqslant 1$ such that

$$x_{k,2n(k)}^{(k)} \leqslant \frac{3}{4} x_{k-1,2n(k-1)}^{(k-1)} \leqslant \frac{1}{2}\left(\frac{3}{4}\right)^k$$

for $k \geqslant 1$. Using the facts that $x^{(k)}_{k,2n(k)-2} > 3x^{(k-1)}_{k-1,2n(k-1)}/4$, $x^{(k)}_{k,2n(k)-1} \leqslant x^{(k)}_1 = x^{(k-1)}_{k-1,2n(k-1)+1}$, and $f_k(x) = 1/2^{k+2}$ for $x \leqslant x^{(k-1)}_{k-1,2n(k-1)+1}$, it is easy to check that

$$x^{(k)}_{k,2n(k)} = \frac{x^{(k)}_{k,2n(k)-2}}{1 + f_k(x^{(k)}_{k,2n(k)-1})} > \frac{2}{3}x^{(k-1)}_{k-1,2n(k-1)} \geqslant \frac{4}{3}x^{(k-1)}_{k-1,2n(k-1)+1}, \tag{20}$$

where we used Lemma 9 in the final inequality in (20). We show by induction that $x^{(k)}_{k+1,2n} \leqslant x^{(k)}_{k,2n}$ and $x^{(k)}_{k+1,2n+1} \geqslant x^{(k)}_{k,2n+1}$ for all $n \leqslant n(k)$. We are given $x^{(k)}_{k+1,0} = x^{(k)}_{k,0} = x^{(k)}_0 = 1/2$ and $x^{(k)}_{k+1,1} = x^{(k)}_{k,1} = x^{(k)}_1$. Let us assume the induction hypothesis for a given $n \leqslant n(k) - 1$. We have

$$\begin{aligned} x^{(k)}_{k+1,2n+2} &= \frac{x^{(k)}_{k+1,2n}}{1 + f_{k+1}(x^{(k)}_{k+1,2n+1})} \leqslant \frac{x^{(k)}_{k+1,2n}}{1 + f_k(x^{(k)}_{k+1,2n+1})} \\ &= \frac{x^{(k)}_{k+1,2n}}{1 + f_k(x^{(k)}_{k,2n+1})} \leqslant \frac{x^{(k)}_{k,2n}}{1 + f_k(x^{(k)}_{k,2n+1})} = x^{(k)}_{k,2n+2} \end{aligned}$$

since $f_{k+1}(x) \geqslant f_k(x) = 1/2^{k+2}$ for $x^{(k)}_{k,2n(k)+1} < x \leqslant x^{(k)}_1$. We also have

$$x^{(k)}_{k+1,2n+3} = \frac{x^{(k)}_{k+1,2n+1}}{1 + f_{k+1}(x^{(k)}_{k+1,2n+2})} \geqslant \frac{x^{(k)}_{k+1,2n+1}}{1 + f_k(x^{(k)}_{k,2n+2})}. \tag{21}$$

This is due to the facts that if $x^{(k)}_{k+1,2n+2} > x^{(k)}_1$, then $f_{k+1}(x^{(k)}_{k+1,2n+2}) = f_k(x^{(k)}_{k+1,2n+2}) \leqslant f_k(x^{(k)}_{k,2n+2})$, while if $x^{(k)}_{k+1,2n+2} \leqslant x^{(k)}_1$, then $f_{k+1}(x^{(k)}_{k+1,2n+2}) \leqslant 1/2^{k+1} \leqslant f_k(x^{(k)}_{k,2n+2})$ since $x^{(k)}_{k,2n+2} \geqslant x^{(k)}_{k,2n(k)} \geqslant 4x^{(k-1)}_{k-1,2n(k-1)+1}/3 = 4x^{(k)}_1/3$ by (20) for $n \leqslant n(k) - 1$. By the induction hypothesis $x^{(k)}_{k+1,2n+1} \geqslant x^{(k)}_{k,2n+1}$ and so by (21) we obtain $x^{(k)}_{k+1,2n+3} \geqslant x^{(k)}_{k,2n+3}$ as desired. Putting everything together gives (18) and (19). Observe by the definitions of f and f_j that $f(x) = f_j(x) = f_{k+1}(x)$ for all $j \geqslant k + 1$ whenever $x^{(k)}_{k,2n(k)+1} \leqslant x \leqslant 1/2$. Since $x^{(k)}_{k,2n(k)+1} \to 0$ as $k \to \infty$, then clearly $f(x)$ is a continuous increasing function on $[0, \infty)$ with $\lim_{x\to 0} f(x) = 0$.

Now consider any initial condition $(1/2, x_1)$ with $0 < x_1 \leqslant 1/4$. Choose k sufficiently large so that the initial condition $(x^{(k)}_0, x^{(k)}_1)$ satisfies $x^{(k)}_1 \leqslant x_1$ and recall that we defined $x^{(k)}_0 = 1/2$. Iterate the initial conditions $(1/2, x_1)$ and $(x^{(k)}_0, x^{(k)}_1)$ with the difference equation $x_{n+1} = x_{n-1}/[1 + f(x_n)]$ where f is defined by (17). By Lemma 2, we obtain $\lim_{n\to\infty}(1/2, x_1) \succcurlyeq \lim_{n\to\infty}(x^{(k)}_0, x^{(k)}_1)$. If $(x^{(k)}_{2n}, x^{(k)}_{2n+1})$ are the $2n$th and $(2n + 1)$st terms of the solution to the difference equation with the function f and initial condition $(x^{(k)}_0, x^{(k)}_1)$, then $(x^{(k)}_{2n(k)}, x^{(k)}_{2n(k)+1}) = (x^{(k)}_{k+1,2n(k)}, x^{(k)}_{k+1,2n(k)+1})$. Since

$$0 \leqslant x^{(k)}_{k+1,2n(k)+1} \leqslant x^{(k)}_{k+1,2n(k)} \leqslant x^{(k)}_{k,2n(k)} \leqslant \frac{1}{2}\left(\frac{3}{4}\right)^k$$

and we may choose k to be arbitrarily large, we must have $\lim_{n\to\infty}(1/2, x_1) \succcurlyeq (0, 0)$. Clearly, $\lim_{n\to\infty}(1/2, x_1) \neq (0, b)$ for any $b > 0$, since $x^{(k)}_{2n(k)+1} \leqslant x^{(k)}_{2n(k)}/2$. Thus $\lim_{n\to\infty}(1/2, x_1) = (0, 0)$.

Finally, consider any x_0 with $0 < x_0 < 1/2$. Fix x_0. Then by our construction, we have

$$x^{(I+1)}_{I+1,2n(I+1)} \leqslant x_0 < x^{(I)}_{I,2n(I)}$$

for some $I \geqslant 0$. Thus for any (x_0, x_1) with $x_1 \leqslant x^{(I+1)}_{I+1,2n(I+1)+1}$, we must have

$$(0,0) = \lim_{n\to\infty}\big(x^{(I+1)}_{I+1,0}, x^{(I+1)}_{I+1,1}\big) = \lim_{n\to\infty}\big(x^{(I+1)}_{I+1,2n(I+1)}, x^{(I+1)}_{I+1,2n(I+1)+1}\big) \succcurlyeq \lim_{n\to\infty}(x_0, x_1).$$

Furthermore, it is obvious that $x^{(i)}_{i,2m(i)+2} \leqslant x_0 < x^{(i)}_{i,2m(i)}$ for some $m(i)$ and any $i \geqslant I$. Fix x_1 with $0 < x_1 < x^{(I+1)}_{I+1,2n(I+1)+1}$. Choose i large enough, say $J \geqslant I$, so that $x_1 > x^{(J)}_{J,2m(J)+1}$. Since $x_0 < x^{(J)}_{J,2m(J)}$, then

$$\lim_{n\to\infty}(x_0, x_1) \succcurlyeq \lim_{n\to\infty}\big(x^{(J)}_{J,2m(J)}, x^{(J)}_{J,2m(J)+1}\big) = \lim_{n\to\infty}\big(x^{(J)}_{J,0}, x^{(J)}_{J,1}\big) = (0,0).$$

Thus $\lim_{n\to\infty}(x_0, x_1) = (0,0)$ as we wished to show. □

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