ORE EXTENSIONS AND POLYCYCLIC GROUP RINGS

ALFRED GOLDIE AND GERHARD MICHLER

This paper studies some aspects of the relationship between a ring R and the extension ring $S = R[x, \alpha]$ obtained by adjoining an indeterminate x subject to the equation

$$xa = a^{\alpha} x, \quad a \in R,$$

where α is an assigned automorphism of the ring R.

In particular, it will be proved that when R is right Noetherian and is a Jacobson ring then S has the same properties. Now S can be localised at powers of the indeterminate x and the ring S_X, so obtained, inherits the same properties. This essentially proves a theorem due to Roseblade [**12**], that the group ring of a finitely generated polycyclic group over a commutative Jacobson ring is also a Jacobson ring.

Finally, some remarks are added regarding what can be proved for $R[x, \alpha]$, without supposing that R is right Noetherian. Asterisks are added to those results throughout the paper which appear to depend on R being right Noetherian (or some positive weaker condition).

The ring R is supposed to have unit element and be right Noetherian (max$-r$). It is a Jacobson ring when each of its prime ideals is an intersection of primitive ideals.

The Ore extension $S = R[x, \alpha]$, where α is an automorphism of R, also has max$-r$. A typical element of S has the forms

$$f(x) = f_0 + f_1 x + \ldots + f_n x^n = f_0 + x f_1^{\alpha^{-1}} + \ldots + x^n f_n^{\alpha^{-n}},$$

where $n \geqslant 0$ and $f_i \in R$. The automorphism is extended to S by setting $x = x^{\alpha}$ so that

$$f^{\alpha}(x) = f_0^{\alpha} + f_1^{\alpha} x + \ldots + f_n^{\alpha} x^n.$$

As $xf(x) = f^{\alpha}(x)x$, it follows that xS is an ideal of S.

A right ideal I of R (or of S) is *α-invariant* (or *α-right ideal*) when $I \supseteq I^{\alpha}$. Since $I \subseteq I^{\alpha^{-1}} \subseteq I^{\alpha^{-2}} \ldots$, the Noetherian property implies that $I = I^{\alpha}$.

An α-ideal T of S (or of R) is *α-prime* when $AB \subseteq T$ where A, B are α-ideals implies that either $A \subseteq T$ or $B \subseteq T$. The ring S is an *α-prime* ring, when 0 is an α-prime ideal.

The following elementary properties are easy to prove and useful.

1. An annihilator ideal of S is an α-ideal.

For let $AB = 0$, where A, B are ideals of S. Then

$$0 = AxB = xA^{\alpha^{-1}}B = AB^{\alpha}x.$$

Hence $0 = A^{\alpha}B = AB^{\alpha}$, so that an annihilator ideal is α-invariant.

2. Let T be an ideal of S (or R) and $N(T)$, $J(T)$ be the nil (Jacobson) radicals of T. Here $N(T)$ is the inverse image in S of the nil radical $N(T)/T$ of the ring S/T

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and $J(T)$ is defined likewise. Clearly $J(T^{\alpha}) = (J(T))^{\alpha}$ and $N(T^{\alpha}) = (N(T))^{\alpha}$. It follows that $N(T)$ and $J(T)$ are α-ideals when T is an α-ideal.

3. Let H be a right ideal of S such that $RH = H$ and $H = H^{\alpha}$. Then H is an ideal of S.

For $xH = H^{\alpha}x = Hx \subseteq H$.

4*. An α-prime ideal P of S (or R) is a semi-prime ideal.

For $(N(P))^{\rho} \subseteq P$ implies that $N(P) = P$ since $N(P)$ is an α-ideal.

5*. An α-prime ideal Q of S (or R) has the form $P \cap P^{\alpha} \cap P^{\alpha^2} \cap \ldots$, where P is a prime ideal and the intersection is finite.

For Q is a semi-prime ideal and $Q = P_1 \cap \ldots \cap P_n$, where the P_i are the minimal primes of Q. Clearly $P_i^{\alpha k}$ are also minimal primes of Q and have to be in the list $P_1, \ldots, P_n$. Grouping together, we have $Q = W_1 \cap \ldots \cap W_k$, where each W_i has the form $P \cap P^{\alpha} \cap P^{\alpha^2} \cap \ldots$ (a finite number of terms). Now Q is α-prime, as are the W_i, hence $Q = W_i$, for some index i.

6. When T is an α-ideal of S, then the automorphism α induces naturally an automorphism α in the ring R/T given by $[r+T]^{\alpha} = [r^{\alpha}+T]$, $r \in R$.

Examples

1. Let $S = R[x]$, $\alpha = idy$. The main theorem is classical (see Goldman [**4**]) when R is commutative and may well be known for general R. The proof given here is valid for commutative R, even non-Noetherian.

2. Let R be a division ring and $S = R[x, \alpha]$. Then S is a principal right (and left) ideal domain and its structure depends on whether α has infinite or finite order. In the former case S is a primitive ring (but not of finite dimension over its centre) and the only ideals are $x^k S$, $k = 0, 1, 2, \ldots$. In the latter case the ideal structure is a little more complicated. (See Jacobson [**6**; p. 40]).

3. $R = F \oplus F$, where F is a field and define $(f_1, f_2)^{\alpha} = (f_2, f_1)$. Note that S is a prime ring but R is only an α-prime ring.

4. Let G be a group, H a normal subgroup and G/H be cyclic generator x. The group ring ΦG over a ring Φ can be written as

$$\Phi G = \sum_{i=-\infty}^{\infty} \Phi H x^i.$$

Define $\alpha : \Phi H \to \Phi H$ by means of

$$y^{\alpha} = xyx^{-1}; \quad y \in \Phi H.$$

Then $\Phi G = S_X$, where $S = R[x, \alpha]$ and $R = \Phi H$, the subscript X denoting localisation at powers of x. This is of classical form since $xS = Sx$ and the elements of S_X have the form sx^{-k}, $s \in S$, $k \geqslant 0$.

Set

$$\mathscr{C}'(0) = (c \in R / ca = 0, \quad a \in R \Rightarrow a = 0)$$

$$'\mathscr{C}(0) = (c \in R / ac = 0, \quad a \in R \Rightarrow a = 0).$$

When R is a semi-prime ring with max-r we know that $\mathscr{C}'(0) \subseteq {}'\mathscr{C}(0)$. The regular elements of R are in $\mathscr{C}'(0) \cap {}'\mathscr{C}(0) = \mathscr{C}(0)$. In like manner these can be defined relative to an ideal T of R. Thus

$$\mathscr{C}'(T) = (c \in R/ca \in T, \quad a \in R \Rightarrow a \in T)$$

and so on*. Corresponding notation is needed for the ring S and the subscript S is used, thus $\mathscr{C}'_S(T)$, when T is an ideal of S.

Lemma (1.1). *Let A be an α-ideal of $S = R[x, \alpha]$. Then*

$$\frac{S}{(A \cap R)S} \approx \frac{R}{A \cap R}[x, \alpha].$$

Proof. The isomorphism is

$$[a_0 + a_1 x + \dots + a_n x^n + (A \cap R)S] \leftrightarrow [a_0 + (A \cap R)] + [a_1 + (A \cap R)]x + \dots.$$

Lemma (1.2). *Let P be a prime ideal of S. Then, either*

(i) *$x \in P$ and $P = (P \cap R) + xS$, or*

(ii) *$x \in \mathscr{C}_S(P)$ and $P = P^\alpha$.*

Proof. When $x \in P$ then (i) is clear. When $x \notin P$ then $xf(x) \in P$ implies that

$$Sxf(x) = xsf(x) \subseteq P.$$

As P is a prime and $x \notin P$ then $f(x) \in P$. Also $x \in \mathscr{C}_S(P)$.
Now $P^\alpha x = xP \subseteq P$ and the primeness of P gives $P^\alpha \subseteq P$ and $P^\alpha \supseteq P$.

Lemma (1.3). *Let P be a prime ideal of S, such that $P = P^\alpha$; then $(P \cap R)$ is an α-prime ideal of R.*

Proof. When $x \in P$ then

$$S/P \approx \frac{R}{P \cap R}$$

and $(P \cap R)$ is both prime and α-invariant in R.

When $x \in \mathscr{C}_S(P)$ then $P = P^\alpha$. Let A, B be ideals of R with $A = A^\alpha$, $B = B^\alpha$ and $AB \subseteq (P \cap R)$. Then $SB = BS$ and

$$ASB = ABS \subseteq P.$$

Thus either $A \subseteq P$ or $B \subseteq P$ and $A \subseteq P \cap R$ or $B \subseteq P \cap R$.

Lemma (1.4)*. *Let R be a semi-prime ring. Then S satisfies the right Ore condition with respect to $\mathscr{C}(0)$.*

Proof. Let $c \in \mathscr{C}(0)$ and $f(x) = f_0 + f_1 x + \ldots + f_n x^n \in S$. Now $d_0 \in \mathscr{C}(0)$ and $b_0 \in R$ exist with $f_0 d_0 = cb_0$. Thus

$$f(x) d_0 = cb_0 + f_1 d_0^{\alpha} x + \ldots + f_n d_0^{\alpha^n} x^n.$$

Repetition gives

$$f(x) d_0 d_1^{\alpha^{-1}} \ldots d_n^{\alpha^{-n}} = cg(x),$$

say, where $d_i \in \mathscr{C}(0)$ and $g(x) \in S$. As $\mathscr{C}(0)$ is α-invariant the element $d_0 d_1^{\alpha^{-1}} \ldots d_n^{\alpha^{-n}}$ is regular and the Ore condition holds.

LEMMA (1.5)*. *Let R be an α-prime ring and T be a non-zero α-ideal of R. Then T has a regular element of R.*

Proof. Let I be a right ideal of R and $T \cap I = 0$. Then $IT = 0$. If $I \neq 0$ then $l(T) \neq 0$ and is an α-ideal. Yet R is an α-prime ring. It follows that T is an essential right ideal of R and contains a regular element.

LEMMA (1.6)*. *Let R be a semi-prime ring and P be a prime ideal of S such that $P \cap R = 0$. Then $\mathscr{C}(0) \subseteq \mathscr{C}_S(P)$.*

Proof. Let $A = (f(x) \in S \mid f(x)c \in P$, some $c \in \mathscr{C}(0))$. Since S has the right Ore condition with respect to $\mathscr{C}(0)$, it follows that A is an ideal. If $A \nsubseteq P$ then A meets $\mathscr{C}_S(P)$, which contradicts the definition of A with $P \cap R = 0$. Hence $\mathscr{C}(0) \subseteq {}'\mathscr{C}_S(P)$.

Let $c \in \mathscr{C}(0)$ and observe that the ascending sequence $c^{-n} P$, where $n = 0, 1, 2, \ldots$ is stationary from $n = k$, say. Here $c^{-n} P = (\phi(x) \in S \mid c^n \phi(x) \in P)$. Let $c^k \phi(x) \in P$. By Lemma (1.4) there are $\psi(x) \in S$ and $d \in \mathscr{C}(0)$ such that $\phi(x) d = c\psi(x)$. Then $c^{k+1} \psi(x) \in P$, hence $c^k \psi(x) \in P$, and $c^{k-1} \phi(x) d \in P$. Using the first part of the proof, we deduce that $c^{k-1} \phi(x) \in P$. Hence $c^{-k} P = c^{-k+1} P$ and this type of reduction continued eventually gives $c^{-1} P = P$, so that $c \in \mathscr{C}'_S(P)$ and the lemma is proved.

Let A be an ideal of S. Define $\tau(A)$ to be the set of elements of R which are leading coefficients of elements of A, together with zero. Define $\tau_0(A)$ to be the subset of $\tau(A)$ for polynomials of minimal degree in A. Clearly $\tau(A)$ and $\tau_0(A)$ are ideals of R. Moreover if B is an α-ideal of S then

$$\tau(A) \cap \tau(B) \supseteq \tau(AB) \supseteq \tau(A)\, \tau(B).$$

LEMMA (1.7). *Let P be a prime ideal of S with $P = P^{\alpha}$; then $(P \cap R)S$ is a prime ideal of S.*

Proof. As $(P \cap R) = (P \cap R)^{\alpha}$ then $x(P \cap R) = (P \cap R)x$, so that $(P \cap R)S$ is an ideal of S. Using

$$\frac{S}{(P \cap R)S} \approx \frac{R}{P \cap R} [x, \alpha],$$

we can regard R as an α-prime ring, P is prime in S and $P \cap R = 0$, with $S = R[x, \alpha]$. It remains to prove that S is a prime ring.

Let $AB = 0$, where A, B are ideals of S. Taking $A = l(B)$ and $B = r(A)$, we can suppose that A and B are α-ideals. Their coefficient ideals satisfy

$$0 = \tau(AB) \supseteq \tau(A)\tau(B).$$

As $\tau(A)$ and $\tau(B)$ are α-ideals of the α-prime ring R, either $\tau(A) = 0$ and $A = 0$, or $\tau(B) = 0$ and $B = 0$. S is a prime ring.

Lemma (1.7) is essentially due to Jategaonkar [8].

Note that when $x \in P$ it may happen that $P \neq P^{\alpha}$ and the lemma fails because $(P \cap R)S$ is not a left ideal. Thus in Example 2, set $P = xS + F_1$, $R = F_1 \oplus F_2$, $F_1 \approx F_2$, then $F_1 S \not\supseteq SF_1$.

LEMMA (1.8). *Let P be an α-ideal of S with $P \cap R = 0$, and set $T = \tau_0(P)$. For any $\phi(x) \in S$ and $a \in \tau_0(P)$ there exists $r(x) \in S$ and $k > 0$ such that*

$$\phi(x)a^k \equiv r(x) \pmod{P}$$

where degree $r(x) <$ minimal degree of polynomials in P and also $k <$ degree $\phi(x)$.

Proof. Let $f(x) = a_0 + a_1 x + \dots + a_s x^s$ be of minimal degree in P and let $\phi(x) = \phi_0 + \phi_1 x + \dots + \phi_n x^n$. Take $n \geqslant s$. Set

$$\phi(x)\, a - \phi_n x^{n-s} f(x) = \psi(x).$$

Now degree $\psi(x) <$ degree $\phi(x)$ when $a_s = a^{\alpha^{-s}}$. Make this choice and apply the process again to $\psi(x)$. We obtain an algorithm of the form

$$\phi(x)\, a^k = \sum_{i=1}^{t} q_i(x) f_i(x) + r(x)$$

where $f_i(x) \in P$ and $f_i(x) = f(x)^{\alpha^{\nu}}$ for some integer ν dependent on i. The lemma follows.

LEMMA (1.9)*. *Let R be an α-prime ring and $f(x) = a_0 + ax + \dots + ax^s$, where $a \in \mathscr{C}(0)$ and $\phi(x) \in S$. There exists $b \in \mathscr{C}(0)$ and $q(x)$, $r(x) \in S$ such that*

$$\phi(x)b = f(x)q(x) + r(x),$$

where degree $r(x) <$ degree $f(x)$.

Proof. Let $\phi(x) = \phi_0 + \phi_1 x + \dots + \phi_n x^n$. There exist $b \in \mathscr{C}(0)$ and $q_{n-s} \in R$ such that $\phi_n b^{\alpha^n} = a q_{n-s}^{\alpha^s}$, using the right Ore condition. Then

$$\phi(x)b - f(x)\, q_{n-s} x^{n-s} = \psi(x)$$

has degree less than degree $\phi(x)$. The process repeated on $\psi(x)$, and so on, gives the result.

LEMMA (1.10)*. *Let P be a prime ideal of the prime ring S such that $P \cap R = 0$; then any ideal properly containing P meets $\mathscr{C}(0)$.*

Proof. Certainly $P = P^\alpha$ since either $x \notin P$ or $P = xS$. As $\mathscr{C}(0)$ is a multiplicatively closed set of elements there is a prime ideal $M \supseteq P$ which is maximal among those ideals which do not meet $\mathscr{C}(0)$. Clearly $M = M^\alpha$.

Let $f(x) \in P$ be of minimal degree in P and $g(x)$ be of minimal degree in M. Let $M \supsetneq P$ and $\phi(x) \in M \cap \mathscr{C}_S(P)$. Now

$$\phi(x)b = g(x)q(x) + r(x),$$

where $b \in \mathscr{C}(0)$ and degree $r(x) <$ degree $g(x)$. Also $b \in \mathscr{C}(0) \subset \mathscr{C}_S(P)$; hence $g(x)q(x) \in \mathscr{C}_S(P)$ and then $g(x) \in \mathscr{C}_S(P)$.

Now, again using the algorithm,

$$f(x)c = g(x)\psi(x) + r(x),$$

where degree $r(x) <$ degree $g(x)$ and $c \in \mathscr{C}(0)$. Now degree $r(x) <$ degree $g(x)$ yet $r(x) \in M$. It follows that $r(x) = 0$ and that

$$\text{degree} f(x) = \text{degree}\, g(x) + \text{degree}\, \psi(x),$$

because the highest coefficient of $g(x)$ is regular. However, $g(x)\psi(x) \in P$ and $g(x) \in \mathscr{C}_S(P)$. Hence $\psi(x) \in P$ and so degree $\psi(x) \geqslant$ degree $f(x)$. Thus degree $g(x) \leqslant 0$ and $M \cap R \neq 0$.

THEOREM (1.11)*. *Let R be a Jacobson ring; then $S = R[x, \alpha]$ is a Jacobson ring.*

Proof. Let P be a prime ideal of S. If $x \in P$, then

$$S/P \approx \frac{R}{P \cap R};$$

hence S/P is a Jacobson ring and $P = J(P)$.

If $x \notin P$, then $P = P^\alpha$ and

$$\frac{S}{(P \cap R)S} \approx \frac{R}{P \cap R}[x, \alpha].$$

Replace R by $R/(P \cap R)$ and reduce to the case when S is a prime ring and P is a prime ideal which satisfies $P \cap R = 0$.

Now suppose that $P = 0$. Let $a(x) = a_0 + a_1 x + \dots + a_n x^n \in J(0)$ and $r \in R$; then, for some $\phi(x) \in S$,

$$(1 + a(x)r)\phi(x) = 1.$$

Comparing constant coefficients gives $(1 + a_0 r)\phi_0 = 1$ and $a_0 \in J_R(0)$, the Jacobson radical of R. Hence $a_0 = 0$, $n > 0$ and a_n is a zero divisor in R. Thus $\tau(J(0))$ has only zero divisors, yet $\tau(J(0))$ is an α-ideal of the α-prime ring R and has a regular element by Lemma (1.5).

It follows that $\tau(J(0)) = 0$ and $J(0) = 0$.

Now let $P \neq 0$ and $P \cap R = 0$; then again $P = p^\alpha$. Suppose that $J(P) \supsetneq P$; then $T = J(P) \cap R \neq 0$ by Lemma (1.10). Evidently T and $\tau_0(P)$ are α-ideals.

As R is semi-primitive the set $\mathscr{N}$, consisting of those primitive ideals Q of R with $T \nsubseteq Q$, is non-empty. Let $Q \in \mathscr{N}$ and Q be the bound of the maximal right ideal

M of R. Choose $a \in T, a \notin M$. Then

$$1 = m + ab, \quad m \in M, \quad b \in R.$$

Since $a \in J(P)$ there exists $\phi(x)$ such that

$$(1-ab)\,\phi(x) \equiv 1 \pmod{P}.$$

Let $c \in \tau_0(P)$ and, using Lemma (1.8), let $\phi(x)\,c^n \equiv r(x) \pmod{P}$ for some $r(x) \in S$ having degree less than the minimal degree of elements in P. Then $mr(x) - c^n \in P$; consequently $r(x) = r_0$ has degree zero and $mr_0 = c^n$ follows. Thus $c^n \in M$.

Suppose that $c \notin M$, and $1 = ct + m$ for some $t \in R$. Now $ct \in \tau_0(P)$ and the previous argument for c applies to ct.

Suppose that $(ct)^s \in M$; then

$$(ct)^{s-1} = m(ct)^{s-1} + (ct)^s \in M,$$

so that $(ct)^{s-1} \in M$. Consequently, $ct \in M$, which is a contradiction. It follows that $c \in M$; hence $\tau_0(P) \subseteq M$ and $\tau_0(P) \subseteq Q$. Since R is an α-prime ring we have

$$0 = (\textstyle\bigcap Q;\, Q \in \mathcal{N}).$$

Hence $\tau_0(P) = 0$, a contradiction. Hence $J(P) \cap R = 0$ and $J(P) = P$, which proves the theorem.

THEOREM (1.12)*. *Let S_X be obtained from S by localising at the powers of x. Then S_X is a Jacobson ring.*

Proof. The automorphism α is taken as extended to S using $x^\alpha = x$. Let $\beta = \alpha^{-1}$. Form $T = S[y, \beta]$, where $ys = s^\beta y$. Since

$$T(xy-1) = (xy-1)T,$$

it follows that

$$\frac{T}{(xy-1)T} \approx S_X.$$

Now T is a Jacobson ring by Theorem (1.11); hence S_X is a Jacobson ring.

COROLLARY. *The group ring of a polycyclic group over a right Noetherian Jacobson ring is a Jacobson ring.*

It was proved by Roseblade [**13**] that the group ring of a polycyclic-by-finite group over a commutative Noetherian Jacobson ring is a Jacobson ring. Our corollary follows by induction from Theorem (1.12) and the observation that, for extension by a finite cycle, the group ring is a factor ring of one for an infinite cycle.

So far we have readily assumed that the ground ring R has max-r, but examination of the methods used shows that they apply to the case when R is any commutative ring and α is the identity automorphism. For an arbitrary ring R with unit the problem still holds some interest, as is shown by the following theorem.

THEOREM (1.13). *Let $S = R[x, \alpha]$, where R is any ring with unit. Let $\mathscr{P}$ be the set of prime ideals P of S such that $P \cap R$ is fixed. Then any chain of prime ideals of $\mathscr{P}$ is of finite length.*

Proof. Let $x \in P$ then $P = xS+(P \cap R)$ is uniquely defined. Let $x \notin P$; then $P = P^{\alpha}$ and passing to the factor ring

$$\frac{S}{(P \cap R)S} \approx \frac{R}{P \cap R}[x, \alpha]$$

we can assume that $P \cap R = 0$. The ideal $(0) \in \mathscr{P}$ by Lemma (1.7).

Suppose that $P \subseteq Q$ are primes of S such that $P \cap R = Q \cap R = 0$ and the minimal degrees of polynomials in P and Q are equal. Let $f(x)$ be of minimal degree in P and $g(x)$ be likewise in Q, where

$$f(x) = f_0+f_1x+ \dots +f_n x^n; \quad g(x) = g_0+g_1x+ \dots +g_n x^n.$$

Then $f(x)\, g_n{}^{\alpha^{-n}} = f_n g(x)$ by minimality of degree. Now $rg(x)$, where $r \in R$, is either zero or of minimal degree in Q. Hence $f_n Rg(x) \subseteq P$. Also $g^{\alpha}(x) \in Q$ and so

$$f_n Rxg(x) \subseteq P$$

and so on. Hence $f_n Sg(x) \subseteq P$ with the consequence that $g(x) \in P$, since $f_n \notin P$.

Let P_0, Q_0 be the sets of polynomials of minimal degree n in P, Q respectively and let P_k, Q_k be the sets of polynomials of degree $\leqslant (n+k)$ in P, Q respectively. We have proved that $P_0 = Q_0$. Now $P_i = P_i{}^{\alpha}$ and P_i is an R-submodule of P, with R acting on both sides. Hence $SP_i = P_iS$. Corresponding remarks hold for Q_i. Let $\phi(x) \in Q_i$ but $\phi(x) \notin Q_{i-1}$. Since

$$\phi(x)\, g_n{}^{\alpha^{-n}} \equiv \phi_n x^i g(x) \pmod{Q_{i-1}},$$

we have that $\phi(x)\, g_n{}^{\alpha^{-n}} \in P$, provided that $P_{i-1} = Q_{i-1}$. Hence $Q_i\, g_n{}^{\alpha^{-n}} \subset P$ and so $Q_i\, Sg_n{}^{\alpha^{-n}} \subseteq P$. Hence $Q_i \subseteq P$ and $Q_i = P_i$. It follows that $Q = P$.

Let $P_1 \subset P_2 \subset \dots$ be a strictly ascending chain of primes with $P_k \cap R = 0$ for $k = 1, 2, \dots$. Since the minimal degrees of polynomials must decrease as k increases, the chain is finite and the theorem is proved.

Let Q be maximal in $\mathscr{P}$. Any α-ideal T of S strictly larger than Q satisfies $T \cap R \neq 0$. For the set of α-ideals K of S such that $K \cap R = 0$ has maximal elements by Zorn's lemma and these are prime ideals, hence lie in $\mathscr{P}$. Thus if $J(Q) \neq Q$ then $J(Q) \cap R \neq 0$ and the main part of the proof of Theorem (1.12) can be started. It can only be completed, however, when the α-prime ring R is known to be semi-prime so that it can be taken to be semi-primitive. This is not an obstacle when α is the identity and Theorem (1.12) holds in this case. The part when $Q = 0$ is due to Herstein [5] and the rest may be known.

In Theorem (1.12) a situation is being considered where $\mathscr{P}$ consists only of the zero prime and the maximal primes Q subject to $Q \cap R = 0$. This part depends on Lemma (1.10) and apparently depends on some weak maximum condition for R. However, some special situations lead to the same result.

THEOREM (1.14). *Let R be a simple ring with unit. Then $R[x, \alpha]$ is a Jacobson ring.*

Proof. The algorithm given in Lemma (1.9) holds with $a = b = 1$. The ideal. of $R[x, \alpha]$ are principal as right ideals and the non-zero members of $\mathscr{P}$ are maximal The main argument of Theorem (1.12) easily applies.

Added in proof. G. Bergmann has now proved that in Theorem (1.13) the length is $\leqslant 2$.

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University of Leeds. University of Giessen.