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Poisson Hopf algebra related to a twisted quantum group

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ABSTRACT

A Poisson algebra $\mathbb{C}[G]$ considered as a Poisson version of the twisted quantized coordinate ring $\mathbb{C}_{q,p}[G]$, constructed by Hodges et al. [11], is obtained and its Poisson structure is investigated. This establishes that all Poisson prime and primitive ideals of $\mathbb{C}[G]$ are characterized. Further it is shown that $\mathbb{C}[G]$ satisfies the Poisson Dixmier-Moeglin equivalence and that Zariski topology on the space of Poisson primitive ideals of $\mathbb{C}[G]$ agrees with the quotient topology induced by the natural surjection from the maximal ideal space of $\mathbb{C}[G]$ onto the Poisson primitive ideal space.

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Introduction

There are many evidences that Poisson structures of Poisson algebras are heavily related to algebraic structures of their quantized algebras. Hodges and Levasseur [10], Hodges et al. [11] and Joseph [14] constructed bijections between the primitive ideal space of the quantized coordinate ring of a semisimple Lie group G and the set of symplectic leaves in G with a Poisson bracket arising from the quantization process. In this case, symplectic leaves correspond to the Poisson primitive ideals in the coordinate ring of G. The author constructed the Poisson symplectic algebra and the multiparameter Poisson Weyl algebra such that their Poisson spectra are homeomorphic to the spectra of the quantized symplectic algebra and the multiparameter quantized Weyl algebra, respectively [17] [20]. Moreover, we find that Poisson structures of Poisson algebras are almost the same as the algebraic structures of their quantized algebras [5, 7–9, 21], [6, Section 2] and [12, Section 3.3].

Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra associated with a connected semisimple Lie group G and let $r = \sum_{\alpha\in\mathbf{R}^+} x_\alpha \wedge x_{-\alpha} \in \bigwedge^2 \mathfrak{g}$, where x_α are root vectors of $\mathfrak{g}$ such that $(x_\alpha \mid x_{-\beta}) = \delta_{\alpha\beta}$. Then r is called the standard classical r-matrix, the quantized universal enveloping algebra $U_q(\mathfrak{g})$ is a quantization of the enveloping algebra $U(\mathfrak{g})$ by r and the quantized function algebra $\mathbb{C}_q[G]$ is obtained as an algebra of functions on $U_q(\mathfrak{g})$. Let $\mathfrak{b}^\pm$ be the Borel subalgebras of $\mathfrak{g}$. Then the pair of subalgebras $U_q(\mathfrak{b}^+)$ and $U_q(\mathfrak{b}^-)$ of $U_q(\mathfrak{g})$ is a dual pair of Hopf algebras. Let u be an alternating form on the dual space $\mathfrak{h}^*$, where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. Hodges et al. [11] constructed a dual pair of Hopf algebras $U_{q,p^{-1}}(\mathfrak{b}^+)$ and $U_{q,p^{-1}}(\mathfrak{b}^-)$ by twisting that pair in the case when u is algebraic and obtained the Drinfeld double $D_{q,p^{-1}}(\mathfrak{g}) = U_{q,p^{-1}}(\mathfrak{b}^+) \bowtie U_{q,p^{-1}}(\mathfrak{b}^-)$. Finally they found a Hopf algebra $\mathbb{C}_{q,p}[G]$, called the multiparameter quantized coordinate ring, as an algebra of functions on $D_{q,p^{-1}}(\mathfrak{g})$, and proved that the prime and primitive ideals of $\mathbb{C}_{q,p}[G]$ are indexed by the elements of the double Weyl group $W \times W$ using a natural action of the torus H associated with $\mathfrak{h}$ and the adjoint action of a Hopf algebra. From this Hopf algebra $\mathbb{C}_{q,p}[G]$ arises the question: Is there a Poisson Hopf algebra related to $\mathbb{C}_{q,p}[G]$? A main aim of this paper is to give a solution for this question. Here we find a Poisson Hopf algebra $\mathbb{C}[G]$ considered as a Poisson version of $\mathbb{C}_{q,p}[G]$ and establish that the Poisson structure of $\mathbb{C}[G]$ is an analog to the algebraic structure of $\mathbb{C}_{q,p}[G]$.

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In Section 1, we construct a Lie algebra $\mathfrak{d}$ such that the enveloping algebra $U(\mathfrak{d})$ may be considered as a classical algebra of $D_{q,p^{-1}}(\mathfrak{g})$. In Section 2, we prove that the standard r-matrix r still makes $\mathfrak{d}$ a Lie bialgebra, find the Poisson bracket $\{\cdot,\cdot\}_r$ on an algebra $\mathbb{C}[G]$ of functions on $U(\mathfrak{d})$, and get finally a Poisson bracket $\{\cdot,\cdot\}$ on $\mathbb{C}[G]$ by twisting $\{\cdot,\cdot\}_r$ using a skew symmetric bilinear form u on $\mathfrak{h}^*$, that makes $\mathbb{C}[G]$ a Poisson Hopf algebra. In Section 3, we establish that the Poisson prime ideals of $\mathbb{C}[G]$ are indexed by the elements of the double Weyl group $W \times W$. In Section 4, we define the Poisson adjoint action of a Poisson Hopf algebra and prove that the Poisson central elements are the fixed elements under the Poisson adjoint action. Finally, in Section 5, we show that the Poisson structure of $\mathbb{C}[G]$ is an analog to the algebraic structure of $\mathbb{C}_{q,p}[G]$ using the Poisson adjoint action defined in the Section 4. Moreover, we prove that $\mathbb{C}[G]$ satisfies the Poisson Dixmier-Moeglin equivalence and that the Poisson primitive ideal space of $\mathbb{C}[G]$ is a quotient space of its classical space.

Several parts of the paper are modified from those of [11] using Poisson terminologies because $\mathbb{C}[G]$ is a good Poisson analogue of the quantum group $\mathbb{C}_{q,p}[G]$ in [11], all fields are of characteristic zero, and vector spaces are over the complex number field $\mathbb{C}$ unless stated otherwise. Moreover, all Poisson algebras considered here are commutative.

A Poisson ideal P of A is said to be *Poisson prime* if, for all Poisson ideals I and J, $IJ \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$. Note that if A is noetherian, then a Poisson prime ideal of A is a prime ideal [5, Lemma 1.1(d)]. A Poisson ideal P is said to be *Poisson primitive* if there exists a maximal ideal M such that P is the largest Poisson ideal contained in M. Note that Poisson primitive is Poisson prime.

1. Algebra of functions

1.1. Notation

Here we recall well-known facts [15], which are summarized [12, Chapter 2]. Let $C = (a_{ij})_{n\times n}$ be an indecomposable and symmetrizable generalized Cartan matrix of finite type. Hence there exists positive integers $\{d_i\}_{1\le i\le n}$ such that the matrix DC is symmetric positive definite, where $D = \operatorname{diag}(d_i)$ is the diagonal matrix. (In [12, Chapter 2], each d_i is denoted by s_i.) Let $\mathfrak{g} = (\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ be the finite dimensional Lie algebra over the complex number field $\mathbb{C}$ associated to $C = (a_{ij})_{n\times n}$.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ with simple roots $\alpha_1, \ldots, \alpha_n$, $\mathbf{R}$ the root system, $\mathbf{R}^+$ the set of positive roots, and W the Weyl group. Choose $h_i \in \mathfrak{h}$, $1 \le i \le n$, such that

$$\alpha_j : \mathfrak{h} \longrightarrow \mathbb{C}, \qquad \alpha_j(h_i) = a_{ij} \qquad \text{for all } j = 1, \ldots, n. \tag{1.1}$$

Then $\{h_i\}_{i=1}^n$ forms a basis of $\mathfrak{h}$ since C has rank n, and $\mathfrak{g}$ is generated by h_i and $x_{\pm\alpha_i}$, $i = 1, \ldots, n$, with relations

$$[h_i, h_j]_{\mathfrak{g}} = 0,\ [h_i, x_{\pm\alpha_j}]_{\mathfrak{g}} = \pm a_{ij} x_{\pm\alpha_j},\ [x_{\alpha_i}, x_{-\alpha_j}]_{\mathfrak{g}} = \delta_{ij} h_i,$$
$$(\mathrm{ad}_{x_{\pm\alpha_i}})^{1-a_{ij}}(x_{\pm\alpha_j}) = 0, \qquad i \ne j$$

[12, Definition 2.1.3]. x_{α_i} and $x_{-\alpha_i}$ are denoted by e_i and f_i respectively [12, Definition 2.1.3]. Denote by $\mathfrak{n}^+$ and $\mathfrak{n}^-$, the subspaces of $\mathfrak{g}$ spanned by root vectors with positive and negative roots, respectively, and set $\mathfrak{n} = \mathfrak{n}^+ \oplus \mathfrak{n}^-$. Hence

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-.$$

For each $\beta \in \mathbf{R}$, we will write x_β for a root vector with root β, hence $\mathfrak{g}_\beta = \mathbb{C}x_\beta \subseteq \mathfrak{n}$, where $\mathfrak{g}_\beta$ is the root space of $\mathfrak{g}$ with root β.

There exists a nondegenerate symmetric bilinear form $(\cdot\,|\,\cdot)$ on $\mathfrak{h}^*$ given by [12, Section 2.3]

$$(\alpha_i \,|\, \alpha_j) = d_i a_{ij} \tag{1.2}$$

for all $i, j = 1, \ldots, n$. This form $(\cdot \mid \cdot)$ induces the isomorphism $\mathfrak{h}^* \longrightarrow \mathfrak{h}, \lambda \mapsto h_\lambda$, where h_λ is defined by

$$(\alpha_i \mid \lambda) = \alpha_i(h_\lambda) \qquad \text{for all } i = 1, \ldots, n.$$

(The isomorphism $\mathfrak{h}^* \longrightarrow \mathfrak{h}, \lambda \mapsto h_\lambda$ is denoted by ν^{-1} in [12, Section 2.3].) Note that, by (1.1) and (1.2),

$$h_{\alpha_i} = d_i h_i, \qquad i = 1, \ldots, n.$$

Identifying $\mathfrak{h}^*$ to $\mathfrak{h}$ via $\lambda \mapsto h_\lambda$, $\mathfrak{h}$ has a nondegenerate symmetric bilinear form $(\cdot \mid \cdot)$ given by

$$(\lambda \mid \mu) = (h_\lambda \mid h_\mu) = \lambda(h_\mu).$$

For example, $(h_i \mid h_j) = (d_i^{-1} h_{\alpha_i} \mid d_j^{-1} h_{\alpha_j}) = d_i^{-1} d_j^{-1} (\alpha_i \mid \alpha_j) = d_j^{-1} a_{ij}$. This is extended to a nondegenerate $\mathfrak{g}$-invariant symmetric bilinear form on $\mathfrak{g}$ by [12, (2.7) and Proposition 2.3.6] and [15, Theorem 2.2 and its proof]:

$$(h_i \mid h_j) = d_j^{-1} a_{ij}, \qquad (h \mid x_\alpha) = 0, \qquad (x_\alpha \mid x_\beta) = 0 \text{ if } \alpha + \beta \neq 0, \qquad (x_{\alpha_i} \mid x_{-\alpha_j}) = d_i^{-1} \delta_{ij} \quad (1.3)$$

for $h \in \mathfrak{h}$ and $\alpha, \beta \in \mathbf{R}$.

Lemma 1.1. *For each $\alpha \in \mathbf{R}^+$,*

$$[x_\alpha, x_{-\alpha}]_{\mathfrak{g}} = (x_\alpha \mid x_{-\alpha}) h_\alpha. \qquad (1.4)$$

Proof. By [12, Proposition 2.3.6],

$$(h_\lambda \mid [x_\alpha, x_{-\alpha}]_{\mathfrak{g}}) = ([h_\lambda, x_\alpha]_{\mathfrak{g}} \mid x_{-\alpha}) = (\lambda \mid \alpha)(x_\alpha \mid x_{-\alpha}) = (x_\alpha \mid x_{-\alpha})(h_\lambda \mid h_\alpha) = (h_\lambda \mid (x_\alpha \mid x_{-\alpha}) h_\alpha)$$

for each $h_\lambda \in \mathfrak{h}$. Hence the result holds. □

1.2.

Let $u \in \bigwedge^2 \mathfrak{h}$. We may write

$$u = \sum_{1 \le i,j \le n} u_{ij} h_i \otimes h_j$$

for some skew symmetric matrix (u_{ij}) since $\{h_i\}_{i=1}^n$ forms a basis of $\mathfrak{h}$. The element u can be considered as a skew symmetric (alternating) form on $\mathfrak{h}^*$ by

$$u(\lambda, \mu) = \sum u_{ij} \lambda(h_i) \mu(h_j)$$

for any $\lambda, \mu \in \mathfrak{h}^*$. Hence there exists a unique linear map $\Phi : \mathfrak{h}^* \longrightarrow \mathfrak{h}^*$ such that

$$u(\lambda, \mu) = (\Phi(\lambda) \mid \mu)$$

for any $\lambda, \mu \in \mathfrak{h}^*$ since the form $(\cdot \mid \cdot)$ on $\mathfrak{h}^*$ is nondegenerate. Set

$$\Phi_+ = \Phi + I, \qquad \Phi_- = \Phi - I,$$

where I is the identity map on $\mathfrak{h}^*$. Thus

$$\begin{aligned} (\Phi_+ \lambda \mid \mu) &= u(\lambda, \mu) + (\lambda \mid \mu) \\ (\Phi_- \lambda \mid \mu) &= u(\lambda, \mu) - (\lambda \mid \mu) \end{aligned} \qquad (1.5)$$

for all $\lambda, \mu \in \mathfrak{h}^*$.

1.3.

Fix a vector space $\mathfrak{k}$ isomorphic to $\mathfrak{h}$ and let $\varphi : \mathfrak{h} \longrightarrow \mathfrak{k}$ be an isomorphism of vector spaces. For each $\lambda \in \mathfrak{h}^*$, denote by $k_\lambda \in \mathfrak{k}$ the element $\varphi(h_\lambda)$. Let

$$\mathfrak{g}' = \mathfrak{k} \oplus \mathfrak{n}$$

and define a skew symmetric bilinear product $[\cdot,\cdot]_{\mathfrak{g}'}$ on $\mathfrak{g}'$ by

$$\begin{aligned} &[k_\lambda, k_\mu]_{\mathfrak{g}'} = 0, && [k_\lambda, x_\alpha]_{\mathfrak{g}'} = (\alpha \mid \lambda) x_\alpha, \\ &[x_\alpha, x_\beta]_{\mathfrak{g}'} = [x_\alpha, x_\beta]_{\mathfrak{g}} \ \ (\alpha \neq -\beta), && [x_\alpha, x_{-\alpha}]_{\mathfrak{g}'} = \varphi([x_\alpha, x_{-\alpha}]_{\mathfrak{g}}) \end{aligned} \tag{1.6}$$

for all $\lambda, \mu \in \mathfrak{h}^*$ and $x_\alpha, x_{-\alpha}, x_\beta \in \mathfrak{n}$. Since $[k_\lambda, x_\alpha]_{\mathfrak{g}'} = [h_\lambda, x_\alpha]_{\mathfrak{g}}$ for each $\lambda \in \mathfrak{h}^*$, $\mathfrak{g}'$ is the Lie algebra isomorphic to $\mathfrak{g}$ such that each element $k_\lambda \in \mathfrak{k}$ corresponds to the element h_λ of $\mathfrak{g}$. That is, $\mathfrak{g}'$ is the Lie algebra $\mathfrak{g}$ such that the elements $h_\lambda \in \mathfrak{h}$ are replaced by $k_\lambda \in \mathfrak{k}$. Note that

$$[x_\alpha, x_\beta]_{\mathfrak{g}'} = [x_\alpha, x_\beta]_{\mathfrak{g}} \in \mathfrak{n} \tag{1.7}$$

for all $\alpha, \beta \in \mathbf{R}$ with $\alpha \neq -\beta$.

Let

$$\mathfrak{d} = \mathfrak{h} \oplus \mathfrak{k} \oplus \mathfrak{n}.$$

Hence $\mathfrak{d} = \mathfrak{k} \oplus \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}'$. Define a skew symmetric bilinear product $[\cdot,\cdot]$ on $\mathfrak{d}$ by

$$\begin{gathered} [h_\lambda, h_\mu] = 0, \qquad [h_\lambda, k_\mu] = 0, \qquad [k_\lambda, k_\mu] = 0, \\ [h_\lambda, x_\alpha] = -(\Phi_- \lambda \mid \alpha) x_\alpha, \qquad [k_\lambda, x_\alpha] = (\Phi_+ \lambda \mid \alpha) x_\alpha, \\ [x_\alpha, x_\beta] = 2^{-1}([x_\alpha, x_\beta]_{\mathfrak{g}} + [x_\alpha, x_\beta]_{\mathfrak{g}'}) \end{gathered} \tag{1.8}$$

for all $h_\lambda, h_\mu \in \mathfrak{h}$, $k_\lambda, k_\mu \in \mathfrak{k}$ and $x_\alpha, x_\beta \in \mathfrak{n}$.

Lemma 1.2. (1) *For any elements $x_\alpha, x_\beta \in \mathfrak{n}$ such that $\alpha + \beta \neq 0$,*

$$[x_\alpha, x_\beta] = [x_\alpha, x_\beta]_{\mathfrak{g}} = [x_\alpha, x_\beta]_{\mathfrak{g}'}. \tag{1.9}$$

(2) *For any elements $x_\alpha, x_\beta, x_\gamma \in \mathfrak{n}$ such that $\alpha + \beta + \gamma \neq 0$,*

$$[[x_\alpha, x_\beta], x_\gamma] = [[x_\alpha, x_\beta]_{\mathfrak{g}}, x_\gamma]_{\mathfrak{g}} = [[x_\alpha, x_\beta]_{\mathfrak{g}'}, x_\gamma]_{\mathfrak{g}'}. \tag{1.10}$$

(3) *For any elements $x_\alpha, x_\beta, x_\gamma \in \mathfrak{n}$,*

$$[[x_\alpha, x_\beta], x_\gamma] = 2^{-1}([[x_\alpha, x_\beta]_{\mathfrak{g}}, x_\gamma]_{\mathfrak{g}} + [[x_\alpha, x_\beta]_{\mathfrak{g}'}, x_\gamma]_{\mathfrak{g}'}). \tag{1.11}$$

Proof.

(1) It follows by (1.7) and (1.8).

(2) If $\alpha + \beta \neq 0$ then $[x_\alpha, x_\beta] = [x_\alpha, x_\beta]_{\mathfrak{g}} = [x_\alpha, x_\beta]_{\mathfrak{g}'} \in \mathfrak{n}$ by (1.9) and thus

$$[[x_\alpha, x_\beta], x_\gamma] = [[x_\alpha, x_\beta]_{\mathfrak{g}}, x_\gamma]_{\mathfrak{g}} = [[x_\alpha, x_\beta]_{\mathfrak{g}'}, x_\gamma]_{\mathfrak{g}'}$$

by (1.9) since $(\alpha + \beta) + \gamma \neq 0$.

Suppose $\alpha + \beta = 0$. We may assume that α is a positive root. By (1.4), $[x_\alpha, x_{-\alpha}]_{\mathfrak{g}} = a h_\alpha \in \mathfrak{h}$ and $[x_\alpha, x_{-\alpha}]_{\mathfrak{g}'} = a k_\alpha \in \mathfrak{k}$, where $a = (x_\alpha \mid x_{-\alpha})$, and

$$\begin{aligned} [[x_\alpha, x_{-\alpha}], x_\gamma] &= 2^{-1}([[x_\alpha, x_{-\alpha}]_{\mathfrak{g}} + [x_\alpha, x_{-\alpha}]_{\mathfrak{g}'}, x_\gamma] = 2^{-1} a([h_\alpha, x_\gamma] + [k_\alpha, x_\gamma]) \\ &= 2^{-1} a(-(\Phi_- \alpha \mid \gamma) + (\Phi_+ \alpha \mid \gamma)) x_\gamma = a(\alpha \mid \gamma) x_\gamma. \end{aligned}$$

Since

$$[[x_\alpha, x_{-\alpha}]_{\mathfrak{g}}, x_\gamma]_{\mathfrak{g}} = a[h_\alpha, x_\gamma]_{\mathfrak{g}} = a(\alpha \mid \gamma)x_\gamma$$
$$[[x_\alpha, x_{-\alpha}]_{\mathfrak{g}'}, x_\gamma]_{\mathfrak{g}'} = a[k_\alpha, x_\gamma]_{\mathfrak{g}'} = a(\alpha \mid \gamma)x_\gamma,$$

we have the result $[[x_\alpha, x_{-\alpha}], x_\gamma] = [[x_\alpha, x_{-\alpha}]_{\mathfrak{g}}, x_\gamma]_{\mathfrak{g}} = [[x_\alpha, x_{-\alpha}]_{\mathfrak{g}'}, x_\gamma]_{\mathfrak{g}'}$.

(3) If $\alpha+\beta+\gamma \neq 0$ then (1.11) holds by (1.10). Suppose that $\alpha+\beta+\gamma = 0$. Then $\alpha+\beta = -\gamma \neq 0$. Hence $[x_\alpha, x_\beta] = [x_\alpha, x_\beta]_{\mathfrak{g}} = [x_\alpha, x_\beta]_{\mathfrak{g}'} \in \mathfrak{n}$ by (1.9). It follows that

$$\begin{aligned}[[x_\alpha, x_\beta], x_\gamma] &= 2^{-1}([[x_\alpha, x_\beta], x_\gamma]_{\mathfrak{g}} + [[x_\alpha, x_\beta], x_\gamma]_{\mathfrak{g}'}) \\ &= 2^{-1}([[x_\alpha, x_\beta]_{\mathfrak{g}}, x_\gamma]_{\mathfrak{g}} + [[x_\alpha, x_\beta]_{\mathfrak{g}'}, x_\gamma]_{\mathfrak{g}'}).\end{aligned}$$

□

Theorem 1.3. *The vector space $\mathfrak{d} = \mathfrak{h} \oplus \mathfrak{k} \oplus \mathfrak{n}$ is a Lie algebra with Lie bracket (1.8).*

Proof. It is enough to show that (1.8) satisfies the Jacobi identity,

$$[[a, b], c] + [[b, c], a] + [[c, a], b] = 0$$

for all nonzero elements $a, b, c \in \mathfrak{d}$. If $a, b, c \in \mathfrak{h} \oplus \mathfrak{k}$ then $[[a, b], c] + [[b, c], a] + [[c, a], b] = 0$ trivially. If one of a, b, c is an element of $\mathfrak{n}$, say $a = h_\lambda, b = k_\mu$, and $c = x_\alpha$, then

$$\begin{aligned}[[a, b], c] + [[b, c], a] + [[c, a], b] &= [[h_\lambda, k_\mu], x_\alpha] + [[k_\mu, x_\alpha], h_\lambda] + [[x_\alpha, h_\lambda], k_\mu] \\ &= (\Phi_+\mu \mid \alpha)(\Phi_-\lambda \mid \alpha)x_\alpha - (\Phi_-\lambda \mid \alpha)(\Phi_+\mu \mid \alpha)x_\alpha \\ &= 0.\end{aligned}$$

If two of a, b, c are elements of $\mathfrak{n}$, say $a = h_\lambda, b = x_\alpha$, and $c = x_\beta$, then

$$\begin{aligned}&[[a, b], c] + [[b, c], a] + [[c, a], b] \\ &\quad = [[h_\lambda, x_\alpha], x_\beta] + [[x_\alpha, x_\beta], h_\lambda] + [[x_\beta, h_\lambda], x_\alpha] \\ &\quad = \begin{cases} -(\Phi_-\lambda \mid \alpha)[x_\alpha, x_\beta] + (\Phi_-\lambda \mid \alpha+\beta)[x_\alpha, x_\beta] + (\Phi_-\lambda \mid \beta)[x_\beta, x_\alpha], & \text{if } \alpha+\beta \neq 0, \\ -(\Phi_-\lambda \mid \alpha)[x_\alpha, x_{-\alpha}] + (\Phi_-\lambda \mid -\alpha)[x_{-\alpha}, x_\alpha], & \text{if } \alpha+\beta = 0, \end{cases} \\ &\quad = 0.\end{aligned}$$

Finally, suppose that all of a, b, c are elements of $\mathfrak{n}$, say $a = x_\alpha$, $b = x_\beta$ and $c = x_\gamma$. Then, by (1.11),

$$\begin{aligned}[[a, b], c] + [[b, c], a] + [[c, a], b] &= [[x_\alpha, x_\beta], x_\gamma] + [[x_\beta, x_\gamma], x_\alpha] + [[x_\gamma, x_\alpha], x_\beta] \\ &= 2^{-1}([[x_\alpha, x_\beta]_{\mathfrak{g}}, x_\gamma]_{\mathfrak{g}} + [[x_\alpha, x_\beta]_{\mathfrak{g}'}, x_\gamma]_{\mathfrak{g}'}) \\ &\quad + 2^{-1}([[x_\beta, x_\gamma]_{\mathfrak{g}}, x_\alpha]_{\mathfrak{g}} + [[x_\beta, x_\gamma]_{\mathfrak{g}'}, x_\alpha]_{\mathfrak{g}'}) \\ &\quad + 2^{-1}([[x_\gamma, x_\alpha]_{\mathfrak{g}}, x_\beta]_{\mathfrak{g}} + [[x_\gamma, x_\alpha]_{\mathfrak{g}'}, x_\beta]_{\mathfrak{g}'}) \\ &= 0\end{aligned}$$

since $\mathfrak{g}$ and $\mathfrak{g}'$ are Lie algebras. It completes the proof. □

1.4.

Let $\mathfrak{m}$ be a Lie algebra and let $s = \sum_i a_i \otimes b_i \in \mathfrak{m} \otimes \mathfrak{m}$. We give a notation:

$$[[s, s]] = [s_{12}, s_{13}] + [s_{12}, s_{23}] + [s_{13}, s_{23}],$$

where

$$[s_{12}, s_{13}] = \sum_{i,j} [a_i, a_j] \otimes b_i \otimes b_j,$$
$$[s_{12}, s_{23}] = \sum_{i,j} a_i \otimes [b_i, a_j] \otimes b_j,$$
$$[s_{13}, s_{23}] = \sum_{i,j} a_i \otimes a_j \otimes [b_i, b_j].$$

Let

$$r = \sum_{\alpha\in\mathbf{R}^+} (x_\alpha \otimes x_{-\alpha} - x_{-\alpha} \otimes x_\alpha) \in \mathfrak{n} \otimes \mathfrak{n}, \tag{1.12}$$

where $(x_\alpha \mid x_{-\alpha}) = 1$. Note that r is skew symmetric. It is well known that $[[r, r]]$ is a $\mathfrak{g}$-invariant element of $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$, that is, $x.[[r, r]] = 0$ for all $x \in \mathfrak{g}$, where the action of $\mathfrak{g}$ on $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ is by the adjoint representation in each factor:

$$x.(a \otimes b \otimes c) = [x, a]_\mathfrak{g} \otimes b \otimes c + a \otimes [x, b]_\mathfrak{g} \otimes c + a \otimes b \otimes [x, c]_\mathfrak{g}.$$

Thus $[[r, r]]$ is also a $\mathfrak{g}'$-invariant element of $\mathfrak{g}' \otimes \mathfrak{g}' \otimes \mathfrak{g}'$ by (1.6). Here we prove the following statement [2, Proposition 2.1.2, Example 2.3.7 and Section 2.1 B] and [11, 1.2].

Proposition 1.4. *The element* $[[r, r]] \in \mathfrak{d} \otimes \mathfrak{d} \otimes \mathfrak{d}$ *is also* $\mathfrak{d}$*-invariant.*

Proof. It is easy to check that

$$x.[r_{12}, r_{13}] = 0, \qquad x.[r_{12}, r_{23}] = 0, \qquad x.[r_{13}, r_{23}] = 0$$

for each $x \in \mathfrak{h} \oplus \mathfrak{k}$. For instance, let $x = h_\lambda \in \mathfrak{h}$ and note that

$$[r_{12}, r_{13}] = \sum_{\alpha,\beta\in\mathbf{R}^+} [x_\alpha, x_\beta] \otimes x_{-\alpha} \otimes x_{-\beta} - [x_\alpha, x_{-\beta}] \otimes x_{-\alpha} \otimes x_\beta$$
$$- [x_{-\alpha}, x_\beta] \otimes x_\alpha \otimes x_{-\beta} + [x_{-\alpha}, x_{-\beta}] \otimes x_\alpha \otimes x_\beta.$$

If Y is the first term of $[r_{12}, r_{13}]$, then

$$\begin{aligned} x.Y &= \sum h_\lambda.([x_\alpha, x_\beta] \otimes x_{-\alpha} \otimes x_{-\beta}) \\ &= \sum([h_\lambda, [x_\alpha, x_\beta]] \otimes x_{-\alpha} \otimes x_{-\beta} + [x_\alpha, x_\beta] \otimes [h_\lambda, x_{-\alpha}] \otimes x_{-\beta} + [x_\alpha, x_\beta] \otimes x_{-\alpha} \otimes [h_\lambda, x_{-\beta}]) \\ &= \sum(-(\Phi_-\lambda \mid \alpha + \beta) + (\Phi_-\lambda \mid \alpha) + (\Phi_-\lambda \mid \beta))[x_\alpha, x_\beta] \otimes x_{-\alpha} \otimes x_{-\beta} \\ &= 0. \end{aligned}$$

It remains to prove that $x_\gamma.[[r, r]] = 0$ for all $x_\gamma \in \mathfrak{n}$. Observe that

$$\begin{aligned} x_\gamma.[[r, r]] &= x_\gamma.([r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]) \\ &= 2^{-1}[x_\gamma.([r_{12}, r_{13}]_\mathfrak{g} + [r_{12}, r_{23}]_\mathfrak{g} + [r_{13}, r_{23}]_\mathfrak{g}) + x_\gamma.([r_{12}, r_{13}]_{\mathfrak{g}'} + [r_{12}, r_{23}]_{\mathfrak{g}'} + [r_{13}, r_{23}]_{\mathfrak{g}'})] \end{aligned}$$

by (1.11) and the third equation of (1.8), where the last two actions are the actions of $\mathfrak{g}$ and $\mathfrak{g}'$, respectively. Hence $x_\gamma.[[r, r]] = 0$ for all $x_\gamma \in \mathfrak{n}$ since $x_\gamma.[[r, r]] = 0$ in $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ and $\mathfrak{g}' \otimes \mathfrak{g}' \otimes \mathfrak{g}'$. □

1.5.

Let $\mathbf{Q}$ and $\mathbf{P}$ denote the root and the weight lattices, respectively, G a connected complex semisimple algebraic group associated with $\mathfrak{g}$, $H \subset G$ a torus associated with $\mathfrak{h}$ and let $\mathbf{L}$ be the group of characters of H [11, 1.1]. Note that $\mathbf{L}$ is a lattice such that $\mathbf{Q} \subseteq \mathbf{L} \subseteq \mathbf{P}$ by [11, 1.2] and that $\mathbf{L}$ spans $\mathfrak{h}^*$ since the set of simple roots $\{\alpha_i\}_{i=1}^n$ forms a basis of $\mathfrak{h}^*$.

Let $\mathcal{C}(\mathfrak{g})$ be the category of $\mathfrak{g}$-module homomorphisms as morphisms and the following (left) $\mathfrak{g}$-modules as objects: the finite-dimensional $\mathfrak{g}$-modules consisting of finite direct sums of finite dimensional irreducible highest weight $\mathfrak{g}$-modules $V(\Lambda)$ with highest weight $\Lambda \in \mathbf{L}^+ = \mathbf{L} \cap \mathbf{P}^+$, where $\mathbf{P}^+ = \{\lambda \in \mathbf{P} \mid \lambda(h_i) \in \mathbb{Z}_{\geq 0}$ for all $i = 1, \ldots, n\}$ the set of dominant integral weights. Here, by $M \in \mathcal{C}(\mathfrak{g})$, we mean that M is an object of $\mathcal{C}(\mathfrak{g})$. Note that $\mathcal{C}(\mathfrak{g})$ is closed under finite direct sums, finite tensor products, and the formation duals (if $M, N \in \mathcal{C}(\mathfrak{g})$ then $M \otimes N$ and M^* have the left $\mathfrak{g}$-module structure with

$$x(y \otimes z) = (xy) \otimes z + y \otimes (xz),$$
$$(xf)(y) = -f(xy)$$

for all $f \in M^*$, $x \in \mathfrak{g}$, $y \in M$, and $z \in N$). Moreover, every $M \in \mathcal{C}(\mathfrak{g})$ can be written by a direct sum of weight spaces $M = \oplus_{\mu \in \mathbf{L}} M_\mu$, where

$$M_\mu = \{v \in M \mid h_\lambda v = (\lambda \mid \mu)v \text{ for } h_\lambda \in \mathfrak{h}\}. \tag{1.13}$$

Given $M = \oplus_{\mu \in \mathbf{L}} M_\mu \in \mathcal{C}(\mathfrak{g})$, note that each element $f \in (M_\mu)^*$ is considered as the element $f' \in M^*$ defined by

$$f' \mid_{M_\mu} = f, \qquad f'(M_\nu) = 0 \qquad \text{for all } \mu \neq \nu \in \mathbf{L}.$$

For instance, for $0 \neq x \in M_\mu$, the dual map x^* defined in M_μ is identified with the dual map x^* defined in M.

Lemma 1.5. *Let $M = \oplus_{\nu \in \mathbf{L}} M_\nu \in \mathcal{C}(\mathfrak{g})$.*
(1) *Let $f \in (M_\mu)^*$, then $f \in (M^*)_{-\mu}$. In particular, if $f \in (M^*)_\mu$, $x \in M_\nu$, and $f(x) \neq 0$ then $-\mu = \nu$.*
(2) *There exists a right $\mathfrak{g}$-module structure in M^*: For $f \in M^*$, $x \in \mathfrak{g}$, and $y \in M$,*

$$(fx)(y) = f(xy).$$

In particular, if $f \in (M_\mu)^*$ and $x_\alpha \in \mathfrak{n}$ then $fx_\alpha \in (M^*)_{-\mu+\alpha}$.

Proof. (1) For any $h_\lambda \in \mathfrak{h}$ and $y \in M_\mu$,

$$(h_\lambda f)(y) = -f(h_\lambda y) = -(\lambda \mid \mu)f(y) = (\lambda \mid -\mu)f(y)$$

and thus $h_\lambda f = (\lambda \mid -\mu)f$. It follows that $f \in (M^*)_{-\mu}$.

Let $\{y_{\nu i}\}_i$ be a basis of M_ν for each $\nu \in \mathbf{L}$. Thus the set $\{y_{\nu i}\}_{\nu,i}$ is a basis of $M = \oplus_{\nu \in \mathbf{L}} M_\nu$ and $f \in (M^*)_\mu$ is uniquely expressed by a $\mathbb{C}$-linear combination $f = \sum_{\nu,i} a_{\nu i} y_{\nu i}^*$ for some $a_{\nu i} \in \mathbb{C}$. Since $f \in (M^*)_\mu$ and the dual map $y_{\nu i}^*$ is an element of $(M^*)_{-\nu}$ by the previous result, we have that $a_{\nu i} = 0$ for all νi with $-\nu \neq \mu$ and thus $f = \sum_i a_{-\mu i} y_{-\mu i}^*$. It follows that $f(M_\nu) = 0$ for all $-\mu \neq \nu$. The final statement is now clear: since $f(x) \neq 0$, $x \in M_{-\mu}$ and thus $\nu = -\mu$.

(2) For any $f \in M^*$, $x, x' \in \mathfrak{g}$, and $y \in M$,

$$\big((fx)x' - (fx')x\big)(y) = f([x, x']_{\mathfrak{g}} y) = (f[x, x']_{\mathfrak{g}})(y).$$

Hence M^* is a right $\mathfrak{g}$-module. If $f \in (M_\mu)^*$, $x_\alpha \in \mathfrak{n}$, $h_\lambda \in \mathfrak{h}$ and $y \in M_{-\alpha+\mu}$, then

$$\big(h_\lambda(fx_\alpha)\big)(y) = -f(x_\alpha h_\lambda y) = -(\lambda \mid -\alpha + \mu)(fx_\alpha)(y).$$

Hence $fx_\alpha \in (M^*)_{-\mu+\alpha}$. □

The following proposition shows that $M \in \mathcal{C}(\mathfrak{g})$ has a (left) $\mathfrak{d}$-module structure.

Proposition 1.6. *Let $M = \oplus_{\mu \in \mathbf{L}} M_\mu$ be a finite direct sum of finite dimensional irreducible highest weight $\mathfrak{g}$-modules $V(\Lambda)$ with highest weight $\Lambda \in \mathbf{L}^+$. Define an action of $\mathfrak{d}$ on M as follows: For $z_\nu \in M_\nu$ and $h_\lambda \in \mathfrak{h}$, $k_\lambda \in \mathfrak{k}$, $x_\alpha \in \mathfrak{n}$,*

$$\begin{aligned} h_\lambda \cdot z_\nu &= (\Phi_+ \nu \,|\, \lambda) z_\nu, \\ k_\lambda \cdot z_\nu &= -(\Phi_- \nu \,|\, \lambda) z_\nu, \\ x_\alpha \cdot z_\nu &= x_\alpha z_\nu, \end{aligned} \tag{1.14}$$

where the right-hand side $x_\alpha z_\nu$ of the third equation is the action inside $\mathfrak{g}$. Then M is a $\mathfrak{d}$-module with action (1.14).

Proof. By (1.8) and (1.14), it is easy to check that

$$\begin{aligned} [h_\lambda, h_\mu] \cdot z_\nu &= 0 = h_\lambda \cdot (h_\mu \cdot z_\nu) - h_\mu \cdot (h_\lambda \cdot z_\nu), \\ [h_\lambda, k_\mu] \cdot z_\nu &= 0 = h_\lambda \cdot (k_\mu \cdot z_\nu) - k_\mu \cdot (h_\lambda \cdot z_\nu), \\ [k_\lambda, k_\mu] \cdot z_\nu &= 0 = k_\lambda \cdot (k_\mu \cdot z_\nu) - k_\mu \cdot (k_\lambda \cdot z_\nu). \end{aligned}$$

Now observe that

$$\begin{aligned} [h_\lambda, x_\alpha] \cdot z_\nu &= -(\Phi_- \lambda \,|\, \alpha) x_\alpha \cdot z_\nu \qquad \text{(by (1.8))} \\ &= (\Phi_+ \alpha \,|\, \lambda) x_\alpha z_\nu \qquad \text{(by (1.5) and (1.14))} \\ &= h_\lambda \cdot (x_\alpha z_\nu) - x_\alpha (h_\lambda \cdot z_\nu), \qquad \text{(by (1.14))} \\ &= h_\lambda \cdot (x_\alpha \cdot z_\nu) - x_\alpha \cdot (h_\lambda \cdot z_\nu). \qquad \text{(by (1.14))} \end{aligned}$$

Similarly, we have

$$[k_\lambda, x_\alpha] \cdot z_\nu = (\Phi_+ \lambda \,|\, \alpha) x_\alpha z_\nu = -(\Phi_- \alpha \,|\, \lambda) x_\alpha z_\nu = k_\lambda \cdot (x_\alpha \cdot z_\nu) - x_\alpha \cdot (k_\lambda \cdot z_\nu).$$

Let $x_\alpha, x_\beta \in \mathfrak{n}$. If $\alpha + \beta \neq 0$ then $[x_\alpha, x_\beta] \in \mathfrak{n}$ and

$$[x_\alpha, x_\beta] \cdot z_\nu = [x_\alpha, x_\beta]_{\mathfrak{g}} z_\nu = x_\alpha \cdot (x_\beta \cdot z_\nu) - x_\beta \cdot (x_\alpha \cdot z_\nu)$$

by (1.9). Suppose that $\alpha + \beta = 0$. We may assume that $\alpha \in \mathbf{R}^+$. Then $[x_\alpha, x_{-\alpha}]_{\mathfrak{g}} = (x_\alpha \,|\, x_{-\alpha}) h_\alpha \in \mathfrak{h}$ and $[x_\alpha, x_{-\alpha}]_{\mathfrak{g}'} = (x_\alpha \,|\, x_{-\alpha}) k_\alpha \in \mathfrak{k}$ by (1.4) and

$$\begin{aligned} [x_\alpha, x_{-\alpha}] \cdot z_\nu &= 2^{-1} (x_\alpha \,|\, x_{-\alpha})(h_\alpha + k_\alpha) \cdot z_\nu \\ &= (x_\alpha \,|\, x_{-\alpha})(\alpha \,|\, \nu) z_\nu = [x_\alpha, x_{-\alpha}]_{\mathfrak{g}} z_\nu \\ &= x_\alpha \cdot (x_{-\alpha} \cdot z_\nu) - x_{-\alpha} \cdot (x_\alpha \cdot z_\nu). \end{aligned}$$

It completes the proof. □

Notation 1.7.

(1) Denote by $\mathcal{C}(\mathfrak{d})$ the category $\mathcal{C}(\mathfrak{g})$ whenever all objects $M \in \mathcal{C}(\mathfrak{g})$ are considered as the $\mathfrak{d}$-modules M with action (1.14). For simplicity, by $M \in \mathcal{C}(\mathfrak{d})$, we mean that M is an object of $\mathcal{C}(\mathfrak{d})$ and we will omit the dot '$\cdot$' denoting the module action of $\mathfrak{d}$ on $M \in \mathcal{C}(\mathfrak{d})$.

(2) Let $U(\mathfrak{d})$ be the universal enveloping algebra of $\mathfrak{d}$. Note that a vector space M is a $\mathfrak{d}$-module if and only if M is a $U(\mathfrak{d})$-module. For $M \in \mathcal{C}(\mathfrak{d})$, $v \in M$, and $f \in M^*$, define

$$c_{f,v} = c^M_{f,v} : U(\mathfrak{d}) \longrightarrow \mathbb{C}, \qquad c_{f,v}(z) = f(zv)$$

and let $\mathbb{C}[G]$ be the $\mathbb{C}$-vector space spanned by all elements of the form

$$c^M_{f,v}, \qquad M \in \mathcal{C}(\mathfrak{d}), \quad v \in M, \quad f \in M^*.$$

Recall the definitions of a bigraded Hopf algebra and a (commutative) Poisson Hopf algebra given in [11, 2.1] and [3, Definition 1.1]. Let K be an (additive) abelian group. A Hopf algebra

$A = (A, \iota, m, \epsilon, \Delta, S)$ over a field $\mathbf{k}$ is said to be a *K-bigraded Hopf algebra* if it is equipped with a $K \times K$-grading $A = \bigoplus_{(\lambda,\mu)\in K\times K} A_{\lambda,\mu}$ such that
(i) $\mathbf{k} \subseteq A_{0,0}$, $A_{\lambda,\mu}A_{\lambda',\mu'} \subseteq A_{\lambda+\lambda',\mu+\mu'}$.
(ii) $\Delta(A_{\lambda,\mu}) \subseteq \sum_{\nu\in K} A_{\lambda,\nu} \otimes A_{-\nu,\mu}$.
(iii) $\lambda \neq -\mu$ implies $\epsilon(A_{\lambda,\mu}) = 0$.
(iv) $S(A_{\lambda,\mu}) \subseteq A_{\mu,\lambda}$.
A Hopf algebra $A = (A, \iota, m, \epsilon, \Delta, S)$ is said to be a *Poisson Hopf algebra* if there exists a skew symmetric bilinear product $\{\cdot,\cdot\}$ on A if $(A, \{\cdot,\cdot\})$ is a Poisson algebra such that
(v) Δ is a Poisson algebra homomorphism. Note that the Poisson bracket in $A \otimes A$ is given as follows:

$$\{a_1 \otimes a_2, b_1 \otimes b_2\} = \{a_1, b_1\} \otimes a_2b_2 + a_1b_1 \otimes \{a_2, b_2\}.$$

A Poisson Hopf algebra $A = (A, \iota, m, \epsilon, \Delta, S, \{\cdot,\cdot\})$ is said to be *K-bigraded* if A satisfies (i)–(v) and
(vi) $\{A_{\lambda,\mu}, A_{\lambda',\mu'}\} \subseteq A_{\lambda+\lambda',\mu+\mu'}$.

Theorem 1.8. *The vector space $\mathbb{C}[G]$ is an $\mathbf{L}$-bigraded commutative Hopf algebra with the following structure: For $M, N \in \mathcal{C}(\mathfrak{d})$,*

$$\begin{gathered} c^M_{f,v} + c^N_{g,w} = c^{M\oplus N}_{(f,g),(v,w)}, \qquad c^M_{f,v}c^N_{g,w} = c^{M\otimes N}_{f\otimes g, v\otimes w}, \\ \epsilon(c^M_{f,v}) = f(v), \qquad S(c^M_{f,v}) = c^{M^*}_{v,f}, \\ \Delta(c^M_{f,v}) = \sum_i c^M_{f,v_i} \otimes c^M_{g_i,v}, \end{gathered} \tag{1.15}$$

where $\{v_i\}$ and $\{g_i\}$ are bases for M and M^* such that $g_i(v_j) = \delta_{ij}$ for all i, j.

For $M \in \mathcal{C}(\mathfrak{d})$, set

$$C(M) = \mathbb{C}\langle c_{f,v} \mid f \in M^*, v \in M\rangle, \qquad C(M)_{\lambda,\mu} = \mathbb{C}\langle c_{f,v} \mid f \in (M^*)_\lambda, v \in M_\mu\rangle.$$

Then, for $\lambda, \mu \in \mathbf{L}$,

$$\mathbb{C}[G]_{\lambda,\mu} = \sum_{M\in\mathcal{C}(\mathfrak{d})} C(M)_{\lambda,\mu}, \qquad \mathbb{C}[G] = \bigoplus_{(\lambda,\mu)\in\mathbf{L}\times\mathbf{L}} \mathbb{C}[G]_{\lambda,\mu}. \tag{1.16}$$

Moreover $\mathbb{C}[G]$ is finitely generated as a $\mathbb{C}$-algebra.

Proof. Note that $c^M_{f,v}$ is an element of the Hopf dual $U(\mathfrak{d})^\circ$ since $M \in \mathcal{C}(\mathfrak{d})$ is finite dimensional. (Refer to [1, I.9.5] for Hopf dual.) Since $\mathcal{C}(\mathfrak{d})$ is closed under finite direct sums, finite tensor products and the formation duals, $\mathbb{C}[G]$ is a sub-Hopf algebra of $U(\mathfrak{d})^\circ$ with Hopf structure (1.15) [1, I.7.3 and I.7.4]. Since $U(\mathfrak{d})$ is a co-commutative Hopf algebra, $U(\mathfrak{d})^\circ$ is a commutative Hopf algebra and thus $\mathbb{C}[G]$ is a commutative algebra. By (1.15) and (1.16), $\mathbb{C}[G]$ is $\mathbf{L}$-bigraded.

Let $\{\omega_1, \dots, \omega_n\}$ be a basis of $\mathbf{L}$ such that $\mathbf{L}^+ = \sum_i \mathbb{Z}_{\geq 0}\omega_i$. For $\Lambda \in \mathbf{L}^+$, if $\Lambda = \sum_i m_i\omega_i$ then $V(\Lambda)$ is isomorphic to a submodule of $\bigotimes_{i=1}^n \underbrace{V(\omega_i) \otimes \dots \otimes V(\omega_i)}_{m_i}$ and thus $\mathbb{C}[G]$ is generated by all elements $c^{V(\omega_i)}_{f,v}$, $1 \leq i \leq n$. Hence $\mathbb{C}[G]$ is finitely generated since each $V(\omega_i)$ is finite dimensional. □

2. Poisson bracket on $\mathbb{C}[G]$

2.1. Hopf dual of co-Poisson Hopf algebra

Refer to [2, Definition 6.2.2] and [16, Definition 2.2.9] for the definition of co-Poisson Hopf algebra. There is a statement that the Hopf dual of a co-Poisson Hopf algebra is a Poisson Hopf algebra [16, Proposition 3.1.5]. Here we give a complete proof for this statement.

Lemma 2.1. *Let $I_1, \ldots, I_r$ be cofinite dimensional subspaces of a vector space V. Then $\cap_{i=1}^r I_i$ is also cofinite.*

Proof. The kernel of the linear map

$$\psi : V \longrightarrow V/I_1 \times \ldots \times V/I_r, \qquad \psi(x) = (x + I_1, \ldots, x + I_r)$$

is $\cap_{i=1}^r I_i$. Hence $\cap_{i=1}^r I_i$ is co-finite since $V/I_1 \times \ldots \times V/I_r$ is finite dimensional. □

Theorem 2.2 ([16, Proposition 3.1.5]). *Let $A = (A, \iota, m, \epsilon, \Delta, S)$ be a co-Poisson co-commutative Hopf algebra with co-bracket δ. Then its Hopf dual A° is a Poisson Hopf algebra with Poisson bracket*

$$\{f, g\} = (f \otimes g)\delta, \qquad (f, g \in A^\circ). \tag{2.1}$$

Proof. Note that A° is commutative since A is co-commutative. Let I and J be co-finite dimensional ideals of A such that $f(I) = 0$ and $g(J) = 0$. Then $K = I \cap J$ is co-finite by Lemma 2.1 and thus $K \otimes A + A \otimes K$ is a co-finite ideal of $A \otimes A$ since $A \otimes A/(K \otimes A + A \otimes K)$ is isomorphic to $(A/K) \otimes (A/K)$. Let L be the ideal of A generated by $\delta^{-1}(K \otimes A + A \otimes K) \cap \Delta^{-1}(K \otimes A + A \otimes K)$. Then L is a co-finite ideal of A by Lemma 2.1 and $\delta(L) \subseteq K \otimes A + A \otimes K$ since $\delta(xy) = \delta(x)\Delta(y) + \Delta(x)\delta(y)$ for all $x, y \in A$. Hence $(f \otimes g)\delta(L) = 0$. It follows that $\{f, g\} \in A^\circ$.

By the co-Jacobi identity and the co-Leibniz identity, A° is a Poisson algebra. For $x, y \in A$,

$$\begin{aligned} m^*(\{f, g\})(x \otimes y) &= (f \otimes g)\delta(xy) \\ &= (f \otimes g)(\delta(x)\Delta(y) + \Delta(x)\delta(y)) \\ &= \{m^*(f), m^*(g)\}(x \otimes y). \end{aligned}$$

Hence A° is a Poisson Hopf algebra. □

Refer to [2, Definition 1.3.1] for the definition of Lie bialgebra. Let $\mathfrak{m}$ be a Lie algebra and let $U(\mathfrak{m})$ be the universal enveloping algebra of $\mathfrak{m}$. It is well known that $U(\mathfrak{m})$ has a Hopf structure $(U(\mathfrak{m}), \iota, m, \epsilon, \Delta, S)$, where

$$\epsilon(x) = 0, \qquad \Delta(x) = x \otimes 1 + 1 \otimes x, \qquad S(x) = -x$$

for all $x \in \mathfrak{m}$.

Corollary 2.3. *Let $\mathfrak{m}$ be a Lie bialgebra with co-commutator δ' and let $U(\mathfrak{m})$ be the universal enveloping algebra of $\mathfrak{m}$.*

(1) *In $U(\mathfrak{m}) = (U(\mathfrak{m}), \iota, m, \epsilon, \Delta, S)$, there exists a unique co-Poisson co-commutative Hopf algebra structure with co-bracket δ such that $\delta\,|_{\,\mathfrak{m}} = \delta'$.*

(2) *The Hopf dual $U(\mathfrak{m})^\circ$ is a Poisson Hopf algebra with Poisson bracket (2.1).*

Proof. (1) (The co-bracket δ is a unique extension $\delta : U(\mathfrak{m}) \longrightarrow U(\mathfrak{m}) \otimes U(\mathfrak{m})$ of δ' such that $\delta(xy) = \Delta(x)\delta(y) + \delta(x)\Delta(y)$ for all $x, y \in U(\mathfrak{m})$ [2, Proposition 6.2.3].)

(2) It follows by (1) and Theorem 2.2. □

2.2. Poisson bracket of $U(\mathfrak{d})^\circ$

Let

$$r = \sum_{\alpha \in \mathbf{R}^+} (x_\alpha \otimes x_{-\alpha} - x_{-\alpha} \otimes x_\alpha) \in \mathfrak{n} \otimes \mathfrak{n} \qquad ((x_\alpha \,|\, x_{-\alpha}) = 1 \text{ for each } \alpha \in \mathbf{R}^+)$$

be the one given in (1.12). Then $\mathfrak{d}$ is a Lie bialgebra with Lie bracket (1.8) and co-commutator

$$\delta'(x) = x.r \tag{2.2}$$

for all $x \in \mathfrak{d}$ by Proposition 1.4 and [2, Proposition 2.1.2], where the action $x.r$ is by the adjoint action of x on each factor of r.

Lemma 2.4. *The co-commutator δ' given in (2.2) is as follows:*

$$\begin{array}{ll} \delta'(h_\lambda) = 0, & h_\lambda \in \mathfrak{h}, \\ \delta'(k_\lambda) = 0, & k_\lambda \in \mathfrak{k}, \\ \delta'(x_\beta) = 2^{-1}x_\beta \wedge (h_\beta + k_\beta), & \beta \in \mathbf{R}^+, \\ \delta'(x_{-\beta}) = 2^{-1}x_{-\beta} \wedge (h_\beta + k_\beta), & \beta \in \mathbf{R}^+, \end{array} \tag{2.3}$$

where $a \wedge b = a \otimes b - b \otimes a$.

Proof. For each $\alpha \in \mathbf{R}^+, h_\lambda.(x_\alpha \wedge x_{-\alpha}) = [h_\lambda, x_\alpha] \wedge x_{-\alpha} + x_\alpha \wedge [h_\lambda, x_{-\alpha}] = 0$. Similarly, $k_\lambda.(x_\alpha \wedge x_{-\alpha}) = 0$. Hence $\delta'(h_\lambda) = 0$ and $\delta'(k_\lambda) = 0$.

Let $\alpha, \beta, \alpha+\beta \in \mathbf{R}$. Thus $-(\alpha+\beta) \in \mathbf{R}$. Since the root spaces of $\mathfrak{n}$ are one dimensional [4, Proposition 10.9],

$$[x_\beta, x_\alpha] = a_{\beta,\alpha} x_{\alpha+\beta}, \qquad [x_\beta, x_{-(\alpha+\beta)}] = b_{\beta,\alpha} x_{-\alpha}$$

for some $a_{\beta,\alpha}, b_{\beta,\alpha} \in \mathbb{C}$. Then

$$a_{\beta,\alpha} = -b_{\beta,\alpha} \tag{2.4}$$

since

$$\begin{aligned} a_{\beta,\alpha} &= (a_{\beta,\alpha} x_{\alpha+\beta} \mid x_{-(\alpha+\beta)}) = ([x_\beta, x_\alpha] \mid x_{-(\alpha+\beta)}) = ([x_\beta, x_\alpha]_{\mathfrak{g}} \mid x_{-(\alpha+\beta)}) \\ &= -(x_\alpha \mid [x_\beta, x_{-(\alpha+\beta)}]_{\mathfrak{g}}) = -(x_\alpha \mid [x_\beta, x_{-(\alpha+\beta)}]) = -(x_\alpha \mid b_{\beta,\alpha} x_{-\alpha}) = -b_{\beta,\alpha}. \end{aligned}$$

Hence $[x_\beta, x_\alpha] \wedge x_{-\alpha} = -x_{\alpha+\beta} \wedge [x_\beta, x_{-(\alpha+\beta)}]$. It follows that

$$\begin{aligned} x_\beta.r &= \sum_{\alpha \in \mathbf{R}^+} [x_\beta, x_\alpha] \wedge x_{-\alpha} + x_\alpha \wedge [x_\beta, x_{-\alpha}] \\ &= \begin{cases} x_\beta \wedge [x_\beta, x_{-\beta}], & \beta \in \mathbf{R}^+, \\ [x_\beta, x_{-\beta}] \wedge x_\beta, & \beta \in \mathbf{R}^-, \end{cases} \\ &= \begin{cases} 2^{-1}x_\beta \wedge (h_\beta + k_\beta), & \beta \in \mathbf{R}^+, \\ 2^{-1}x_\beta \wedge (h_{-\beta} + k_{-\beta}), & \beta \in \mathbf{R}^- \end{cases} \end{aligned}$$

by (1.4) and (1.8). □

By Corollary 2.3, there exists a unique extension δ of δ' given in (2.2) and the Hopf dual $U(\mathfrak{d})^\circ$ is a Poisson Hopf algebra with Poisson bracket

$$\{a, b\}_r = (a \otimes b)\delta$$

for all $a, b \in U(\mathfrak{d})^\circ$. Let us find $\{\cdot, \cdot\}_r$ in the sub-Hopf algebra $\mathbb{C}[G]$ of $U(\mathfrak{d})^\circ$.

Lemma 2.5. *Let $M, N \in \mathcal{C}(\mathfrak{d})$ and let $f \in (M^*)_\beta, g \in (N^*)_\gamma, p \in M_\eta, v \in N_\rho$ for $\beta, \gamma, \eta, \rho \in \mathbf{L}$. Then*

$$\begin{array}{ll} \{c_{f,p}, c_{g,v}\}_r(h_\lambda) = 0, & h_\lambda \in \mathfrak{h}, \\ \{c_{f,p}, c_{g,v}\}_r(k_\lambda) = 0, & k_\lambda \in \mathfrak{k}, \\ \{c_{f,p}, c_{g,v}\}_r(x_\alpha) = (\alpha \mid \rho) f(x_\alpha p) g(v) - (\alpha \mid \eta) f(p) g(x_\alpha v), & \alpha \in \mathbf{R}. \end{array} \tag{2.5}$$

Proof. By (2.3), $\{c_{f,p}, c_{g,v}\}_r(h_\lambda) = 0$, $\{c_{f,p}, c_{g,v}\}_r(k_\lambda) = 0$ and

$$\begin{aligned}\{c_{f,p}, c_{g,v}\}_r(x_\alpha) &= (c_{f,p} \otimes c_{g,v})\delta'(x_\alpha)\\ &= 2^{-1}\big(f(x_\alpha p)g((h_\alpha + k_\alpha)v) - f((h_\alpha + k_\alpha)p)g(x_\alpha v)\big)\\ &= (\alpha \mid \rho)f(x_\alpha p)g(v) - (\alpha \mid \eta)f(p)g(x_\alpha v)\end{aligned}$$

since $(h_\alpha + k_\alpha)v = 2(\alpha \mid \rho)v$ and $(h_\alpha + k_\alpha)p = 2(\alpha \mid \eta)p$ by (1.14) and (1.5). □

Lemma 2.6. *For $M, N \in \mathcal{C}(\mathfrak{d})$, let $\{f_i\}_{i=1}^r, \{u_i\}_{i=1}^r, \{g_j\}_{j=1}^s, \{v_j\}_{j=1}^s$ be $\mathbb{C}$-bases of M^*, M, N^*, N, respectively, such that $f_i(u_k) = \delta_{ik}$ and $g_j(v_\ell) = \delta_{j\ell}$. For any $\alpha \in \mathbf{R}^+$, suppose that*

$$x_\alpha u_i = \sum_{k=1}^r a_{ik}u_k, \qquad x_{-\alpha}v_j = \sum_{\ell=1}^s b_{j\ell}v_\ell,$$

where $a_{ik}, b_{j\ell} \in \mathbb{C}$. Then

$$f_i x_\alpha = \sum_{k=1}^r a_{ki}f_k, \qquad g_j x_{-\alpha} = \sum_{\ell=1}^s b_{\ell j}g_\ell.$$

Proof. Let $f_i x_\alpha = \sum_{k=1}^r d_k f_k$, for some $d_k \in \mathbb{C}$. Then

$$d_t = \left(\sum_{k=1}^r d_k f_k\right)(u_t) = (f_i x_\alpha)(u_t) = f_i(x_\alpha u_t) = f_i\left(\sum_{k=1}^r a_{tk}u_k\right) = a_{ti}.$$

Hence $f_i x_\alpha = \sum_{k=1}^r a_{ki}f_k$. Similarly, we have $g_j x_{-\alpha} = \sum_{\ell=1}^s b_{\ell j}g_\ell$. □

Proposition 2.7. *The function algebra $\mathbb{C}[G]$ is an $\mathbf{L}$-bigraded Poisson Hopf algebra with Poisson bracket*

$$\{c_{f,p}, c_{g,v}\}_r = [(\eta \mid \rho) - (\beta \mid \gamma)]c_{f,p}c_{g,v} + 2\sum_{\nu \in \mathbf{R}^+}(c_{f,x_\nu p}c_{g,x_{-\nu}v} - c_{fx_\nu,p}c_{gx_{-\nu},v}) \tag{2.6}$$

for weight vectors $f \in (M^*)_\beta, g \in (N^*)_\gamma, p \in M_\eta, v \in N_\rho$ of $M, N \in \mathcal{C}(\mathfrak{d})$, where $\beta, \gamma, \eta, \rho \in \mathbf{L}$.

Proof. Let $\lambda_1, \ldots, \lambda_n$ be a $\mathbb{C}$-basis of $\mathfrak{h}^*$. Then $\mathfrak{B} = \{h_{\lambda_i}, k_{\lambda_i}, x_\alpha \mid i = 1, \ldots, n, \alpha \in \mathbf{R}\}$ is a $\mathbb{C}$-basis of $\mathfrak{d}$. For convenience, we set $\mathfrak{B} = \{y_1, \ldots, y_r\}$. Hence each y_i is one of h_λ, k_λ and x_α, $0 \neq \lambda \in \mathfrak{h}^*, \alpha \in \mathbf{R}$. By the Poincare–Birkhoff–Witt theorem, $U(\mathfrak{d})$ has the $\mathbb{C}$-basis consisting of the standard monomials $Y = y_1^{s_1}\cdots y_r^{s_r}$ that is a product of $y_i \in \mathfrak{B}$. Let $\ell(Y)$ be the number $s_1 + \ldots + s_r$ of elements of $\mathfrak{B}$ appearing in Y. We will call $\ell(Y)$ the length of Y. Denote by RH the right-hand side of (2.6). We proceed by induction on $\ell(Y)$ to show $\{c_{f,p}, c_{g,v}\}_r(Y) = \mathrm{RH}(Y)$.

If $\ell(Y) = 0$ then $Y = 1$, $\{c_{f,p}, c_{g,v}\}_r(1) = 0$ and $\mathrm{RH}(1) = [(\eta \mid \rho) - (\beta \mid \gamma)]f(p)g(v)$. If $f(p)g(v) \neq 0$ then $-\beta = \eta$ and $-\gamma = \rho$ by Lemma 1.5(1) and thus $(\eta \mid \rho) - (\beta \mid \gamma) = 0$. It follows that

$$[(\eta \mid \rho) - (\beta \mid \gamma)]f(p)g(v) = 0. \tag{2.7}$$

Hence $\{c_{f,p}, c_{g,v}\}_r(1) = 0 = \mathrm{RH}(1)$.

Suppose that $\ell(Y) = 1$. Then $Y = y_i$ for some i and thus $Y = h_\lambda$, $Y = k_\lambda$ or $Y = x_\alpha$. If $Y = h_\lambda$ then

$$\begin{aligned}\mathrm{RH}(Y) = [(\eta \mid \rho) &- (\beta \mid \gamma)](f(h_\lambda p)g(v) + f(p)g(h_\lambda v))\\ &+ 2\sum(f(h_\lambda x_\nu p)g(x_{-\nu}v) + f(x_\nu p)g(h_\lambda x_{-\nu}v))\\ &- 2\sum(f(x_\nu h_\lambda p)g(x_{-\nu}v) + f(x_\nu p)g(x_{-\nu}h_\lambda v))\\ = [(\eta \mid \rho) &- (\beta \mid \gamma)](f(h_\lambda p)g(v) + f(p)g(h_\lambda v))\end{aligned}$$

$$
\begin{aligned}
&+2\sum f([h_\lambda,x_\nu]p)g(x_{-\nu}v)+f(x_\nu p)g([h_\lambda,x_{-\nu}]v)\\
&=[(\Phi_+\eta\mid\lambda)+(\Phi_+\rho\mid\lambda)][(\eta\mid\rho)-(\beta\mid\gamma)]f(p)g(v) \qquad (\text{by } (1.14), 1.8))\\
&=0. \qquad (\text{by } (2.7))
\end{aligned}
$$

Hence $\{c_{f,p},c_{g,v}\}_r(h_\lambda)=0=\mathrm{RH}(h_\lambda)$ by (2.5). Similarly we have $\{c_{f,p},c_{g,v}\}_r(k_\lambda)=\mathrm{RH}(k_\lambda)$. Let $Y=x_\alpha$, $\alpha\in\mathbf{R}$. Suppose that $\alpha\in\mathbf{R}^+$. Then

$$
\begin{aligned}
\mathrm{RH}(x_\alpha)&=[(\eta\mid\rho)-(\beta\mid\gamma)](f(x_\alpha p)g(v)+f(p)g(x_\alpha v))\\
&\quad+2\sum f([x_\alpha,x_\nu]p)g(x_{-\nu}v)+f(x_\nu p)g([x_\alpha,x_{-\nu}]v)\\
&\quad(\text{replace } [x_\alpha,x_\nu] \text{ by } a_{\alpha\nu}x_{\alpha+\nu}, \text{ then } [x_\alpha,x_{-(\alpha+\nu)}]=-a_{\alpha\nu}x_{-\nu} \text{ by } (2.4))\\
&=[(\eta\mid\rho)-(\beta\mid\gamma)](f(x_\alpha p)g(v)+f(p)g(x_\alpha v))+2f(x_\alpha p)g([x_\alpha,x_{-\alpha}]v)\\
&=[(\eta\mid\rho)-(\beta\mid\gamma)](f(x_\alpha p)g(v)+f(p)g(x_\alpha v))+f(x_\alpha p)g((h_\alpha+k_\alpha)v)\\
&=[(\eta\mid\rho)-(\beta\mid\gamma)+2(\alpha\mid\rho)]f(x_\alpha p)g(v)+[(\eta\mid\rho)-(\beta\mid\gamma)]f(p)g(x_\alpha v).
\end{aligned}
$$

If $f(x_\alpha p)g(v)\neq 0$ then $-\beta=\alpha+\eta$ and $-\gamma=\rho$ by Lemma 1.5(1), thus $f(p)g(x_\alpha v)=0$ and

$$
\begin{aligned}
\mathrm{RH}(x_\alpha)&=[(\eta\mid\rho)-(\beta\mid\gamma)+2(\alpha\mid\rho)]f(x_\alpha p)g(v)=(\alpha\mid\rho)f(x_\alpha p)g(v)\\
&=(\alpha\mid\rho)f(x_\alpha p)g(v)-(\alpha\mid\eta)f(p)g(x_\alpha v).
\end{aligned}
$$

Similarly, if $f(p)g(x_\alpha v)\neq 0$ then $-\beta=\eta$ and $-\gamma=\alpha+\rho$ by Lemma 1.5(1), thus $f(x_\alpha p)g(v)=0$ and

$$
\begin{aligned}
\mathrm{RH}(x_\alpha)&=[(\eta\mid\rho)-(\beta\mid\gamma)]f(p)g(x_\alpha v)=-(\alpha\mid\eta)f(p)g(x_\alpha v)\\
&=(\alpha\mid\rho)f(x_\alpha p)g(v)-(\alpha\mid\eta)f(p)g(x_\alpha v).
\end{aligned}
$$

It follows that

$$
\mathrm{RH}(x_\alpha)=(\alpha\mid\rho)f(x_\alpha p)g(v)-(\alpha\mid\eta)f(p)g(x_\alpha v)
$$

and thus $\{c_{f,p},c_{g,v}\}_r(x_\alpha)=\mathrm{RH}(x_\alpha)$ for $\alpha\in\mathbf{R}^+$ by (2.5). Similarly, we have $\{c_{f,p},c_{g,v}\}_r(x_{-\alpha})=\mathrm{RH}(x_{-\alpha})$ for $\alpha\in\mathbf{R}^+$.

Suppose that $\ell(Y)=s>1$ and that $\{c_{f,p},c_{g,v}\}_r(Z)=\mathrm{RH}(Z)$ for all monomials Z with $\ell(Z)<s$. Note that $Y=Xy$ for some monomial X with $\ell(X)=s-1$ and $y\in\mathfrak{B}$. Note that

$$
\Delta(c_{f,p})=\sum_i c_{f,u_i}\otimes c_{f_i,p}, \qquad \Delta(c_{g,v})=\sum_i c_{g,v_i}\otimes c_{g_i,v}, \tag{2.8}
$$

where $\{f_i\}_{i=1}^a,\{u_i\}_{i=1}^a,\{g_i\}_{i=1}^b,\{v_i\}_{i=1}^b$ are $\mathbb{C}$-bases of M^*,M,N^*,N such that $f_i(u_j)=\delta_{ij}$ and $g_i(v_j)=\delta_{ij}$. We may assume that f_i,u_i,g_i,v_i are all weight vectors and denote by η_i and ρ_i the weights of u_i and v_i, respectively. Hence the weights of f_i and g_i are $-\eta_i$ and $-\rho_i$, respectively, by Lemma 1.5(1). Then

$$
\begin{aligned}
\{c_{f,p},c_{g,v}\}_r(Xy)&=\sum_{i,j}\{c_{f,u_i},c_{g,v_j}\}_r(X)(c_{f_i,p}c_{g_j,v})(y)+(c_{f,u_i}c_{g,v_j})(X)\{c_{f_i,p},c_{g_j,v}\}_r(y)\\
&\quad\Big(\text{since } \Delta(\{c_{f,p},c_{g,v}\}_r)=\sum_{i,j}\{c_{f,u_i},c_{g,v_j}\}_r\otimes c_{f_i,p}c_{g_j,v}+c_{f,u_i}c_{g,v_j}\otimes\{c_{f_i,p},c_{g_j,v}\}_r\Big)\\
&=\sum_{i,j}[(\eta_i\mid\rho_j)-(\beta\mid\gamma)](c_{f,u_i}c_{g,v_j})(X)(c_{f_i,p}c_{g_j,v})(y)\\
&\quad+2\sum_{i,j}\sum_{\alpha\in\mathbf{R}^+}(c_{f,x_\alpha u_i}c_{g,x_{-\alpha}v_j}-c_{fx_\alpha,u_i}c_{gx_{-\alpha},v_j})(X)(c_{f_i,p}c_{g_j,v})(y)\\
&\quad+\sum_{i,j}[(\eta\mid\rho)-(\eta_i\mid\rho_j)](c_{f,u_i}c_{g,v_j})(X)(c_{f_i,p}c_{g_j,v})(y)
\end{aligned}
$$

$$+2\sum_{i,j}\sum_{\alpha\in\mathbf{R}^+}(c_{f,u_i}c_{g,v_j})(X)(c_{f_i,x_\alpha p}c_{g_j,x_{-\alpha}v}-c_{f_ix_\alpha,p}c_{g_jx_{-\alpha},v})(y)$$

(by induction hypothesis)

$$=\sum_{i,j}[(\eta\mid\rho)-(\beta\mid\gamma)](c_{f,u_i}c_{g,v_j})(X)(c_{f_i,p}c_{g_j,v})(y)$$

$$+2\sum_{\alpha\in\mathbf{R}^+}\sum_{i,j}(c_{f,u_i}c_{g,v_j})(X)(c_{f_i,x_\alpha p}c_{g_j,x_{-\alpha}v})(y)-(c_{fx_\alpha,u_i}c_{gx_{-\alpha},v_j})(X)(c_{f_i,p}c_{g_j,v})(y)$$

$$+2\sum_{\alpha\in\mathbf{R}^+}\sum_{i,j}(c_{f,x_\alpha u_i}c_{g,x_{-\alpha}v_j})(X)(c_{f_i,p}c_{g_j,v})(y)-(c_{f,u_i}c_{g,v_j})(X)(c_{f_ix_\alpha,p}c_{g_jx_{-\alpha},v})(y)$$

$$=\mathrm{RH}(Xy)$$

$$+2\sum_{\alpha\in\mathbf{R}^+}\sum_{i,j}(c_{f,x_\alpha u_i}c_{g,x_{-\alpha}v_j})(X)(c_{f_i,p}c_{g_j,v})(y)-(c_{f,u_i}c_{g,v_j})(X)(c_{f_ix_\alpha,p}c_{g_jx_{-\alpha},v})(y)$$

(by (2.8))

$$=\mathrm{RH}(Xy)$$

$$+2\sum_{\alpha\in\mathbf{R}^+}\sum_{i,j}\sum_{k,\ell}a_{ik}b_{j\ell}(c_{f,u_k}c_{g,v_\ell})(X)(c_{f_i,p}c_{g_j,v})(y)-a_{ki}b_{\ell j}(c_{f,u_i}c_{g,v_j})(X)(c_{f_k,p}c_{g_\ell,v})(y)$$

$$\left(\text{replace } x_\alpha u_i \text{ and } x_{-\alpha}v_j \text{ by } \sum_{k=1}^{r}a_{ik}u_k \text{ and } \sum_{\ell=1}^{s}b_{j\ell}v_\ell \text{ and use Lemma 2.6}\right)$$

$$=\mathrm{RH}(Xy).$$

Hence we derive (2.6).

It is easily observed that the Poisson bracket of $\mathbb{C}[G]$ preserves the $\mathbf{L}$-bigrading by (2.6) and Lemma 1.5. It follows that $\mathbb{C}[G]$ is an $\mathbf{L}$-bigraded Poisson Hopf algebra by Theorems 1.8 and 2.2. □

Lemma 2.8. *Let $A_1=(A,\iota,m,\epsilon,\Delta,S,\{\cdot,\cdot\}_1)$ and $A_2=(A,\iota,m,\epsilon,\Delta,S,\{\cdot,\cdot\}_2)$ be K-bigraded Poisson Hopf algebras over a field $\mathbf{k}$. Define a bilinear product $\{\cdot,\cdot\}$ on A by*

$$\{a,b\}=\{a,b\}_1+\{a,b\}_2$$

for $a,b\in A$. Then $A=(A,\iota,m,\epsilon,\Delta,S,\{\cdot,\cdot\})$ is also a K-bigraded Poisson Hopf algebra.

Proof. It is easy to see that A is a K-bigraded Poisson algebra with the Poisson bracket $\{\cdot,\cdot\}$. Since Δ is a Poisson algebra homomorphism in A_1 and A_2, it is also a Poisson algebra homomorphism in A. Hence $A=(A,\iota,m,\epsilon,\Delta,S,\{\cdot,\cdot\})$ is also a K-bigraded Poisson Hopf algebra. □

Theorem 2.9. *The function algebra $\mathbb{C}[G]$ is an $\mathbf{L}$-bigraded Poisson Hopf algebra with Poisson bracket*

$$\{c_{f,w},c_{g,v}\}=[(\Phi_+\eta\mid\rho)-(\Phi_+\beta\mid\gamma)]c_{f,w}c_{g,v}+2\sum_{\nu\in\mathbf{R}^+}(c_{f,x_\nu w}c_{g,x_{-\nu}v}-c_{fx_\nu,w}c_{gx_{-\nu},v}) \tag{2.9}$$

for weight vectors $f\in(M^*)_\beta, g\in(N^*)_\gamma, w\in M_\eta, v\in N_\rho$ of $M,N\in\mathcal{C}(\mathfrak{d})$,

Remark 2.10.

(1) Suppose that the skew-symmetric bilinear form u in 1.2 is equal to zero. Then we obtain (2.7) from (2.9) by (1.5). Moreover, the subspace $\langle h_\lambda-k_\lambda\mid\lambda\in\mathfrak{h}^*\rangle$ of $\mathfrak{d}$ is a Lie ideal of $\mathfrak{d}$ and $\mathfrak{d}/\langle h_\lambda-k_\lambda\mid\lambda\in\mathfrak{h}^*\rangle\cong\mathfrak{g}$ by (1.8). Thus $\mathfrak{d}$ and (2.9) are generalizations of $\mathfrak{g}$ and (2.7), respectively.

(2) One should compare (2.7) and (2.9) with [1, (6) of I.8.16 Theorem] and [11, Corollary 3.10], respectively. Namely, the Poisson Hopf algebras $\mathbb{C}[G]$ with Poisson brackets (2.7) and (2.9)

can be considered as Poisson versions of the quantized algebras $\mathcal{O}_q(G)$ and $\mathbb{C}_{q,p}[G]$ [1, I.8.16 Theorem] [11].

Proof of Theorem 2.9. Note that $\mathbb{C}[G]$ is an $\mathbf{L}$-bigraded commutative Hopf algebra and that u in 1.2 is a skew symmetric bilinear form on $\mathbf{L}$. Define a bilinear product $\{\cdot,\cdot\}_u$ on $\mathbb{C}[G]$ by

$$\{a_{\lambda,\mu}, b_{\lambda',\mu'}\}_u = [u(\mu,\mu') - u(\lambda,\lambda')]a_{\lambda,\mu}b_{\lambda',\mu'}$$

for $a_{\lambda,\mu} \in \mathbb{C}[G]_{\lambda,\mu}$ and $b_{\lambda',\mu'} \in \mathbb{C}[G]_{\lambda',\mu'}$. Then it is easy to check that $\mathbb{C}[G]$ is an $\mathbf{L}$-bigraded Poisson Hopf algebra with Poisson bracket $\{\cdot,\cdot\}_u$ since u is skew-symmetric. Hence the result follows by Proposition 2.7 and Lemma 2.8 since

$$u(\eta,\rho) + (\eta\,|\,\rho) = (\Phi_+\eta\,|\,\rho),\;\; u(\beta,\gamma) + (\beta\,|\,\gamma) = (\Phi_+\beta\,|\,\gamma)$$

by (1.5). □

3. Poisson prime ideals of $\mathbb{C}[G]$

In this section, we restate some basic definitions and properties for $\mathbb{C}[G]$ using Poisson terminologies that appear in the quantized algebra $\mathbb{C}_{q,p}[G]$ of [11, Section 4].

Assume throughout the section that, for each $\Lambda \in \mathbf{L}^+$, $V(\Lambda)$, denotes the highest weight $\mathfrak{g}$-module with the highest weight Λ and thus $V(\Lambda) \in \mathcal{C}(\mathfrak{d})$ satisfies the action (1.14). Let $M \in \mathcal{C}(\mathfrak{d})$. Then $M = \oplus_{\mu\in\mathbf{L}} M_\mu$. We often write v_μ for $v \in M_\mu$.

Theorem 3.1. *Set*

$$\mathbb{C}[G]^+ = \sum_{\Lambda\in\mathbf{L}^+}\sum_{f\in V(\Lambda)^*}\mathbb{C}c^{V(\Lambda)}_{f,v_\Lambda},\quad \mathbb{C}[G]^- = \sum_{\Lambda\in\mathbf{L}^+}\sum_{f\in V(\Lambda)^*}\mathbb{C}c^{V(\Lambda)}_{f,v_{w_0\Lambda}},$$

where w_0 is the longest element of the Weyl group W of $\mathfrak{g}$. The multiplication map $\mathbb{C}[G]^+\otimes\mathbb{C}[G]^- \longrightarrow \mathbb{C}[G]$ is surjective.

Proof. It is proved by mimicking the proof of [16, 2.2.1 of Chapter 3] or [14, 9.2.2 Proposition]. □

Denote by $U(\mathfrak{b}^+)$ and $U(\mathfrak{b}^-)$ the subalgebras of $U(\mathfrak{d})$ generated by

$$h_\lambda \in \mathfrak{h}, \qquad x_\alpha, \qquad \alpha \in \mathbf{R}^+$$

and

$$k_\lambda \in \mathfrak{k}, \qquad x_{-\alpha}, \qquad \alpha \in \mathbf{R}^+,$$

respectively.

Let $M \in \mathcal{C}(\mathfrak{d})$. Given a subset $X \subseteq M$, write $X^\perp$ for the orthogonal space of X in M^*, that is,

$$X^\perp = \{f \in M^*\,|\,f(X) = 0\}.$$

Let w_0 be the longest element of the Weyl group W. For $\Lambda \in \mathbf{L}^+$ and $y \in W$, define the ideals I^+_y and I^-_y of the $\mathbf{L}$-bigraded Poisson Hopf algebra $\mathbb{C}[G]$ with Poisson bracket (2.9):

$$I^+_y = \langle c^{V(\Lambda)}_{f,v_\Lambda}\,|\,f \in (U(\mathfrak{b}^+)V(\Lambda)_{y\Lambda})^\perp\rangle, \qquad I^-_y = \langle c^{V(\Lambda)}_{f,v_{w_0\Lambda}}\,|\,f \in (U(\mathfrak{b}^-)V(\Lambda)_{yw_0\Lambda})^\perp\rangle.$$

For $w = (w_+, w_-) \in W\times W$, define

$$I_w = I^+_{w_+} + I^-_{w_-}.$$

Lemma 3.2. *For any $w = (w_+, w_-) \in W\times W$, I_w is a homogeneous Poisson ideal of $\mathbb{C}[G]$.*

Proof. Since $I^+_{w_+}$ and $I^-_{w_-}$ are generated by homogeneous elements, they are homogeneous ideals. Moreover $I^+_{w_+}$ and $I^-_{w_-}$ are Poisson ideals by (2.9). Hence I_w is a homogeneous Poisson ideal. □

For any $w = (w_+, w_-) \in W \times W$, define the following homogeneous elements in the Poisson algebra $\mathbb{C}[G]/I_w$:

$$c_{w\Lambda} = c^{V(\Lambda)}_{f_{-w_+\Lambda}, v_\Lambda} + I_w, \qquad \tilde{c}_{w\Lambda} = c^{V(\Lambda)^*}_{v_{w_-\Lambda}, f_{-\Lambda}} + I_w, \qquad \Lambda \in \mathbf{L}^+.$$

Note that $\tilde{c}_{w\Lambda}$ is of the form $c^{V(\Lambda)}_{f_{-w_-w_0\Lambda}, v_{w_0\Lambda}} + I_w$ since $w_0(\mathbf{R}^+) = \mathbf{R}^-$ by [14, A.1.4].

An element a of a Poisson algebra A is said to be *Poisson normal* if $\{a, A\} \subseteq aA$. We should compare the following lemma with [11, Lemma 4.2].

Lemma 3.3. *Let $\Lambda \in \mathbf{L}^+$.*
(1) $\{c_{w\Lambda}, c_{f_{-\lambda}, v_\mu} + I_w\} = [(\Phi_+\Lambda \mid \mu) - (\Phi_+ w_+\Lambda \mid \lambda)](c_{f_{-\lambda}, v_\mu} + I_w)c_{w\Lambda}$.
(2) $\{\tilde{c}_{w\Lambda}, c_{f_{-\lambda}, v_\mu} + I_w\} = [(\Phi_- w_-\Lambda \mid \lambda) - (\Phi_-\Lambda \mid \mu)](c_{f_{-\lambda}, v_\mu} + I_w)\tilde{c}_{w\Lambda}$.
In particular, both $c_{w\Lambda}$ and $\tilde{c}_{w\Lambda}$ are Poisson normal elements of $\mathbb{C}[G]/I_w$.

Proof. (1) It is proved immediately by (2.9).

(2) It is proved immediately by (2.9) and (1.5). □

Observe that the sets

$$\mathcal{E}^+_w = \{\alpha c_{w\Lambda} \mid \alpha \in \mathbb{C}^\times,\ \Lambda \in \mathbf{L}^+\}, \qquad \mathcal{E}^-_w = \{\alpha \tilde{c}_{w\Lambda} \mid \alpha \in \mathbb{C}^\times,\ \Lambda \in \mathbf{L}^+\}$$
$$\mathcal{E}_w = \mathcal{E}^+_w \mathcal{E}^-_w$$

are multiplicatively closed sets in $\mathbb{C}[G]/I_w$. Denote the localization by

$$\mathbb{C}[G]_w = (\mathbb{C}[G]/I_w)_{\mathcal{E}_w}.$$

Theorem 3.4. *For any Poisson prime ideal P of $\mathbb{C}[G]$, there exists a unique $w = (w_+, w_-) \in W \times W$ such that $I_w \subseteq P$ and $(P/I_w) \cap \mathcal{E}_w = \varnothing$.*

Proof. The proof is parallel to that of [11, Theorem 4.4]. We repeat it for completion. For $\Lambda \in \mathbf{L}^+$, define an order relation on the weight vectors of $V(\Lambda)^*$ by $f \le f'$ if $f' \in fU(\mathfrak{b}^+)$. This induces a partial ordering on the set of one dimensional weight spaces. For $\Lambda \in \mathbf{L}^+$, set

$$\mathcal{D}(\Lambda) = \{f \in (V(\Lambda)^*)_{-\mu} \mid c^{V(\Lambda)}_{f, v_\Lambda} \notin P\}.$$

We claim $\mathcal{D}(\Lambda) \neq \varnothing$ for all $\Lambda \in \mathbf{L}^+$. Let $\{g_i\}$ and $\{v_i\}$ be dual bases for $V(\Lambda)^*$ and $V(\Lambda)$ that consist of weight vectors. Since we have

$$1 = \epsilon(c^{V(\Lambda)}_{f_{-\Lambda}, v_\Lambda}) = \sum_{i \in I} S(c^{V(\Lambda)}_{f_{-\Lambda}, v_i}) c^{V(\Lambda)}_{g_i, v_\Lambda}$$

there exists an index i such that $c^{V(\Lambda)}_{g_i, v_\Lambda} \notin P$ and thus $g_i \in \mathcal{D}(\Lambda)$ as claimed.

Suppose that $f, f' \in \mathcal{D}(\Lambda)$ are both maximal elements and set $f \in (V(\Lambda)^*)_{-\mu}, f' \in (V(\Lambda)^*)_{-\mu'}$. By (2.9) and the maximality of f in $\mathcal{D}(\Lambda)$, we have

$$\{c^{V(\Lambda)}_{f, v_\Lambda}, c^{V(\Lambda)}_{f', v_\Lambda}\} = [(\Phi_+\Lambda \mid \Lambda) - (\Phi_+\mu \mid \mu')] c^{V(\Lambda)}_{f, v_\Lambda} c^{V(\Lambda)}_{f', v_\Lambda} \qquad (\text{mod } P).$$

Using the same argument for the maximality of f', we obtain

$$\{c^{V(\Lambda)}_{f', v_\Lambda}, c^{V(\Lambda)}_{f, v_\Lambda}\} = [(\Phi_+\Lambda \mid \Lambda) - (\Phi_+\mu' \mid \mu)] c^{V(\Lambda)}_{f, v_\Lambda} c^{V(\Lambda)}_{f', v_\Lambda}, \qquad (\text{mod } P).$$

Thus, we have $(\Lambda \mid \Lambda) = (\mu \mid \mu')$ by adding the above equations since $c^{V(\Lambda)}_{f,v_\Lambda} c^{V(\Lambda)}_{f',v_\Lambda} \notin P$ and u are skew symmetric. It follows that there exists $w_\Lambda \in W$ such that $\mu = w_\Lambda \Lambda = \mu'$ by [15, Proposition 11.4]. That is, there exists a unique (up to scalar multiplication) maximal element g_Λ in $\mathcal{D}(\Lambda)$ with weight $-w_\Lambda \Lambda$.

For $\Lambda, \Lambda' \in \mathbf{L}^+$, consider the maximal elements $g_\Lambda \in \mathcal{D}(\Lambda)$ and $g_{\Lambda'} \in \mathcal{D}(\Lambda')$ with weights $-w_\Lambda \Lambda$ and $-w_{\Lambda'} \Lambda'$, respectively. Applying the argument above to a pair of such elements $c^{V(\Lambda)}_{g_\Lambda, v_\Lambda}$ and $c^{V(\Lambda')}_{g_{\Lambda'}, v_{\Lambda'}}$, we get $(w_\Lambda \Lambda \mid w_{\Lambda'} \Lambda') = (\Lambda \mid \Lambda')$ for all $\Lambda, \Lambda' \in \mathbf{L}^+$. It follows that $w_\Lambda = w_{\Lambda'}$ and thus there exists a unique $w_+ \in W$ such that $f_{-w_+\Lambda}$ is the maximal element in $\mathcal{D}(\Lambda)$ for all $\Lambda \in \mathbf{L}^+$ [16, Lemma 5.1.6 of Chapter 3]. For each $\Lambda \in \mathbf{L}^+$ and weight vector $g \in V(\Lambda)^*$, if $c^{V(\Lambda)}_{g,v_\Lambda} \in I^+_{w_+} \setminus P$ then $g \in \mathcal{D}(\Lambda)$ and thus $g \le f_{-w_+\Lambda}$. But $g \in (U(\mathfrak{b}^+)V(\Lambda)_{w_+\Lambda})^\perp$. Thus $g(U(\mathfrak{b}^+)V(\Lambda)_{w_+\Lambda}) = 0$ and thus $f_{-w_+\Lambda}(V(\Lambda)_{w_+\Lambda}) = 0$, that is a contradiction. It follows that $I^+_{w_+} \subseteq P$. Using the same argument as above we deduce the existence of a unique $w_- \in W$ such that $I^-_{w_-} \subseteq P$. Now it is easy to check $(P/I_w) \cap \mathcal{E}_w = \varnothing$ since P is a prime ideal. □

For a Poisson algebra A, denote by P.Spec A (respectively, P.Prim A) the set of all Poisson prime ideals (respectively, Poisson primitive ideals) of A. By Theorem 3.4, we have the following result.

Corollary 3.5. *For any $w = (w_+, w_-) \in W \times W$, set*

$$\mathrm{P.Spec}_w \mathbb{C}[G] = \{P \in \mathrm{P.Spec}\, \mathbb{C}[G] \mid I_w \subseteq P \text{ and } (P/I_w) \cap \mathcal{E}_w = \varnothing\}$$
$$\mathrm{P.Prim}_w \mathbb{C}[G] = \mathrm{P.Prim}\, \mathbb{C}[G] \cap \mathrm{P.Spec}_w \mathbb{C}[G].$$

Then

$$\mathrm{P.Spec}\, \mathbb{C}[G] = \bigsqcup_{w \in W \times W} \mathrm{P.Spec}_w \mathbb{C}[G]$$
$$\mathrm{P.Prim}\, \mathbb{C}[G] = \bigsqcup_{w \in W \times W} \mathrm{P.Prim}_w \mathbb{C}[G]$$

and, for each $w \in W \times W$,

$$\mathrm{P.Spec}_w \mathbb{C}[G] \cong \mathrm{P.Spec}\, \mathbb{C}[G]_w, \qquad \mathrm{P.Prim}_w \mathbb{C}[G] \cong \mathrm{P.Prim}\, \mathbb{C}[G]_w.$$

4. Poisson adjoint action

Let A be a Poisson algebra. Recall the definition of Poisson A-module M [18, Definition 1]: A vector space M is said to be a Poisson A-module if

(i) M is a module over the commutative algebra A with module structure

$$M \times A \longrightarrow M, (z, a) \mapsto za,$$

(ii) M is a module over the Lie algebra $(A, \{\cdot,\cdot\})$ with module structure

$$A \times M \longrightarrow M, (a, z) \mapsto a * z$$

such that

$$z\{a, b\} = a * (zb) - (a * z)b \tag{4.1}$$

and

$$(ab) * z = (b * z)a + (a * z)b \tag{4.2}$$

for all $a, b, \in A$ and $z \in M$.

Definition 4.1. Let M be a Poisson module over a Poisson Hopf algebra $A = (A, \iota, m, \epsilon, \Delta, S)$. Then *the Poisson adjoint action* on M is defined by

$$\mathrm{ad}_a(z) = \sum_{(a)} (a_1 * z)S(a_2), \qquad a \in A, z \in M,$$

where $\Delta(a) = \sum_{(a)} a_1 \otimes a_2$.

There exists a *canonical Poisson adjoint action* on A given by

$$\mathrm{ad}_a(z) = \sum_{(a)} \{a_1, z\}S(a_2), \qquad a, z \in A$$

since A is a Poisson A-module with Poisson module structure such that $a * z = \{a, z\}$ and za is the multiplication in A for all $a, z \in A$. Moreover the canonical Poisson adjoint action is a derivation on A, that is,

$$\mathrm{ad}_a(zy) = \mathrm{ad}_a(z)y + \mathrm{ad}_a(y)z, \qquad a, z, y \in A. \tag{4.3}$$

Lemma 4.2. *Let M be a Poisson module over a Poisson Hopf algebra $A = (A, \iota, m, \epsilon, \Delta, S)$.*
(1) *$ad_a(z) = -\sum_{(a)}[S(a_2) * z]a_1$ for $a \in A$ and $z \in M$.*
(2) *$ad_{ab} = \epsilon(a)ad_b + \epsilon(b)ad_a$ for all $a, b \in A$.*
(3) *$ad_{\{a,b\}} = (ad_a)(ad_b) - (ad_b)(ad_a)$ for all $a, b \in A$.*
(4) *Define*

$$\begin{aligned} M \times A &\longrightarrow M, \qquad (z, a) \mapsto z \cdot a = \epsilon(a)z \\ A \times M &\longrightarrow M, \qquad (a, z) \mapsto a *' z = ad_a(z). \end{aligned}$$

Then $(M, \cdot, *')$ is a Poisson A-module.

Proof. Let $a, b \in A$ and $z \in M$.

(1) It follows immediately by the fact that

$$0 = \epsilon(a)1 * z = \sum (a_1 S(a_2)) * z = \sum (S(a_2) * z)a_1 + \sum (a_1 * z)S(a_2)$$

by (4.2).

(2) It follows immediately by the fact that

$$\begin{aligned} \mathrm{ad}_{ab}(z) &= \sum ((a_1b_1) * z)S(a_2b_2) \\ &= \sum [(b_1 * z)a_1 + (a_1 * z)b_2]S(a_2)S(b_2) \qquad \text{(by (4.2))} \\ &= \epsilon(a)\mathrm{ad}_b(z) + \epsilon(b)\mathrm{ad}_a(z). \end{aligned}$$

(3) We show that S is a Poisson anti-homomorphism, namely

$$S(\{a, b\}) = -\{S(a), S(b)\}. \tag{4.4}$$

Since

$$\begin{aligned} 0 = \{\epsilon(a)1, b\} &= \left\{\sum S(a_1)a_2, b\right\} = \sum \{S(a_1), b\}a_2 + \sum S(a_1)\{a_2, b\} \\ 0 = \{a, \epsilon(b)1\} &= \left\{a, \sum S(b_1)b_2\right\} = \sum \{a, S(b_1)\}b_2 + \sum S(b_1)\{a, b_2\} \end{aligned}$$

we have

$$\sum \{S(a_1), b\}a_2 = -\sum S(a_1)\{a_2, b\}, \qquad \sum \{a, S(b_1)\}b_2 = -\sum S(b_1)\{a, b_2\}. \tag{4.5}$$

Note that

$$\Delta(\{a,b\}) = \sum a_1b_1 \otimes \{a_2,b_2\} + \sum \{a_1,b_1\} \otimes a_2b_2 \tag{4.6}$$

since A is a Poisson Hopf algebra. Since $\epsilon(\{a,b\}) = 0$ for all $a, b \in A$ by [3, 1.2],

$$0 = \epsilon(\{a,b\})1 = \sum S(\{a,b\}_1)\{a,b\}_2 = \sum S(\{a_1,b_1\})a_2b_2 + \sum S(a_1b_1)\{a_2,b_2\}$$

by (4.6), thus

$$\sum S(\{a_1,b_1\})a_2b_2 = -\sum S(a_1b_1)\{a_2,b_2\} = -\{S(a_1),S(b_1)\}a_2b_2 \tag{4.7}$$

by (4.5). Therefore

$$\begin{aligned} S(\{a,b\}) &= S\left(\left\{\sum a_1\epsilon(a_2), \sum b_1\epsilon(b_2)\right\}\right) = \sum S(\{a_1,b_1\})\epsilon(a_2)\epsilon(b_2) \\ &= \sum S(\{a_1,b_1\})a_2S(a_3)b_2S(b_3) \\ &= -\sum \{S(a_1),S(b_1)\}a_2S(a_3)b_2S(b_3) \qquad \text{(by (4.7))} \\ &= -\sum \{S(a_1)\epsilon(a_2), S(b_1)\epsilon(b_2)\} = -\{S(a),S(b)\}, \end{aligned}$$

as claimed.

Now (3) follows by the fact that

$$\begin{aligned} \mathrm{ad}_{\{a,b\}}(z) &= \sum (\{a,b\}_1 * z)S(\{a,b\}_2) \\ &= \sum [(a_1b_1) * z]S(\{a_2,b_2\}) + \sum [\{a_1,b_1\} * z]S(a_2b_2) \qquad \text{(by (4.6))} \\ &= \sum [(a_1 * z)b_1 + (b_1 * z)a_1]\{S(b_2),S(a_2)\} \qquad \text{(by (4.2), (4.4))} \\ &\quad + \sum [a_1 * (b_1 * z) - b_1 * (a_1 * z)]S(a_2)S(b_2) \\ &= \sum \left(S(b_2) * [(a_1 * z)S(a_2)] - [S(b_2) * (a_1 * z)]S(a_2)\right) b_1 \qquad \text{(by (4.1))} \\ &\quad - \sum \left(S(a_2) * [(b_1 * z)S(b_2)] - [S(a_2) * (b_1 * z)]S(b_2)\right) a_1 \qquad \text{(by (4.1))} \\ &\quad + \sum [a_1 * (b_1 * z) - b_1 * (a_1 * z)]S(a_2)S(b_2) \\ &= -\mathrm{ad}_b\mathrm{ad}_a(z) + \mathrm{ad}_a\mathrm{ad}_b(z) \qquad \text{(by (1))} \\ &\quad + \sum [(a_1S(a_2)) * (b_1 * z)]S(b_2) - \sum [(b_1S(b_2)) * (a_1 * z)]S(a_2) \qquad \text{(by (4.2))} \\ &= (\mathrm{ad}_a\mathrm{ad}_b - \mathrm{ad}_b\mathrm{ad}_a)(z) \end{aligned}$$

since $\sum a_1S(a_2) = \epsilon(a)1$ and $\sum b_1S(b_2) = \epsilon(b)1$.

(4) Clearly $(M,\cdot)$ is a module over the commutative algebra A and $(M,*')$ is a module over the Lie algebra $(A,\{\cdot,\cdot\})$ by (3). Hence it is enough to prove that

$$z\cdot\{a,b\} = a *' (z\cdot b) - (a *' z)\cdot b, \qquad (ab) *' z = (b *' z)\cdot a + (a *' z)\cdot b.$$

The first equation follows from the fact $\epsilon(\{a,b\}) = 0$ by [3, 1.2] and the second equation follows from (2). □

Theorem 4.3. *Let A be a Poisson Hopf algebra and let M be a Poisson A-module. Set*

$$\begin{aligned} \mathcal{Z}(M) &= \{z \in M \mid a * z = 0 \text{ for all } a \in A\} \\ M^{ad} &= \{z \in M \mid ad_a(z) = 0 \text{ for all } a \in A\}. \end{aligned}$$

Then $\mathcal{Z}(M) = M^{ad}$.

Proof. If $z \in \mathcal{Z}(M)$ then $\mathrm{ad}_a(z) = \sum_{(a)}(a_1 * z)S(a_2) = 0$ for all $a \in A$. Thus $\mathcal{Z}(M) \subseteq M^{\mathrm{ad}}$. Conversely, if $z \in M^{\mathrm{ad}}$ then

$$a * z = \sum_{(a)}(a_1 * z)\epsilon(a_2) = \sum_{(a)}(a_1 * z)S(a_2)a_3 = \sum_{(a)} \mathrm{ad}_{a_1}(z)a_2 = 0$$

for all $a \in A$. Thus $M^{\mathrm{ad}} \subseteq \mathcal{Z}(M)$. □

Let $\mathfrak{m}$ be a Lie algebra and let $\mathcal{S}(\mathfrak{m})$ be the symmetric algebra of $\mathfrak{m}$. It is well known that $\mathcal{S}(\mathfrak{m})$ is a Poisson Hopf algebra with

$$\Delta(z) = z \otimes 1 + 1 \otimes z, \qquad \epsilon(z) = 0, \qquad S(z) = -z, \qquad \{z, y\} = [z, y]$$

for all $z, y \in \mathfrak{m}$. The canonical Poisson adjoint action on $\mathcal{S}(\mathfrak{m})$ is given by

$$\mathrm{ad}_a(z) = \sum_{(a)}\{a_1, z\}S(a_2) = \{a, z\}$$

for all $a \in \mathfrak{m}$ and $z \in \mathcal{S}(\mathfrak{m})$. Thus it is clear that $\mathcal{S}(\mathfrak{m})^{\mathrm{ad}} = \mathcal{Z}(\mathcal{S}(\mathfrak{m}))$.

5. *H*-action and Poisson primitive ideals of $\mathbb{C}[G]$

Here, we find the Poisson center of $\mathbb{C}[G]_w$, $w \in W \times W$, by modifying the statements and proofs [11, Section 4.2] using Poisson terminologies instead of those of noncommutative algebra and prove that $\mathbb{C}[G]$ satisfies the Poisson Dixmier-Moeglin equivalence.

5.1.

Recall, as given in 1.5, that H is a torus associated with the Cartan subalgebra $\mathfrak{h}$ and that $\mathbf{L}$ is the character group of H. There is an action of H on $\mathbb{C}[G] = \bigoplus_{(\lambda,\mu)\in\mathbf{L}\times\mathbf{L}} \mathbb{C}[G]_{\lambda,\mu}$ by

$$h \cdot z = \mu(h)z, \qquad h \in H, z \in \mathbb{C}[G]_{\lambda,\mu}.$$

Lemma 5.1. (1) *For all $h \in H$ and $a, z \in \mathbb{C}[G]$,*

$$ad_a(h \cdot z) = h \cdot ad_a(z),$$

where ad_a is the canonical Poisson adjoint action of $\mathbb{C}[G]$.

(2) *For each $w \in W \times W$, $\mathbb{C}[G]_w$ is a Poisson module over $\mathbb{C}[G]$ with module structure*

$$(z + I_w)a = za + I_w, \qquad a * (z + I_w) = \{a, z\} + I_w$$

for $a, z \in \mathbb{C}[G]$.

(3) *For each $w \in W \times W$, there is a Poisson adjoint action on $\mathbb{C}[G]_w$ defined by*

$$ad_a(z + I_w) = \sum_{(a)}\{a_1, z\}S(a_2) + I_w,$$

that satisfies

$$ad_a(h \cdot (z + I_w)) = h \cdot ad_a(z + I_w)$$

for $a, z \in \mathbb{C}[G]$, $h \in H$.

Proof. (1) For $a = c^M_{f,\nu} \in \mathbb{C}[G]_{\lambda,\mu}$, $z \in \mathbb{C}[G]_{\nu,\eta}$ and $h \in H$, we have

$$\mathrm{ad}_a(h \cdot z) = \eta(h)\mathrm{ad}_a(z) = \eta(h)\sum\{c^M_{f,\nu_i}, z\}S(c^M_{g_i,\nu})$$
$$h \cdot \mathrm{ad}_a(z) = h \cdot \left(\sum\{c^M_{f,\nu_i}, z\}S(c^M_{g_i,\nu})\right) = \eta(h)\sum\{c^M_{f,\nu_i}, z\}S(c^M_{g_i,\nu})$$

by (1.15) and Theorem 2.9 since we may assume that the dual bases $\{v_i\}$ and $\{g_i\}$ of M and M^* are weight vectors for all i. It follows that $\mathrm{ad}_a(h\cdot z)=h\cdot\mathrm{ad}_a(z)$ for all $h\in H$ and $a,z\in\mathbb{C}[G]$.

(2) Since $\mathbb{C}[G]$ is a Poisson module with the canonical Poisson module structure and I_w is a Poisson ideal by Lemma 3.2, $\mathbb{C}[G]/I_w$ is a Poisson module. Hence the localization $\mathbb{C}[G]_w$ is a Poisson module.

(3) Since I_w is a Poisson ideal by Lemma 3.2, the canonical Poisson adjoint action acts on $\mathbb{C}[G]/I_w$. Moreover, the canonical Poisson adjoint action is a derivation by (4.3) and thus the canonical Poisson adjoint action extends uniquely on the localization $\mathbb{C}[G]_w$.

Since I_w is a homogeneous Poisson ideal by Lemma 3.2, $\mathbb{C}[G]/I_w$ is also $\mathbf{L}$-bigraded. Note that there exists a generating set of I_w consisting of H-eigenvectors and each element of $\mathcal{E}_w$ is an H-eigenvector. Thus the action of H on $\mathbb{C}[G]$ induces an action on $\mathbb{C}[G]_w$, and thus the result follows immediately by (1). □

Notation 5.2. Fix $w\in W\times W$. For $\Lambda\in\mathbf{L}^+, f\in V(\Lambda)^*$ and $v\in V(\Lambda)$, we set

$$z_f^+=c_{w\Lambda}^{-1}(c_{f,v_\Lambda}^{V(\Lambda)}+I_w),\qquad z_v^-=\tilde{c}_{w\Lambda}^{-1}(c_{v,f_{-\Lambda}}^{V(\Lambda)^*}+I_w).$$

Let $\{\omega_1,\dots,\omega_n\}$ be a basis of $\mathbf{L}$ such that $\omega_i\in\mathbf{L}^+$ for all i. For each $\lambda=\sum_i s_i\omega_i\in\mathbf{L}$, we define Poisson normal elements of $\mathbb{C}[G]_w$ by

$$c_{w\lambda}=\prod_{i=1}^n c_{w\omega_i}^{s_i},\qquad \tilde{c}_{w\lambda}=\prod_{i=1}^n \tilde{c}_{w\omega_i}^{s_i},\qquad d_\lambda=(\tilde{c}_{w\lambda}c_{w\lambda})^{-1}.$$

Note that

$$c_{w(\lambda+\mu)}=c_{w\lambda}c_{w\mu},\qquad \tilde{c}_{w(\lambda+\mu)}=\tilde{c}_{w\lambda}\tilde{c}_{w\mu}\tag{5.1}$$

for any $\lambda,\mu\in\mathbf{L}$.

Define subalgebras of $\mathbb{C}[G]_w$ by

$$C_w=\mathbb{C}[z_f^+,z_v^-,c_{w\lambda}\mid f\in V(\Lambda)^*, v\in V(\Lambda),\Lambda\in\mathbf{L}^+,\lambda\in\mathbf{L}]$$

$$C_w^+=\mathbb{C}[z_f^+\mid f\in V(\Lambda)^*,\Lambda\in\mathbf{L}^+],\qquad C_w^-=\mathbb{C}[z_v^-\mid v\in V(\Lambda),\Lambda\in\mathbf{L}^+].$$

Lemma 5.3. (1) *For any $\Lambda,\Gamma\in\mathbf{L}^+$ and $f\in V(\Lambda)^*$, there exists an element $g\in V(\Lambda+\Gamma)^*$ such that $z_f^+=z_g^+$.*
(2) *For any $\Lambda,\Gamma\in\mathbf{L}^+$ and $v\in V(\Lambda)$, there exists an element $u\in V(\Lambda+\Gamma)$ such that $z_v^-=z_u^-$.*

Proof. It is proved by mimicking the proof of [11, 4.8]. □

Lemma 5.4. *For each $w\in W\times W$, the algebras C_w and $C_w^\pm$ are H-stable Poisson subalgebras of $\mathbb{C}[G]_w$.*

Proof. Note that $c_{w\Lambda}\in(\mathbb{C}[G]_w)_{-w_+\Lambda,\Lambda}$, $\tilde{c}_{w\Lambda}\in(\mathbb{C}[G]_w)_{w_-\Lambda,-\Lambda}$ and thus

$$z_{f_{-\lambda}}^+\in(\mathbb{C}[G]_w)_{w_+\Lambda-\lambda,0},\qquad z_{v_\lambda}^-\in(\mathbb{C}[G]_w)_{-w_-\Lambda+\lambda,0},$$

$$c_{w\lambda}\in(\mathbb{C}[G]_w)_{-w_+\lambda,\lambda},\qquad \tilde{c}_{w\lambda}\in(\mathbb{C}[G]_w)_{w_-\lambda,-\lambda}.$$

Hence

$$h\cdot z_f^+=z_f^+,\qquad h\cdot z_v^-=z_v^-,\qquad h\cdot c_{w\lambda}=\lambda(h)c_{w\lambda},\qquad h\cdot\tilde{c}_{w\lambda}=\lambda(h)^{-1}\tilde{c}_{w\lambda}\tag{5.2}$$

for all $h\in H$ and thus the algebras C_w and $C_w^\pm$ are H-stable.

Observe by (2.9) and Lemma 3.3 that, for any $\Lambda \in \mathbf{L}^+, f_{-\lambda} \in (V(\Lambda)^*)_{-\lambda}$ and $f_{-\mu} \in (V(\Lambda)^*)_{-\mu}$,

$$\{z^+_{f_{-\lambda}}, z^+_{f_{-\mu}}\} = az^+_{f_{-\lambda}} z^+_{f_{-\mu}} - 2\sum_{\alpha\in\mathbf{R}^+} z^+_{f_{-\lambda}x_\alpha} z^+_{f_{-\mu}x_{-\alpha}},$$

where $a = (\Phi_+\Lambda \mid \Lambda) - (\Phi_+\lambda \mid \mu) + (\Phi_+w_+\Lambda \mid \mu) - (\Phi_+w_+\Lambda \mid \lambda) \in \mathbb{C}$. Hence C^+_w is a Poisson subalgebra by Lemma 5.3. Similarly, C^-_w is a Poisson subalgebra.

Observe that, for any $\Lambda \in \mathbf{L}^+$,

$$\{z^+_{f_{-\lambda}}, z^-_{v_\mu}\} = bz^+_{f_{-\lambda}} z^-_{v_\mu} + 2\sum_{\alpha\in\mathbf{R}^+} z^+_{f_{-\lambda}x_\alpha} z^-_{x_{-\alpha}v_\mu} \tag{5.3}$$

for some $b \in \mathbb{C}$ by (2.9) and Lemma 3.3. Thus C_w is also a Poisson subalgebra by Lemma 5.3. □

Theorem 5.5. (1) *Let C^H_w be the set of all fixed elements in C_w under the action of H. Then $C^H_w = \mathbb{C}[z^+_f, z^-_v \mid f \in V(\Lambda)^*, v \in V(\Lambda), \Lambda \in \mathbf{L}^+]$, that is a Poisson subalgebra.*

(2) *The set $\mathcal{D} = \{d_\lambda \mid \lambda \in \mathbf{L}\}$ is a multiplicatively closed subset of C^H_w. Moreover $\mathbb{C}[G]_w$ and $\mathbb{C}[G]^H_w$ are localizations of C_w and C^H_w at $\mathcal{D}$, respectively, that is,*

$$\mathbb{C}[G]_w = (C_w)_\mathcal{D}, \qquad \mathbb{C}[G]^H_w = (C^H_w)_\mathcal{D},$$

where $\mathbb{C}[G]^H_w$ is the set of all fixed elements in $\mathbb{C}[G]_w$ under the action of H.

(3) *For each $\lambda \in \mathbf{L}$, let $(\mathbb{C}[G]_w)_\lambda = \{a \in \mathbb{C}[G]_w \mid h \cdot a = \lambda(h)a \text{ for all } h \in H\}$. Then*

$$\mathbb{C}[G]_w = \bigoplus_{\lambda\in\mathbf{L}} (\mathbb{C}[G]_w)_\lambda, \qquad (\mathbb{C}[G]_w)_\lambda = \mathbb{C}[G]^H_w c_{w\lambda}.$$

Moreover each $(\mathbb{C}[G]_w)_\lambda$ is an invariant subspace of $\mathbb{C}[G]_w$ under the Poisson adjoint action.

Proof. (1) It is proved by (5.2), (5.3), and Lemma 5.4.

(2) and (3) These are proved by mimicking the proof of [11, Theorem 4.7] using Poisson terminologies. We repeat it for completion. Let $\{v_i\}$ and $\{g_i\}$ be dual bases of $V(\Lambda)$ and $V(\Lambda)^*$. Then

$$1 = \epsilon\left(c^{V(\Lambda)}_{f_{-\Lambda},v_\Lambda}\right) = \sum_i S\left(c^{V(\Lambda)}_{f_{-\Lambda},v_i}\right) c^{V(\Lambda)}_{g_i,v_\Lambda} = \sum_i c^{V(\Lambda)^*}_{v_i,f_{-\Lambda}} c^{V(\Lambda)}_{g_i,v_\Lambda}.$$

Multiplying both sides of the equation by d_Λ yields $d_\Lambda = \sum z^-_{v_i} z^+_{g_i}$. Thus $\mathcal{D} \subseteq C^H_w$ by (1) and $\mathcal{D}$ is a multiplicatively closed set by Lemma 5.3. Now, by Theorem 3.1, any element of $\mathbb{C}[G]_w$ is a sum of elements of the form $c^{V(\Lambda)}_{f,v_\Lambda} c^{V(\Gamma)}_{g,v_{-\Gamma}} d_\Upsilon$, where $\Lambda, \Gamma, \Upsilon \in \mathbf{L}^+$. This element lies in $(\mathbb{C}[G]_w)_\lambda$ if and only if $\Lambda - \Gamma = \lambda$. In this case, $c^{V(\Lambda)}_{f,v_\Lambda} c^{V(\Gamma)}_{g,v_{-\Gamma}} d_\Upsilon$ is equal to the element $z^+_f z^-_g d_\Upsilon d^{-1}_\Gamma c_{w\lambda} \in [(C^H_w)_\mathcal{D}]c_{w\lambda}$. Since $\mathbb{C}[G]_w$ is invariant under the Poisson adjoint action and the Poisson adjoint action on $\mathbb{C}[G]_w$ commutes with the action of H by Lemma 5.1(3), $(\mathbb{C}[G]_w)_\lambda$ is invariant under the Poisson adjoint action. (The fact that $(\mathbb{C}[G]_w)_\lambda$ is invariant under the Poisson adjoint action is also proved by Lemmas 5.8 and 3.3.) Hence the remaining assertions follow. □

5.2.

Let $N \in \mathcal{C}(\mathfrak{d})$. For $v \in N$ and $g \in N^*$, define $\psi_{v\otimes g} \in \mathrm{End}(N)$ by $\psi_{v\otimes g}(z) = g(z)v$ for $z \in N$. Let us show that

$$\psi : N \otimes N^* \longrightarrow \mathrm{End}(N), \qquad v \otimes g \mapsto \psi_{v\otimes g}$$

is an isomorphism of vector spaces. Fix bases $\{v_i\}^r_{i=1}$ and $\{g_i\}^r_{i=1}$ of N and N^* such that $g_i(v_j) = \delta_{ij}$. Every element $w \in N \otimes N^*$ can be expressed by $w = \sum_i w_i \otimes g_i$ for some $w_i \in N$. If $\psi_w = \psi(w) = 0$ then

$$0 = \psi_w(v_j) = \sum_i g_i(v_j)w_i = w_j$$

for all j and thus $w = 0$. It follows that ψ is injective and surjective since the dimension of $N \otimes N^*$ is equal to that of $\mathrm{End}(N)$.

Since $\psi_{[\sum_i v_i \otimes g_i]}(v_j) = v_j$ for all j, we have

$$\psi_{[\sum_i v_i \otimes g_i]} = \mathrm{id}_N, \tag{5.4}$$

where id_N is the identity map on N. Denote by ζ the canonical embedding

$$\zeta : \mathbb{C} \longrightarrow N \otimes N^*, \qquad \zeta(1) = \sum_{i=1}^{r} v_i \otimes g_i. \tag{5.5}$$

Note that $\mathbb{C}$ and $N \otimes N^*$ are $U(\mathfrak{d})$-modules with module structures

$$U(\mathfrak{d}) \times \mathbb{C} \longrightarrow \mathbb{C}, \qquad (x, 1) \mapsto x.1 := \epsilon(x)1$$
$$U(\mathfrak{d}) \times (N \otimes N^*) \longrightarrow N \otimes N^*, \qquad (x, a \otimes b) \mapsto x.(a \otimes b) := \Delta(x)(a \otimes b)$$

since the counit map ϵ and the comultiplication map Δ in $U(\mathfrak{d})$ are algebra homomorphisms.

Lemma 5.6. *The map ζ is a homomorphism of $U(\mathfrak{d})$-modules.*

Proof. Let $x, y \in U(\mathfrak{d})$, $v \in N$, and let $S(y)v = \sum_{k=1}^{r} a_k v_k$, $a_k \in \mathbb{C}$. Then

$$\psi_{[\sum_i (xv_i)\otimes(yg_i)]}(v) = \sum_i g_i(S(y)v)xv_i = \sum_{i,k} a_k g_i(v_k)xv_i = x\sum_k a_k v_k = xS(y)v.$$

Hence, for $z \in U(\mathfrak{d})$,

$$\begin{aligned}\psi_{z\zeta(1)} = \psi_{z\sum_i v_i\otimes g_i} &= \psi_{[\sum_{i,(z)}(z_1 v_i)\otimes(z_2 g_i)]}\\ &= \sum_{(z)} z_1 S(z_2)\mathrm{id}_N = \epsilon(z)\mathrm{id}_N\\ &= \epsilon(z)\psi_{[\sum_i v_i\otimes g_i]} = \psi_{\epsilon(z)\zeta(1)}\end{aligned}$$

by (5.4) and (5.5). Since ψ is an isomorphism, we have that $z\zeta(1) = \epsilon(z)\zeta(1) = \zeta(\epsilon(z)1)$ for every $z \in U(\mathfrak{d})$. $\square$

Lemma 5.7. *Let $N \in \mathcal{C}(\mathfrak{d})$, $g \in N^*$, $v \in N$. Let $\{v_i\}_{i=1}^{r}$ and $\{g_i\}_{i=1}^{r}$ be bases of N and N^* such that $g_i(v_j) = \delta_{ij}$. Then, for any $c_{f,p} = c_{f,p}^N \in \mathbb{C}[G]$,*

$$c_{(1\otimes\zeta)^*(f\otimes g\otimes v),p} = ac_{f,p}$$

where $a = \sum_i g(v_i)g_i(v) \in \mathbb{C}$.

Proof. For any $z \in U(\mathfrak{d})$,

$$\begin{aligned}c_{(1\otimes\zeta)^*(f\otimes g\otimes v),p}(z) &= (1\otimes\zeta)^*(f\otimes g\otimes v)(zp) = (f\otimes g\otimes v)(1\otimes\zeta)(zp)\\ &= (f\otimes g\otimes v)\left(\sum_i zp\otimes v_i\otimes g_i\right)\\ &= \sum_i f(zp)g(v_i)g_i(v) = ac_{f,p}(z).\end{aligned}$$

Hence $c_{(1\otimes\zeta)^*(f\otimes g\otimes v),p} = ac_{f,p}$. $\square$

A $\mathbb{C}[G]$-module M is said to be *locally closed* if, for each $z \in M$, $\mathbb{C}[G]z$ is finite dimensional.

Lemma 5.8. *Let $\Lambda \in \mathbf{L}^+, f \in (V(\Lambda)^*)_{-\lambda}, p \in V(\Lambda)_\lambda, N \in \mathcal{C}(\mathfrak{d})$. Let $\{v_i\}_{i=1}^r$ and $\{g_i\}_{i=1}^r$ be bases of N and N^* such that $g_i(v_j) = \delta_{ij}$. Then, for $c = c^N_{g,v} \in \mathbb{C}[G]_{-\eta,\gamma}$,*

(1) $ad_c(z^+_f) = a_0 z^+_f + \sum_{\alpha\in\mathbf{R}^+} a_\alpha z^+_{fx_\alpha}$, *where*

$$a_0 = [(\Phi_+\lambda \mid \eta) - (\Phi_+w_+\Lambda \mid \eta)] \sum_i g(v_i)g_i(v),$$

$$a_\alpha = 2\sum_i (gx_{-\alpha})(v_i)g_i(v).$$

In particular, if $\eta \neq \gamma$ then $a_0 = 0$ and if $\eta = \gamma$ then $a_\alpha = 0$ for all $\alpha \in \mathbf{R}^+$.

(2) $ad_c(z^-_p) = b_0 z^-_p + \sum_{\alpha\in\mathbf{R}^+} b_\alpha z^-_{x_{-\alpha}p}$, *where*

$$b_0 = [(\Phi_-w_-\Lambda \mid \eta) - (\Phi_-\lambda \mid \eta)] \sum_i g(v_i)g_i(v),$$

$$b_\alpha = 2\sum_i (gx_\alpha)(v_i)g_i(v).$$

In particular, if $\eta \neq \gamma$ then $b_0 = 0$ and if $\eta = \gamma$ then $b_\alpha = 0$ for all $\alpha \in \mathbf{R}^+$.

(3) *If $a_\alpha \neq 0$ for some $\alpha \in \mathbf{R}^+$, then $ad_c(z^-_p) = 0$. Conversely, if $b_\alpha \neq 0$ for some $\alpha \in \mathbf{R}^+$, then $ad_c(z^+_f) = 0$. In particular, if $\eta > \gamma$ then $ad_c(z^+_f) = 0$ and if $\eta < \gamma$ then $ad_c(z^-_p) = 0$.*

(4) *$ad_c(C^+_w) \subseteq C^+_w$, $ad_c(C^-_w) \subseteq C^-_w$, $ad_c(C^H_w) \subseteq C^H_w$ for any $c \in \mathbb{C}[G]$ and the Poisson adjoint actions on $C^\pm_w$ and C^H_w are locally finite.*

Proof. For convenience, we will write z for $z + I_w \in \mathbb{C}[G]_w$. We may assume that all v_i and g_i are weight vectors with weights γ_i and $-\gamma_i$ respectively.

(1) We have

$$\begin{aligned}
\mathrm{ad}_c(z^+_f) &= \sum_i \{c_{g,v_i}, c^{-1}_{w\Lambda} c_{f,v_\Lambda}\} S(c_{g_i,v}) \\
&= \sum_i (-c^{-1}_{w\Lambda}\{c_{f,v_\Lambda}, c_{g,v_i}\} + c^{-2}_{w\Lambda} c_{f,v_\Lambda}\{c_{w\Lambda}, c_{g,v_i}\}) c_{v,g_i} \\
&= \sum_i [(\Phi_+\lambda \mid \eta) - (\Phi_+w_+\Lambda \mid \eta)] c^{-1}_{w\Lambda} c_{f\otimes g\otimes v, v_\Lambda\otimes v_i\otimes g_i} + 2\sum_{\alpha\in\mathbf{R}^+, i} c^{-1}_{w\Lambda} c_{fx_\alpha\otimes gx_{-\alpha}\otimes v, v_\Lambda\otimes v_i\otimes g_i} \\
&\qquad \text{(by (2.9) and Lemma 3.3)} \\
&= [(\Phi_+\lambda \mid \eta) - (\Phi_+w_+\Lambda \mid \eta)] c^{-1}_{w\Lambda} c_{f\otimes g\otimes v, (1\otimes\zeta)(v_\Lambda)} + 2\sum_{\alpha\in\mathbf{R}^+} c^{-1}_{w\Lambda} c_{fx_\alpha\otimes gx_{-\alpha}\otimes v, (1\otimes\zeta)(v_\Lambda)} \\
&\qquad \text{(by (5.5))} \\
&= [(\Phi_+\lambda \mid \eta) - (\Phi_+w_+\Lambda \mid \eta)] c^{-1}_{w\Lambda} c_{(1\otimes\zeta)^*(f\otimes g\otimes v), v_\Lambda} + 2\sum_{\alpha\in\mathbf{R}^+} c^{-1}_{w\Lambda} c_{(1\otimes\zeta)^*(fx_\alpha\otimes gx_{-\alpha}\otimes v), v_\Lambda} \\
&\qquad \text{(by Lemma 5.6)} \\
&= a_0 z^+_f + \sum_{\alpha\in\mathbf{R}^+} a_\alpha z^+_{fx_\alpha}. \qquad \text{(by Lemma 5.7)}
\end{aligned}$$

If $\eta \neq \gamma$ then $g(v_i)g_i(v) = 0$ for each $1 \le i \le r$ by Lemma 1.5, thus $a_0 = 0$. If $\eta = \gamma$ then $gx_{-\alpha}(v_i)g_i(v) = 0$ for each $1 \le i \le r$ by Lemma 1.5, thus $a_\alpha = 0$ for each $\alpha \in \mathbf{R}^+$.

(2) Similar to (1).

(3) Let z be a weight vector with weight μ. Then we write $\mu = \mathrm{wt}(z)$. Observe that

$$\begin{aligned}(gx_{-\alpha})(v_i)g_i(v) \neq 0 &\Rightarrow \mathrm{wt}(gx_{-\alpha}) = -\mathrm{wt}(v_i), \mathrm{wt}(g_i) = -\mathrm{wt}(v) \\ &\Rightarrow \eta + \alpha = \gamma \end{aligned} \tag{5.6}$$

and

$$\begin{aligned}(gx_{\alpha})(v_i)g_i(v) \neq 0 &\Rightarrow \mathrm{wt}(gx_{\alpha}) = -\mathrm{wt}(v_i), \mathrm{wt}(g_i) = -\mathrm{wt}(v) \\ &\Rightarrow \eta - \alpha = \gamma \end{aligned} \tag{5.7}$$

by Lemma 1.5.

If $a_\alpha \neq 0$ for some $\alpha \in \mathbf{R}^+$ then $gx_{-\alpha}(v_i)g_i(v) \neq 0$ for some i. Thus $\eta \neq \gamma$ and $(gx_\alpha)(v_i)g_i(v) = 0$ for all i and all $\alpha \in \mathbf{R}^+$ by (5.6) and (5.7). Hence $b_0 = 0$ and $b_\alpha = 0$ for all $\alpha \in \mathbf{R}^+$ by (2), and thus $\mathrm{ad}_c(z_p^-) = 0$. In particular, if $\eta > \gamma$ then $\mathrm{ad}_c(z_f^+) = 0$ by (5.6).

Similarly, if $b_\alpha \neq 0$ for some $\alpha \in \mathbf{R}^+$, then $\mathrm{ad}_c(z_f^+) = 0$, and if $\eta < \gamma$ then $\mathrm{ad}_c(z_p^-) = 0$ by (5.7).

(4) It is clear by (1), (2), and Theorem 5.5 since every object of $\mathcal{C}(\mathfrak{d})$ is finite dimensional. □

5.3.

Henceforth we denote by U the Poisson enveloping algebra of $\mathbb{C}[G]$. (See [18, Definition 3] for the definition of Poisson enveloping algebra.) Note that U is an associative algebra. Since $\mathbb{C}[G]_w$ is a Poisson module over $\mathbb{C}[G]$ by Lemma 5.1(2), C_w^+ and C_w^- are Poisson modules over $\mathbb{C}[G]$ with module structure

$$za = \epsilon(a)z, \qquad a * z = \mathrm{ad}_a z$$

for $a \in \mathbb{C}[G], z \in C_w^\pm$, by Lemma 4.2(4) and Lemma 5.8(1), (2). Thus $C_w^\pm$ are U-modules [18, Corollary 6]. Recall that the socle of a module M, denoted by $\mathrm{Soc}(M)$, is the sum of all minimal submodules of M.

Lemma 5.9. *$\mathrm{Soc}(C_w^+) = \mathbb{C}$ and $\mathrm{Soc}(C_w^-) = \mathbb{C}$.*

Proof. For $\Lambda \in \mathbf{L}^+$, let $T_\Lambda = \{z_f^+ \mid f \in V(\Lambda)^*\}$. Then $C_w^+ = \cup_{\Lambda \in \mathbf{L}^+} T_\Lambda$ by Lemma 5.3. Let M be a minimal submodule of C_w^+ and choose $0 \neq z_f^+ \in M$. Applying ad_c on z_f^+ for a suitable element $c = c_{g,v} \in \mathbb{C}[G]$, we may assume that $f \in V(\Lambda)^*_{-w_+\Lambda}$ by Lemma 5.8(1). Hence $1 = z_f^+ \in M$, and thus $\mathbb{C} = Uz_f^+ = M$ since $\mathrm{ad}_c(1) = 0$ for all $c \in \mathbb{C}[G]$ by (4.3). It follows that $\mathrm{Soc}(C_w^+) = \mathbb{C}$. Analogously we have $\mathrm{Soc}(C_w^-) = \mathbb{C}$. □

Theorem 5.10. *There is no nontrivial Poisson ideal I of C_w^H such that $\mathrm{ad}_c(I) \subseteq I$ for all $c \in \mathbb{C}[G]$.*

Proof. Observe that $C_w^+ \otimes C_w^-$ is a Poisson module over $\mathbb{C}[G]$ with module structure

$$\begin{aligned} (C_w^+ \otimes C_w^-) \times \mathbb{C}[G] &\longrightarrow C_w^+ \otimes C_w^-, \qquad (z \otimes y, a) \mapsto (z \otimes y) \cdot a = \epsilon(a)(z \otimes y) \\ \mathbb{C}[G] \times (C_w^+ \otimes C_w^-) &\longrightarrow C_w^+ \otimes C_w^-, \qquad (a, z \otimes y) \mapsto a * (z \otimes y) = \mathrm{ad}_a(z) \otimes y + x \otimes \mathrm{ad}_a(y) \end{aligned}$$

by Lemma 4.2 and that, by Lemma 5.8, C_w^H is also a Poisson module over $\mathbb{C}[G]$ with module structure

$$za = \epsilon(a)z, \qquad a * z = \mathrm{ad}_a z$$

for $a \in \mathbb{C}[G], z \in C_w^H$. Let $c = c_{f_{-\lambda}, v_\mu} \in \mathbb{C}[G]_{-\lambda,\mu}, \lambda \neq \mu$. We verify that $c * z = 0$ for any $z \in \mathrm{Soc}(C_w^+ \otimes C_w^-)$. Express $z \in \mathrm{Soc}(C_w^+ \otimes C_w^-)$ by $z = \sum_i z_{g_i}^+ \otimes z_{v_i}^-$, where g_i and v_i are weight vectors. By Lemmas 5.3 and 5.8, we may assume that weights of g_i and v_i are $-w_+\Lambda$ and $w_-\Lambda$, $\Lambda \in \mathbf{L}^+$, respectively, by considering the elements $c_{g_{-\eta}, v_\gamma} * z \in Uz$, $\eta \neq \gamma$. Thus $c * z = 0$ since both $\mathrm{ad}_c(z_{f_i}^+) = 0$ and $\mathrm{ad}_c(z_{v_i}^-) = 0$ by Lemma 5.8(1), (2).

Let us show that $\mathrm{Soc}(C_w^+ \otimes C_w^-) = \mathbb{C}$. Every element of $C_w^+ \otimes C_w^-$ may be written by $\sum_i a_i \otimes b_i$, where the b_i are linearly independent elements of the form $z_{v_\mu}^-$, $\mu \in \mathbf{L}$. Suppose that $\sum_i a_i \otimes b_i \in \mathrm{Soc}(C_w^+ \otimes C_w^-)$. Replacing each a_i by $\sum_j z_{g_{\mu_{ij}}}^+$ and acting $c_{f_{-\lambda}, v_\lambda}$ on $\sum_i a_i \otimes b_i$, we may assume that each a_i is a common eigenvector of all $c_{f_{-\lambda}, v_\lambda}$, $\lambda \in \mathbf{L}$, by Lemma 5.8(1), (2). For any $c_{f_{-\lambda}, v_\mu}$ such that $\lambda < \mu$, $\mathrm{ad}_{c_{f_{-\lambda}, v_\mu}}(b_i) = 0$ by Lemma 5.8(3). Thus

$$0 = c_{f_{-\lambda}, v_\mu} * \left(\sum_i a_i \otimes b_i \right) = \sum_i \mathrm{ad}_{c_{f_{-\lambda}, v_\mu}}(a_i) \otimes b_i.$$

It follows that $\mathrm{ad}_{c_{f_{-\lambda}, v_\mu}}(a_i) = 0$ for all $\lambda \neq \mu$ by Lemma 5.8(3), and thus $Ua_i = \mathbb{C}a_i$ by Lemma 5.8(1). Hence $a_i \in \mathrm{Soc}(C_w^+) = \mathbb{C}$ by Lemma 5.9. Therefore $\sum a_i \otimes b_i \in \mathrm{Soc}(\mathbb{C} \otimes C_w^-) = \mathbb{C} \otimes \mathbb{C}$, as claimed.

The multiplication map $\sigma : C_w^+ \otimes C_w^- \longrightarrow C_w^H$ is a U-module epimorphism by Theorem 5.5(1) since the canonical Poisson adjoint action is a derivation by (4.3). If I is a nonzero Poisson ideal of C_w^H invariant under the Poisson adjoint action, then I is a submodule of C_w^H and thus $\sigma^{-1}(I)$ contains a minimal submodule since $C_w^+ \otimes C_w^-$ is locally finite by Lemma 5.8(4). Therefore $\sigma^{-1}(I)$ has a nonzero element of $\mathrm{Soc}(C_w^+ \otimes C_w^-) = \mathbb{C}$. It follows that $I = C_w^H$. □

Corollary 5.11. *The Poisson algebra $\mathbb{C}[G]_w^H$ has no nontrivial Poisson ideal I such that $ad_c(I) \subseteq I$ for all $c \in \mathbb{C}[G]$. Moreover $(\mathbb{C}[G]_w^H)^{ad} = \mathbb{C}$.*

Proof. Since $\mathbb{C}[G]_w^H$ is a localization of C_w^H by Theorem 5.5(2), $\mathbb{C}[G]_w^H$ has no nontrivial Poisson ideal invariant under the Poisson adjoint action by Theorem 5.10. Suppose that there exists an element $y \in (\mathbb{C}[G]_w^H)^{\mathrm{ad}} \setminus \mathbb{C}$. Then y is transcendental over $\mathbb{C}$ and the ideal $(y-a)\mathbb{C}[G]_w^H$, $a \in \mathbb{C}$, is a nonzero Poisson ideal of $\mathbb{C}[G]_w^H$ invariant under the Poisson adjoint action by Theorem 4.3. Thus $y - a$ is invertible for each $a \in \mathbb{C}$, that is a contradiction since $\{(y-a)^{-1} \mid a \in \mathbb{C}\}$ is an uncountably infinite and linearly independent set but $\mathbb{C}[G]_w^H$ has a countable dimension over $\mathbb{C}$. □

Theorem 5.12. *Fix $w \in W \times W$. Let $\mathcal{Z}_w$ be the Poisson center of $\mathbb{C}[G]_w$, that is,*

$$\mathcal{Z}_w = \{a \in \mathbb{C}[G]_w \mid \{a, b\} = 0 \text{ for all } b \in \mathbb{C}[G]_w\}.$$

Then

(1) *$\mathcal{Z}_w = \mathbb{C}[G]_w^{ad}$.*
(2) *$\mathcal{Z}_w = \oplus_{\lambda \in \mathbf{L}} \mathcal{Z}_\lambda$, where $\mathcal{Z}_\lambda = \mathcal{Z}_w \cap \mathbb{C}[G]_w^H c_{w\lambda}$.*
(3) *If $\mathcal{Z}_\lambda \neq 0$ then $\mathcal{Z}_\lambda = \mathbb{C}u_\lambda$ for some unit u_λ.*
(4) *$\mathcal{Z}_w$ is isomorphic to a group algebra of a free abelian group with finite rank over $\mathbb{C}$.*
(5) *The group H acts transitively on the maximal ideals of $\mathcal{Z}_w$.*

Proof.

(1) It is proved by Theorem 4.3.

(2) It is proved by Theorem 5.5(3).

(3) Let u_λ be a nonzero element of $\mathcal{Z}_\lambda$. Then $u_\lambda = ac_{w\lambda}$ for some $a \in \mathbb{C}[G]_w^H$. This implies that $a = u_\lambda c_{w\lambda}^{-1}$ is a Poisson normal element by Lemma 3.3. Hence the ideal of $\mathbb{C}[G]_w^H$ generated by a is a nonzero Poisson ideal invariant under the Poisson adjoint action. It follows that a is a unit by Corollary 5.11 and thus u_λ is a unit. If $z \in \mathcal{Z}_\lambda$ then $zu_\lambda^{-1} \in \mathcal{Z}_0$. Thus $\mathcal{Z}_\lambda = \mathbb{C}u_\lambda$ since $\mathcal{Z}_0 = \mathbb{C}$ by Corollary 5.11.

(4) Let $\mathbf{M} = \{\lambda \in \mathbf{L} \mid \mathcal{Z}_\lambda \neq 0\}$. Then $\mathbf{M}$ is a subgroup of $\mathbf{L}$ by (2), (3) and (5.1). Thus $\mathcal{Z}_w$ is isomorphic to the group algebra of the free abelian group $\mathbf{M}$ of finite rank over $\mathbb{C}$ by (2), (3).

(5) By the Hilbert's Nullstellensatz and (4), H acts transitively on the maximal ideals of $\mathcal{Z}_w$. □

Theorem 5.13. *For each $w \in W \times W$, the Poisson ideals of $\mathbb{C}[G]_w$ are generated by their intersection with the Poisson center $\mathcal{Z}_w$.*

Proof. Note that $\mathbb{C}[G]_w$ and $\mathbb{C}[G]_w^H$ are Poisson modules over $\mathbb{C}[G]$ with module structure

$$za = \epsilon(a)z, \qquad a * z = \mathrm{ad}_a z$$

for $a \in \mathbb{C}[G]$ by Lemma 5.1(2), Lemma 4.2(4), Theorem 5.5(1), and Lemma 5.8(1), (2). The following argument is a modification of [11, Proof of Theorem 4.15]. Any element $f \in \mathbb{C}[G]_w$ can be written uniquely in the form $f = \sum a_\lambda c_{w\lambda}$ by Theorem 5.5(3), where $a_\lambda \in \mathbb{C}[G]_w^H$. Define $\pi : \mathbb{C}[G]_w \longrightarrow \mathbb{C}[G]_w^H$ to be the projection given by $\pi(\sum a_\lambda c_{w\lambda}) = a_0$. Since $\mathrm{ad}_c(a_\lambda c_{w\lambda}) \in (\mathbb{C}[G]_w)_\lambda$ for $c \in \mathbb{C}[G]$ by Theorem 5.5(3),

$$\pi(\mathrm{ad}_c(f)) = \mathrm{ad}_c(a_0) = \mathrm{ad}_c\pi(f)$$

for $c \in \mathbb{C}[G]$. Thus π is a homomorphism of Poisson modules. Define the support of f to be $\mathrm{Supp}(f) = \{\lambda \in \mathbf{L} \mid a_\lambda \neq 0\}$. Let I be a Poisson ideal of $\mathbb{C}[G]_w$. For any set $Y \subseteq \mathbf{L}$ such that $0 \in Y$, define

$$I_Y = \{b \in \mathbb{C}[G]_w^H \mid b = \pi(f) \text{ for some } f \in I \text{ such that } \mathrm{Supp}(f) \subseteq Y\}.$$

Since I is a Poisson ideal of $\mathbb{C}[G]_w$, I is invariant under the Poisson adjoint action. Let $b \in I_Y$. Then $b = \pi(f)$ for some $f = \sum a_\lambda c_{w\lambda} \in I$ such that $\mathrm{Supp}(f) \subseteq Y$. For any $a \in \mathbb{C}[G]_w^H$,

$$\begin{aligned} I \ni af &= \sum (aa_\lambda)c_{w\lambda}, \\ I \ni \{f, a\} &= \sum(\{a_\lambda, a\}c_{w\lambda} + a_\lambda\{a, c_{w\lambda}\}) = \sum(\{a_\lambda, a\}c_{w\lambda} + caa_\lambda c_{w\lambda}), \qquad c \in \mathbb{C} \end{aligned}$$

by Lemma 3.3 and thus $\mathrm{Supp}(\{f, a\}) \subseteq \mathrm{Supp}(f) \subseteq Y$ and $\{b, a\} \in I_Y$. Hence I_Y is also a Poisson ideal of $\mathbb{C}[G]_w^H$ invariant under the Poisson adjoint action since π is a Poisson module homomorphism and $\mathrm{Supp}(\mathrm{ad}_c(f)) \subseteq Y$ for $\mathrm{Supp}(f) \subseteq Y$ and $c \in \mathbb{C}[G]$. Hence I_Y is either zero or $\mathbb{C}[G]_w^H$ for each $Y \subseteq \mathbf{L}$ by Corollary 5.11.

Now let $I' = (I \cap \mathcal{Z}_w)\mathbb{C}[G]_w$ and suppose $I \neq I'$. Choose an element $f = \sum a_\lambda c_{w\lambda} \in I \setminus I'$ whose support S has the smallest cardinality. We may assume without loss of generality that $0 \in S$. Suppose that there exists $g \in I'$ with $\mathrm{Supp}(g) \subseteq S$ and fix $\lambda \in \mathrm{Supp}(g)$. Then $gc_{w\lambda}^{-1} \in I'$ and $0 \in \mathrm{Supp}(gc_{w\lambda}^{-1})$. Thus there exists an element $g' \in I'$ with $\mathrm{Supp}(g') \subseteq \mathrm{Supp}(gc_{w\lambda}^{-1})$ and $\pi(g') = 1$ by the above paragraph. But then $f - a_\lambda g' c_{w\lambda}$ is an element of $I \setminus I'$ with smaller support than f. Thus there can be no elements in I' whose support is contained in S.

Since $0 \neq a_0 \in I_S$, $I_S = \mathbb{C}[G]_w^H$ by the first paragraph. Hence we may assume that $\pi(f) = a_0 = 1$. Then $\mathrm{ad}_c(f) \in I'$ for any $c \in \mathbb{C}[G]$ since $|\mathrm{Supp}(\mathrm{ad}_c(f))| < |\mathrm{Supp}(f)|$ and $\mathrm{ad}_c(f) \in I$, thus $\mathrm{ad}_c(f) = 0$ for any $c \in \mathbb{C}[G]$ by the second paragraph since $\mathrm{Supp}(\mathrm{ad}_c(f)) \subseteq S$. It follows that $f \in I \cap \mathbb{C}[G]_w^{\mathrm{ad}} = I \cap \mathcal{Z}_w \subseteq I'$ by Theorem 5.12(1), that is a contradiction. This completes the proof. □

5.4.

Recall the Poisson Dixmier-Moeglin equivalence [19, Theorem 2.4 and preceding comment]. Let $\mathbf{k}$ be an algebraically closed field with characteristic zero. A Poisson $\mathbf{k}$-algebra A is said to satisfy the *Poisson Dixmier-Moeglin equivalence* if the following conditions are equivalent: For a Poisson prime ideal P of A,

(i) P is Poisson primitive (i.e., there exists a maximal ideal M of A such that P is the largest Poisson ideal contained in M).
(ii) P is rational (i.e., the Poisson center of the quotient field of A/P is equal to $\mathbf{k}$).
(iii) P is locally closed (i.e., the intersection of all Poisson prime ideals properly containing P is strictly larger than P).

Note that $\mathbf{L}$ is the character group of the torus H. Hence there is an action of $H \times H$ on $\mathbb{C}[G]$ defined by

$$(H \times H) \times \mathbb{C}[G] \longrightarrow \mathbb{C}[G], \ \ (h, h').c_{f,v}^M = \lambda(h)\mu(h')c_{f,v}^M, \tag{5.8}$$

where $f \in (M^*)_\lambda, v \in M_\mu$. Since $(\lambda + \mu)(h) = \lambda(h)\mu(h)$, for all $\lambda, \mu \in \mathbf{L}$ and $h \in H$, each element $(h, h') \in H \times H$ acts by Poisson automorphism by Theorem 2.9. Let $\{\omega_1, \dots, \omega_n\}$ be a basis of $\mathbf{L}$ such that $\mathbf{L}^+ = \sum_i \mathbb{Z}_{\geq 0}\omega_i$, as in the proof of Theorem 1.8. Then $H \times H$ acts rationally on $\mathbb{C}[G]$ since $\mathbb{C}[G]$ is generated by finitely many eigenvectors of the form $c^{V(\omega_i)}_{f_{-\lambda}, v_\mu}$, $v_\mu \in V(\omega_i)_\mu, f_{-\lambda} \in (V(\omega_i)^*)_{-\lambda}$, and each $V(\omega_i)$ is finite dimensional (see the proof of Theorem 1.8.)

An ideal I of $\mathbb{C}[G]$ is said to be $H \times H$-*ideal* (or $H \times H$-stable) if $(H \times H)(I) \subseteq I$. An $H \times H$-ideal P is said to be $H \times H$-*prime* if, for $H \times H$-ideals I and J, $IJ \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$. A Poisson ideal I of $\mathbb{C}[G]$ is said to be *Poisson* $H \times H$-*prime ideal* if I is $H \times H$-prime ideal. For an ideal J of $\mathbb{C}[G]$, we denote

$$(J : H \times H) = \bigcap_{(h,h') \in H \times H} (h, h')(J).$$

Note that $(J : H \times H)$ is a $H \times H$-ideal and that $(J : H \times H)$ is $H \times H$-prime if J is prime.

Lemma 5.14. *Let $a \in \mathbb{C}[G]_{\lambda,\mu}$ and $a' \in \mathbb{C}[G]_{\lambda',\mu'}$. If $(\lambda, \mu) \neq (\lambda', \mu')$ then there exists an element $(h, h') \in H \times H$ such that the eigenvalues of a and a' with respect to the action of (h, h') are distinct.*

Proof. Since $(\lambda, \mu) \neq (\lambda', \mu')$, we have that $\lambda \neq \lambda'$ or $\mu \neq \mu'$, say $\lambda \neq \lambda'$. Then λ and λ' are linearly independent [13, Lemma of Section 16.1]. Choose any $h' \in H$. There exists an element $h \in H$ such that $\lambda(h) \neq \lambda'(h)\mu(h')^{-1}\mu'(h')$ [13, Lemma C of Section 16.2]. Hence $(h, h').a = \lambda(h)\mu(h')a$, $(h, h').a' = \lambda'(h)\mu'(h')a'$ and $\lambda(h)\mu(h') \neq \lambda'(h)\mu'(h')$. □

Let $w \in W \times W$. Since I_w is homogeneous by Lemma 3.2 and all elements of the multiplicatively closed set $\mathcal{E}_w$ are $H \times H$-eigenvectors, (5.8) induces an action of $H \times H$ on $\mathbb{C}[G]_w$.

Lemma 5.15. *Every Poisson $H \times H$-prime ideal of $\mathbb{C}[G]_w$ is zero. In particular, I_w is a $H \times H$-stable Poisson prime ideal of $\mathbb{C}[G]$.*

Proof. Let Q be a Poisson $H \times H$-prime ideal of $\mathbb{C}[G]_w$. Then Q is a Poisson prime ideal by [1, II.1.12. Corollary]. Suppose that $Q \neq 0$. Then there exists a nonzero element $a \in Q \cap \mathcal{Z}_w$ by Theorem 5.13. Acting $H \times H$ on a, we may assume that $a = c^{V(\Lambda)}_{f_{-\lambda}, v_\Lambda}$ or $a = c^{V(\Lambda)^*}_{v_\lambda, f_{-\Lambda}}$ for some λ and Λ by Theorem 5.5 and Lemma 5.14, say $a = c^{V(\Lambda)}_{f_{-\lambda}, v_\Lambda} \in Q$. Applying Poisson adjoint action on a, we have that Q contains an element $c_{w\Lambda}$ by Lemma 5.8(1), which is a contradiction since $c_{w\Lambda}$ is invertible. Hence $Q = 0$.

Let Q be a minimal prime ideal of $\mathbb{C}[G]_w$. Then Q is a Poisson ideal [5, Lemma 1.1(c)] and the $H \times H$-prime ideal $(Q : H \times H)$ is prime [1, II.1.12. Corollary]. Hence $Q = (Q : H \times H)$, that is a Poisson $H \times H$-prime ideal of $\mathbb{C}[G]_w$. Therefore $Q = 0$ by the above paragraph. It follows that I_w is $H \times H$-stable Poisson prime. □

Corollary 5.16. *The Poisson $H \times H$-prime ideals of $\mathbb{C}[G]$ are only the ideals I_w, $w \in W \times W$.*

Proof. It follow by Lemma 5.15. □

Theorem 5.17.

(1) *The Poisson algebra $\mathbb{C}[G]$ satisfies the Poisson Dixmier-Moeglin equivalence. More precisely,*

$$\begin{aligned} \mathrm{P.Prim}\,\mathbb{C}[G] &= \{\textit{locally closed Poisson prime ideals}\} \\ &= \{\textit{rational Poisson prime ideals}\} \\ &= \bigsqcup_{w \in W \times W} \{\textit{maximal elements of}\ \mathrm{P.Spec}_w \mathbb{C}[G]\} \end{aligned}$$

(2) $\mathrm{P.Spec}\,\mathbb{C}[G]$ *and* $\mathrm{P.Prim}\,\mathbb{C}[G]$ *are topological quotients of* $\mathrm{Spec}\,\mathbb{C}[G]$ *and* $\max\mathbb{C}[G]$, *respectively.*

Proof.

(1) It follows by Theorem 1.8, Corollary 3.5, Corollary 5.16 and [5, Theorem 4.3].

(2) It follows by [5, Theorem 4.1]. □

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