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On the Primitivity of Prime Rings

CHARLES LANSKI*

University of Southern California, Los Angeles, California 90007

RICHARD RESCO

University of Southern California, Los Angeles, California 90007

AND

LANCE SMALL*

University of California, San Diego, La Jolla, California 92093

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In this paper we obtain some conditions which force prime rings to be primitive. Our main theorems are converses to well-known results on the primitivity of certain subrings of primitive rings. Applications are given to the case of primitive domains, and a tensor product theorem is proved which answers a question of Herstein on the primitivity of $E[x_1, ..., x_n]$, for E the endomorphism ring of a vector space over a division ring.

Throughout the paper, all modules are right (unital) modules and "primitive" will mean right primitive. When R has an identity, the existence of a faithful irreducible R module is equivalent to the existence in R of a proper right ideal T satisfying $T + I = R$ for every nonzero ideal I of R [2, Theorem 1, p. 508]. We begin by stating a useful and well-known result, the proof of which is straightforward using the existence of a faithful irreducible module.

LEMMA 1. *Let R be a prime ring and I a nonzero ideal of R. Then R is a primitive ring if and only if I is a primitive ring.*

Our next result will allow us to assume that the prime rings under consideration have identity.

LEMMA 2. *Let K be a commutative domain with identity, and R a prime K-algebra. Then there exists a prime K-algebra R_1, which has an identity and in which R embeds as an ideal.*

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Proof. Let $A = R \times K$, the Cartesian product of R with K, with addition defined componentwise and multiplication given by $(r, k)(r', k') = (rr' + kr' + k'r, kk')$. Set $T = \{y \in A \mid y(I, 0) = (0, 0)$ for I a nonzero ideal of $R\}$. Since R is a prime K-algebra, T is a prime K-algebra ideal of A and $(R, 0) \cap T = (0, 0)$. Thus $R_1 = A/T$ is the desired K-algebra.

We can now prove the main result of the paper. One direction is well-known and trivial.

THEOREM 1. *If R is a prime ring containing a nonzero idempotent e, then R is a primitive ring if and only if eRe is a primitive ring.*

Proof. If R is a primitive ring with faithful irreducible module M, then Me is a faithful irreducible eRe module. To prove the other direction we may assume that R has an identity. For considering R as an ideal in the ring R_1 of Lemma 2, $eRe = eR_1e$, so it suffices to prove the theorem for R_1, using Lemma 1. Let V be a proper right ideal of eRe comaximal with each nonzero ideal of eRe, and set $T = VR + (1 - e)R$. Clearly, T is a right ideal of R and is proper, for $e \in T$ would mean $e = \sum v_i r_i + (1 - e)x$ so that $e = \sum ev_i er_i e \in V$, contradicting $V \neq eRe$. If I is a nonzero ideal of R, eIe is nonzero since R is prime, so $T + I$ contains $V + eIe = eRe$ by choice of V. Thus $R = eR + (1 - e)R \subset T + I$, T is comaximal with each nonzero ideal of R, and so R is a primitive ring.

It is possible to construct a faithful irreducible module for R in Theorem 1 from one for eRe. If N is a faithful irreducible eRe module, $M = N \otimes_{eRe} eR$, and $L = \{m \in M \mid mRe = 0\}$, then it can be shown that M/L is a faithful irreducible R module.

We now apply Theorem 1 to obtain a theorem on tensor products of primitive rings. Recall that if R is a primitive ring with nonzero socle, each minimal right ideal of R has the form eR for e an idempotent; these are all isomorphic as R modules, and the division rings $eRe \cong \mathrm{End}_R(eR)$ are all isomorphic. For such an idempotent in R, call $D = eRe$ the division ring associated to R.

THEOREM 2. *Let F be a field, R a primitive F-algebra with nonzero socle, and D the division ring associated to R. For any F-algebra A, $R \otimes_F A$ is a primitive ring if and only if $D \otimes_F A$ is a primitive ring.*

Proof. For either implication A must be a prime algebra, so by Lemma 2, A embeds as an ideal in a prime algebra A_1 with identity. For any F-algebra S, $S \otimes_F A$ is an ideal of $S \otimes_F A_1$, so Lemma 1 allows us to assume that A has identity. Let $e \in R$ be an idempotent with $D \cong eRe$. Then $f = e \otimes 1$ is an idempotent in $R \otimes_F A$ and $f(R \otimes_F A)f \cong D \otimes_F A$. Applying Theorem 1 shows that it suffices to prove that $R \otimes_F A$ is a prime ring when $D \otimes_F A$ is primitive. We do this by showing that for T an ideal of $R \otimes_F A$, $T \neq 0$ implies $fTf \neq 0$. If $\sum r_i \otimes a_i \in T - (0)$ with $\{a_i\}$ F-independent in A and $r_1 \neq 0$, then the primeness of R implies that $eRr_1Re \neq 0$, so $exr_1ye \neq 0$ for some $x, y \in R$. Thus $t = \sum xr_iy \otimes a_i \in T$ and $ftf \neq 0$ by the independence of the $\{a_i\}$.

We record as a corollary to Theorem 2 the answer to the question of Herstein mentioned in the Introduction. Information on when the conditions in the corollary are satisfied can be found in [1].

COROLLARY. *Let D be a division algebra, V a vector space over D, and set $R = \mathrm{End}_D(V)$. Then $R[x_1, \ldots, x_n]$ is a primitive ring exactly when $D[x_1, \ldots, x_n]$ is a primitive ring. In particular, if D is a field, $R[x_1, \ldots, x_n]$ is never primitive.*

Proof. Let F be the center of D and take $A = F[x_1, \ldots, x_n]$ in Theorem 2.

We remark that Theorem 2 holds when F is replaced by a commutative domain K. However, in this case one must assume that R is a faithful K-algebra and that A is a flat K module. These assumptions guarantee that $R \otimes_K A$ is a torsion-free K module so that our proof works by choosing $\{a_i\}$ independent over the quotient field of K.

We turn to another application of Theorem 1. As with that result, this one is essentially the converse of a well-known fact. Recall that for any nonempty subset S of R, $l(S) = \{x \in R \mid xS = 0\}$.

THEOREM 3. *Let R be a prime ring and V a right ideal of R. Then R is a primitive ring exactly when $V/(V \cap l(V))$ is a primitive ring.*

Proof. That $V/(V \cap l(V))$ is a primitive ring when R is a primitive ring is well-known and follows by taking a faithful irreducible R module M, considering $N = M/L$ for $L = \{m \in M \mid mV = 0\}$, and observing that N is a faithful irreducible $V/(V \cap l(V))$ module. Assume now that $V/(V \cap l(V))$ is a primitive ring. Using Lemma 2 and Lemma 1 shows that we may assume that R has identity, since V is a right ideal in the R_1 given by Lemma 2, and $V \cap l(V)$ is independent of the overring containing V.

Let $S = \{r \in R \mid rV \subset V\}$ be the idealizer of V in R and let T be the subring of $M_2(R)$ given by $T = \left(\begin{smallmatrix} R & RV \\ V & S \end{smallmatrix}\right)$. We claim that $U = \left(\begin{smallmatrix} 0 & l(V) \cap RV \\ 0 & l(V) \end{smallmatrix}\right)$ is a prime ideal of T. That U is an ideal of T follows easily. It is straightforward to show that U is a prime ideal. Briefly, if $xTy \subset U$ for $x, y \in T$ with $y = \left(\begin{smallmatrix} y_1 & y_2 \\ y_3 & y_4 \end{smallmatrix}\right)$, then $x\left(\begin{smallmatrix} R & 0 \\ 0 & 0 \end{smallmatrix}\right) y \subset U$ implies that either $y_1 = 0$ or that the first column of x is zero. Assuming $y_1 \neq 0$, $x\left[\begin{smallmatrix} 0 & 0 \\ VR & 0 \end{smallmatrix}\right] y \subset U$ forces $x \in U$. If $x \notin U$, then $y_1 = 0$, but now considering $xT(\left[\begin{smallmatrix} 0 & 0 \\ 0 & RV \end{smallmatrix}\right] y) \subset U$ and then $xT(y\left[\begin{smallmatrix} 0 & 0 \\ V & 0 \end{smallmatrix}\right]) \subset U$ yields $y \in U$. Therefore, $W = T/U$ is a prime ring with identity containing $e = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and $eWe \cong S/l(V)$. Now $V/(V \cap l(V))$ embeds in eWe as an ideal, so our hypothesis and Lemma 1 show that eWe is a primitive ring. Using Theorem 1 yields first that W is a primitive ring, and then that $R \cong (1 - e)\, W(1 - e)$ is a primitive ring, completing the proof of the theorem.

One consequence of Theorem 3 is a result applying to domains and demonstrating that many subrings of primitive rings are primitive.

COROLLARY. *Let R be a semi-prime ring and V a right ideal of R with $l(V) = 0$. Then the following statements are equivalent:*

(i) *V is a primitive ring;*

(ii) *Every subring of R containing V is primitive;*

(iii) *Some subring of R containing V is primitive.*

Proof. By Theorem 3 it suffices to show that when V is a primitive ring, any subring S of R containing V is a prime ring. Now R is a semi-prime ring, so for any nonzero ideal I of R, $IV \neq 0$ implies that $0 \neq VI \subset V \cap I$. Hence, R is a prime ring when V is primitive. But AV is a nonzero right ideal of R for any nonzero ideal A of S, so it follows that S must be a prime ring.

An amusing consequence of the last corollary is that one can prove the primitivity of a free algebra in a finite or countable set of indeterminates from the classical result for two variables. Specifically, for any field F, let $R = F\langle x, y\rangle$ be the free algebra over F in noncommuting indeterminates. Then R is a primitive ring [3, p. 36]. Let $V = xR$ and $S = \{r \in R \mid rV \subset V\}$, the idealizer of V. Clearly, $S = F + xR$, and as an F-algebra S is generated by $\{xy^i \mid i \geqslant 0\}$. These generators are free since $Rx \cap Ry = 0$ forces $\sum Rxy^i$ to be direct. Thus $S \cong F\langle X\rangle$, for X a countably infinite set of noncommuting indeterminates, and S is a primitive ring by the corollary. Finally, if $T = F\langle x_1, ..., x_n\rangle$ for $n > 1$, $U = x_1T$, and W is the idealizer of U in T, then $W = F + x_1T$. If $\{m_i\}$ is the standard monomial F-basis for $F\langle x_2, ..., x_n\rangle$, then $\{x_1m_i\}$ is a free set of generators for the F-algebra W. Thus $W \cong S$ is a primitive ring, so T is a primitive ring by the corollary.

Note added in proof. Theorem 1 can be obtained from Theorem 27 in S. A. Amitsur, Rings of quotients and Morita context, *J. Algebra* **17** (1971), 273–298. Also there is some overlap of this paper with a forthcoming paper by W. K. Nicholson and J. Watters to appear in the *J. Lond. Math. Soc.*

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