

POISSON DIXMIER-MOEGLIN EQUIVALENCE FROM A TOPOLOGICAL POINT OF VIEW

JUAN LUO, XINGTING WANG, AND QUANSHUI WU

ABSTRACT. A complex affine Poisson algebra A is said to satisfy the Poisson Dixmier-Moeglin equivalence if the Poisson cores of maximal ideals of A are precisely those Poisson prime ideals that are locally closed in the Poisson prime spectrum $\text{P.spec } A$ and if, moreover, these Poisson prime ideals are precisely those whose extended Poisson centers are exactly the complex numbers. In this paper, we provide a purely topological criterion for Poisson Dixmier-Moeglin equivalence in terms of the poset $\text{P.spec } A$ with respect to the inclusion of Poisson prime ideals. In particular, we prove that the Zariski topology of $\text{P.spec } A$ can detect the Poisson Dixmier-Moeglin equivalence for any complex affine Poisson algebra A . The results of this paper, where the full structure of a Poisson algebra is not necessary, are developed in the general context of a commutative algebra equipped with a set of derivations over a large base field compared with the vector space dimension of the algebra and the size of the derivation set.

1. INTRODUCTION

For a (left) noetherian k -algebra S , one of the most beautiful results in the direction of understanding the representation theory of S is through the work of Dixmier [Dix77a] and Moeglin [Moe80] when $S = U(\mathfrak{g})$, the enveloping algebra of a finite dimensional complex Lie algebra \mathfrak{g} , by characterizing primitive ideals of S . More precisely, Dixmier and Moeglin showed the following types of prime ideals in S coincide:

- the set of locally closed prime ideals (those constitute locally closed points in the prime spectrum of S);
- the set of primitive ideals (those being annihilators of simple (left) modules over S); and
- the set of rational prime ideals (those prime ideals P in S such that the center of the Goldie quotient ring of S/P is algebraic over the base field k).

In their honor, nowadays we refer to S as satisfying the *Dixmier-Moeglin equivalence* if the equivalence of three above properties holds for primes in the spectrum of S . Ever since, the Dixmier-Moeglin equivalence has been recognized as a very general phenomenon that happens for a wide range of algebras, see for example [Von96, Zal71, GL00, BL14, BSM18, BRS10, GZ10]. Very recently, it is proved in [BWY19] that the Dixmier-Moeglin equivalence can be totally captured by the Zariski topology of the prime spectrum of S under the cardinality assumption $\dim_k S < |k|$.

The notion of Poisson bracket, first introduced by Siméon Denis Poisson, arises naturally in Hamiltonian mechanics and differential geometry. Poisson algebras have become deeply entangled with noncommutative geometry, integrable systems, topological field theories and representation theory of noncommutative algebras. They are essential in the study of the noncommutative discriminant [BY18, NTY17] and representation theory of noncommutative algebras [WWY17,

2010 *Mathematics Subject Classification.* 16D60, 17B63, 13N15.

Key words and phrases. Dixmier-Moeglin equivalence, commutative differential algebras, Poisson algebras, primitive ideals, prime spectrum.

WWY18]. In addition, there has been renewed interest in enveloping algebras of Poisson algebras [LWZ15, LWZ17].

The aim of this paper is to study an analogue of the Dixmier-Moeglin equivalence for affine complex Poisson algebras. A *Poisson algebra* is a commutative k -algebra A together with a bilinear map $\{-, -\} : A \times A \rightarrow A$ that is both a Lie bracket and a bi-derivation. The *Poisson center* of A is the subalgebra of A defined by $Z_P(A) := \{a \in A \mid \{a, b\} = 0, \forall b \in A\}$. An ideal I of A is called a *Poisson ideal* if $\{I, A\} \subseteq I$. For each ideal J of A , there is a largest Poisson ideal contained in J , which is called the *Poisson core* of J . The *Poisson primitive ideals* of A are the Poisson cores of the maximal ideals in A . A Poisson prime ideal of A is a Poisson ideal of A that is also a prime ideal. The Poisson prime spectrum of A is denoted by $\text{P.spec } A$ consisting of all Poisson prime ideals in A , which is induced with the subspace topology from the prime spectrum $\text{spec } A$. We say that A satisfies the *Poisson Dixmier-Moeglin equivalence* provided the following types of Poisson prime ideals in $\text{P.spec } A$ coincide:

- the set of locally closed Poisson prime ideals;
- the set of Poisson primitive ideals; and
- the set of Poisson rational ideals (those Poisson prime ideals P in A such that the Poisson center of the fractional quotient ring A/P is algebraic over the base field k).

Some examples in which this equivalence holds have been given by [Oh99, Theorem 2.4, Proposition 2.13]. Later, this equivalence has been established for Poisson algebras with suitable rational torus actions by Goodearl [Go06] in the general context of a commutative differential k -algebra and for any affine complex Poisson algebra with only finitely many Poisson primitive ideals by Brown and Gordon [BG03, Lemma 3.4]. On the other side, the tool of model theory and differential algebraic geometry has been recently employed to study the Poisson Dixmier-Moeglin equivalence for affine complex Poisson algebras; see references [BLLM17, LL19].

In this paper, our first main result is to assert that, for any complex affine Poisson algebra A , there is a purely topological characterization of the Poisson Dixmier-Moeglin equivalence. In particular, we show that the poset of Poisson prime ideals $(\text{P.spec } A, \subseteq)$ can detect whether or not the Poisson Dixmier-Moeglin equivalence holds.

Theorem A (Theorem 6.2). *Let A be a complex affine Poisson algebra. Then A satisfies the Poisson Dixmier-Moeglin equivalence if and only if any Poisson prime ideal that has a set of minimal Poisson prime ideals over it with cardinality less than $|\mathbb{C}|$ has only finitely many minimal Poisson prime ideals.*

This result is somewhat interesting, because it says that for complex affine Poisson algebras the underlying Zariski topology on the Poisson prime spectrum “sees” the Poisson Dixmier-Moeglin equivalence. In Definition 2.6, the notion of κ -separability is introduced to accommodate this new topological characterization mentioned above for a fixed cardinality κ in an arbitrary Zariski space. In particular, it is proved that a point in a Zariski space is locally closed if and only if it is \aleph_0 -separable (Lemma 2.3). This concept is motivated by the work in [BWY19] on Dixmier-Moeglin equivalence for left noetherian algebras over large base fields and we expect it will play a role in further study of the (Poisson) Dixmier-Moeglin equivalence.

One consequence of Theorem A is that the Poisson Dixmier-Moeglin equivalence can be detected locally in the Poisson prime spectrum with the Zariski topology since the notion of κ -separability is a local property according to Lemma 2.4.

Theorem B (Theorem 6.3). *Let A be a complex affine Poisson algebra. Suppose*

$$\text{P.spec } A = \bigcup X_i$$

is a union of locally closed subsets with each X_i , when endowed with the subspace topology, homeomorphic to $\text{P.spec } A_i$ for some complex affine Poisson algebra A_i . Then A satisfies the Poisson Dixmier-Moeglin equivalence if and only if each A_i satisfies the Poisson Dixmier-Moeglin equivalence.

The relevance of this theorem is seen in the fact that the semiclassical limits of many quantum algebras have Poisson prime spectra of this form [Go06], although generally in these cases work of Goodearl [Go06], using additional information about the stratification coming from the rational action of an algebraic torus on the Poisson algebra, allows one to deduce that the Poisson Dixmier-Moeglin equivalence holds. Our Theorem B again strengthens the fact that for complex affine Poisson algebras an abstract stratification with parts homeomorphic to complex affine schemes of finite type immediately yields the Poisson Dixmier-Moeglin equivalence without having any underlying action of an algebraic group.

It is important to notice that the tool of stratification is a key ingredient in the important work of Goodearl and Letzter [GL00] in obtaining the Dixmier-Moeglin equivalence for many classes of quantum algebras and the corresponding Poisson Dixmier-Moeglin equivalence for their semiclassical limits [Go06]. When we turn to the maximal (or prime) spectrum of a complex affine Poisson algebra A , there are several stratifications. One algebraic stratification is to use the Poisson core, where two maximal (or prime) ideals in A are set to be in the same *symplectic core* if they share the same Poisson core. Our next result provides another topological characterization of the Poisson Dixmier-Moeglin equivalence for complex affine Poisson algebras in terms of their symplectic core stratification.

Theorem C (Theorem 6.7). *Let A be a complex affine Poisson algebra. Then A satisfies the Poisson Dixmier-Moeglin equivalence if and only if all symplectic cores in its maximal (or prime) spectrum are locally closed.*

Moreover in the case of complex affine Poisson algebras, we can consider the symplectic foliation on the underlying complex Poisson varieties in the sense of [Wein83] for smooth Poisson manifolds and generally follow the work of [BG03] for the singular case. In [BG03], the Poisson bracket of a complex affine Poisson algebra A is said to be *algebraic* if all the symplectic leaves are locally closed. Now suppose the Poisson bracket of A is algebraic. It is further shown that the symplectic leaf stratification coincides with the symplectic core stratification [BG03, Proposition 3.6(2)]. Therefore, we know A satisfies the Poisson Dixmier-Moeglin equivalence (Corollary 6.8).

In general, it is difficult to show the Poisson bracket of A is algebraic if there are infinitely many symplectic leaves. But when there are finitely many symplectic leaves, one result of [BG03, Proposition 3.7(1)] ensures that the Poisson bracket of A is algebraic. Our last result provides a G -equivariant version of the aforementioned result when we have a rational action of an algebraic group G on A with its induced action on the symplectic leaves and cores in the corresponding stratifications.

Theorem D (Theorem 6.10). *Let G be an algebraic group acting rationally by Poisson automorphism on a complex affine Poisson algebra A . If the set of G -orbits of symplectic leaves (or symplectic cores) is finite, then A satisfies the Poisson Dixmier-Moeglin equivalence.*

The outline of the paper is as follows. In Section 2, we discuss the topological criterion used in Theorem A and study some of its basic properties in an arbitrary Zariski space. In Section 3, we review some facts of a commutative differential k -algebra and state the corresponding Dixmier-Moeglin equivalence. Later in Section 4, we develop our main theorems in the general context of a commutative differential k -algebra over a large base field k , where we prove Theorem A and Theorem B within that context. In Section 5, we extend the symplectic core stratification to a

commutative differential k -algebra and study its relations with the Dixmier-Moeglin equivalence together with a possible rational action of an algebraic group. Finally in Section 6, we apply our previous results to complex affine Poisson algebras and prove Theorem C and Theorem D there.

Throughout the paper, k will denote a field of characteristic zero.

2. κ -SEPARABILITY FOR ZARISKI SPACES AND POSETS

A *Zariski space* is a topological space X that every open subset is quasicompact and every nonempty closed irreducible subset Y is the closure of a unique point p , where p is called the *generic point* of Y . Every every open subset of X is quasicompact is equivalent to that X is a noetherian space, that is, X satisfies the ascending chain condition for open subsets. For instance, the prime spectrum $\text{spec } R$ of any noetherian ring with identity is a Zariski space.

This following equivalent descriptions of locally closed subsets in a topological space are standard. We include the proofs for the convenience of the reader.

Definition-Lemma 2.1. *Let X be a topological space. For any subset $S \subseteq X$, the following are equivalent:*

- (i) S is the intersection of an open and a closed subsets of X .
- (ii) Every point $s \in S$ has an open neighborhood Y in X such that $S \cap Y$ is closed in Y .
- (iii) S is open in its closure \bar{S} in X .

If the above conditions hold, we say that the subset S is locally closed in X .

Proof. (i) \Rightarrow (ii) Let $S = U \cap V$, where U is open and V is closed. Take $Y = U$ and $S \cap Y = (U \cap V) \cap U = V \cap U$, which is closed in $U = Y$.

(ii) \Rightarrow (iii) It suffices to show that, for any point $s \in S$, there is an open neighborhood Y of s such that $S \cap Y = \bar{S} \cap Y$. From (ii), we can choose such a Y that $S \cap Y$ is closed in Y . So we have

$$\begin{aligned} \bar{S} \cap Y &\subseteq \overline{(S \cap Y)} \cap Y \subseteq \overline{(S \cap Y)} \cap Y \\ &\subseteq \overline{(S \cap Y)} \cap Y \subseteq \overline{(S \cap Y)} \cap Y = S \cap Y. \end{aligned}$$

(iii) \Rightarrow (i) Since S is open in \bar{S} , there is some open subset U such that $S = U \cap \bar{S}$. \square

Throughout the paper, we will employ the above equivalent definitions of locally closed subset implicitly.

Definition 2.2. Let X be a Zariski space, and κ be a cardinal number. A point $p \in X$ is said to be κ -separable if we can write

$$\overline{\{p\}} \setminus \{p\} = \bigcup_{q \in \Lambda \subseteq \overline{\{p\}} \setminus \{p\}} \overline{\{q\}},$$

where the subset Λ of $\overline{\{p\}} \setminus \{p\}$ has cardinality strictly less than κ .

We can use this separable condition to describe when a point is locally closed in a Zariski space. The following result is probably well-known in algebraic geometry. We provide a proof in order to show that a point is locally closed in a Zariski space is equivalent to the fact that it is \aleph_0 -separable.

Lemma 2.3. *Let X be a Zariski space. Then a point $p \in X$ is locally closed if and only if it is \aleph_0 -separable.*

Proof. Suppose p is locally closed. By Definition-Lemma 2.1(iii), $Y := \overline{\{p\}} \setminus \{p\}$ is closed. We first show that Y is a finite union of its closed irreducible components. If Y is irreducible, then we are done. If not, we can write $Y = Y_1 \cup Y_2$, where Y_1, Y_2 are proper closed subsets of Y . Now if either Y_1 or Y_2 is not irreducible, we can repeat the process to write it as a union of two proper closed subsets. But we notice that this decomposition process has to stop eventually since X is quasicompact so every descending chain of closed subsets has to stabilize. Thus we can write $Y = Y_1 \cup \dots \cup Y_n$, where Y_1, \dots, Y_n are all closed irreducible subsets. Next since X is Zariski, there are generic points $q_1, \dots, q_n \in Y$ such that $Y_1 = \overline{\{q_1\}}, \dots, Y_n = \overline{\{q_n\}}$. So we have

$$\overline{\{p\}} \setminus \{p\} = Y = \bigcup_{1 \leq i \leq n} Y_i = \bigcup_{1 \leq i \leq n} \overline{\{q_i\}}.$$

This shows that p is \aleph_0 -separable.

Conversely, suppose p is \aleph_0 -separable. So there is a finite subset $\Lambda \subseteq \overline{\{p\}} \setminus \{p\}$ such that

$$\overline{\{p\}} \setminus \{p\} = \bigcup_{q \in \Lambda} \overline{\{q\}}.$$

Since finite union of closed sets is still closed, we get $\overline{\{p\}} \setminus \{p\}$ is closed and p is open in $\overline{\{p\}}$. So p is locally closed by Lemma 2.1(iii). \square

Our next observation is that whether a point is κ -separable or not can be detected locally in a Zariski space.

Lemma 2.4. *Let X be a Zariski space, and κ be an infinite cardinal number. A point $p \in X$ is κ -separable in X if and only if p is κ -separable in some locally closed subset Y containing p , where Y is endowed with the subspace topology.*

Proof. First, it is routine to check that any locally closed subset Y in X with the subspace topology is again Zariski. Suppose p is κ -separable in X . Then there is a subset $\Lambda \subseteq \overline{\{p\}} \setminus \{p\}$ of cardinality strictly less than κ such that

$$\overline{\{p\}} \setminus \{p\} = \bigcup_{q \in \Lambda} \overline{\{q\}}.$$

Hence for any locally closed subset Y containing p we have

$$(\overline{\{p\}} \cap Y) \setminus \{p\} = \bigcup_{q \in \Lambda'} (\overline{\{q\}} \cap Y),$$

where $\Lambda' \subseteq \Lambda$ consists of those $q \in \Lambda$ such that $\overline{\{q\}} \cap Y \neq \emptyset$. It is easy to see that $\overline{\{q\}} \cap Y$ is the closure of the point $q \in Y$. So the above equality shows that p is κ -separable in Y .

Conversely, suppose p is κ -separable in some locally closed subset Y containing p . So

$$(\overline{\{p\}} \cap Y) \setminus \{p\} = \bigcup_{q \in \Lambda} (\overline{\{q\}} \cap Y),$$

for some subset $\Lambda \subseteq (\overline{\{p\}} \cap Y) \setminus \{p\}$ of cardinality strictly less than κ . By Lemma 2.1(i), we can write $Y = U \cap V$ with U open and V closed. So we get $\overline{\{q\}} \cap Y = \overline{\{q\}} \cap U$ for any $q \in Y$. Decompose the closed subset $\overline{\{p\}} \setminus U$ into

$$\overline{\{p\}} \setminus U = Y_1 \cup \dots \cup Y_n = \overline{\{q_1\}} \cup \dots \cup \overline{\{q_n\}},$$

where q_1, \dots, q_n are generic points of the closed irreducible components Y_1, \dots, Y_n . Then we have

$$\overline{\{p\}} \setminus \{p\} = \left((\overline{\{p\}} \cap U) \setminus \{p\} \right) \cup (\overline{\{p\}} \setminus U) = \left(\bigcup_{q \in \Lambda} (\overline{\{q\}} \cap U) \right) \cup \left(\bigcup_{1 \leq i \leq n} \overline{\{q_i\}} \right) = \bigcup_{q \in \Lambda'} \overline{\{q\}},$$

where $\Lambda' = \Lambda \cup \{q_1, \dots, q_n\}$ has cardinality strictly less than κ since $\aleph_0 \leq |\Lambda| < \kappa$. \square

It is important to point out that the criterion of κ -separability for a Zariski space X is uniquely determined by the poset (X, \leq) , where the partial order \leq is defined by the relation $x \leq y$ if and only if $y \in \overline{\{x\}}$ for any $x, y \in X$. Recall for an element s in a poset (S, \leq) , a *cover* of s is a minimal element in the subset $\{t \in S \mid s \not\leq t\}$.

Definition 2.5. Let (S, \leq) be a poset, and κ be a cardinal number. An element $s \in S$ is said to be κ -separable if the set of all covers of s has cardinality strictly less than κ .

Lemma 2.6. Let X be a Zariski space, and κ be a cardinal number. A point of $p \in X$ is κ -separable in X if and only if it is κ -separable in the poset (X, \leq) , where $x \leq y$ if and only if $y \in \overline{\{x\}}$ for any $x, y \in X$.

Proof. Since X is Zariski, it satisfies the descending chain condition on closed subsets. By definition, it implies that for any $r \not\geq p$ in the poset (X, \leq) there could be only finitely many elements between them. So for any p in the poset (X, \leq) , having a subset Λ in X satisfying the properties: $|\Lambda| < \kappa$ and for any $r \not\geq p$ there is some $q \in \Lambda$ such that $r \geq q$, is the same as the set of all covers of p having cardinality strictly less than κ . Now suppose we can write

$$\overline{\{p\}} \setminus \{p\} = \bigcup_{q \in \Lambda \subseteq \overline{\{p\}} \setminus \{p\}} \overline{\{q\}}.$$

This is equivalent to say that for any $r \in \overline{\{p\}} \setminus \{p\}$ there is some $q \in \Lambda$ such that $r \in \overline{\{q\}}$. By definition in the poset (X, \leq) , this is equivalent to the fact that for any $r \not\geq p$ there is some $q \in \Lambda$ such that $r \geq q$. So our result follows from direct translation. \square

Remark 2.7. In general, in an arbitrary poset (S, \leq) for any κ -separable point p there may not exists a subset Λ of cardinality less than κ in S satisfying for any $r \not\geq p$ there is some $q \in \Lambda$ such that $r \geq q$, e.g., consider the interval $[1, \gamma]$ with reversed order where γ is the first uncountable ordinal, then γ has no covers, it is automatically \aleph_0 -separable but there does not exist a countable subset Λ in $[1, \gamma]$ satisfying for any $a \not\geq \gamma$ there is some $b \in \Lambda$ such that $a \geq b$.

3. COMMUTATIVE DIFFERENTIAL ALGEBRAS

In this section, we follow the notations in [Go06] to discuss Dixmier-Moeglin equivalence in the context of a commutative algebra together with a set of derivations.

A *commutative differential k -algebra* is a pair (R, Δ) , where R is a commutative k -algebra and $\Delta \subseteq \text{Der}_k R$ is a subset of k -linear derivations of R . Throughout, let (R, Δ) be a commutative differential k -algebra.

A Δ -ideal of R is any ideal I of R satisfying $\delta(I) \subseteq I$ for any $\delta \in \Delta$. A prime ideal which is also a Δ -ideal is called a *prime Δ -ideal*. There is another notion of Δ -prime ideal referring to any Δ -ideal that is prime among all Δ -ideals. These two concepts coincide when R is noetherian; see reference [Go06, Lemma 1.1(d)]. We use $\Delta\text{-spec } R$ to denote the set of all Δ -prime ideals in R , which is a Zariski space with the induced Zariski topology via the obvious embedding $\Delta\text{-spec } R \subseteq \text{spec } R$.

Since derivations are extendable to localizations, any localization of R is still a commutative differential k -algebra.

For any ideal J of R , the Δ -core of J is the largest Δ -ideal contained in J , denoted by

$$(J : \Delta) := \{r \in R \mid \delta_1 \cdots \delta_n(r) \in J, \forall \delta_1, \dots, \delta_n \in \Delta, n \geq 0\}.$$

In particular, Δ -cores of maximal ideals of R are called Δ -primitive ideals. We use $\Delta\text{-prim } R$ to denote the set of all Δ -primitive ideals in R .

The Δ -center of R is the subalgebra

$$Z_\Delta(R) := \{r \in R \mid \delta(r) = 0, \forall \delta \in \Delta\}.$$

For any Δ -prime ideal P of R , all derivations in Δ can pass to R/P , which then extend uniquely to the quotient field $\text{Frac } R/P$. We say that P is Δ -rational if the field $Z_\Delta(\text{Frac } R/P)$ is algebraic over the base field k .

By analogy with the Dixmier-Moeglin equivalence for algebras, we say that R satisfies the Δ -Dixmier-Moeglin equivalence provided the following equivalences hold for any $P \in \Delta\text{-spec } R$

$$P \text{ locally closed in } \Delta\text{-spec } R \iff P \text{ } \Delta\text{-primitive} \iff P \text{ } \Delta\text{-rational}.$$

We recall some basic properties about $\Delta\text{-spec } R$. In properties below, (i) is Dixmier's result [Dix77, Lemma 3.3.2]; (iv) is proved in [Go06, Lemma 1.1(e)] when R is affine over k , where we prove the same result for those R that are noetherian over large base fields.

Lemma 3.1. *Let (R, Δ) be a commutative differential k -algebra.*

- (i) *The Δ -core of any prime ideal of R is prime. In particular, every Δ -primitive ideal of R is prime.*
- (ii) *There is a natural homeomorphism between $\Delta\text{-spec } RS^{-1}$ for the localization of R at any multiplicative closed set S in R and the subset of $\Delta\text{-spec } R$ containing those Δ -prime ideals that do not intersect with S equipped with the subspace topology.*
- (iii) *If R is noetherian, then the set of all minimal Δ -prime ideals containing any Δ -ideal is finite. In particular, every Δ -semiprime ideal is the intersection of finitely many Δ -prime ideals.*
- (iv) *If R is noetherian and $\dim_k R < |k|$, then every Δ -prime ideal is an intersection of Δ -primitive ideals.*

Proof. (i) For any finite tuple $\Lambda = (\delta_1, \dots, \delta_n)$ of Δ , we define an ideal $P(\Lambda)$ in R by

$$P(\Lambda) := (\cdots ((P : \delta_1) : \delta_2)) \cdots : \delta_n = \left\{ r \in R \mid \delta_1^{i_1} \delta_2^{i_2} \cdots \delta_n^{i_n}(r) \in P, \forall i_1, \dots, i_n \geq 0 \right\}.$$

Then

$$(P : \Delta) = \bigcap_{\text{finite tuples } \Lambda \text{ in } \Delta} P(\Lambda).$$

For any two finite tuples Λ and $\Lambda' = (\delta'_1, \dots, \delta'_m)$ of Δ , it is clear that $P(\Lambda), P(\Lambda') \supseteq P(\tilde{\Lambda})$, where $\tilde{\Lambda} = \Lambda \cup \Lambda' = (\delta_1, \dots, \delta_n, \delta'_1, \dots, \delta'_m)$. So the set of ideals $P(\Lambda)$ for Λ ranges over all finite tuples of Δ forms a directed set with respect to the inclusion of ideals. Since the intersection of any directed set of prime ideals is again prime, it suffices to prove that $P(\Lambda)$ is prime for any finite tuple Λ of Δ . By iteration we may reduce to the case that $\Delta = \{\delta\}$.

Now let P be prime and $(P : \Delta) = P(\delta) = \{r \in R \mid \delta^n(r) \in P, \forall n \geq 0\}$. Suppose we have $a, b \in R$ such that $ab \in (P : \Delta)$ with $a \notin (P : \Delta)$. So we can find $\delta^s(a) \notin P$ with s minimal. Then

for any $n \geq s$, we have

$$\begin{aligned} \sum_{s \leq i \leq n} \binom{n}{i} \delta^i(a) \delta^{n-i}(b) &= \sum_{0 \leq i \leq n} \binom{n}{i} \delta^i(a) \delta^{n-i}(b) - \sum_{0 \leq i \leq s-1} \binom{n}{i} \delta^i(a) \delta^{n-i}(b) \\ &= \delta^n(ab) - \sum_{0 \leq i \leq s-1} \binom{n}{i} \delta^i(a) \delta^{n-i}(b) \in P. \end{aligned}$$

An easy induction on n using the fact that $\delta^s(a)$ is not in the prime ideal P implies that $\delta^i(b) \in P$ for all $i \geq 0$. Hence $b \in (P : \Delta)$ and $(P : \Delta)$ is prime.

(ii) It follows from the well-known one-to-one correspondence for prime ideals in localizations.

(iii) One can see from (i) that every prime ideal minimal over a Δ -ideal is a Δ -prime ideal of R . So it follows from Noether's theorem.

(iv) By [BG01, Proposition II.7.12], we know R is a Jacobson ring. So every prime ideal of R is the intersection of maximal ideals containing it. Then for any Δ -prime ideal J of R we have

$$J = (J : \Delta) = \bigcap_{J \subseteq P \in \max R} P = \left(\left(\bigcap_{J \subseteq P \in \max R} P \right) : \Delta \right) = \bigcap_{J \subseteq P \in \max R} (P : \Delta) = \bigcap_{J \subseteq Q \in \Delta\text{-prim } R} Q.$$

□

Remark 3.2. As a consequence of Lemma 3.1(i), we see that

$$\Delta\text{-prim } R \subseteq \Delta\text{-spec } R \subseteq \text{spec } R,$$

which can be considered as inclusions of Zariski spaces.

The following result is well-known for the prime spectrum of any associative algebra, e.g., [BG01, Lemma II.7.11], we adapt the version for Δ -prime spectrums here.

Lemma 3.3. *Let (R, Δ) be a commutative differential k -algebra. For any Δ -prime ideal P of R , the following are equivalent:*

- (i) P is locally closed in $\Delta\text{-spec } R$;
- (ii) There exists $f \neq 0$ in the quotient algebra R/P such that the localization $(R/P)_f$ has no proper nontrivial Δ -prime ideals;
- (iii) The intersection of all Δ -prime ideals properly containing P is a Δ -ideal properly containing P ;
- (iv) P is \aleph_0 -separable in $\Delta\text{-spec } R$ assuming that R is noetherian.

Proof. (i) \Rightarrow (ii) By Definition-Lemma 2.1(iii), we can write

$$\{P\} = \{Q \in \Delta\text{-spec } R \mid Q \supseteq P\} \setminus \{Q \in \Delta\text{-spec } R \mid Q \supseteq I\}$$

for some Δ -ideal I strictly containing P . Pick any $f \in I \setminus P$ and still write f as its image in I/P . Since every Δ -prime ideal Q strictly containing P must contain I , we have $f \in I \subseteq Q$. So by Lemma 3.1(ii), $(R/P)_f$ has no proper nontrivial Δ -prime ideals.

(ii) \Rightarrow (iii) Choose any preimage of f in R , which we still denote by f such that $f \in R \setminus P$. Again by Lemma 3.1(ii), every Δ -prime ideal strictly containing P has to contain f . Thus f is contained in the intersection of all Δ -prime ideals properly containing P , which then is a Δ -ideal properly containing P .

(iii) \Rightarrow (iv) Let I be the intersection of all Δ -prime ideals properly containing P , which is a Δ -ideal properly containing P . Since R is noetherian, by Lemma 3.1(iii) the set of all minimal Δ -prime ideals containing I is finite. So P is \aleph_0 -separable in Δ -spec R by Lemma 2.6.

(iv) \Rightarrow (i) It is Lemma 2.3. \square

The next result was first observed in the proof of [Go06, Proposition 1.2] for the embedding of Δ -center of fractional quotient ring of Δ -primitive ideal into quotient ring of the corresponding maximal ideal of R , we generalize it in the case of any Δ -prime ideal of R .

Lemma 3.4. *Let (R, Δ) be a commutative differential k -algebra. For any prime ideal P of R , there exists a field extension*

$$\phi : Z_{\Delta}(\text{Frac } R/(P : \Delta)) \longrightarrow \text{Frac } R/P$$

over the base field k .

Proof. We may assume that $(P : \Delta) = (0)$. By Lemma 3.1(i), R is an integral domain. Set $Q = \text{Frac } R$. We first show that $Z_{\Delta}(Q)$ is a k -subalgebra of the localization R_P . For any fraction $rs^{-1} \in Z_{\Delta}(Q)$ and any $\delta \in \Delta$, we have $\delta(rs^{-1}) = (\delta(r)s - r\delta(s))s^{-2} = 0$. This implies that $\delta(r)s = r\delta(s)$ or equivalently $rs^{-1} = \delta(r)\delta(s)^{-1}$. Since $s \notin (P : \Delta) = (0)$, there is a finite set of derivations $\delta_1, \dots, \delta_n \in \Delta$ such that $\delta_1 \cdots \delta_n(s) \notin P$. By above argument, we then have $rs^{-1} = \delta_n(r)\delta_n(s)^{-1} = \cdots = [\delta_1 \cdots \delta_n(r)][\delta_1 \cdots \delta_n(s)]^{-1} \in R_P$. Therefore, we have a k -algebra map

$$\phi : Z_{\Delta}(Q) \xrightarrow{\subseteq} R_P \xrightarrow{\text{quot.}} R_P/PR_P \xrightarrow{\cong} \text{Frac } R/P.$$

We can see that ϕ is an extension of fields over k since $Z_{\Delta}(Q)$ is a field over k . \square

The most difficult case in Δ -Dixmier-Moeglin equivalence is to show that every Δ -rational ideal of R is locally closed in Δ -spec R due to the following result, which was first given in the complex affine Poisson case by Oh [Oh99, Propositions 1.7, 1.1] and later proved for any commutative affine differential k -algebra by Goodearl [Go06, Proposition 1.2]. We show the same result holds for any commutative noetherian differential k -algebra over large base field.

Proposition 3.5. *Let (R, Δ) be a commutative differential k -algebra. Suppose R is noetherian and $\dim_k R < |k|$. Then we have the following implications for any $P \in \Delta$ -spec R :*

$$P \text{ locally closed in } \Delta\text{-spec } R \implies P \text{ } \Delta\text{-primitive} \implies P \text{ } \Delta\text{-rational}.$$

Proof. Suppose P is locally closed in Δ -spec R . By Lemma 3.1(iv), we have $P = \bigcap_{P \subseteq P_i \in \Delta\text{-prim } R} P_i$. Now we must have some $P_i = P$ since Lemma 3.3(iii) implies that

$$P \subsetneq \bigcap_{P \subsetneq P_i \in \Delta\text{-spec } R} P_i \subseteq \bigcap_{P \subsetneq P_i \in \Delta\text{-prim } R} P_i.$$

This shows that P is Δ -primitive.

Next let P be Δ -primitive and we can write $P = (M : \Delta)$ for some $M \in \max R$. By Lemma 3.4, we have a field extension $Z_{\Delta}(\text{Frac } R/P) \rightarrow \text{Frac}(R/M)$ over k . Since $\dim_k R < |k|$, we have that R satisfies the Nullstellensatz [BG01, Proposition II.7.16]. So $Z_{\Delta}(\text{Frac } R/P) \subseteq \text{Frac}(R/M)$ is algebraic over k by Definition [BG01, Definition II.7.14]. So P is Δ -rational and it completes the proof. \square

4. MAIN RESULTS

We need some more lemmas before we can assert our main theorems. The next lemma is originated from the well-known results in ring theory concerning ideals in rings obtained by extending scalars in centrally closed algebras. In particular, we modify [BWY19, Lemma 2.1] into the setting of any commutative differential k -algebra.

Lemma 4.1. *Let (R, Δ) be a commutative differential k -algebra. Suppose K/k is any field extension and both R and $(\text{Frac } R) \otimes_k K$ are integral domains. If the zero ideal of R is Δ -rational, then every nonzero Δ -ideal of $R \otimes_k K$ contains an element of the form $r \otimes 1$ for some $0 \neq r \in R$.*

Proof. Let I be a nonzero Δ -ideal of $R \otimes_k K$ and choose a nonzero element $x \in I$ such that $x = \sum_{i=1}^d r_i \otimes \lambda_i$ with $r_i \in R$, $\lambda_i \in K$ and d minimal. We claim that $d = 1$. If not, suppose that $d > 1$. Clearly all r_1, \dots, r_d are nonzero in R . So in $(\text{Frac } R) \otimes_k K$, we may write $x = (r_1 \otimes 1)y$, where $y = \sum_{i=1}^d z_i \otimes \lambda_i$ with $z_1 = r_1/r_1, \dots, z_d = r_d/r_1 \in \text{Frac } R$. Set $F := k(z_1, \dots, z_d)$ to be the extension of k in $\text{Frac } R$ generated by z_1, \dots, z_d . We claim that F is algebraic over k . We note, for any $\delta \in \Delta$ which can be extended to a K -linear derivation of $R \otimes_k K$, the following element

$$\delta(x)(r_1 \otimes 1) - x(\delta(r_1) \otimes 1) = \sum_{i=2}^d (\delta(r_i)r_1 - r_i\delta(r_1)) \otimes \lambda_i$$

is again in I . By minimality of d , we have $\lambda_2, \dots, \lambda_d$ are k -linearly independent and $\delta(r_i)r_1 = r_i\delta(r_1)$ for all $2 \leq i \leq d$. In terms of $\text{Frac } R$, the above equality can be read as $\delta(r_i/r_1) = 0$ for all $\delta \in \Delta$ or equivalently $z_i \in Z_\Delta(\text{Frac } R)$ for all $1 \leq i \leq d$. Since (0) is rational, we have that all z_i 's are algebraic over k . This proves the claim.

Now since $[F : k] < \infty$, we see that $[F \otimes_k K : K] < \infty$ and $y \in F \otimes_k K$ is algebraic over $K = k \otimes_k K$. In particular, there is a non-trivial relation $y^m(1 \otimes c_m) + y^{m-1}(1 \otimes c_{m-1}) + \dots + (1 \otimes c_0) = 0$ for some integer $m \geq 1$ and $c_i \in K$. Furthermore, we may assume c_0 is nonzero since $F \otimes_k K \subseteq (\text{Frac } R) \otimes_k K$ is an integral domain. Without loss of generality, set $c_0 = 1$. Then by construction

$$r_1^m = \sum_{j=0}^m (r_1 \otimes 1)^{m-j} x^j (1 \otimes c_j) - \sum_{j=1}^m (r_1 \otimes 1)^{m-j} x^j (1 \otimes c_j) = - \sum_{j=1}^m (r_1 \otimes 1)^{m-j} x^j (1 \otimes c_j) \in I.$$

Since r_1 is nonzero in the integral domain R , we have r_1^m is nonzero and the result follows. \square

Remark 4.2. We note that this result need not hold if $(\text{Frac } R) \otimes_k K$ is not an integral domain. For example, if $R = K = \mathbb{Q}(\sqrt{2})$ and $k = \mathbb{Q}$. Then $R \otimes_k K$ is not an integral domain and hence there is some nonzero prime ideal I of $R \otimes_k K$. But R is a field, so $I \cap (R \otimes 1)$ is necessarily zero.

The following key lemma is inspired by [BWY19, Lemma 2.3], which reveals the fact that rationality of Δ -prime ideals can be captured purely in terms of the poset of Δ -prime ideals when the base field k is sufficiently large compared with the dimension of the commutative differential k -algebra R and the size of the derivation set Δ .

Lemma 4.3. *Let (R, Δ) be a commutative differential k -algebra. Suppose R is noetherian and both $\dim_k R$ and $|\Delta| < |k|$. Then any Δ -prime ideal is Δ -rational if and only if it is $|k|$ -separable in $\Delta\text{-spec } R$.*

Proof. Let P be a Δ -prime ideal of R . By replacing R with R/P , we may assume that $P = (0)$ and R is a noetherian integral domain. Notice that if $|k| \leq \aleph_0$ then $\dim_k R < \infty$ and so R is just a finite field extension of k . Then (0) is Δ -rational, maximal ideal and the claim is vacuously true in this case. Thus we assume henceforth that k is uncountable.

Let $\mathcal{B} = \{r_\alpha : \alpha \in \Lambda\}$ be a k -basis for R , where Λ is an index set with $|\Lambda| = \dim_k R < |k|$. Then for any $\alpha, \beta \in \Lambda$ and $\delta \in \Delta$, we can write

$$r_\alpha r_\beta = \sum_{\gamma \in \Lambda} c_{\alpha, \beta}^\gamma r_\gamma \quad \text{and} \quad \delta(r_\alpha) = \sum_{\gamma \in \Lambda} d_{\delta, \alpha}^\gamma r_\gamma,$$

where the coefficients $c_{\alpha, \beta}^\gamma, d_{\delta, \alpha}^\gamma \in k$ are zero for all but finitely many $\gamma \in \Lambda$. Consider the subset

$$T := \left\{ c_{\alpha, \beta}^\gamma, d_{\delta, \alpha}^\gamma \mid \forall \alpha, \beta, \gamma \in \Lambda, \delta \in \Delta \right\}$$

of k . Since for any $(\alpha, \beta, \delta) \in \Lambda \times \Lambda \times \Delta$ there are only finitely many values of γ for which $c_{\alpha, \beta}^\gamma$ and $d_{\delta, \alpha}^\gamma$ are nonzero, we can construct an injection from $T \hookrightarrow \Lambda \times \Lambda \times \Delta \times \mathbb{N}$. In particular, $|T| \leq |\Lambda|^2 |\Delta| |\mathbb{N}| < |k|$, since k is uncountable and $|\Lambda|, |\Delta| < |k|$.

Now let k_0 be the prime subfield of k and let $F = k_0(T)$ be the extension of k_0 generated by T . Set $R_0 := \sum_{\alpha \in \Lambda} F r_\alpha$ to be the F -subspace of R . We claim that R_0 is a commutative differential F -subalgebra of R that is a noetherian integral domain satisfying $R_0 \otimes_F k \cong R$ and $|\text{Frac } R_0| < |k|$. By construction, R_0 is closed under multiplication of R and is invariant under the actions of Δ . So R_0 is a commutative differential F -subalgebra of R satisfying $R_0 \otimes_F k \cong R$. Now since R is a free R_0 -module, we have an inclusion-preserving injection from the set of left R_0 -modules to the set of left R -modules. In particular, R_0 is noetherian since R is. Clearly $R_0 \subset R$ is an integral domain since R is. Moreover, since $\dim_F R_0 < |k|$, $|F| \leq \aleph_0 |T| < |k|$, and k is uncountable, we have $|R_0| < |k|$. Since every element of $\text{Frac } R_0$ can be expressed in the form sr^{-1} with $s, r \in R_0$, we have that $|\text{Frac } R_0| < |k|$. This proves the claim.

We first show that $|k|$ -separability implies Δ -rationality. Suppose that (0) is not Δ -rational. Then there is some $z = ab^{-1} \in Z_\Delta(\text{Frac } R)$ that is not algebraic over k for some $a, b \in R$. Since z is not in k , we have a and b are k -linearly independent and we may assume without loss of generality that $a, b \in \mathcal{B} \subset R_0$ in the above construction. So $z = ab^{-1} \in \text{Frac } R_0$. Now for any $\lambda \in k$, consider the element $z_\lambda := z \otimes 1 - 1 \otimes \lambda$ in $(\text{Frac } R_0) \otimes_F k$ and the subset

$$U := \{\lambda \in k \mid z_\lambda \text{ is a unit in } (\text{Frac } R_0) \otimes_F k\}$$

of k . We show that $|U| < |k|$ by the Amitsur trick. If not, suppose $|U| = |k|$. Since $(\text{Frac } R_0) \otimes_F k$ has dimension strictly less than $|k|$, there is necessarily a (finite) k -dependence of z_λ^{-1} with $\lambda \in U$; after clearing denominators in this dependence, we get that z is algebraic over k , which is a contradiction.

We see, as a localization of R , $(\text{Frac } R_0) \otimes_F k$ is also a noetherian integral domain. Let $\lambda \in k \setminus U$. By Krull's principal ideal theorem, we have (z_λ) is contained in some height one prime ideal P_λ of $(\text{Frac } R_0) \otimes_F k$. Moreover, the Δ -core of P_λ is nonzero since it contains $z_\lambda \in Z_\Delta(\text{Frac } R)$ and is again prime by Lemma 3.1(i). This forces $P_\lambda = (P_\lambda : \Delta)$ to be a height one Δ -prime ideal of $(\text{Frac } R_0) \otimes_F k$. By Lemma 3.1(ii), $Q_\lambda := P_\lambda \cap R$ is a height one Δ -prime ideal of R . We notice that $Q_\lambda \neq Q_{\lambda'}$ if $\lambda \neq \lambda'$ in $k \setminus U$. Otherwise, it would imply $0 \neq \lambda' - \lambda = z_\lambda - z_{\lambda'} \in Q_\lambda = Q_{\lambda'}$ and we get a contradiction.

Now let us assume that (0) is $|k|$ -separable in $\Delta\text{-spec } R$, namely there exists an index set \mathcal{S} with $|\mathcal{S}| < |k|$ and a set of nonzero Δ -prime ideals $\{P_s \mid s \in \mathcal{S}\}$ such that every nonzero Δ -prime ideal of R contains some P_s . Then since $|\mathcal{S}| < |k|$ and $|k \setminus U| = |k|$, there is some $s \in \mathcal{S}$ such that $P_s \subseteq Q_\lambda$ for infinitely many $\lambda \in k \setminus U$. Since Q_λ 's are height one primes of R , we have that $0 \subsetneq P_s \subseteq Q_\lambda$ implies that $P_s = Q_\lambda$. But $Q_\lambda \neq Q_{\lambda'}$ if $\lambda \neq \lambda'$, so we get a contradiction. Thus we have shown one direction of the lemma.

For the other direction such that Δ -rationality implies $|k|$ -separability, we suppose that (0) is Δ -rational. Retaining the above notations, for any $0 \neq a \in R_0$, let \mathcal{S}_a denote the set of Δ -prime

ideals of R containing $a \otimes 1$. By Lemma 4.1, since (0) is rational and R_0 and $(\text{Frac } R_0) \otimes_F k$ are integral domains, every nonzero Δ -ideal of $R_0 \otimes_F k \cong R$ contains an element of the form $a \otimes 1$ with some nonzero a in R_0 . Then

$$\Delta\text{-spec } R \setminus \{(0)\} = \bigcup_{a \in R_0 \setminus \{0\}} \mathcal{S}_a.$$

Now for each nonzero a in R_0 , let $P_a = \bigcap_{P \in \mathcal{S}_a} P$. Then P_a is a nonzero Δ -semiprime ideal of R since $a \otimes 1 \in P_a$. Thus by Lemma 3.1(iii), P_a is a finite intersection of nonzero Δ -prime ideals, say $P = P_{a,1} \cap \cdots \cap P_{a,n_a}$ and every nonzero Δ -prime ideal containing P_a contains some $P_{a,i}$. Finally, we let

$$\mathcal{S} = \{P_{a,i} \mid a \in R_0 \setminus \{0\}, 1 \leq i \leq n_a\}.$$

Then \mathcal{S} is a set of nonzero Δ -prime ideals of R of cardinality at most $|R_0| \times |\mathbb{N}| < |k|$ and by construction, every nonzero Δ -prime ideal of R contains some Δ -prime ideal from \mathcal{S} . So (0) is $|k|$ -separable in $\Delta\text{-spec } R$. This completes the proof. \square

Now we are able to give our main theorems. The first main result was proved in [BLLM17, Theorem 7.1] for any complex affine Poisson algebra as a weak version of the Poisson Dixmier-Moeglin equivalence, where our third topological condition was replaced by the finiteness condition of Poisson prime ideals of height exactly increased by one.

Theorem 4.4. *Let (R, Δ) be a commutative differential k -algebra. Suppose R is noetherian and both $\dim_k R$ and $|\Delta| < |k|$. For a Δ -prime ideal P of R , the following are equivalent:*

- (i) P is Δ -primitive;
- (ii) P is Δ -rational;
- (iii) P is $|k|$ -separable in $\Delta\text{-spec } R$.

Proof. (i) \Rightarrow (ii) is Proposition 3.5 and (ii) \Leftrightarrow (iii) is Lemma 4.3. So it suffices to show (iii) \Rightarrow (i). We may assume that k is uncountable and $P = (0)$ where R becomes a noetherian integral domain. By definition, there is a subset $\{Q_s \in \Delta\text{-spec } R \setminus \{(0)\} \mid s \in \Lambda\}$, where Λ is an index set with $|\Lambda| < |k|$, such that every nonzero Δ -prime ideal of R must contain some Q_s . For each $s \in \Lambda$, choose some nonzero $f_s \in Q_s$. Denote by T the multiplicative closed subset of R generated by all f_s , and $B := RT^{-1}$ the localization of R at T . It is clear that $|T| \leq \aleph_0 |\Lambda| < |k|$ since $|k|$ is uncountable. So B is noetherian with $\dim_k B < |k|$. Then B satisfies the Nullstellensatz [BG01, Proposition II.7.16] such that for any maximal ideal J of B , B/J is algebraic over k . Let $I := J \cap R$ for any maximal ideal J of B . Since R/I embeds into B/J , we can see that R/I too is an algebraic extension of k and hence I is maximal in R . By the construction of T , we have $I = J \cap R$ does not intersect T , which implies that I does not contain any Q_s . But then we must have $(0) = (I : \Delta)$ is Δ -primitive since otherwise $f_s \in Q_s \subseteq (I : \Delta) \subseteq I$ for some Q_s for $(I : \Delta)$ is Δ -prime by Lemma 3.1(i). This completes the proof. \square

Our next main result reveals a purely topological nature of the Δ -Dixmier-Moeglin equivalence for any commutative noetherian differential k -algebra (R, Δ) whenever both $\dim_k R$ and $|\Delta| < |k|$. In particular, we show that the Zariski topology of $\Delta\text{-spec } R$ can detect the Δ -Dixmier-Moeglin equivalence for any such commutative differential k -algebra (R, Δ) .

Theorem 4.5. *Let (R, Δ) be a commutative differential k -algebra. Suppose that R is noetherian and both $\dim_k R$ and $|\Delta| < |k|$. Then the following are equivalent:*

- (i) R satisfies the Δ -Dixmier-Moeglin equivalence;
- (ii) Every Δ -rational ideal of R is locally closed in $\Delta\text{-spec } R$;

- (iii) Every Δ -primitive ideal of R is locally closed in $\Delta\text{-spec } R$;
- (iv) In the Zariski space $\Delta\text{-spec } R$, that a Δ -prime ideal P is $|k|$ -separable implies that it is \aleph_0 -separable;
- (v) In the poset $(\Delta\text{-spec } R, \subseteq)$, that a Δ -prime ideal P is $|k|$ -separable implies that it is \aleph_0 -separable.

Proof. The above equivalences are direct consequences of Proposition 3.5, Theorem 4.4, and Lemma 2.6. Note that for $\Delta\text{-spec } R$, the partial order \leq of Lemma 2.6 amounts to inclusion of Δ -prime ideals. \square

Now we give some consequences of our main results.

Brown and Gordon showed the Poisson Dixmier-Moeglin equivalence holds for any affine complex Poisson algebra with only finitely many Poisson primitive ideals [BG03, Lemma 3.4]. Our main theorems yield the similar result in terms of any commutative noetherian differential algebra over large base field with finitely many Δ -primitive ideals.

Corollary 4.6. *Let (R, Δ) be a commutative differential k -algebra. Suppose that R is noetherian with both $\dim_k R$ and $|\Delta| < |k|$. If every point in $\Delta\text{-spec } R$ is \aleph_0 -separable, then R satisfies the Δ -Dixmier-Moeglin equivalence. In particular if $\Delta\text{-prim } R$ is finite, then R satisfies the Δ -Dixmier-Moeglin equivalence.*

Proof. If any point in $\Delta\text{-spec } R$ is \aleph_0 -separable, we know R satisfies the Δ -Dixmier-Moeglin equivalence by Theorem 4.5. In particular, suppose there are only finitely many Δ -primitive ideals. We notice that every Δ -prime ideal of R is an intersection of Δ -primitive ideals by Lemma 3.1(iv). Since there are only finitely many Δ -primitive ideals, this implies that $\Delta\text{-spec } R$ is finite and hence every point in $\Delta\text{-spec } R$ is \aleph_0 -separable. This completes the proof. \square

Our next corollary is parallel to the similar result in [BWY19, Theorem 1.1] for the Dixmier-Moeglin equivalence, which is somewhat interesting since it suggests that for sufficiently large base fields the underlying topology on the Δ -prime spectrum completely determines the Δ -Dixmier-Moeglin equivalence.

Corollary 4.7. *Let (R, Δ_R) and (S, Δ_S) be two commutative differential k -algebras. Suppose that R, S are noetherian with $\dim_k R, \dim_k S, |\Delta_R|, |\Delta_S| < |k|$. If there is an isomorphism between the two posets $(\Delta\text{-spec } R, \subseteq)$ and $(\Delta\text{-spec } S, \subseteq)$. Then R satisfies the Δ -Dixmier-Moeglin equivalence if and only if S satisfies the Δ -Dixmier-Moeglin equivalence. In particular, if $\Delta\text{-spec } R$ and $\Delta\text{-spec } S$ are homeomorphic, then R satisfies the Δ -Dixmier-Moeglin equivalence if and only if S does.*

Proof. The result straight follows from Theorem 4.5(i) \Leftrightarrow (v) since the condition (v) is preserved under any inclusion-preserving bijection or homeomorphism between $\Delta\text{-spec } R$ and $\Delta\text{-spec } S$. \square

5. STRATIFICATION

In this section, we show that the Δ -Dixmier-Moeglin equivalence is a local property for the Zariski space $\Delta\text{-spec } R$ while working over large base fields. Thus by considering proper stratification of the Zariski space $\Delta\text{-spec } R$, we can establish the Δ -Dixmier-Moeglin equivalence by verifying it locally in each stratification.

Theorem 5.1. *Let (R, Δ) be a commutative noetherian differential k -algebra with both $\dim_k R$ and $|\Delta| < |k|$. Suppose that $\Delta\text{-spec } R = \bigcup X_i$ is a union of locally closed subsets with each X_i , when endowed with the subspace topology, homeomorphic to $\Delta\text{-spec } R_i$ for some noetherian*

commutative differential k -algebra (R_i, Δ_i) with both $\dim_k R_i$ and $|\Delta_i| < |k|$. Then R satisfies the Δ -Dixmier-Moeglin equivalence if and only if each R_i satisfies the Δ_i -Dixmier-Moeglin equivalence.

Proof. By Theorem 4.5, we know R satisfies the Δ -Dixmier-Moeglin equivalence if and only if that any point P in $\Delta\text{-spec } R$ is $|k|$ -separable implies that it is \aleph_0 -separable. Now by Lemma 2.4, we know that the $|k|$ -separability (resp. \aleph_0 -separability) can be determined in any locally closed subset of $\Delta\text{-spec } R$. In particular, P is $|k|$ -separable (resp. \aleph_0 -separable) if and only if P is $|k|$ -separable (resp. \aleph_0 -separable) in any locally closed subset X_i containing P . Thus the result follows. \square

The Δ -Dixmier-Moeglin equivalence has been established for any differential commutative k -algebra R with suitable torus actions by Goodearl [Go06]. In particular, in these cases it is shown that the corresponding $\Delta\text{-spec } R$ is a finite disjoint union of locally closed subsets X_i , where each X_i is homeomorphic to the prime spectrum of some Laurent polynomial ring. We summarize the above illustration by asserting a local topological criterion when the topological space $\Delta\text{-spec } R$ is formed by glueing together finitely many noetherian affine schemes as locally closed subsets.

Corollary 5.2. *Let (R, Δ) be a commutative noetherian differential k -algebra with both $\dim_k R$ and $|\Delta| < |k|$. Suppose that $\Delta\text{-spec } R = \bigcup X_i$ is a union of locally closed subsets with each X_i , when endowed with the subspace topology, homeomorphic to $\text{spec } R_i$ for some noetherian commutative k -algebra R_i with $\dim_k R_i < |k|$. Then R satisfies the Δ -Dixmier-Moeglin equivalence.*

Proof. We notice that any commutative noetherian k -algebra R_i with $\dim_k R_i < |k|$ satisfies Dixmier-Moeglin equivalence. Moreover, when R_i is considered as a differential commutative k -algebra with $\Delta = \{0\}$, it automatically satisfies the Δ -Dixmier-Moeglin equivalence. So the result follows from Theorem 5.1. \square

Next, we extend the symplectic core stratification on the maximal spectrum of a complex affine Poisson algebra discussed in [BG03, §3.3] to the Δ -prime spectrum of a commutative differential k -algebra (R, Δ) .

We define a relation \sim on $\text{spec } R$ by:

$$P \sim Q \iff (P : \Delta) = (Q : \Delta).$$

We can easily check that \sim is an equivalence relation on $\text{spec } R$, where we denote the corresponding equivalence class of any $P \in \text{spec } R$ by $\mathcal{C}_{\text{spec}}(P)$, so that

$$\text{spec } R = \bigsqcup_{Q \in \Delta\text{-spec } R} \mathcal{C}_{\text{spec}}(Q) = \{P \in \text{spec } R \mid (P : \Delta) = Q\}.$$

The set $\mathcal{C}_{\text{spec}}(P)$ is called the *prime differential core* (of P). Moreover, the (*maximal*) *differential core* of P is denoted by $\mathcal{C}(P) := \mathcal{C}_{\text{spec}}(P) \cap \max R$ and we have

$$\max R = \bigsqcup_{Q \in \Delta\text{-prim } R} \mathcal{C}(Q) = \{P \in \max R \mid (P : \Delta) = Q\}.$$

Notice that there is a continuous retraction $\pi : \text{spec } R \rightarrow \Delta\text{-spec } R$ given by the Δ -core such that $\pi(P) = (P : \Delta)$ for any $P \in \text{spec } R$ and $\Delta\text{-spec } R$ is a topological quotient of $\text{spec } R$ via π [Go06, Theorem 1.3]. Moreover, the restriction of π on $\max R$ gives a continuous surjection denoted by $\pi_m : \max R \rightarrow \Delta\text{-prim } R$ [Go06, Theorem 1.5]. In general, $\Delta\text{-prim } R$ is not a topological quotient of $\max R$ via π_m unless R satisfies the Δ -Dixmier-Moeglin equivalence.

Proposition 5.3. *Let (R, Δ) be a commutative noetherian differential k -algebra with both $\dim_k R$ and $|\Delta| < |k|$. Then the following are equivalent:*

- (i) R satisfies the Δ -Dixmier-Moeglin equivalence.
- (ii) The prime differential core of any $P \in \Delta\text{-prim } R$ is locally closed in $\text{spec } R$.
- (iii) The differential core of any $P \in \Delta\text{-prim } R$ is locally closed, and

$$\overline{\mathcal{C}(P)} = \{Q \in \max R \mid P \subseteq Q\}$$

in $\max R$.

If the above conditions hold, then each (prime) differential core of Δ -primitive ideals is smooth in its closure.

Proof. (i) \Rightarrow (ii) We know P is locally closed in $\Delta\text{-spec } R$. Since $\Delta\text{-spec } R$ is a topological quotient of $\text{spec } R$ via π , we have $\pi^{-1}(P) = \mathcal{C}_{\text{spec}}(P)$ is locally closed in $\text{spec } R$.

(ii) \Rightarrow (i) Since $P \in \mathcal{C}_{\text{spec}}(P)$, we know that $\overline{\mathcal{C}_{\text{spec}}(P)} = \{Q \in \text{spec } R \mid P \subseteq Q\}$. By Lemma 3.3, that $\mathcal{C}_{\text{spec}}(P)$ is locally closed implies that there is some ideal I properly containing P such that

$$\{Q \in \text{spec } R \mid P \subseteq Q\} \setminus \{Q \in \text{spec } R \mid (Q : \Delta) = P\} = \{Q \in \text{spec } R \mid I \subseteq Q\}.$$

Let Q be any Δ -prime ideal properly containing P . Since $P \subsetneq (Q : \Delta) = Q$, we have $I \subseteq Q$. By Lemma 3.3, we know P is locally closed in $\Delta\text{-spec } R$. Hence R satisfies the Δ -Dixmier-Moeglin equivalence by Theorem 4.5.

(i) \Rightarrow (iii) Let I be the intersection of all Δ -prime ideals properly containing P . Since P is locally closed in $\Delta\text{-spec } R$, we have $P \subsetneq I$ and

$$\begin{aligned} \mathcal{C}(P) &= \{Q \in \max R \mid P \subseteq (Q : \Delta)\} \setminus \{Q \in \max R \mid P \subsetneq (Q : \Delta)\} \\ &= \{Q \in \max R \mid P \subseteq Q\} \setminus \{Q \in \max R \mid I \subseteq Q\} \end{aligned}$$

So $\mathcal{C}(P)$ is locally closed in $\max R$. Moreover since R is a Jacobson ring, we have

$$P = \bigcap_{P \subseteq Q \in \max R} Q = \left(\bigcap_{Q \in \mathcal{C}(P)} Q \right) \cap \left(\bigcap_{I \subseteq Q \in \max R} Q \right).$$

Since P is prime and $P \subsetneq I \subseteq \bigcap_{I \subseteq Q \in \max R} Q$, we have $\bigcap_{Q \in \mathcal{C}(P)} Q = P$ and

$$\overline{\mathcal{C}(P)} = \left\{ Q \in \max R \mid \left(\bigcap_{M \in \mathcal{C}(P)} M \right) \subseteq Q \right\} = \{Q \in \max R \mid P \subseteq Q\}.$$

(iii) \Rightarrow (i) By Theorem 4.5, it suffices to show that any Δ -primitive ideal P of R is locally closed in $\Delta\text{-spec } R$. Since $\mathcal{C}(P)$ is open in $\overline{\mathcal{C}(P)} = \{Q \in \max R \mid P \subseteq Q\}$, there is some Δ -ideal I properly containing P such that

$$\overline{\mathcal{C}(P)} \setminus \mathcal{C}(P) = \{Q \in \max R \mid P \subsetneq (Q : \Delta)\} = \{Q \in \max R \mid I \subseteq Q\}.$$

For any Δ -prime ideal L properly containing P , by Lemma 3.1(iv), we have

$$L = \bigcap_{L \subseteq Q \in \Delta\text{-prim } R} Q = \bigcap_{L \subseteq M \in \max R} (M : \Delta) \supseteq (I : \Delta) = I \supsetneq P.$$

So P is locally closed in $\Delta\text{-spec } R$.

Finally, let J be the defining ideal of the singular locus of $\overline{\mathcal{C}(P)}$ (resp. $\overline{\mathcal{C}_{\text{spec}}(P)}$). Since J is a Δ -ideal properly containing P , it follows from the definition of Δ -core that any maximal ideal Q in $\mathcal{C}(P)$ does not contain J . Hence the point corresponding to Q belongs to the smooth locus of $\overline{\mathcal{C}(P)}$, as required. \square

Finally, suppose G is an affine algebraic group over k . Let G act on R by k -algebra automorphisms. An G -ideal of R is any ideal I of R that is stable under G . For any ideal I in R , let $(I : G)$ denote the largest G -ideal of R contained in I , that is,

$$(I : G) = \bigcap_{g \in G} g(I) = \{r \in R \mid g(r) \in I, \forall g \in G\}.$$

An G -prime ideal of R is any proper G -ideal of R that is prime in the sense of all G -ideals. In particular, $(P : G)$ is G -prime for any prime ideal P . A (G, Δ) -ideal of R is any ideal that is both a G -ideal and a Δ -ideal. We recall the notion of (G, Δ) -prime ideal where the primeness is defined in the reign of all (G, Δ) -ideals, and we write $(G, \Delta)\text{-spec } R$ for the (G, Δ) -spectrum of R , that is, the set of all (G, Δ) -prime ideals of R , equipped with the natural Zariski topology.

Now we introduce the concept of rational action of G on (R, Δ) .

Definition 5.4. Let (R, Δ) be a differential k -algebra, and G an affine algebraic group over k . Suppose G acts on R by k -algebra automorphisms. The action of G on (R, Δ) is *rational* if

- (i) R is a directed union of finite-dimensional G -invariant k -subspaces V_i such that the restriction maps $G \rightarrow \text{Aut}(R) \rightarrow \text{GL}(V_i)$ are morphisms of algebraic varieties.
- (ii) The k -linear span of Δ is a directed union of finite dimensional G -invariant k -subspaces of $\text{Der}_k R$ under the induced G -action such that $g.\delta = g \circ \delta \circ g^{-1}$ for any $\delta \in \text{Der}_k R$ and $g \in G$.

The following lemma is based on [Go06, Lemma 3.1], where we extend any rational action of an algebraic torus to that of an affine connected algebraic group.

Lemma 5.5. Let (R, Δ) be a commutative noetherian differential k -algebra and G an affine algebraic group acting rationally on (R, Δ) .

- (i) If I is a Δ -ideal of R , then $(I : G)$ is a (G, Δ) -ideal. Similarly, if I is a G -ideal, then $(I : \Delta)$ is a (G, Δ) -ideal.
- (ii) $(P : G) \in (G, \Delta)\text{-spec } R$ for all $P \in \Delta\text{-spec } R$.
- (iii) Assume that G is connected. The following sets coincide:
 - (a) $(G, \Delta)\text{-spec } R$;
 - (b) The set of all G -prime Δ -ideals in R ;
 - (c) The set of all Δ -prime G -ideals in R ;
 - (d) The set of all prime (G, Δ) -ideals in R .

Proof. (i) Let I be a Δ -ideal of R . Then we have

$$g(\Delta(I : G)) = g \circ \Delta \circ g^{-1}(g(I : G)) \subseteq g.\Delta(I) \subseteq \Delta(I) \subseteq I$$

for all $g \in G$. So $\Delta(I : G) \subseteq (I : G)$ and $(I : G)$ is a (G, Δ) -ideal. Similarly for any G -ideal I , we have

$$\delta_1 \cdots \delta_n g(I : \Delta) = g(g^{-1}\delta_1 g) \cdots (g^{-1}\delta_n g)(I : \Delta) = g(g^{-1}.\delta_1) \cdots (g^{-1}.\delta_n)(I : \Delta) \subseteq g(I : \Delta) \subseteq g(I) \subseteq I$$

for any $\delta_1, \dots, \delta_n \in \Delta$ and $g \in G$. This implies that $g(I : \Delta) \subseteq (I : \Delta)$ for any $g \in G$. So $(I : \Delta)$ is a (G, Δ) -ideal.

(ii) For any $P \in \Delta\text{-spec } R$, it is clear that $(P : G)$ is a G -prime ideal. By (i), $(P : G)$ is a (G, Δ) -ideal, and we can see that it is also (G, Δ) -prime since it is G -prime.

(iii) It is important to notice that the proof of [BG01, Proposition II.2.9] works for any affine algebraic group G whose coordinate ring is an integral domain. In particular, when G is connected,

any G -prime ideal of R is prime. Moreover, by [Go06, Lemma 1.1(d)], any Δ -prime ideal of R is also prime. So the sets (b), (c), and (d) coincide. It is clear that (d) \subseteq (a).

Finally, let Q be a (G, Δ) -prime ideal. By Lemma 3.1(iii), there are only finite many minimal prime ideals over Q and all of them are Δ -ideals. Say Q_1, \dots, Q_n are those minimal primes over Q . Since G must permute those Q_i 's, we know their G -orbits are finite. Then [BG01, Proposition II.2.9] again implies that Q_1, \dots, Q_n are all G -ideals and hence are all (G, Δ) -ideals. Since $Q_1 \cdots Q_n \subseteq Q$, we have $Q = Q_i$ for some i since Q is (G, Δ) -prime. Therefore (a) \subseteq (d). \square

As a consequence, there exists a corresponding G -stratification of $\Delta\text{-spec } R$ such that

$$\Delta\text{-spec } R = \bigsqcup_{J \in (G, \Delta)\text{-spec } R} \Delta\text{-spec}_J R,$$

where

$$\Delta\text{-spec}_J R = \{P \in \Delta\text{-spec } R : (P : G) = J\} \text{ for all } J \in (G, \Delta)\text{-spec } R.$$

Now suppose $(G, \Delta)\text{-spec } R$ is finite. We can list the (G, Δ) -prime ideals as J_1, \dots, J_n . Therefore, we have

$$\Delta\text{-spec}_{J_i} R = \{P \in \Delta\text{-spec } R : P \supseteq J_i\} \setminus \bigcup_{J_i \subsetneq J_j} \{P \in \Delta\text{-spec } R : P \supseteq J_j\}.$$

Hence each $\Delta\text{-spec}_{J_i} R$ is locally closed in $\Delta\text{-spec } R$. So we can apply our local criterion of Δ -Dixmier-Moeglin equivalence.

Proposition 5.6. *Let (R, Δ) be a commutative noetherian differential k -algebra with both $\dim_k R$ and $|\Delta| < |k|$, and G an affine connected algebraic group acting rationally on (R, Δ) . Suppose the set $(G, \Delta)\text{-spec } R$ is finite and for each $J \in (G, \Delta)\text{-spec } R$, we have $\Delta\text{-spec}_J R$ is homeomorphic to $\text{spec } R_J$ for some noetherian commutative k -algebra R_J with $\dim_k R_J < |k|$. Then R satisfies the Δ -Dixmier-Moeglin equivalence.*

Proof. It follows from Theorem 5.1 together with the discussion above. \square

Example 5.7. Let $G = (k^\times)^r$ be an algebraic torus acting rationally on a commutative noetherian differential k -algebra (R, Δ) with both $\dim_k R$ and $|\Delta| < |k|$. By [Go06, Theorem 3.2], we know each $\Delta\text{-spec}_J R$ is homeomorphic to $\text{spec } R_J$, where R_J is a Laurent polynomial ring in at most r indeterminates over the fixed field $Z_\Delta(\text{Frac } R/J)^G$. Now suppose R has only finitely many (G, Δ) -ideals. Our Proposition 5.6 then yields that R satisfies the Δ -Dixmier-Moeglin equivalence with a slightly different assumption on (R, Δ) compared to [Go06, Theorem 3.3].

6. APPLICATIONS TO THE POISSON DIXMIER-MOEGLIN EQUIVALENCE

In our last section, we give some applications to Poisson algebras. Let $(A, \{-, -\})$ be a Poisson k -algebra, where the bilinear map $\{-, -\} : A \times A \rightarrow A$ is both a Lie bracket and a bi-derivation. We can see that (A, Δ_A) is indeed a commutative differential k -algebra with $\Delta_A := \{\{a, -\} \mid a \in A\}$ consisting of all Hamiltonian derivations of A . When A is a finitely generated algebra over k , we can always choose Δ_A to be a finite set only consisting of those Hamiltonian derivations corresponding to a finite set of generators of A . Moreover, Δ -ideals, Δ -prime ideals, and Δ -primitive ideals of A are exactly Poisson ideals, Poisson prime ideals, and Poisson primitive ideals of A , respectively. In particular, $\Delta\text{-spec } A = \text{P.spec } A$ is the Poisson prime spectrum of A and $\Delta\text{-prim } A = \text{P.prim } A$ is the subset of $\text{P.spec } A$ consisting of Poisson primitive ideals with the subspace topology. We use $\mathcal{P}(I) := (I : \Delta_A)$ to denote the Poisson core of any ideal I in A . Finally, the Δ -Dixmier-Moeglin equivalence holds for the commutative differential k -algebra (A, Δ_A) if and only if the Poisson Dixmier-Moeglin equivalence holds for the Poisson algebra A .

Our main result Theorem 4.4 yields another proof of [BLLM17, Theorem 3.2].

Theorem 6.1. *Let A be a complex affine Poisson algebra. For any Poisson prime ideal P , the following are equivalent:*

- (i) P is Poisson primitive;
- (ii) P is Poisson rational;
- (iii) P is $|k|$ -separable in $\text{P.spec } A$.

Proof. As discussed above, (A, Δ_A) is a commutative differential complex affine algebra, where Δ_A can be chosen as a finite set. Hence we know $|\Delta_A| \leq \dim_{\mathbb{C}} A \leq \aleph_0 < \aleph = |\mathbb{C}|$. As a consequence, the equivalences follow from Theorem 4.4. \square

Theorem 6.2. *Let A be a complex affine Poisson algebra. Then the following are equivalent.*

- (i) A satisfies the Poisson Dixmier-Moeglin equivalence.
- (ii) Every Poisson primitive ideal of A is locally closed in $\text{P.spec } A$.
- (iii) Every Poisson rational ideal of A is locally closed in $\text{P.spec } A$.
- (iv) In the poset $(\text{P.spec } A, \subseteq)$, that any Poisson prime ideal is $|\mathbb{C}|$ -separable implies that it is \aleph_0 -separable.

Proof. As discussed above, (A, Δ_A) is a commutative differential complex affine algebra, where Δ_A can be chosen as a finite set. Hence we know $|\Delta_A| \leq \dim_{\mathbb{C}} A \leq \aleph_0 < \aleph = |\mathbb{C}|$. As a consequence, the equivalences follow from Theorem 4.5. \square

Theorem 6.3. *Let A be a complex affine Poisson algebra. Suppose $\text{P.spec } A = \bigcup X_i$ is a union of locally closed subsets with each X_i , when endowed with the subspace topology, homeomorphic to $\text{P.spec } A_i$ for some complex affine Poisson algebra A_i . Then A satisfies the Poisson Dixmier-Moeglin equivalence if and only if each A_i satisfies the Poisson Dixmier-Moeglin equivalence. In particular, if each X_i is homeomorphic to $\text{spec } A_i$ for some commutative noetherian complex algebra essentially of finite type, then A satisfies the Poisson Dixmier-Moeglin equivalence.*

Proof. It is Theorem 5.1 and Corollary 5.2 with the suitable assumptions on A and A_i 's. \square

Now we recall the symplectic core stratification on the maximal spectrum \mathcal{A} of a complex affine Poisson algebra A as a special case of the differential core stratification on a commutative differential algebra.

The equivalence relation \sim on $\text{spec } A$ defined by the symplectic core stratification is given by:

$$\mathfrak{p} \sim \mathfrak{q} \iff \mathcal{P}(\mathfrak{p}) = \mathcal{P}(\mathfrak{q}),$$

where we denote the corresponding equivalence class of any $\mathfrak{p} \in \text{spec } A$ by $\mathcal{C}_{\text{spec}}(\mathfrak{p})$, so that

$$\text{spec } A = \bigsqcup_{\mathfrak{q} \in \text{P.spec } A} \mathcal{C}_{\text{spec}}(\mathfrak{q}) = \{\mathfrak{p} \in \text{spec } A \mid \mathcal{P}(\mathfrak{p}) = \mathfrak{q}\}.$$

The set $\mathcal{C}_{\text{spec}}(\mathfrak{p})$ is called the *prime symplectic core* (of \mathfrak{p}). Moreover, the *(maximal) symplectic core* of \mathfrak{p} is denoted by $\mathcal{C}(\mathfrak{p}) := \mathcal{C}_{\text{spec}}(\mathfrak{p}) \cap \mathcal{A}$ and we have

$$\mathcal{A} = \bigsqcup_{\mathfrak{q} \in \text{P.prim } A} \mathcal{C}(\mathfrak{q}) = \{\mathfrak{m} \in \mathcal{A} \mid \mathcal{P}(\mathfrak{m}) = \mathfrak{q}\}.$$

Let A be a complex affine Poisson algebra, and $\mathcal{A} = \max A$ the maximal spectrum of A as a complex Poisson variety with respect to the Poisson bracket on A . Next, we discuss the stratification on \mathcal{A} by symplectic leaves following [BG03, §3.5].

First of all, we may assume that A is smooth. In this case, \mathcal{A} can be thought as a complex analytic manifold equipped with the complex topology. Now let \hat{A} be the ring of complex analytic functions on \mathcal{A} . Then \hat{A} is a Poisson algebra by considering the closed algebraic 2-form on \mathcal{A} associated to the Poisson bracket of A as complex analytic, which we also denote by $\{-, -\}$. Now the *symplectic leaf* $\mathcal{L}(\mathfrak{m})$ containing the point \mathfrak{m} of \mathcal{A} is the maximal connected complex analytic manifold in \mathcal{A} such that $\mathfrak{m} \in \mathcal{L}(\mathfrak{m})$ and $\{-, -\}$ is nondegenerate at every point of $\mathcal{L}(\mathfrak{m})$. According to [Wein83, Proposition 1.3], symplectic leaves exist and afford a stratification of \mathcal{A} .

Secondly, we extend this stratification to an arbitrary complex affine Poisson algebra A by setting inductively an ascending chain of ideals of A

$$0 \subseteq I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots$$

such that $I_0 = \text{rad } A$ and I_i/I_{i-1} is the vanishing ideal on the singular locus of A/I_{i-1} for all $i \geq 1$. We fix some integer m such that $I_m = A$. Now let $\mathcal{A}_i = \max A/I_i$, and $(\mathcal{A}_i)^{\text{smooth}}$ the smooth locus of \mathcal{A}_i for all $0 \leq i \leq m$. By [MR01, Proposition 15.2.14(i)] or [Polish97, Corollary 2.4], each I_i is indeed a semiprime Poisson ideal of A . So each $(\mathcal{A}_i)^{\text{smooth}} = \mathcal{A}_i \setminus \mathcal{A}_{i+1}$ is a complex analytic Poisson manifold and has, as above, a foliation by symplectic leaves. So there is again a stratification of \mathcal{A} :

$$\mathcal{A} = \bigsqcup_{0 \leq i \leq m-1} (\mathcal{A}_i \setminus \mathcal{A}_{i+1}) = \bigsqcup_{0 \leq i \leq m-1} (\mathcal{A}_i)^{\text{smooth}} = \bigsqcup_{\substack{0 \leq i \leq m-1 \\ j \in \mathcal{S}_i}} \mathcal{L}_{i,j},$$

where each \mathcal{S}_i is an index set corresponding to the symplectic leaf stratification $(\mathcal{A}_i)^{\text{smooth}} = \bigsqcup_{j \in \mathcal{S}_i} \mathcal{L}_{i,j}$ on $(\mathcal{A}_i)^{\text{smooth}}$.

Definition 6.4. [BG03, §3.6] The Poisson bracket of A is said to be *algebraic* if each symplectic leaf appearing in the symplectic leaf stratification is a locally closed subvariety of \mathcal{A} .

Our next lemma is based on [BG03, Lemma 3.5] and [BG03, Proposition 3.6]. We keep the proofs for the sake of completeness.

Lemma 6.5. *Let A be a complex affine Poisson algebra, and $\mathcal{L}(\mathfrak{m})$ be the symplectic leaf containing some maximal ideal $\mathfrak{m} \in \mathcal{A}$. Then we have*

- (i) $\overline{\mathcal{L}(\mathfrak{m})} = \{\mathfrak{q} \in \mathcal{A} \mid \mathcal{P}(\mathfrak{m}) \subseteq \mathfrak{q}\}$;
- (ii) $\mathcal{L}(\mathfrak{m}) \subseteq \mathcal{C}(\mathfrak{m})$ and $\overline{\mathcal{L}(\mathfrak{m})} = \overline{\mathcal{C}(\mathfrak{m})}$;
- (iii) *Each symplectic core is a disjoint union of symplectic leaves;*
- (iv) *The Zariski closure of each symplectic leaf (resp. symplectic core) is a disjoint union of symplectic leaves (resp. symplectic cores).*

Further suppose the Poisson bracket of A is algebraic. Then the symplectic core stratification coincides with the symplectic leaf stratification.

Proof. We write $\mathfrak{p} = \mathcal{P}(\mathfrak{m})$ as the Poisson core of $\mathfrak{m} \in \mathcal{L} := \mathcal{L}(\mathfrak{m})$.

- (i) Denote by \mathfrak{k} the defining ideal of $\overline{\mathcal{L}}$. It suffices to show that $\mathfrak{p} = \mathfrak{k}$.

We first show that $\mathfrak{p} \subseteq \mathfrak{k}$. Write $\overline{\mathcal{A}} = \max(A/\mathfrak{p}A)$. Recall \hat{A} is the ring of complex analytic functions on \mathcal{A} . We use $\overline{(\)}$ to denote the corresponding images of \hat{A} and \mathcal{A} in $\hat{A}/\mathfrak{p}\hat{A}$ and $\overline{\mathcal{A}}$, respectively. Let $B(\epsilon)$ be the complex analytic ball with radius ϵ . For any $H \in \hat{A}$, let $\sigma_m(z) : B(\epsilon) \rightarrow \mathcal{A}$ and $\bar{\sigma}_m(z) : B(\epsilon) \rightarrow \overline{\mathcal{A}}$ be the integral curves with respect to $\{H, -\}$ and $\{\overline{H}, -\}$ with initial conditions $\sigma_m(0) = \mathfrak{m}$ and $\bar{\sigma}_m(0) = \bar{\mathfrak{m}}$. By definition of integral curve, we have

$$\frac{d}{dz}(f \circ \sigma_m) = \{H, f\} \circ \sigma_m, \quad \frac{d}{dz}(\bar{f} \circ \bar{\sigma}_m) = \{\overline{H}, \bar{f}\} \circ \bar{\sigma}_m,$$

for any $f \in \widehat{A}$ in a neighborhood of 0 in $B(\epsilon)$. Now consider $\overline{\mathcal{A}}$ as a closed subset of \mathcal{A} . Since $\mathfrak{p}\widehat{A}$ is a Poisson ideal of \widehat{A} , the second equation above can be read as

$$\frac{d}{dz}(f \circ \overline{\sigma}_m) = \{H, f\} \circ \overline{\sigma}_m.$$

Then by the uniqueness of flows we get $\sigma_m = \overline{\sigma}_m$ in a neighborhood of 0 in $B(\epsilon)$. Since symplectic leaves are by definition obtained by traveling along integral curves to Hamiltonians, the leaf \mathcal{L} through \mathfrak{m} is contained in $\overline{\mathcal{A}}$. This proves $\mathfrak{p} \subseteq \mathfrak{k}$.

Conversely for $\mathfrak{k} \subseteq \mathfrak{p}$, without loss of generality, we may assume that \mathcal{A} is smooth since the symplectic leaf \mathcal{L} is defined in some smooth subvariety containing \mathfrak{m} . Let $\sigma_m(z) : B(\epsilon) \rightarrow \mathcal{A}$ be an integral curve to $\{H, -\}$ with $\sigma_m(0) = \mathfrak{m}$ for some $H \in A$. Then, by definition of an integral curve, we have $\frac{d}{dz}(f \circ \sigma_m) = \{H, f\} \circ \sigma_m$ for any $f \in \mathfrak{k}$. Since σ_m has image in \mathcal{L} and f vanishes on \mathcal{L} , we have

$$0 = \frac{d}{dz}(f \circ \sigma_m) = \{H, f\} \circ \sigma_m(0) = \{H, f\}(\mathfrak{m}).$$

This implies that $\{A, \mathfrak{k}\} = \bigcap_{m \in \mathcal{L}} m = \overline{\mathcal{L}} = \mathfrak{k}$. So \mathfrak{k} is a Poisson ideal and hence $\mathfrak{k} \subseteq \mathfrak{p} = \mathcal{P}(\mathfrak{m}) \subseteq \mathfrak{m}$.

(ii) Pick any $\mathfrak{n} \in \mathcal{L}(\mathfrak{m})$. By (i), we get

$$\overline{\mathcal{L}(\mathfrak{m})} = \{\mathfrak{q} \in \mathcal{A} \mid \mathcal{P}(\mathfrak{m}) \subseteq \mathfrak{q}\} = \{\mathfrak{q} \in \mathcal{A} \mid \mathcal{P}(\mathfrak{n}) \subseteq \mathfrak{q}\}.$$

This implies that $\mathcal{P}(\mathfrak{m}) = \mathcal{P}(\mathfrak{n})$. Hence $\mathcal{L}(\mathfrak{m}) \subseteq \mathcal{C}(\mathfrak{m})$. Moreover, we have

$$\{\mathfrak{q} \in \mathcal{A} \mid \mathcal{P}(\mathfrak{m}) \subseteq \mathfrak{q}\} = \overline{\mathcal{L}(\mathfrak{m})} \subseteq \overline{\mathcal{C}(\mathfrak{m})} \subseteq \{\mathfrak{q} \in \mathcal{A} \mid \mathcal{P}(\mathfrak{m}) \subseteq \mathfrak{q}\}.$$

So $\overline{\mathcal{L}(\mathfrak{m})} = \overline{\mathcal{C}(\mathfrak{m})}$.

(iii) It is a consequence of (ii) since $\mathfrak{n} \in \mathcal{C}(\mathfrak{m})$ if and only if $\mathcal{C}(\mathfrak{n}) = \mathcal{C}(\mathfrak{m})$ if and only if $\mathcal{L}(\mathfrak{n}) \subseteq \mathcal{C}(\mathfrak{m})$ since $\mathcal{L}(\mathfrak{n}) \subseteq \mathcal{C}(\mathfrak{n})$. Hence $\mathcal{C}(\mathfrak{m})$ is a disjoint union of symplectic leaves contained in it.

(iv) For any symplectic leaf $\mathcal{L} = \mathcal{L}(\mathfrak{m})$, one can conclude from (i) that for any $\mathfrak{n} \in \mathcal{A}$ such that $\mathfrak{n} \in \overline{\mathcal{L}}$ if and only if $\mathcal{P}(\mathfrak{m}) \subseteq \mathfrak{n}$ if and only if $\mathcal{P}(\mathfrak{m}) \subseteq \mathcal{P}(\mathfrak{n})$ if and only if $\overline{\mathcal{L}(\mathfrak{n})} \subseteq \overline{\mathcal{L}}$ if and only if $\mathcal{L}(\mathfrak{n}) \subseteq \overline{\mathcal{L}}$. So $\overline{\mathcal{L}}$ is a disjoint union of symplectic leaves. The argument for any symplectic core is similar.

Finally assume the Poisson bracket of A is algebraic. By (i), we know

$$\overline{\mathcal{L}} = \{\mathfrak{q} \in \mathcal{A} \mid \mathfrak{p} \subseteq \mathfrak{q}\} = \{\mathfrak{q} \in \mathcal{A} \mid \mathfrak{p} = \mathcal{P}(\mathfrak{q})\} \bigsqcup \{\mathfrak{q} \in \mathcal{A} \mid \mathfrak{p} \subsetneq \mathcal{P}(\mathfrak{q})\}.$$

Since the Poisson bracket of A is algebraic, we know $\overline{\mathcal{L}} \setminus \mathcal{L}$ is closed. So there is some Poisson ideal I properly containing \mathfrak{p} such that for any $\mathfrak{n} \in \mathcal{A}$ such that $\mathfrak{n} \in \overline{\mathcal{L}} \setminus \mathcal{L}$ if and only if $I \subseteq \mathcal{P}(\mathfrak{n})$. This implies that $\mathcal{L} = \{\mathfrak{q} \in \mathcal{A} \mid \mathfrak{p} = \mathcal{P}(\mathfrak{q})\} = \mathcal{C}(\mathfrak{m})$. \square

There is a continuous retraction $\pi : \text{spec } A \rightarrow \text{P.spec } A$ given by the Poisson core such that $\pi(\mathfrak{p}) = \mathcal{P}(\mathfrak{p})$ for any $\mathfrak{p} \in \text{spec } A$. Moreover, $\text{P.spec } A$ is a topological quotient of $\text{spec } A$ via π [Go06, Theorem 1.3]. The restriction of π on $\max A$ gives a continuous surjection denoted by $\pi_m : \max A \rightarrow \text{P.prim } A$ [Go06, Theorem 1.5].

The following proposition gives an affirmative answer to [Go06, Question 1.4] asking whether or not the Δ -primitive spectrum is a topological quotient of the maximal spectrum for any commutative differential algebra in the special case of complex affine Poisson algebra.

Proposition 6.6. *Let A be a complex affine Poisson algebra. Then $\text{P.prim } A$ is a topological quotient of $\max A$ via $\pi_m : \max A \rightarrow \text{P.prim } A$.*

Proof. By definition, it is routine to check that $\pi_m : \mathcal{A} \rightarrow \text{P.prin } A$ is a continuous surjective map. Now suppose $\pi_m^{-1}(X)$ is closed in \mathcal{A} for some subset $X \subseteq \text{P.prin } A$. In order to show that $\text{P.prin } A$ is a topological quotient of \mathcal{A} via π_m , we need to show that X is closed in $\text{P.prin } A$. We know there exists some ideal I of A such that $\pi_m^{-1}(X) = \{\mathfrak{m} \in \mathcal{A} \mid I \subseteq \mathfrak{m}\}$. It suffices to show that $X = \{\mathfrak{m} \in \text{P.prin } A \mid I \subseteq \mathfrak{m}\}$.

In one direction, take any $\mathfrak{q} \in \text{P.prin } A$ such that $I \subseteq \mathfrak{q}$. Write $\mathfrak{q} = \mathcal{P}(\mathfrak{m})$ for some $\mathfrak{m} \in \mathcal{A}$. It is clear that $I \subseteq \mathfrak{q} = \mathcal{P}(\mathfrak{m}) \subseteq \mathfrak{m}$. So $\mathfrak{m} \in \pi_m^{-1}(X)$ and $\mathfrak{q} = \pi_m(\mathfrak{m}) \in \pi_m(\pi_m^{-1}(X)) = X$. So $X \supseteq \{\mathfrak{m} \in \text{P.prin } A \mid I \subseteq \mathfrak{m}\}$.

In the other direction, take any $\mathfrak{q} \in X$. It is clear that $\pi_m^{-1}(\mathfrak{q}) = \mathcal{C}(\mathfrak{q}) \subseteq \pi_m^{-1}(X)$. By Lemma 6.5(i)&(ii), we get

$$\{\mathfrak{m} \in \mathcal{A} \mid \mathfrak{q} \subseteq \mathfrak{m}\} = \overline{\mathcal{C}(\mathfrak{q})} \subseteq \pi_m^{-1}(X) = \{\mathfrak{m} \in \mathcal{A} \mid I \subseteq \mathfrak{m}\}.$$

This implies that $I \subseteq \mathfrak{q}$ and $\mathfrak{q} \in \{\mathfrak{m} \in \text{P.prin } A \mid I \subseteq \mathfrak{m}\}$. So $X \subseteq \{\mathfrak{m} \in \text{P.prin } A \mid I \subseteq \mathfrak{m}\}$. This completes our proof. \square

Our following result gives another description of the Poisson Dixmier-Moeglin equivalence for any complex affine Poisson algebra A in terms of the symplectic core stratification on its maximal spectrum \mathcal{A} . It turns out that the Poisson Dixmier-Moeglin equivalence holds for A if and only if each symplectic core in \mathcal{A} is locally closed, or by definition the Poisson bracket of A is weakly algebraic.

Theorem 6.7. *Let A be a complex affine Poisson algebra. Then the following are equivalent.*

- (i) *A satisfies the Poisson Dixmier-Moeglin equivalence;*
- (ii) *$\mathcal{C}_{\text{spec}}(\mathfrak{p})$ is locally closed in $\text{spec } A$ for any $\mathfrak{p} \in \text{spec } A$;*
- (iii) *$\mathcal{C}(\mathfrak{m})$ is locally closed in $\max A$ for any $\mathfrak{m} \in \max A$.*

Proof. The equivalences come from Proposition 5.3 by considering the complex affine Poisson algebra A as a special commutative differential algebra (A, Δ_A) . In particular, for (iii) the condition $\overline{\mathcal{C}(\mathfrak{m})} = \{\mathfrak{q} \in \mathcal{A} \mid \mathcal{P}(\mathfrak{m}) \subseteq \mathfrak{q}\}$ is automatically satisfied by Lemma 6.5(i)&(ii). \square

Corollary 6.8. *Let A be a complex affine Poisson algebra. If the Poisson bracket of A is algebraic, then A satisfies the Poisson Dixmier-Moeglin equivalence.*

Proof. The result follows from Lemma 6.5 and Theorem 6.7. \square

Remark 6.9. In general, the inverse statement of Corollary 6.8 does not hold. The following example is taken from [Van96, Example 2.37] (also see [BG03, Remarks §3.6(1)]). Let $A = \mathbb{C}[x, y, z]$ with Poisson bracket given by $\{x, y\} = 0$, $\{x, z\} = \alpha x$ and $\{y, z\} = y$ for some $\alpha \in \mathbb{R}$. Then the symplectic leaves have small dimensions: the single points $(0, 0, c)$ for $c \in \mathbb{C}$ and the 2-dimensional symplectic leaves defined by the equation $xy^\alpha = \text{constant}$. If α is rational, then the leaves are algebraic; but when α is irrational, then the leaves are not algebraic varieties. On the other side, the symplectic cores are given by points $(0, 0, c)$ for $c \in \mathbb{C}$ and $(\mathbb{C}^\times)^2 \times \mathbb{C}$. So they are always locally closed.

Let G be an algebraic group. Suppose that G acts rationally by Poisson automorphisms on A . It is easy to check that G then acts rationally on the commutative differential algebra (A, Δ_A) where Δ_A can be chosen to be a finite set of Hamiltonian derivations $\{a_i, -\}_{1 \leq i \leq n}$ where $\{a_i\}_{1 \leq i \leq n}$ is a basis for some finite-dimensional G -invariant generating space of A . As a consequence, G acts on the symplectic leaves and symplectic cores of A , and we label the G -orbits of symplectic leaves (or symplectic cores) \mathcal{G}_i , so that we have a stratification

$$\mathcal{A} = \bigsqcup \mathcal{G}_i,$$

where $\mathcal{G}_i = \sqcup_{g \in G/\text{stab}(\mathcal{L})} g \cdot \mathcal{L}$ for some symplectic leaf (or symplectic core) \mathcal{L} of \mathcal{A} . The following result is a G -equivariant version of [BG03, Proposition 3.7].

Theorem 6.10. *Let G be an algebraic group acting rationally by Poisson automorphism on a complex affine Poisson algebra A . If the G -equivariant stratification of A into G -orbits of symplectic leaves (or symplectic cores) is finite, then A satisfies the Poisson Dixmier-Moeglin equivalence.*

Proof. According to Lemma 6.5(iii), any G -orbit of symplectic cores is a disjoint union of G -orbits of symplectic leaves. So if there are only finitely many G -orbits of symplectic leaves, there are only finitely many G -orbits of symplectic cores. So without loss of generality, we may assume that there are only finitely many G -orbits of symplectic cores, which are listed as follows

$$\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_t.$$

We argue by noetherian induction to prove that any symplectic core contained in \mathcal{G}_i is locally closed. We may assume that any symplectic core contained in \mathcal{G}_i , where $\mathcal{G}_i \neq \mathcal{A}$, is locally closed. Without loss of generality, suppose $\mathcal{G}_1, \dots, \mathcal{G}_n$ are the symplectic cores whose Zariski closure is exactly \mathcal{A} . Write $X = \sqcup_{1 \leq i \leq n} \mathcal{G}_i$. We can assume that \mathcal{A} is irreducible. We claim that X is a non-empty open set of \mathcal{A} . Notice that we can prove a G -equivariant version of Lemma 6.5(iv) such that each $\overline{\mathcal{G}_i}$ is a disjoint union of G -orbits of symplectic cores contained in it. Suppose $\mathcal{G}_i = \bigcup_{g \in G} g \cdot \mathcal{C}$ for some symplectic core \mathcal{C} . By Lemma 6.5, we know $\mathcal{C} = \{\mathfrak{m} \in \mathcal{A} \mid \mathfrak{p} \subseteq \mathfrak{m}\}$ for some Poisson primitive ideal \mathfrak{p} . Then it is easy to see that, for any $g \in G$, we have $g \cdot \mathcal{C} = \{\mathfrak{m} \in \mathcal{A} \mid g(\mathfrak{p}) \subseteq \mathfrak{m}\}$. So

$$\overline{\mathcal{G}_i} = \overline{\bigcup_{g \in G} g \cdot \mathcal{C}} = \left\{ \mathfrak{m} \in \mathcal{A} \mid \left(\bigcap_{g \in G} g(\mathfrak{p}) \right) \subseteq \mathfrak{m} \right\} = \{ \mathfrak{m} \in \mathcal{A} \mid (\mathfrak{p} : G) \subseteq \mathfrak{m} \}.$$

Since $(\mathfrak{p} : G)$ is a Poisson G -ideal by Lemma 5.5(i), for any $\mathfrak{m} \in \mathcal{A}$, we know $\mathfrak{m} \in \overline{\mathcal{G}_i}$ if and only if $G \cdot \mathcal{C}(\mathfrak{m}) \subseteq \overline{\mathcal{G}_i}$. So $\overline{\mathcal{G}_i}$ is a disjoint union of G -orbits of symplectic cores. Therefore if $\overline{\mathcal{G}_i} \neq \mathcal{A}$, we have $\overline{\mathcal{G}_i}$ can not contain any $\mathcal{G}_1, \dots, \mathcal{G}_n$ whose closure is \mathcal{A} . Then we know

$$X = \mathcal{A} \setminus \left(\bigsqcup_{\overline{\mathcal{G}_i} \neq \mathcal{A}} \mathcal{G}_i \right) = \mathcal{A} \setminus \left(\bigcup_{\overline{\mathcal{G}_i} \neq \mathcal{A}} \overline{\mathcal{G}_i} \right).$$

Since there are only finitely many \mathcal{G}_i , we see X is a non-empty open set of \mathcal{A} . This proves our claim.

It remains to show that any symplectic core \mathcal{C} in X is locally closed. Since X is open, it suffices to show that $\mathcal{C} = \overline{\mathcal{C}} \cap X$. One direction $\mathcal{C} \subseteq \overline{\mathcal{C}} \cap X$ is clear. For the other direction, take any $\mathfrak{m} \in X$ whose rank is maximal among X , say $\text{rk}(\mathfrak{m}) = j$ and $\mathfrak{m} \in \mathcal{G}_1$. Let \mathcal{B} be the proper closed subvariety of \mathcal{A} consisting of points whose rank is strictly less than j . Since the action of G preserves the rank of point, it is clear that \mathcal{B} is G -invariant. So \mathcal{B} is a disjoint union of G -orbits of symplectic cores by [BG03, Proposition 3.6(1)]. Moreover since $\overline{\mathcal{G}_i} = \mathcal{A}$, we have $\mathcal{G}_i \not\subseteq \mathcal{B}$ for all $1 \leq i \leq n$. Then we get X is an irreducible open subset of \mathcal{A} on which the rank is constant and equal to j . As a consequence, the closure of any symplectic core contained in X is j -dimensional. In particular, \mathcal{C} is j -dimensional. But by Lemma 6.5, one can see that $\overline{\mathcal{C}} \setminus \mathcal{C}$ is a disjoint union of symplectic cores whose closures all have dimension strictly less than j . Hence we have

$$(\overline{\mathcal{C}} \cap X) \setminus \mathcal{C} \subseteq (\overline{\mathcal{C}} \setminus \mathcal{C}) \cap (X \setminus \mathcal{C}) = \emptyset.$$

So $\mathcal{C} = \overline{\mathcal{C}} \cap X$ is locally closed.

Finally, our result follows from Theorem 6.7. □

Now we give some examples by using our main results on stratifications of maximal spectrum of complex affine Poisson algebras.

Example 6.11. Let S be a three-dimensional Sklyanin algebra over the complex numbers. By definition, S is a connected graded noncommutative algebra generated by three indeterminants x, y, z and subject to three quadratic relations:

$$axy + byx + cz^2 = ayz + bzy + cx^2 = azx + bxz + cy^2 = 0$$

for some $[a : b : c] \in \mathbb{P}^2$ satisfying $abc \neq 0$ and $(3abc)^3 \neq (a^3 + b^3 + c^3)^3$. In [ATV90], it is shown that S has a central regular element g of degree three and $S/gS \cong B(E, \mathcal{L}, \sigma)$, the twisted homogeneous coordinate ring of the associated geometric data (E, \mathcal{L}, σ) of S . It is further shown that S is module-finite over its center if and only if $|\sigma| < \infty$, where σ is some translation on the elliptic curve E .

In the following, we assume that S is module-finite over its center Z and denote $n = |\sigma| < \infty$. We know by [WWY17, Proposition 5.5(1)] that there exists a unique nonzero Poisson structure (up to scalars) on Z if g is assumed to be in the Poisson center of Z . More precisely, according to [ST94], the center Z of S can be presented as

$$Z = \mathbb{C}[z_1, z_2, z_3, g]/(F),$$

where z_1, z_2, z_3 are three algebraically independent central elements of degree n and F is a single relation of degree $3n$. The Poisson bracket on Z is then determined (up to scalars) as

$$\{z_1, z_2\} = \partial_{z_3} F, \quad \{z_2, z_3\} = \partial_{z_1} F, \quad \{z_3, z_1\} = \partial_{z_2} F,$$

with g in the Poisson center of the Poisson algebra Z . As a consequence of [WWY17, Theorem 1.3(2)], we know the Poisson bracket of Z is algebraic. Moreover, the group $\Sigma := \mathbb{Z}_3 \times \mathbb{C}^\times$ acts on Z by Poisson automorphisms where \mathbb{Z}_3 permutes z_1, z_2, z_3 and \mathbb{C}^\times scales all the variables z_1, z_2, z_3, g simultaneously. It is shown also in [WWY17, Theorem 1.3(3)] that the Σ -orbits of symplectic cores (and hence symplectic leaves) are finite. Now by our Theorem 6.7 or Theorem 6.10 we know Z satisfies the Poisson Dixmier-Moeglin equivalence.

Example 6.12. Let $n \geq 2$ be an integer and consider $n - 2$ generic polynomials Q_i in \mathbb{C}^n with coordinates x_i , $i = 1, \dots, n$. For any polynomial $\lambda \in \mathbb{C}[x_1, \dots, x_n]$, we can define a bilinear differential operation

$$\{-, -\} : \mathbb{C}[x_1, \dots, x_n] \otimes \mathbb{C}[x_1, \dots, x_n] \rightarrow \mathbb{C}[x_1, \dots, x_n]$$

by the formula

$$\{f, g\} = \lambda \frac{df \wedge dg \wedge dQ_1 \wedge \dots \wedge dQ_{n-2}}{dx_1 \wedge dx_2 \wedge \dots \wedge dx_n}, \quad f, g \in \mathbb{C}[x_1, \dots, x_n].$$

This operation gives a Poisson algebra structure on $\mathbb{C}[x_1, \dots, x_n]$. The polynomials Q_i , $i = 1, \dots, n-2$ are Casimir functions for the Poisson bracket and any Poisson structure in \mathbb{C}^n with $n - 2$ generic Casimirs Q_i are written in this form.

The case $n = 4$ corresponds to the classical (generalized) Sklyanin quadratic Poisson algebra. The very Sklyanin algebra is associated with the following two quadrics in \mathbb{C}^4 :

$$\begin{aligned} Q_1 &= x_1^2 + x_2^2 + x_3^2, \\ Q_2 &= J_{12}x_1^2 + J_{23}x_2^2 + J_{31}x_3^2 + x_4^2, \end{aligned}$$

where we fix a lattice $\Lambda \subseteq \mathbb{C}$ and let $\{\theta_{ab} \mid ab = 00, 01, 10, 11\}$ denote Jacobi's four theta functions with period lattice Λ such that

$$J_{12} = \frac{\theta_{11}(\tau)^2 \theta_{01}(\tau)^2}{\theta_{00}(\tau)^2 \theta_{10}(\tau)^2}, \quad J_{23} = \frac{\theta_{11}(\tau)^2 \theta_{10}(\tau)^2}{\theta_{00}(\tau)^2 \theta_{01}(\tau)^2}, \quad J_{12} = -\frac{\theta_{11}(\tau)^2 \theta_{00}(\tau)^2}{\theta_{01}(\tau)^2 \theta_{10}(\tau)^2}, \quad \text{for some } \tau \in \mathbb{C}/\Lambda.$$

The Poisson brackets with $\lambda = 1$ between the affine coordinates are described as follows

$$\{x_i, x_j\} = (-1)^{i+j} \det \left(\frac{\partial Q_k}{\partial x_l} \right), \quad l \neq i, j \text{ and } i > j.$$

According to [OR02, Proposition 2.2], these Poisson brackets all have symplectic leaves of small dimensions 0 or 2. More precisely, 2-dimensional symplectic leaves are the smooth locus of the complete intersections $(Q_1 - \lambda_1, \dots, Q_n - \lambda_n)$ and the 0-dimensional symplectic singletons are the points in the singular locus of $(Q_1 - \lambda_1, \dots, Q_n - \lambda_n)$ and the hyperplane $\lambda = 0$ for all $(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$. As a consequence, these Poisson brackets are all algebraic and the corresponding Poisson algebras $\mathbb{C}[x_1, \dots, x_n]$ all satisfy the Poisson Dixmier-Moeglin equivalence.

ACKNOWLEDGMENTS

Part of this research work was done during the first and second authors' visit to Shanghai Center for Mathematical Sciences in June-July 2019. They are grateful for the invitation and wish to thank Fudan University for its hospitality. Juan Luo is supported by the NSFC (project 1190010260). Quanshui Wu is supported by the NSFC (project 11771085). The authors want to thank Ken Goodearl for some helpful comments including Remark 2.7.

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MATHEMATICS AND SCIENCE COLLEGE, SHANGHAI NORMAL UNIVERSITY, SHANGHAI 200234, CHINA

E-mail address: luojuan@shnu.edu.cn

DEPARTMENT OF MATHEMATICS, HOWARD UNIVERSITY, 2400 SIXTH ST. NW, WASHINGTON DC, 20059, USA

E-mail address: xingting.wang@howard.edu

SCHOOL OF MATHEMATICAL SCIENCES, FUDAN UNIVERSITY, SHANGHAI 200433, CHINA

E-mail address: qswu@fudan.edu.cn