PRIMITIVE IDEALS IN FINITE EXTENSIONS OF NOETHERIAN RINGS

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Introduction

In this paper connections between the primitive spectrum of a Noetherian k-algebra R, where k is a field, and the primitive spectra of suitably conditioned extension rings of R are presented. These results are then applied to obtain information about the graded-primitive ideals of the enveloping algebra of a finite dimensional Lie superalgebra ($\mathbb{Z}_2$-graded Lie algebra) over k.

We first consider the following situation. Let R be a Noetherian ring, and let S be a ring extension of R finitely generated as left and right R-module. Let P be a prime ideal of S, and let Q be an arbitrary minimal prime ideal of $R/(P \cap R)$. We prove in 1.4 that P is right primitive if and only if Q is right primitive.

The above conclusion is well known (for R and S not necessarily Noetherian) in the special case where the extension is assumed to be normalizing (see [7] for details). However, the proof in the normalizing setting depends on the following fundamental fact: If $R \hookrightarrow S$ is a finite normalizing extension—here again R is not necessarily Noetherian—then a simple right S-module is of finite length as right R-module. However, when R and S are as originally defined, a simple right S-module may be of infinite length as R-module. In [14], Stafford gives an example where R is an image of $U(\mathrm{sl}_2 \times \mathrm{sl}_2)$, where the extension ring S is finite and free as left and right R-module, and where there exists a simple right S-module which is not Artinian as R-module. (There further exists, for this extension, a simple right R-module which induces to an S-module of infinite length.) It follows therefore that any proof in the non-normalizing setting will be fundamentally different from that in the normalizing setting. The approach we take relies on facts about Noetherian bimodules; see [2] or [9] for required background.

Our second interest is in the stability, with respect to finite extensions, of certain characterizations of primitivity. To be more specific, let R be a k-algebra, and let P be a prime ideal of R. Consider the following properties: (A) *P is right primitive*, (B) *P is rational* (*that is, the centre of the classical quotient ring of R/P is algebraic over k*), *and* (C) *P is locally closed in the prime spectrum of R* (*that is, the intersection of those prime ideals properly containing P properly contains P*). By well-known work of Moeglin, Dixmier, and others (see [8] or [12]), these properties are equivalent if R is the enveloping algebra of a finite dimensional Lie algebra over k, when k has characteristic zero. (When k has positive characteristic, the equivalence follows from PI-theory.) Our concern is with the stability of the properties ((A)$\Leftrightarrow$(B)) and ((A)$\Leftrightarrow$(C)).

So let S be a k-algebra extension of R finitely generated as left and right R-module.

Received 12 November 1987.

1980 *Mathematics Subject Classification* (1985 *Revision*) 16A33.

We see in 1.5 that if R has either of the properties ((A) $\Rightarrow$ (B)) or ((B) $\Rightarrow$ (A)), then S has the same property. Further, we prove in 2.3, 2.4 that if S is a finite free extension of R (that is, S_R and ${}_RS$ are finitely generated free modules), and R has finite GK-dimension, then R has the property ((A) $\Rightarrow$ (C)) if and only if S does. (See **[10]** for basic information on GK-dimension.) It follows that if S is a finite free extension of the enveloping algebra of a finite dimensional k-Lie algebra, then S has the property ((A) $\Leftrightarrow$ (B) $\Leftrightarrow$ (C)).

We can now apply the above results. Suppose that the field k has characteristic zero, and let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a finite dimensional k-Lie superalgebra; see **[1, 13]** for background information. Let S now be the enveloping algebra of $\mathfrak{g}$, and let U be the enveloping algebra of the Lie algebra $\mathfrak{g}_0$. The extension $U \hookrightarrow S$ is finite and free, so the above mentioned results hold. However, we would like to obtain information pertinent to the graded representation theory of $\mathfrak{g}$. Our approach combines the above material with facts about prime ideals and semiprimitivity in group graded algebras; see **[4]** for details.

The k-algebra S has a $\mathbb{Z}_2$-grading induced from the grading on $\mathfrak{g}$, say $S = S_0 \oplus S_1$, and the graded representation theory of $\mathfrak{g}$ corresponds with that of S. Also, since S_0 is a finite free extension of U, the above results hold as well for the extension $U \hookrightarrow S_0$.

Let p be a graded prime ideal of S. We see in 3.1 that the following properties are equivalent: (i) *p is right graded-primitive*, (ii) *$p \cap S_0$ is the intersection of two (not necessarily distinct) primitive ideals of S_0*, (iii) *p is the intersection of two (not necessarily distinct) primitive ideals of S*, (iv) *the minimal prime ideals of $U/(p \cap U)$ are primitive*, (v) *p is left graded-primitive*.

The centre of the classical quotient ring of S/p will consist of the direct sum of one or two field extensions of k. We shall say that p is *rational* provided these field extensions are algebraic over k. We show in 3.2 that: *p is graded-primitive $\Leftrightarrow$ p is rational $\Leftrightarrow$ p is locally closed in the graded prime spectrum of S*. In other words, we obtain the natural generalization of ((A) $\Leftrightarrow$ (B) $\Leftrightarrow$ (C)).

Finally, we see in 3.3 that, if $\mathfrak{g}$ is nilpotent, then p is graded primitive if and only if it is a maximal graded ideal.

Section 1 deals with general finite extensions of Noetherian rings, Section 2 treats finite free extensions of Noetherian k-algebras of finite GK-dimension, and Section 3 considers the applications to enveloping algebras of Lie superalgebras. There is one preparatory lemma in Section 1 worth briefly mentioning here. In 1.3 it is shown that if S and T are prime Noetherian rings and there exists a torsion-free Noetherian S-T-bimodule, then T is right primitive if S is right primitive. This gives a partial answer to a question raised in **[9**, 5.2.15].

A word on notation. We denote the set of prime ideals of R by $\operatorname{Spec} R$, and the set of right primitive ideals of R by r-$\operatorname{Prim} R$. The Gelfand–Kirillov dimension of a module M is denoted by $\operatorname{GK}\dim M$. Also, a torsion-free finitely generated bimodule is torsion-free and finitely generated on each side.

This work was completed while the author was a graduate student at the University of Washington. The author is indebted to his advisor R. B. Warfield, Jr. for providing advice and encouragement, and to T. H. Lenagan for offering useful suggestions and uncovering errors in preliminary versions of this note. The author is also grateful to K. A. Brown, to R. S. Irving, and to the referee for their helpful comments.

1. *Finite extensions*

In this section we consider the setting where R and S are Noetherian rings and S is a ring extension of R finitely generated on one or both sides as R-module. Our approach utilizes connections between such extensions and torsion-free Noetherian bimodules. See [**2**] or [**9**] for background information.

The proof of the first lemma is a straightforward adaptation of [**15**, proof of 1.1].

LEMMA 1.1. *Let S be a prime Noetherian ring and let R be a Noetherian subring. Let Q be a minimal prime ideal of R. Then if S_R is finite, there exists a torsion-free finitely generated S-R/Q-bimodule. Moreover, this bimodule may be assumed to be a subfactor of S. (When ${}_RS$ rather than S_R is assumed to be finite, there exists a torsion-free finitely generated R/Q-S-bimodule subfactor of S.)*

Proof. Assume that S_R is finite, and let F be the classical quotient ring of S. Let $0 = F_0 \subset F_1 \subset \ldots \subset F_n = F$ be an F-R-bimodule composition series for F. Then for some i, the subfactor F_i/F_{i-1} has Q as right annihilator. Observe that F_i/F_{i-1} is either torsion or torsion-free as right R/Q-module. However, if the first case holds, since ${}_F(F_i/F_{i-1})$ is finitely generated, $(F_i/F_{i-1})_{R/Q}$ cannot be faithful. Hence F_i/F_{i-1} is torsion-free as right R/Q-module. Since it is immediate that F_i/F_{i-1} is torsion-free as left S-module, it is easy to see that $(F_i \cap S)/(F_{i-1} \cap S)$ is the desired S-R/Q-bimodule. The case when ${}_RS$ is finite is handled similarly.

In view of the above lemma, we may apply to finite extensions two results from [**2, 10**] concerning algebras bonded by a torsion-free Noetherian bimodule. The corollary we obtain will be needed later. (Consult [**10**] for basic details on GK-dimension.)

COROLLARY 1.2. *Let S be a prime Noetherian algebra over a field k, and let R be a Noetherian k-subalgebra of S. Suppose that either ${}_RS$ or S_R is finite, and let Q be a minimal prime ideal of R.*

(i) *If K is the centre of the classical quotient ring of S and F is the centre of the classical quotient ring of R/Q, then* $\operatorname{trdeg}_k K = \operatorname{trdeg}_k F$.

(ii) $\operatorname{GK}\dim(S) = \operatorname{GK}\dim(R/Q)$.

Proof. (i) This follows from 1.1 and [**2**, 2.7].

(ii) This follows from 1.1 and [**10**, 5.4].

The next lemma is crucial for all that follows and is also interesting in its own right.

LEMMA 1.3. *Let S and T be prime Noetherian rings, and let M be a torsion-free finitely generated S-T-bimodule. If S is right primitive then so is T.*

Proof. Let V be a faithful simple right S-module. Then $U = V_S \otimes M$ is a finitely generated right T-module which is non-zero by [**3**, 3.4]. Choose W_T to be a maximal proper submodule of U_T. If $(U/W)_T$ is faithful then the proof is complete. Thus suppose otherwise; this assumption will lead to a contradiction.

Let $I = \operatorname{ann}(U/W)_T$, and consider ${}_S(M/MI)_T$. To see that ${}_S(M/MI)$ is not faithful, note first that since I contains a regular element of T, ${}_SM$ embeds in ${}_SMI$.

Therefore, $K\dim_S(M/MI) < K\dim_S M = K\dim_S S$. (See [**11**] for details on left Krull dimension.) Hence ${}_S(M/MI)$ is torsion by [**11**, 6.3.11]. That ${}_S(M/MI)$ is not faithful now follows easily since $(M/MI)_T$ is finitely generated.

Hence let $J = \operatorname{ann}_S(M/MI)$; we are assuming that $J \neq 0$. Observe that $VJ = V$ and $JM \subseteq MI$. Thus

$$U = V \otimes_S M = \sum (v \cdot j \otimes m) = \sum (v \otimes j \cdot m) \subseteq \sum (v \otimes m \cdot i) = U \cdot I < U,$$

where the above sums are taken over all $v \in V$, $j \in J$, $m \in M$, and $i \in I$. The lemma follows from this contradiction.

The following is now easily deduced from the above lemmas.

THEOREM 1.4. *Let R be a Noetherian ring and let S be a ring extension of R finitely generated on both sides as R-module. Let P be a prime ideal of S, and let Q be a minimal prime ideal of $R/(P \cap R)$. Then P is right primitive if and only if Q is right primitive.*

We conclude this section with two corollaries to the above results. We first turn our attention to rational prime ideals and primitivity. For the rest of this section we let k be a field, and we recall that a prime ideal Q of a k-algebra R is *rational* provided the centre of the classical quotient ring of R/Q is algebraic over k.

COROLLARY 1.5. *Let R be a Noetherian k-algebra and let S be a k-algebra extension of R finitely generated on both sides as R-module. If R satisfies either of the properties,*

(i) *every right primitive ideal is rational,*

(ii) *every rational prime ideal is right primitive,*

then S satisfies the same property.

Proof. Let P be a prime ideal of S and let Q be a prime ideal of R which is minimal over $P \cap R$. By 1.2(i), P is rational if and only if Q is rational. The result now follows from 1.4.

The next result shows that a basic property of the enveloping algebra of a finite dimensional nilpotent Lie algebra extends to a finite extension.

COROLLARY 1.6. *Let R and S be a Noetherian k-algebras of finite* GK*-dimension, and let S be a k-algebra extension of R. If S_R is finite and R satisfies the property that every right primitive ideal of R is maximal, then every right primitive ideal of S is maximal. In particular, if R is assumed to be the enveloping algebra of a finite dimensional nilpotent k-Lie algebra then every right primitive ideal of S is maximal.*

Proof. Let P be a right primitive ideal of S. Then the prime ideals of R minimal over $P \cap R$ are right primitive and hence maximal, by 1.1, 1.3 and our assumptions on R. Suppose that I is an ideal of S such that $I > P$ and $I \cap R \subseteq Q$, for some prime ideal Q of R minimal over $P \cap U$. Thus $\operatorname{GK}\dim(R/(I \cap R)) \geqslant \operatorname{GK}\dim(R/Q)$. (See [**10**].) But $\operatorname{GK}\dim(R/I(\cap R)) = \operatorname{GK}\dim(S/I)$ by [**10**, 5.5], and

$$\operatorname{GK}\dim(S/I) < \operatorname{GK}\dim(S/P) = \operatorname{GK}\dim(U/Q),$$

by 1.2(ii) and [**10**, 3.16]. This contradiction implies, for any ideal I of S with $I > P$, that $I \cap R$ lies inside no maximal ideal of R. It is now easy to see that P must be maximal.

2. *Finite free extensions with finite* GK-*dimension*

Let k be a field. Throughout this section R and S will be Noetherian k-algebras of finite GK-dimension (see [10] for required background), with S being a k-algebra extension of R. Adopting the terminology of normalizing extensions (see, for example, [7]) we shall say that a prime ideal P of S *lies over* a prime ideal Q of R provided that Q is a minimal prime ideal over $P \cap R$.

We first record an unpublished observation of T. H. Lenagan; the author is grateful to him for allowing the inclusion here of this result. The proof provided preserves the essence of the original.

THEOREM 2.1. *Suppose that S is finite and free as either left or right R-module. Then for each prime ideal Q of R there exists a prime ideal P of S lying over Q.*

Proof. Assume that S_R is finite and free, and consider ${}_S(S/SQ)_R$. It is faithful as right R/Q-module. Let $J = \mathrm{ann}_S(S/SQ)$. Then $J \cap R \subseteq Q$, since $J \subseteq SQ$ and $SQ \cap R = Q$. Also, $\mathrm{GK}\dim(R/Q) = \mathrm{GK}\dim(S/J) = \mathrm{GK}\dim(R/(J \cap R))$, by [10, 5.3, 5.4, 5.5]. Thus Q is minimal over $J \cap R$ by [10, 3.16]. Now let P be an ideal of S maximal such that P contains J and $P \cap R \subseteq Q$. It is easily checked that P is a prime ideal lying over Q.

The case when ${}_RS$ is finite and free follows similarly.

REMARK. Observe that in the proof of 2.1, the hypothesis that S_R is free is used only to ensure that $SI \cap R = I$ for every ideal I of R. This last condition will hold under an assumption weaker than the assumption that S_R is free; see [6] for details.

In view of 1.4, we now have the following.

PROPOSITION 2.2. *Suppose that S is finite as left and right R-module and is free on one side. Let Q be a prime ideal of R, and choose P to be a prime ideal of S lying over Q. Then Q is right primitive if and only if P is right primitive.*

Recall that a prime ideal Q of R is *locally closed* in $\mathrm{Spec}\, R$ provided that the intersection of those prime ideals properly containing Q properly contains Q.

THEOREM 2.3. *Suppose that ${}_RS$ is finite and free. Let Q be a prime ideal of R. If the prime ideals of S lying over Q are locally closed in* $\mathrm{Spec}\, S$, *then Q is locally closed in* $\mathrm{Spec}\, R$. *Moreover, if every element of r-*Prim S *is locally closed in* $\mathrm{Spec}\, S$, *then the same holds true for r-*Prim R.

Proof. Assume that the prime ideals lying over Q are locally closed in $\mathrm{Spec}\, S$. To prove that Q is locally closed in $\mathrm{Spec}\, R$, it suffices to find an ideal I of S such that

(i) $I \cap R \nsubseteq Q$,

(ii) $\forall Q' \in \mathrm{Spec}\, R : Q' > Q \Rightarrow Q' \supseteq I \cap R$.

We first find a convenient factor ring of S. Let $M = {}_R(S/QS)_S$, and let $J = \mathrm{ann}\, M_S$. As in the proof of 2.1, Q is a minimal prime ideal over $J \cap R$. The desired factor ring will be S/J, as we shall see.

Observe that since ${}_RS$ is free, ${}_RM \cong {}_R(\bigoplus_{i=1}^{t} R/Q)$ for some t. In particular, M is torsion-free as left R/Q-module. Thus let F be the classical quotient ring of R/Q, and denote by M' the F-S-bimodule $F \otimes_R M$. Identify ${}_RM$ with its natural embedding in M'. Now let

$$0 = M'_0 \subset M'_1 \subset \ldots \subset M'_n = M'$$

be an F-S-bimodule composition series for M'. By letting ${}_R[M_i]_S = {}_R[M'_i \cap M]_S$, for $1 \leqslant i \leqslant n$, we obtain a series $0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$, as in the proof of 1.1, of R/Q-S-bimodule subfactors of M such that for each i there exists a prime ideal P_i which makes M_i/M_{i-1} a torsion-free Noetherian R/Q-S/P_i-bimodule. Note that since M is faithful as right S/J-module, $\{P_i\}$ contains all the prime ideals of S minimal over J. Also note that $\operatorname{GK}\dim(S/P_i) = \operatorname{GK}\dim(R/Q)$, for each i, by [**10**, 5.3, 5.4].

Now let Q' be a prime ideal of R such that $Q' > Q$. Letting $J' = \operatorname{ann}(S/Q'S)_S$, we see as before that Q' is minimal over $J' \cap R$. Thus, if the ideal P' of S is chosen to be maximal such that $P' \supseteq J'$ and $P' \cap R \subseteq Q'$, then P' lies over Q'. Further, $P' \supseteq J' \supseteq J$, and since, by 1.2(ii) and [**10**, 3.16],

$$\operatorname{GK}\dim(S/P') = \operatorname{GK}\dim(R/Q') < \operatorname{GK}\dim(R/Q),$$

we see from the previous paragraph that P' is not minimal over J.

Therefore, let $\bar{S} = S/J$, and let $\bar{R}$ and $\bar{Q}$ be the images in $\bar{S}$ of R and Q respectively. Note that $\bar{Q}$ is a minimal prime ideal of $\bar{R}$. Also, for any prime ideal $\bar{Q}' > \bar{Q}$, there exists a non-minimal prime ideal $\bar{P}'$ of $\bar{S}$ such that $\bar{P}'$ lies over $\bar{Q}'$. Let

$$\bar{I} = \bigcap\{P \in \operatorname{Spec}\bar{S} \mid P \text{ is not minimal}\}.$$

To prove the proposition, it now suffices to show that $\bar{I} \cap \bar{R} \nsubseteq \bar{Q}$. For convenience, we now redefine R, S, I, and Q to be $\bar{R}$, $\bar{S}$, $\bar{I}$, and $\bar{Q}$ respectively. Note however that the module ${}_RS$ need no longer be free.

Let $\{P_i\}$ now be the set of minimal primes of S, and let

$$I_i = \bigcap\{P \in \operatorname{Spec} S \mid P > P_i\}.$$

Note that $\bigcap_i I_i$ is equal to the above ideal I of S. Hence, since Q is prime, it suffices now to show that $I_i \cap R \nsubseteq Q$ for each i.

Therefore, suppose that $I_i \cap R \subseteq Q$ for some i. Then $P_i \cap R \subseteq Q$, and P_i must lie over Q since Q is minimal. Thus, P_i is locally closed by assumption. Hence $I_i > P_i$, and $\operatorname{GK}\dim(S/I_i) < \operatorname{GK}\dim(S/P_i)$. But $\operatorname{GK}\dim(S/I_i) \geqslant \operatorname{GK}\dim(R/Q)$, since $I_i \cap R \subseteq Q$, and we saw above that $\operatorname{GK}\dim(R/Q) = \operatorname{GK}\dim(S/P_i)$. This contradiction implies that $I \cap R \nsubseteq Q$, and therefore Q is locally closed in $\operatorname{Spec} R$. The rest of the result follows easily from 1.1, 1.3, and 2.1.

THEOREM 2.4. *Suppose that S is a finite and free right R-module. If every element of r-Prim R is locally closed in* Spec R, *then the same holds true for r-Prim S.*

Proof. Assume the hypothesis on R and suppose that $\{s_1, \ldots, s_n\}$ is a free basis for S_R. Let S embed in $\operatorname{End}(S_R) \cong M_n(R)$ via left multiplication. We have ${}_SM_n(R) \cong {}_S\operatorname{End}(S_R) \cong \bigoplus_{i=1}^n({}_S\operatorname{Hom}(R_R, S_R)) \cong \bigoplus_{i=1}^n({}_SS)$. Since it is clear that the right primitive ideals of $M_n(R)$ are locally closed, the result now follows from 2.3.

We shall say that a ring extension $R \hookrightarrow S$ is a *finite free* ring extension provided that ${}_RS$ and S_R are finite free modules.

For the remainder of this section, choose k to be of characteristic zero. Let $\mathfrak{g}$ be a finite dimensional Lie algebra over k, and let U be the enveloping algebra of $\mathfrak{g}$. Recall that in U a prime ideal is right primitive if and only if it is rational, and if and only if it is locally closed in Spec U. We close this section by showing that this property of U extends to a finite free extension.

We first require a lemma which is in fact a special case of [**5**, Theorem 1]. We include a proof here for completeness.

LEMMA 2.5. *Let R and S be Noetherian rings (not necessarily of finite* GK-*dimension) such that S is a ring extension of R finitely generated as right R-module. Then S is a Jacobson ring if R is a Jacobson ring. In particular, if $R = U$ then S is a Jacobson ring.*

Proof. Assume that in R every prime ideal is semiprimitive. Let P be a prime ideal of S. By 1.1, there exists a torsion-free finitely generated S/P-R/Q-bimodule, for some prime ideal Q of U. But P must then also be semiprimitive by [**9**, 5.2.15], and the lemma follows.

THEOREM 2.6. *Let S be a finite free extension of U. Then given a prime ideal P of S, the following conditions are equivalent:*
(i) *P is right primitive,*
(ii) *P is rational,*
(iii) *P is locally closed in* Spec S,
(iv) *P is left primitive.*

Proof. ((i)$\Leftrightarrow$(ii)) This follows from 1.5.
((i)$\Rightarrow$(iii)) This follows from 2.4.
((iii)$\Rightarrow$(i)) This follows from 2.5.
((iv)$\Leftrightarrow$(i)) This follows by symmetry.

3. *Applications to Lie superalgebras*

In this section k is a field of characteristic zero, and $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a finite dimensional Lie superalgebra over k. (See [**1**] or [**13**] for required background.) Let S be the enveloping algebra of $\mathfrak{g}$, and let U be the enveloping algebra of the Lie algebra $\mathfrak{g}_0$. Since S is a finite free extension of U, all the results of Sections 1 and 2 are applicable.

Recall that S has a $\mathbb{Z}_2$-grading induced from the grading of $\mathfrak{g}$; say $S = S_0 \oplus S_1$. The graded representation theory of S corresponds with that of $\mathfrak{g}$. Also, since S_0 is a finite free extension of U, the first two sections apply to the extension $U \hookrightarrow S_0$.

Define a graded prime ideal of S to be *right graded-primitive* if it is the annihilator of an irreducible graded right S-module. The next two results help describe the graded-primitive ideals of S. Our approach combines the material of Sections 1 and 2 with facts about group graded algebras (see [**4**]).

Observe that in light of 2.6, we may refer to right or left primitive ideals of S_0 or S as simply being *primitive*.

THEOREM 3.1. *Let p be a graded prime ideal of S. Then the following conditions are equivalent:*
(i) *p is right graded-primitive,*
(ii) *$p \cap S_0$ is the intersection of two (not necessarily distinct) primitive ideals of S_0,*
(iii) *p is the intersection of two (not necessarily distinct) primitive ideals of S,*
(iv) *the minimal prime ideals of $U/(p \cap U)$ are primitive,*
(v) *p is left graded-primitive.*

Proof. ((i) ⇒ (ii)) Let $V = V_0 \oplus V_1$ be a simple right S-module with $\operatorname{ann}(V_S) = p$. It is immediate that V_0 and V_1 are simple S_0-modules. Hence if $Q_0 = \operatorname{ann}(V_0)_{S_0}$ and $Q_1 = \operatorname{ann}(V_1)_{S_0}$, then Q_0 and Q_1 are primitive and $p \cap S_0 = Q_0 \cap Q_1$.

((ii) ⇒ (iii)) Let $p \cap S_0 = Q_0 \cap Q_1$, with Q_0 and Q_1 primitive. Without loss of generality, Q_0 and Q_1 are both minimal over $p \cap S_0$. By [**4**, 6.3], p is the intersection of two prime ideals P_α and P_β which are both minimal over p. By [**4**, 7.3], both P_α and P_β lie over Q_0 and Q_1. Hence P_α and P_β are primitive by 1.4.

((iii) ⇒ (i)) Assume that p is the intersection of two primitive ideals, say P_α and P_β. Hence by [**4**, 4.4], p is an intersection of right graded-primitive ideals. Also by 2.6, P_α and P_β are locally closed in Spec S. Hence by [**4**, 6.3], p is locally closed in the graded prime spectrum of S. Thus p itself must be right graded-primitive.

((iii) ⇒ (iv)) Suppose $p = P_\alpha \cap P_\beta$, with P_α and P_β both primitive ideals of S. Let Q be a prime ideal of U minimal over $p \cap U$. Then one of P_α, P_β lies over Q. Hence Q must be primitive by 1.4.

((iv) ⇒ (ii)) Suppose that the minimal prime ideals of $U/(p \cap U)$ are primitive. Let Q_0 and Q_1 be the prime ideals of S_0 minimal over $p \cap S_0$. Let I be a prime ideal of U minimal over $p \cap U$. Then some prime ideal P of S minimal over p lies over I. By 1.4, P must be primitive. By [**4**, 6.3, 7.3], P lies over both Q_0 and Q_1. Hence Q_0 and Q_1 are both primitive by 1.4.

((v) ⇔ (i)) This follows by symmetry.

Now let p be a graded prime ideal of S. The centre of the classical quotient ring of S/p will consist of the direct sum of one or two field extensions of k, by [**4**, 6.3]. We shall say that p is *rational* provided these field extensions are algebraic over k.

THEOREM 3.2. *Let p be a graded prime ideal of S. Then the following conditions are equivalent:*

(i) *p is (right) graded-primitive,*

(ii) *p is rational,*

(iii) *p is locally closed in the graded prime spectrum of S.*

Proof. ((i) ⇔ (ii)) This follows immediately from 3.1 ((i) ⇔ (iii)).

((iii) ⇒ (i)) Since p is semiprime by [**4**, 5.5], it is semiprimitive by 2.5. Hence by [**4**, 4.4], p is an intersection of graded-primitive ideals. Thus (iii) implies (i).

((i) ⇒ (iii)) Suppose that p is graded-primitive. Then by 3.1, p is an intersection of two primitive ideals of S, say P_α and P_β. Since P_α and P_β will be locally closed in Spec S, it follows from [**4**, 6.3] that p is locally closed in the graded prime spectrum of S.

We close with the following.

PROPOSITION 3.3. *Suppose that $\mathfrak{g}$ is nilpotent. Then the graded-primitive ideals of S are precisely the maximal graded ideals.*

Proof. This follows from 3.1, 1.6, and [**4**, 6.3].

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