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Primitive Ideals of Certain Noetherian Algebras*

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1. Introduction

Let A be a finitely-generated, noetherian algebra over a field K, and consider the following conditions on a prime ideal P of A:

(1) P is primitive.

(2) The center of the quotient ring $Q(A/P)$ is an algebraic field extension of K.

(3) P is a G-ideal; that is, the prime ideals properly containing P intersect in an ideal properly containing P.

(3′) A/P has a countable separating set of ideals: that is, there is a sequence of ideals $I_1, I_2, \ldots$ properly containing P such that if J is an ideal properly containing P, then J contains I_n for some n.

In this paper, we study various restrictions on A which imply that some of these conditions are equivalent. Before describing our results, let us review some background. We note that for A noetherian, the quotient ring of a prime image of A always exists by Goldie's theorem, a fact which we use freely without further comment [5].

The first result of the type which concerns us is a theorem of Dixmier [1, Theorem 4.5.7].

Theorem 1.1. *Let K be a field of characteristic* 0 *and let A be the enveloping algebra over K of a finite-dimensional, solvable Lie algebra. For a prime ideal P of A, the conditions* (1), (2), *and* (3) *are equivalent.*

This theorem characteristizes primitive ideals via information on the center [(2)], and also geometrically, since (3) says that the point corresponding to P in the prime spectrum of A is locally closed. [The set Spec$A$ of prime ideals carries the Jacobson topology, in which the closed sets are those of the form

$$V(I)=\{P\in \operatorname{Spec} A : P\supset I\}.$$

The set of primitive ideals, Priv$A$, is also a topological space with the subspace topology.]

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A similar result has been obtained for certain group rings through the work of Zalesskii, Lorenz, and Passman [15–17, 21]:

Theorem 1.2. *Let A be the group ring of a finitely-generated nilpotent-by-finite group over a field K. For a prime ideal P of A, the conditions (1), (2), and (3) are equivalent, and hold if and only if P is maximal.*

Thus the primitive ideals are the closed points of $\operatorname{Spec} A$.

The implications (3)→(1)→(2) are known to hold for the enveloping algebra of any finite-dimensional Lie algebra and the group ring of any polycyclic-by-finite group [3, 16, 19]. However, Lorenz has constructed a polycyclic group ring for which (2) is satisfied and (3) fails, over non-absolute fields. (Recall that an absolute field is a field which is an algebraic extension of a finite field.) It is unknown whether (2) implies (3) for arbitrary enveloping algebras. Condition (3′) may be viewed as a substitute for the stronger condition (3), and the following results are known:

Theorem 1.3 (Dixmier [2]). *Let K be an uncountable algebraically closed field of characteristic 0 and let A be the enveloping algebra of a finite-dimensional Lie algebra. Then conditions* (1), (2), *and* (3′) *are equivalent for prime ideals of A.*

Theorem 1.4 (Farkas, Lorenz [4, 14]). *Let K be an uncountable field and G a polycyclic group. Then conditions (1), (2), and (3′) are equivalent for prime ideals of $K[G]$.*

Thus conditions (1) and (2) are equivalent for a wider class of algebras, provided we restrict to uncountable fields. Some restriction on the field is required. For by Roseblade's theorem [19], if K is an absolute field and G is a polycyclic group, then KG cannot be primitive, but $Q(K[G])$ may well have center K. Conversely, we have constructed an example of a primitive noetherian algebra over an absolute field K, with center transcendental over K [8]. In addition, (0) is not a G-ideal in this algebra.

In Sect. 2 of this paper, we prove that condition (2) implies (3′) for many algebra, including finitely-presented noetherian algebras. This follows from a base field extension theorem, which also has as a consequence that if A is a primitive Goldie algebra over K and the center of $Q(A)$ is K, then $A \otimes_K L$ is primitive for any field extension L. In addition, the base field extension theorem allows us to partially extend Theorem 1.3, showing that conditions (1) and (2) are equivalent as long as K has infinite transcendence degree.

The remainder of the paper is devoted to proving analogues of the theorems described earlier for certain families of iterated Ore extensions. Given a ring R together with an automorphism φ and a φ-derivation δ, we may form the *Ore extension* $S=R[x;\varphi,\delta]$, consisting of polynomials in x over R which satisfy the rule

$$rx = x\varphi(r) + \delta(r)$$

for all r in R. Let us call an algebra an *iterated Ore extension* if it is obtained from the base field by a finite number of such Ore extensions. In case each automorphism φ is the identity, call the resulting ring a *differential operator ring.*

Alternatively, if each derivation $\delta=0$, we can modify the procedure by adjoining x^{-1} simultaneously with respect to φ^{-1}, and we will call the resulting ring a *twisted Laurent extension.* Observe that enveloping algebras of solvable Lie algebras are examples of differential operator rings, and group rings of polycyclic groups are twisted Laurent extensions.

We have proved [7] that for a prime ideal of an iterated Ore extension, (3)→(1)→(2). In Sect. 4, we show that Dixmier's proof of the implication (2)→(3) in Theorem 1.1 can be imitated to obtain a proof for certain differential operator rings. The key to this procedure is the use of results of Jordan on prime ideals in Ore extensions with respect to a derivation [13]. Goldie and Michler proved the analogous results for Ore extensions with respect to automorphisms [6], and this allows us to prove that (2) implies (3) for certain twisted Laurent extensions. Actually, we prove that a prime ideal P satisfying (2) is maximal. Thus we obtain what could be considered a uniform proof of Theorems 1.1 and 1.2, via Ore extension methods.

Farkas used the work of Goldie and Michler to prove that (3′) implies (1) for arbitrary twisted Laurent extensions over uncountable fields. In Sect. 3, we show that Farkas' argument can be modified, using the work of Jordan, to prove this implication for arbitrary differential operator rings. Since Sect. 2 yields the implication (2)→(3′), we thereby obtain the equivalence of these three conditions for such algebras, over uncountable fields, and the equivalence of (1) and (2) over countable algebraically closed fields of infinite transcendence degree.

We wish to thank the referee for several valuable suggestions.

2. An Intersection Theorem

Theorem 2.1. *Let F be a field and A an algebra over F which is prime and right Goldie. Let L be the center of the quotient ring $Q(A)$, and let K be a field extension of F for which $L\otimes_F K$ is a field. Then any non-zero ideal of $A\otimes_F K$ intersects A in a non-zero ideal. Moreover, if A is left or right primitive, so is $A\otimes_F K$.*

Remark. The condition of the theorem is satisfied for any field extension K if $L=F$. In addition, it is satisfied if L is separable algebraic over F and K is separably algebraically closed, or if L is purely inseparable and K is separable [12].

Proof. Denote $A\otimes_F K$ by A_K, and let $\{c_i\}$ be a basis of K over F. Then the $\{c_i\}$ also form a basis of A_K as a free A-module.

Let I be a non-zero ideal of A_K. Every non-zero element of I is a unique K-linear combination of elements in A, with coefficients among the c_i, and we may choose n to be the fewest number of c_i's required for any such expression. With a suitable ordering of the c_i's, there is an element $x\neq 0$ in I of the form

$$x=c_1a_1+\ldots+c_na_n,$$

with $a_i\in A$. Consider the set of elements

$$\{a\in A: I \text{ contains an element } c_1a+c_2a_2+\ldots+c_na_n\}.$$

This forms a non-zero ideal in A. Since A is a prime Goldie ring, some element of the set is regular, so we may assume a_1 is regular in the expression above for x. If $n=1$, then $a_1 \in I$ and $I \cap A \neq (0)$, so we will assume that $n>1$.

To any element $a \in A$ we may associate the derivation d of A defined by

$$d(z)=za-az.$$

Fix such an a and d. By the Ore condition, there exist elements e and f in A with

$$a_1 e=d(a_1)f,$$

and f regular. Consider the element

$$xe-d(x)f=\sum_{i=1}^{n} c_i(a_i e-d(a_i)f).$$

This lies in I, and involves fewer than n terms, so must equal 0. Therefore

$$a_i e=d(a_i)f \tag{1}$$

for all $i \leqq n$.

The derivation d extends to $Q(A)$ by the usual quotient rule: $d(b^{-1}) = -b^{-1}d(b)b^{-1}$. Using this, for $i>1$ we obtain

$$d(a_i a_1^{-1})=d(a_i)a_1^{-1}-a_i a_1^{-1}d(a_1)a_1^{-1}.$$

Replacing $a_1^{-1}d(a_1)$ by ef^{-1} yields

$$d(a_i a_1^{-1})=d(a_i)a_1^{-1}-a_i e f^{-1}a_1^{-1},$$

so that

$$d(a_i a_1^{-1})a_1 f=d(a_i)f-a_i e,$$

which equals 0 by (1). Since a_1 and f are regular, we conclude that $d(a_i a_1^{-1})=0$, or $a_i a_1^{-1}$ commutes with a. The choice of a was arbitrary, so $a_i a_1^{-1}$ lies in the center L of $Q(A)$.

The element $xa_1^{-1}=c_1+c_2a_2a_1^{-1}+\ldots+c_n a_n a_1^{-1}$ is thus an element of $L\otimes_F K$. By assumption, this is a field, and xa_1^{-1} has an inverse y in $L\otimes_F K$. We may choose a common denominator $b \in A$ for the elements of L involved in y, so that since y is central in A_K, we have

$$yAb \subset A_K.$$

It follows from this and the centrality of xa_1^{-1} that

$$a_1 Ab=a_1 x a_1^{-1} yAb=xyAb \subset xA_K \subset I,$$

which implies that $a_1 Ab \subset I \cap A$. But A is prime, so that $a_1 Ab$ is non-zero, and $I \cap A$ is non-zero.

The primitivity result follows easily. Let $\mathbf{m} \subset A$ be a maximal right ideal which contains no non-zero ideal. Then $\mathbf{m}A_K$ is a proper right ideal of A_K, and we may choose a maximal right ideal $M \supset \mathbf{m}A_K$ in A_K. The A_K-module A_K/M is simple, and contains $A/\mathbf{m}$ as an A-submodule. Let I be the annihilator of A_K/M. Then $I \cap A$ annihilates $A/\mathbf{m}$, which is faithful by assumption, so $I \cap A = (0)$. The first part of the theorem implies that $I = (0)$ and A_K is right primitive. The symmetric argument applies for left primitivity.

Remarks. (i) The referee and Martha Smith both indicated that my original assumptions on L in Theorem 2.1 are unnecessary. I have followed the latter's suggestions in the penultimate paragraph of the proof.

(ii) Smith also has extended Theorem 2.1 to any prime algebra A, provided we let L be the extended centroid [20]. In case the extended centroid $L = F$, Resco has shown that if A is primitive, so is $A \otimes_F B$ for any primitive algebra B [18].

Example. Let $\mathfrak{g}$ be the infinite-dimensional Lie algebra over a field K of characteristic 0 with basis $x, y_1, y_2, \dots$ and relations

$$[y_i, y_j] = 0 \quad \text{and} \quad [y_i, x] = y_{i+1}.$$

Let A be the enveloping algebra of $\mathfrak{g}$. Then A is an Ore domain, the center of $Q(A)$ is K, and A is primitive if K is countable [11]. By Theorem 2.1, we find that A is primitive over any field, and has a countable separating set of ideals. Actually Theorem 2.1 shows more – the proof that A is primitive in [11] involved the construction of a simple, faithful module, and 2.1 implies that this construction works over any field.

As a consequence of 2.1, we obtain the results described in the introduction.

Theorem 2.2. *Let B be a countably-presented, right noetherian algebra over a field K. Let P be a prime ideal of B such that the center Z of $Q(B/P)$ is algebraic over K. Then B/P has a countable separating set of ideals.*

Proof. The ideal P is finitely-generated, so B/P is countably-presented. Let F be the subfield of K generated over the prime field by all the elements of K involved in the defining relations for B and P, and let A be the F-subalgebra of B/P which has the same set of generators as B. Then A is countable and $B/P = A \otimes_F K$.

We claim that A and K satisfy the conditions of Theorem 2.1. The primeness of A follows immediately from A_K being prime, and regular elements of A remain regular in A_K by the freeness of A_K over A. Thus A has a right quotient ring and is right Goldie. Let L be the center of $Q(A)$. Then $L \otimes_F K$ is a subring of Z, an algebraic field extension of K, which implies that $L \otimes_F K$ is a field.

Thus, applying 2.1, we find that any non-zero ideal of B/P contains a non-zero element of A. Since A is countable, the ideals generated in B/P by the non-zero elements of A form a countable separating set.

Proposition 2.3. *Let F be an algebraically closed countable field of infinite transcendence degree and A a prime F-algebra which is finitely-presented and right Goldie with F the center of $Q(A)$. If $A \otimes_F K$ is right or left primitive for some field extension K, then so is A.*

Proof. Since A is defined by a finite number of relations, there is a subfield $F_0 \subset F$ which is finitely generated over the prime field and an F_0-algebra A_0 such that

$A = A_0 \otimes_{F_0} F$. Let $\mathbf{m}$ be a right ideal of $A \otimes_F K$ which is comaximal with every non-zero two-sided ideal. Then for every a in A_0, the right ideal $\mathbf{m}$ contains some element p_a in

$$1 + (A_0 \otimes_{F_0} K) a (A_0 \otimes_{F_0} K).$$

The algebra A_0 is countable, so we may choose a countable extension $F_1 \supset F_0$ in K which contains all the elements of K involved in the family $\{p_a : a \in A_0\}$. Let $A_1 = A_0 \otimes_{F_0} F_1$. Then $\mathbf{m} \cap A_1$ is comaximal with every ideal of A_1 generated by an element of A_0.

Similarly, we can find a sequence of countable extensions $F_1 \subset F_2 \subset F_3 \subset \ldots \subset K$ so that, setting $A_n = A_0 \otimes_{F_0} F_n$, we have $\mathbf{m} \cap A_{n+1}$ comaximal with every ideal of A_{n+1} generated by a non-zero element of A_n. Let $F_\omega = \bigcup F_n$ and $A_\omega = A_0 \otimes_{F_0} F_\omega$. Then $\mathbf{m} \cap A_\omega$ is comaximal with every non-zero two-sided ideal of A_ω, so that A_ω is right primitive.

The field F_ω is a countable extension of F_0, so that we can embed F_ω in F over F_0 and regard A as $A_\omega \otimes_{F_\omega} F$. It follows that A_ω is prime, right Goldie, and $Q(A_\omega) \subset Q(A)$, so that the center of $Q(A_\omega)$ is F_ω. Therefore Theorem 2.1 applies, and A is right primitive. The same argument can be used for left primitivity.

As a consequence, we can extend Theorem 1.3 as indicated:

Corollary 2.4. *Let F be a countable algebraically closed field of characteristic* 0 *which has infinite transcendence degree and let A be the enveloping algebra of a finite-dimensional Lie algebra. A prime ideal P of A is primitive if and only if $Q(A/P)$ has F as its center.*

Proof. We need only prove sufficiency, so assume the center of $Q(A/P)$ is F, and let K be an uncountable, algebraically closed field extension. By Theorem 2.1, every non-zero ideal of $(A/P) \otimes_F K$ has non-zero intersection with A/P, so $(A/P) \otimes_F K$ has a countable separating set of ideals. By Theorem 1.3, the algebra $(A/P) \otimes_F K$ is primitive, and Proposition 2.3 implies that A/P is primitive.

Of course, we may use the same argument for any family of finitely-presented noetherian algebras for which it is known that conditions (1), (2), and (3′) are equivalent over an uncountable field, to deduce that (1) and (2) are equivalent over big countable fields. In particular, by Theorem 1.4, this is true for group rings of polycyclic groups. Actually the equivalence of (1) and (2) is known in this case over any non-absolute field [17].

3. The Baire Property for Differential Operator Rings

A topological space is called a *Baire space* if the intersection of any countable collection of dense open sets is dense. The proofs of Dixmier and Farkas that the existence of a countable separating set of ideals implies primitivity, for prime images of enveloping algebras and twisted Laurent extensions, essentially amount to proving that the corresponding primitive ideal spaces are Baire [2, 4]. This follows from an observation which Farkas credits to Kaplansky:

Proposition 3.1. *Let R be a prime, semiprimitive ring with a countable separating set. If the primitive ideal space* Priv$R$ *is Baire, then R is primitive.*

Proof. Suppose R is not primitive. Then every primitive ideal contains one of a countable collection of non-zero elements $\{t_i\}$, and $\operatorname{Priv} R$ is the union of the closed sets $V(t_i)$. But each of these sets is proper, since R is primitive, and each is nowhere dense, since R is prime. This contradicts the Baire hypothesis.

In this section we shall follow Farkas' ideas to prove that the Baire property is preserved under the adjunction of a variable with respect to a derivation, thereby showing that for differential operator rings, condition (3′) implies (1). Let δ be a derivation of the noetherian ring R, and assume that R contains a characteristic 0 field. An ideal I of R is called a *δ-ideal* if $\delta(I) \subset I$, and is *δ-prime* if $JK \subset I$ for two δ-ideals J and K implies that $J \subset I$ or $K \subset I$. We summarize below the facts which we use about R and its Ore extension $S = R[x, \delta]$, all of which were proved by Jordan in [13].

3.1. (i) Any prime ideal of S intersects R in a δ-prime ideal, and conversely such an ideal extends to a prime ideal of S.

(ii) The δ-prime ideals of R are prime.

(iii) If $P \neq (0)$ is a prime ideal of S and $P \cap R = (0)$, then the regular elements of R have regular images in S/P, and any ideal of S properly containing P contains a regular element of R.

(iv) If (0) is δ-prime in R and I is a non-zero δ-ideal of R, then I contains a regular element of R.

(v) If S is prime, then it satisfies the Ore condition with respect to the regular elements of R.

(vi) If R is a Jacobson ring, so is S.

(vii) Let f, g be elements of S and let c be the leading coefficient of g. Then

$$f c^k = \sum h_i g c^{n_i} + s,$$

for some h_i, n_i and an element s with $\deg s < \deg g$.

We also need the following fact, which is presumably well-known:

Lemma 3.3. *Let R be a simple ring and let δ be a derivation of R which is not inner. Then $R[x;\delta]$ is simple.*

The main result of this section is

Theorem 3.4. *Let R be a noetherian Jacobson algebra over an uncountable characteristic 0 field, and let δ be a derivation of R. If every homomorphic image of R has a primitive ideal space which is Baire, then the same holds for $S = R[x;\delta]$.*

We will prove some preliminary results first. Assume below that R is a noetherian Jacobson ring containing the rationals, and δ is a derivation of R.

Lemma 3.5. *Let R be δ-prime and semiprimitive, with* $\operatorname{Priv} R$ *Baire. Let $J_1, J_2, \ldots$ be a sequence of non-zero δ-ideals. Then R has a primitive ideal which does not contain any J_i.*

Proof. It suffices to show that $\operatorname{Priv} R - V(J_i)$ is dense for all i. For then, by the Baire property, the intersection is non-empty. The ideal J_i contains a regular element t, by 3.2(iv). If $\operatorname{Priv} R - V(J_i)$ is not dense, its closure is the variety $V(I)$ for some non-zero ideal I. Then

$$\operatorname{Priv} R = V(I) \cup V(J_i) = V(I \cap J_i).$$

But since t is regular, $0 \neq tI \subset I \cap J_i$, so every primitive ideal contains tI, a contradiction.

The primes of $R[x;\delta]$ are of two types – extensions of δ-prime ideals of R, and non-extended ideals. The next result, which follows the argument of Farkas' Lemma 7, deals with the second type.

Proposition 3.6. *Let $S = R[x;\delta]$ be prime and let P be a non-zero prime ideal of S with $P \cap R = (0)$. If* Priv$R$ *is Baire, then so is* PrivS/P.

Proof. Let $\mathscr{M}$ be the set of maximal right ideals $\mathbf{m}$ of R such that $\mathbf{m}S + P = S$, and let

$\mathcal{O} = \{Q \in \operatorname{Priv} R : Q$ is the annihilator of $R/\mathbf{m}$ for some $\mathbf{m} \in \mathscr{M}\}$.

Since $P \cap R = (0)$, there is some positive integer n which is the lowest degree assumed by the non-zero elements of P. Let $\mathbf{p}$ be the set of leading coefficients in R of the degree n elements of P, along with 0. Then $\mathbf{p}$ is an ideal, and we claim $\mathbf{p} \subset \mathbf{m}$ for any $\mathbf{m}$ in $\mathscr{M}$.

By the definition of $\mathscr{M}$, there is an element of the form

$$1 + \sum a_i x^i \tag{2}$$

in P, with $a_i \in \mathbf{m}$. Let $c \neq 0$ be an element of $\mathbf{p}$, which we assume is the leading coefficient of some $g \in P$. Then 3.2(vii) implies that for each i,

$$x^i c^k \equiv s_i \,(\operatorname{mod} P)$$

for some $k \geqq 0$ and s_i with $\deg s_i < \deg g = n$. Thus, for k large enough, we find that

$$a_i x^i c^k - a_i s_i$$

lies in P for all i in the sum (2). Multiplying the element in (2) by c^k, we obtain

$$c^k - \sum a_i s_i$$

as an element of P. Since this has degree $< n$, it must be 0, and $a_i \in \mathbf{m}$ implies that $c^k \in \mathbf{m}$.

We conclude that c is in $\mathbf{m}$ as follows: if not, then $1 = ct + m$ for some $t \in R$ and $m \in \mathbf{m}$. But ct is an P, so $(ct)^j$ is in $\mathbf{m}$ for some j, and

$$(ct)^{j-1} = (ct)^j + m(ct)^j$$

is in $\mathbf{m}$. Thus $ct \in \mathbf{m}$, and $1 \in \mathbf{m}$, a contradiction.

Therefore the ideal $\mathbf{p}$ lies in each $\mathbf{m}$, and annihilates each module $R/\mathbf{m}$. But then $\mathbf{p}$ is a non-zero δ-ideal contained in

$$J = \bigcap \{Q : Q \in \mathcal{O}\}.$$

To prove PrivS/P is Baire, it suffices by Lemma 4 of [4] to show, given a sequence of prime ideals P_i in S/P over which lie every primitive ideal of S/P, that some $P_i = (0)$. This will be true if some $P_i \cap R = (0)$, by 3.2(iii). We prove this by showing that every primitive ideal of R contains some $P_i \cap R$ or $\mathbf{p}$. For then, using Lemma 3.5 and the fact that $\mathbf{p} \neq (0)$, some $P_i \cap R = (0)$.

Let Q be a primitive ideal of R. If Q is in $\mathcal{O}$, then Q contains $\mathbf{p}$, so assume otherwise and let Q be the annihilator of $R/\mathbf{m}$. Then $\mathbf{m} \notin \mathcal{M}$, and $\mathbf{m}S+P$ is contained in some maximal right ideal of S. Passing to S/P, we find that Q contains the annihilator in R of a simple S/P-module, and the conclusion follows.

The proof of the main theorem turns out to be easier than the corresponding proof in the twisted Laurent extension case:

Proof of 3.4. A short argument (Lemma 2 of [4]) shows that it suffices to prove PrivS/P is Baire for every prime ideal P. Those primes which are not extended from R are handled by Proposition 3.6. Thus we may assume that R is δ-prime and ’$=(0)$. Also, by noetherian induction, we may assume all proper homomorphic images of S satisfy the Baire property. This leaves two cases. First, every non-zero prime ideal of S may have non-zero intersection with R. An argument similar to that at the end of the proof of 3.6 disposes of this.

Alternatively, S has non-zero prime ideals which meet R in (0). These primes correspond to the primes of $Q(R)[x;\delta]$, where $Q(R)$ is the simple quotient ring of R. In view of Lemma 3.3, the derivation δ must be inner on $Q(R)$, with

$$\delta(r)=ra-ar$$

for some a in $Q(R)$. But then replacing x by $x-a$, we find that $Q(R)[x;\delta]$ is isomorphic to $Q(R)[x]$. Since $Q(R)$ is an algebra over an uncountable field, there is an uncountable collection of primes Q_α in S with the property that $Q_\alpha \cap R=(0)$.

Let $\{P_i\}$ be a countable collection of non-zero prime ideals such that every element of Priv$S$ contains some P_i. If some P_i lies in some Q_α, they must be equal by 3.2(iii). Thus we may find some α for which Q_α contains no P_i. Every primitive ideal of S/Q_α contains some P_i+Q_α/Q_α. But PrivS/Q_α is Baire by assumption, so some $P_i+Q_\alpha=Q_\alpha$, a contradiction. Thus no such family of ideals $\{P_i\}$ can exist, and Priv$S$ is Baire by Lemma 4 of [4].

Corollary 3.7. *Let A be a differential operator ring over an uncountable field of characteristic* 0. *Then the primitive ideal space of any homomorphic image of A is Baire. If P is a prime ideal of A with a countable separating set, then P is primitive.*

Thus, for differential operator rings over uncountable fields, the conditions (1), (2), and (3′) are equivalent. As remarked at the end of Sect. 2, Proposition 2.3 can be used to deduce that (1) and (2) are equivalent over countable algebraically closed fields of infinite transcendence degree. It would be interesting to know whether there is a finitely-generated noetherian algebra over an uncountable field whose primitive ideal space is not Baire. Weakening hypotheses, one can construct right Ore domains without the Baire property. For example, let k be any field of characteristic 0, and let

$$A=k\{x,y\}/(yx-x(y^2+1)).$$

This is an Ore extension of $k[y]$, so it is a right Ore domain. The (0) ideal satisfies conditions (2) and (3′), but we have proved elsewhere that A is not right primitive [9], although A is left primitive. By Proposition 3.1, the space of right primitive ideals is not Baire.

4. Certain Differential Operator Rings

In this section, we use Jordan's results (3.2) to show that Dixmier's proof of Theorem 1.1 can be extended to obtain the following:

Theorem 4.1. *Let K be a characteristic 0 field and let A be a differential operator ring defined over K with respect to derivations $d_1, \ldots, d_n$ and corresponding variables $x_1, \ldots, x_n$. Assume for $1<k<m\leqq n$ that*

$$\begin{aligned} d_m(x_k) &= x_k x_m - x_m x_k \\ &= p_{k,m}(x_1, \ldots, x_{k-1})x_k + q_{k,m}(x_1, \ldots, x_{k-1}), \end{aligned}$$

for some polynomials $p_{k,m}$ and $q_{k,m}$. Then for a prime ideal P, the conditions (1), (2), and (3) are equivalent.

It is evident that the enveloping algebra of a finite-dimensional, completely solvable Lie algebra satisfies the conditions of the theorem, with $p_{k,m}$ lying in K and $q_{k,m}$ linear. As noted in the introduction, the implications (3)→(1)→(2) are known to hold for any iterated Ore extension, so we need only prove (2)→(3). A key step in Dixmier's proof extends directly:

Lemma 4.2. *Let R be a simple noetherian algebra over a field K of characteristic 0, and let $d_1, \ldots, d_n$ be a family of derivations of $R[x]$ such that*

(i) *R is stable under each d_m.*

(ii) *The elements of the quotient ring $Q(R[x])$ which are annihilated by all the d_m are algebraic over K.*

(iii) *For each m, there exist elements a_m, b_m in R for which $d_m(x)=a_m x+b_m$.*

Then at most one non-zero prime ideal of $R[x]$ is invariant under all the d_m.

Proof. Let P be a non-zero prime of $R[x]$ which is d_m-invariant for all m. Using the simplicity of R, we may show that among the non-zero elements of P of minimal degree is a monic polynomial

$$p = x^t + x^{t-1}r_{t-1} + \ldots + r_0,$$

with $t>0$. Applying d_m, we obtain

$$\begin{aligned} d_m(p) &= t x^{t-1}(a_m x + b_m) + (t-1)x^{t-1}a_m r_{t-1} \\ &\quad + x^{t-1}d_m(r_{t-1}) + (\text{lower degree terms}). \end{aligned}$$

Therefore

$$d_m(p) - t a_m p = x^{t-1}(t b_m - a_m r_{t-1} + d_m(r_{t-1})) + (\text{lower degree terms}).$$

Since this is an element of P of lower degree than p, it must be zero, so

$$d_m(p) = t a_m p \tag{3}$$

and

$$d_m(r_{t-1}) = a_m r_{t-1} - t b_m .$$

It follows that

$$d_m\left(x+\frac{1}{t}r_{t-1}\right)=a_m\left(x+\frac{1}{t}r_{t-1}\right). \tag{4}$$

For every r in R, the element $pr-rp$ is in P and has degree $<t$, so it equals 0 and p is in the center of $R[x]$. In particular, r_{t-1} is in the center of R, and the map which fixes R and sends x to $x+\frac{1}{t}r_{t-1}$ is an automorphism. Thus we may assume that the elements b_m are all 0, using (4) and the change of variable.

In the quotient ring $Q(R[x])$, the following equation holds, by (3):

$$d_m(px^{-t})=ta_m px^{-t}-tpx^{-t-1}(a_m x)=0.$$

By hypothesis, px^{-t} is algebraic over K, which forces $p=x^t$. Since P is prime, we conclude that $t=1$ and $P=(x)$.

We need one more preliminary result, which is essentially Theorem 5.1 of [10].

Lemma 4.3. *Let K be a field of characteristic* 0, *and let d be a K-derivation of $K[x]$, sending x to $f(x)$. Then the d-invariant non-zero prime ideals of $K[x]$ are all prime divisors of $(f(x))$, and so are finite in number.*

We can now prove Theorem 4.1, following the approach of Dixmier [1] in proving 1.1:

Proof of 4.1. Let P be a prime ideal of A for which $Q(A/P)$ has center algebraic over K. We must show that P is a G-ideal. Let A_m be the K-subalgebra of A generated by $x_1, \ldots, x_m$, and let $P_m=P\cap A_m$. By 3.2(i) and (ii), each P_m is prime in A_m and d-invariant. (We will call an ideal in A_m d-invariant if it is invariant under all the derivations d_i for $i>m$.) We will show that for each m, the d-invariant prime ideals of A_m which properly contain P_m have an intersection which properly contains P_m. Since every ideal of A is d-invariant, the case $m=n$ yields the theorem.

The case $m=1$ follows from Lemma 4.3. Assume P_{m-1} satisfies the desired property, for $m>1$, and let B_i denote A_i/P_i. The ring B_{m-1} is a prime subalgebra of B_m, and we may view B_m as a homomorphic image of $B_{m-1}[x_m; d_m]$. By 3.2, any prime of $B_{m-1}[x_m; d_m]$ which intersects B_{m-1} in (0) is either (0) or of height one. If P_m is of the second type, then 3.2 implies that every d-invariant prime of A_m properly containing P_m must intersect B_{m-1} in a non-zero d-invariant prime. By the inductive hypothesis, these primes have non-zero intersection in B_{m-1}, so that the d-invariant primes of A_m properly containing P_m have an intersection strictly larger than P_m.

Therefore we may assume that $B_m=B_{m-1}[x_m; d_m]$. The d-invariant primes of B_m which meet B_{m-1} non-trivially have a non-zero intersection by induction, so it suffices to prove that there is at most one other non-zero d-invariant prime of B_m. Let Q be the simple quotient ring of B_{m-1}. By 3.2(v), B_m satisfies the Ore condition with respect to the regular elements of B_{m-1}, and the quotient ring is $Q[x_m; d_m]$. The primes of B_m which contain no regular elements of B_{m-1} are the restrictions of the primes in $Q[x_m; d_m]$, and by 3.2(iii), these are exactly the primes of B_m which meet B_{m-1} in (0). Thus it suffices to prove that $Q[x_m; d_m]$ has at most one non-zero, d-invariant prime ideal.

We may assume that d_m is an inner derivation on Q, for otherwise $Q[x_m; d_m]$ is simple by Lemma 3.3. Let d_m be the inner derivation induced by some element $q \in Q$. Then $x_m - q$ is central in $Q[x_m; d_m]$, and we may change variables so that $d_m = 0$ on Q. The derivations $d_1, \ldots, d_{m-1}$ can be extended to A_{m-1} and B_{m-1} as the inner derivations corresponding to $x_1, \ldots, x_{m-1}$. We may then extend $d_1, \ldots, d_n$ further to $Q[x_m - q]$. By 3.2(iii), the regular elements of $Q[x_m - q]$ are regular in A/P, so the quotient ring of $Q[x_m - q]$ embeds in $Q(A/P)$. Therefore the elements in the quotient ring of $Q[x_m - q]$ which are annihilated by $d_1, \ldots, d_n$ lie in the center of $Q(A/P)$, and are algebraic over K. The hypothesis of Theorem 4.1 now insures that Lemma 4.2 may be applied to $Q[x_m - q]$, so that $Q[x_m - q]$ has at most one non-zero d-invariant prime, and the theorem is proved.

5. Certain Twisted Laurent Extensions

The main result of this section is

Theorem 5.1. *Let K be a field and let A be the twisted Laurent extension defined over K with respect to variables $x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}$ which satisfy the relations*

$$x_m x_k x_m^{-1} = u_{km} x_k \quad \text{or} \quad x_m x_k x_m^{-1} = u_{km} x_k^{-1},$$

for $k < m$ and u_{km} a unit in the subalgebra A_{k-1} generated by $x_1, \ldots, x_{k-1}$ and their inverses. Then for a prime ideal P, the conditions (1), (2), and (3) are equivalent, and hold if and only if P is maximal.

In particular, group rings of finitely-generated nilpotent groups satisfy the hypothesis of Theorem 5.1. The conclusion is known for such algebras (Theorem 1.2), but the proof to be given follows the same lines as the proof of Theorem 4.1. We will need analogues of Jordan's results on differential operator rings in 3.2.

Let R be a noetherian ring, and let φ be an automorphism of R, with associated twisted Laurent extension $S = R[x, x^{-1}; \varphi]$. Goldie and Michler [6] obtained some results on $R[x; \varphi]$ which immediately extend to S, and which we will summarize. An ideal I of R is φ-invariant if $\varphi(I) \subset I$ and φ-prime if, in addition, for any two φ-invariant ideals J, K with $JK \subset I$, either $J \subset I$ or $K \subset I$.

5.2. (i) Any prime ideal of S intersects R in a φ-prime ideal and conversely, any φ-prime ideal of R extends to a prime of S.

(ii) The φ-prime ideals of R have the form $P \cap \varphi(P) \cap \ldots \cap \varphi^{n-1}(P)$, for P a prime ideal with $\varphi^n(P) = P$.

(iii) If $P \neq (0)$ is a prime ideal of S and $P \cap R = (0)$, then the regular elements of R have regular images in S/P, and any ideal of S properly containing P contains a regular element of R.

(iv) If (0) is φ-prime in R, any non-zero φ-invariant ideal of R contains a regular element.

(v) If R is semiprime, then S satisfies the Ore condition with respect to the regular elements of R.

We also need an analogue of Lemma 4.3. Let $Q_1, \ldots, Q_m$ be isomorphic copies of a simple ring, and let $Q = Q_1 x \ldots x Q_m$. Let φ be an automorphism of Q which maps Q_i identically to $Q_{\sigma(i)}$ for some permutation σ of $\{1, \ldots, m\}$ with a single orbit. Denote by L the subring of the center of Q fixed by φ. Then L is a field isomorphic to the center of Q_1.

Lemma 5.3. *Let I be a non-zero ideal of $Q[x, x^{-1}; \varphi]$. Then I is generated by a polynomial in $L[x^m]$.*

Proof. Given an element $\sum_{i=u}^{v} x^i r_i$ of $Q[x, x^{-1}; \varphi]$ with $r_u \neq 0$ and $r_v \neq 0$, we call its degree $v-u$. Let $p = x^n r_n + \ldots + r_0$ be a non-zero element of I of minimal degree n. By multiplying p on the right by an element of Q, we may assume that r_n lies in one of the simple factors Q_i of Q. There exist elements c_j and d_j in Q_i such that

$$\sum_j c_j r_n d_j = 1,$$

and so

$$\sum_{k=1}^{m} x^{-k}\Big(\sum_j \varphi^{-n}(c_j) p d_j\Big) x^k$$

is monic. We may assume, then, that p is monic, and the minimality of the degree of p implies that $I = pQ[x, x^{-1}; \varphi]$.

Suppose r_i is another non-zero coefficient of p. By using the same procedure as above, we can find another element q in I of no greater degree whose i^{th} coefficient is 1. The degree of q must be n, and $q = ps$ with s equal to the degree n coefficient of q. Hence $r_i s = 1$ and all the non-zero coefficients of p are units.

Let $q = pr_0^{-1}$, so that q is an element of I with constant term 1. For any element $s \in Q$, the commutator $sq - qs$ is an element of I of degree $< n$, and must be 0:

$$0 = sq - qs = \sum_{i=0}^{n} x^i(\varphi^i(s) r_i r_0^{-1} - r_i r_0^{-1} s).$$

Each $r_i r_0^{-1} \neq 0$ is a unit, so

$$r_i r_0^{-1} s (r_i r_0^{-1})^{-1} = \varphi^i(s).$$

This means that the automorphism φ^i is inner, but by the definition, this is possible only if m divides i. Thus the non-zero coefficients of p all have indices divisible by m. Also, since φ^{im} is the identity, we find that

$$r_i r_0^{-1} s (r_i r_0^{-1})^{-1} = s,$$

so the elements $r_i r_0^{-1}$ all lie in ther center. This proves that I is generated by an element of $L[x^m]$.

A more complicated situation arises in the proof of the theorem. Let $Q = Q_1 \times \ldots \times Q_m$ be a product of isomorphic copies of a simple ring, let φ be an automorphism of Q which maps Q_i identically to $Q_{\sigma(i)}$ for some permutation σ of $\{1, \ldots, m\}$, and fix a field K lying in the center of Q. Let $\psi_1, \ldots, \psi_n$ be K-automorphisms of $Q[x, x^{-1}; \varphi]$ such that each ψ_i maps any factor Q_j to another factor Q_k, and the corresponding permutations of the indices under all the ψ_i's act transitively. In addition, assume that

$$\psi_i(x)=u_i x \quad \text{or} \quad \psi_i(x)=u_i x^{-1}$$

for some units u_i in Q.

Lemma 5.4. *Suppose, in the setting above, any element of $Q[x, x^{-1}; \varphi]$ which is fixed by $\psi_1, \ldots, \psi_n$ is algebraic over K. Then there are no non-zero proper ideals of $Q[x, x^{-1}; \varphi]$ which are ψ_i-invariant for all i.*

Proof. Call an ideal ψ-invariant if it is ψ_i-invariant for all i, and let I be a non-zero, ψ-invariant ideal of $Q[x, x^{-1}; \varphi]$. Let k be the number of orbits of φ, and set $R_1, \ldots, R_k$ equal to the products of the Q_j's in the distinct orbits. Then $R_i[x, x^{-1}; \varphi]$ is the type of ring handled in Lemma 5.3, and if $I \cap R_i[x, x^{-1}; \varphi]$ is non-zero, it is generated by a central polynomial p_i with 1 as its lowest degree term. However the transitivity assumption on the ψ_j's insures that each such intersection is non-zero. In addition, the transitivity and the assumed action of the ψ_j's on x implies that the p_i's all have the same degree. Hence I is generated by the polynomial $p=p_1+\ldots+p_k$, which is a non-zero element of least degree in I, with 1 as its lowest degree term and all its coefficients central.

We may re-order the automorphisms so that for some $s \leqq n$,

$$\begin{aligned} \psi_i(x) &= u_i x & i &\leqq s \\ &= u_i x^{-1} & i &> s. \end{aligned}$$

Choose $i \leqq s$. Then $\psi_i(p)$ is a polynomial in I of the same degree as p, and with 1 as its lowest degree term. Since I contains no elements of smaller degree, we must have $\psi_i(p)=p$. Alternatively, choose $j, k>s$. Then the elements $\psi_i\psi_j(p)$ and $\psi_j(p)$ both have 1 as their highest degree term and have minimal degree in I, so they are equal. Similarly,

$$\psi_j\psi_k(p)=p,$$

which implies that $\psi_j^{-1}(p)=\psi_k(p)$. In particular, for $j=k$, we obtain $\psi_j^{-1}(p) =\psi_j(p)$ and

$$\psi_j(p)=\psi_k(p)$$

for all $j, k>s$. Choosing a fixed $j>s$, we find that the element

$$p\psi_j(p)$$

is fixed by all $\psi_1, \ldots, \psi_n$. For instance, applying ψ_j yields $\psi_j(p)p$, but the coefficients are central, so this equals $p\psi_j(p)$. By assumption, $p\psi_j(p)$ is therefore algebraic over K. This is only possible if it is a constant polynomial, in which case I is the unit ideal.

Proof of 5.1. The implications (3)→(1)→(2) hold for any iterated Ore extension, so it suffices to show that if P is a prime ideal of A for which the center of $Q(A/P)$ is algebraic over K, then P is maximal. For $i=1, \ldots, n$, let ψ_i be the inner automorphism of A defined by

$$\psi_i(a)=x_i a x_i^{-1},$$

and call an ideal ψ-invariant if it is ψ_i-invariant for all i. Let $P_m = P \cap A_m$ and $P_0 = P \cap K = (0)$. Also set $B_m = A_m/P_m$. We will prove the theorem by showing inductively on m that P_m is maximal among ψ-invariant semiprime ideals of A_m. Since all ideals of $A = A_n$ are ψ-invariant, the case $m = n$ yields the conclusion that P is maximal.

Each ideal P_m is an intersection of prime ideals which are permuted by the automorphisms ψ_i, by 5.2. More precisely, the minimal primes of P_m are fixed by $\psi_1, \ldots, \psi_m$ and $\psi_{m+1}, \ldots, \psi_n$ induce all possible permutations of the minimal primes. Consequently, given a ψ-invariant semiprime ideal I of A_m which strictly contains P_m, any minimal prime of I must strictly contain a minimal prime of P_m. For the automorphisms ψ_i permute the minimal primes of I, so that if a minimal prime coincided with one for P_m, we would have $P_m = I$.

We wish to show P_m is maximal among ψ-invariant semiprime ideals of A_m. This is certainly true for $m = 0$; let us assume so for P_{m-1}, with $m > 0$. We may view B_m as a homomorphic image of $B_{m-1}[x_m, x_m^{-1}; \psi_m]$. Any ψ-invariant semiprime ideal of B_m intersects B_{m-1} in a ψ-invariant ideal which is semiprime, by 5.2, and so is (0) by induction.

Suppose P_m is non-zero in $B_{m-1}[x_m, x_m^{-1}; \psi_m]$. We have seen that any semiprime ψ-invariant ideal I properly containing P_m has minimal primes strictly containing those of P_m. But then 5.2(iii) implies that $I \cap B_{m-1} \neq (0)$, which is impossible by the preceding remark. Therefore, P_m is a maximal ψ-invariant, semiprime ideal.

Alternatively, we may assume $B_m = B_{m-1}[x_m, x_m^{-1}; \psi_m]$. Let Q be the semisimple quotient ring of B_{m-1}. As noted, the ψ_i's act transitively on the minimal primes of (0) in B_{m-1}, so their extensions to Q transitively permute the simple factors. By 5.2(v), we may adjoin to B_m the inverses of regular elements of B_{m-1}, obtaining $Q' = Q[x_m, x_m^{-1}; \psi_m]$. The ψ-invariant semiprime ideals of B_m correspond to those of Q', since they all intersect B_{m-1} in (0). Moreover, by 5.2(iii), the regular elements of B_{m-1} remain regular in A/P, so that $Q' \subset Q(A/P)$ and any element of Q' fixed by all the ψ_i's is algebraic over K. The conditions of Lemma 5.4 are all satisfied, which implies that (0) is the only ψ-invariant ideal of Q'. Therefore (0) is the only ψ-invariant semiprime ideal of B_m, and the theorem is proved.

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Note Added in Proof

The assumption in 1.3 and 2.4 that the base field be algebraically closed in order for conditions (1) and (2) to be equivalent has recently been removed by L. Small and the author. This and related results will appear in a subsequent joint paper.