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On the Characterization of Primitive Ideals in Enveloping Algebras

Ronald S. Irving[⋆,1] and Lance W. Small[⋆,2]

[1] Department of Mathematics, University of Chicago, Chicago, IL 60637, U.S.A.
[2] Department of Mathematics, University of California, La Jolla, CA 92093, U.S.A.

1. Introduction

Let F be a field of characteristic 0, and let $U(L)$ be the enveloping algebra of a finite-dimensional Lie algebra L. Given a prime ideal P of $U(L)$, consider the following two conditions:

(1) *P is primitive*

(2) *the center of the quotient ring $Q(U(L)/P)$, called the heart of P, is algebraic over F.*

By a result of Quillen, it is always the case that (1) implies (2) [7; 1, 4.1.6]. One would like to know that (2) implies (1); for instance, this would imply that there is no difference between left and right primitive ideals. The purpose of this paper is to show that previously known results can be extended to obtain:

Theorem. *Assume either that F has infinite transcendence degree over the rationals or that L is an algebraic Lie algebra. Then the primitive ideals of $U(L)$ are precisely those primes P whose heart is algebraic over F.*

Dixmier has proved that (1) and (2) are equivalent in case L is solvable [1, 4.5.7] or F is uncountable and algebraically closed [2]. In addition, one can pass to algebraically closed fields of infinite transcendence degree over the rationals [5]. In Sect. 2 we show, for a given L and F, that if (1) and (2) are equivalent over the algebraic closure $\bar{F}$, then the equivalence descends to F. This depends on the fact that if the over-ring of a liberal or finite centralizing extension is primitive, so is the base ring [8].

Dixmier's proof that (2) implies (1) in the solvable case makes use of an auxiliary condition:

(3) P is a G-ideal; that is, the prime ideals properly containing P intersect in an ideal properly containing P.

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Equivalently, (3) says that $\{P\}$ is locally closed in $\operatorname{Spec} U(L)$. Since $U(L)$ is a Jacobson ring [3], condition (3) implies condition (1). Recently, Moeglin has proved that if L is an algebraic Lie algebra and F is an uncountable, algebraically closed field, then (1) implies (3) [6]. In Sect. 3 we show that whenever (1) and (3) are known to be equivalent for L over an uncountable, algebraically closed field, they are equivalent over any field F, and in addition, (1) and (2) are equivalent as well. This depends on the fact that if P is primitive and $P \otimes F'$ is a G-ideal for some field extension F' of F, then P is a G-ideal. In particular, for algebraic Lie algebras over any field F, the three conditions are all equivalent.

We have recently learned from Moeglin that she has proved that (1) implies (3) for any Lie algebra L over an uncountable, algebraically closed field F [9]. Thus our results would imply that the three conditions are equivalent for any Lie algebra L over any (characteristic 0) field.

2. Descent Under Algebraic Extensions

Let F be a field and L a Lie algebra over F. We wish to prove in this section that if (2) implies (1) for prime ideals of the enveloping algebra over $\bar{F}$, then the same holds over F. This depends on the following two facts:

Proposition 2.1. *Let $A \subset B$ be a liberal extension; that is, B is generated as A-module by a finite set of elements which centralize A. If B is primitive, then A is primitive.*

Proof. This is Theorem 5.6 of [8]. Alternatively, we can prove it as follows. Since B is prime, A is as well. By [4, Theorem 4], a faithful, simple B-module V decomposes over A as a finite direct sum of simple A-modules $V_1, \ldots, V_n$. Let P_i be the annihilator of V_i in A for $i=1, \ldots, n$. Then $\bigcap P_i$ annihilates V and so is (0). Since A is prime, $P_j=(0)$ for some j, and V_j is a faithful, simple A-module.

Proposition 2.2. *Let A be an algebra over a field F and let K be a field extension of F such that $A \underset{F}{\otimes} K$ is primitive and noetherian. Then for some finitely-generated field extension K' of F inside K, the algebra $A \underset{F}{\otimes} K'$ is primitive.*

Proof. Let M be a maximal right ideal of $A \underset{F}{\otimes} K$ which contains no non-zero ideal, and choose generators $p_1, \ldots, p_n$ for M as right ideal. Let K' be the field extension of F generated by the scalars appearing in $p_1, \ldots, p_n$. The proposition will follow if we show that M intersects $A \underset{F}{\otimes} K'$ in a maximal right ideal containing no non-zero ideal.

The intersection $M' = M \cap (A \underset{F}{\otimes} K')$ certainly contains no non-zero ideal, for if it does, so does M. By the choice of K', we know that $M' \underset{K'}{\otimes} K$ contains M, so that they are equal. Let N' be a right ideal of $A \underset{F}{\otimes} K'$ properly containing M'. Then $N' \underset{K'}{\otimes} K$ properly contains M. Thus $N' \underset{K'}{\otimes} K$ contains 1, and by the freeness of $A \underset{F}{\otimes} K$ over $A \underset{F}{\otimes} K'$, so does N'. Therefore M' is maximal.

We can now prove the main result of this section.

Theorem 2.3. *Let L be a finite-dimensional Lie algebra over a field F of characteristic 0. Assume that in $U(L)\otimes_F \bar{F}$, any prime ideal whose heart equals $\bar{F}$ is primitive. Then in $U(L)$, any prime ideal with heart algebraic over F is primitive.*

Proof. Let P be a prime ideal of $U(L)$ with a heart K which is algebraic over F, and set $A=U(L)/P$. The algebra $A\otimes\bar{F}$ is semiprime [1, 3.4.2], and as a homomorphic image of $U(L\otimes\bar{F})$, it is noetherian. We claim its quotient ring is equal to $Q(A)\otimes\bar{F}$. To see this, note that any regular element of $A\otimes\bar{F}$ lies in $A\otimes F'$ for some finite extension F' of F, and $Q(A)\otimes F'$ is a partial localization of $A\otimes F'$ which is artinian. Thus $Q(A)\otimes F'$ equals $Q(A\otimes F')$, and every regular element of $A\otimes\bar{F}$ has its inverse in $Q(A)\otimes\bar{F}$.

As a result, the center of $Q(A\otimes\bar{F})$ is the algebraic extension $K\otimes\bar{F}$ of $\bar{F}$, and so must be a finite product of copies of $\bar{F}$. This means that $Q(A\otimes\bar{F})$ is itself a finite product of matrix rings over $\bar{F}$. By [1, 3.4.2], there is a minimal prime $\bar{P}$ of $A\otimes\bar{F}$ with $\bar{P}\cap A=(0)$, and the quotient ring of $(A\otimes\bar{F})/\bar{P}$ will be one of the matrix rings over $\bar{F}$, with center $\bar{F}$. Thus, by assumption, $\bar{P}$ is primitive.

Applying Proposition 2.2, we obtain a finite extension F' of F and a prime ideal $P'=(A\otimes F')\cap\bar{P}$ such that $(A\otimes F')/P'$ is primitive and $P'\cap A=(0)$. Let $B=(A\otimes F')/P'$. Then B is a liberal extension of A, as in Proposition 2.1, and we can conclude that A is primitive.

Corollary 2.4. *Let F be any field of infinite transcendence degree over the rationals and L a finite-dimensional Lie algebra over F. Any prime ideal of $U(L)$ with a heart algebraic over F is primitive.*

Proof. This follows from 2.3, since the result is known to hold for $\bar{F}$ [2, 5].

3. Descent of G-ideals

In this section, we prove that if conditions (1) and (3) are equivalent over a large field, they are equivalent over smaller fields, and are implied by (2). We must show first that the property of being a G-ideal descends under purely transcendental field extensions.

Lemma 3.1. *Let A be a prime algebra over a field F such that in $A\otimes F(t)$, the ideal (0) is a G-ideal. Then (0) is a G-ideal in A.*

Proof. Let P be a non-zero prime ideal of A. Then $P'=P\otimes F(t)$ is prime in $A'=A\otimes F(t)$, since A'/P' is a partial localization of the prime ring $(A/P)[t]$. Let

$$a=\sum_{i=0}^{m} a_i t^i$$

be a non-zero element in every non-zero prime of A', where $a_i\in A$ and $a_m\neq 0$. In particular, P' contains a, so that

$$a=(\sum b_j g_j(t))\,h(t)^{-1}$$

for some elements $b_j \in P$ and $g_j(t), h(t) \in F[t]$. Thus we obtain

$$\sum_{i=0}^{m} a_i t^i h(t) = \sum b_j g_j(t),$$

and a_m is the highest degree coefficient on the left side. This implies that a_m is an F-linear combination of the b_j's, so a_m lies in P. We conclude that every non-zero prime of A contains a_m, and (0) is a G-ideal.

Corollary 3.2. *Let A be a prime algebra over a field F and let K be a purely transcendental field extension such that the* (0) *ideal in $A \otimes K$ is a G-ideal. Then* (0) *is a G-ideal in A.*

Proof. Let a be a non-zero element in every non-zero prime of $A \otimes K$. We may choose a subfield K' of K generated over F by a finite subset of the transcendence basis of K over F, such that a lies in $A \otimes K' = A'$. For any non-zero prime P of A', the ideal $P \otimes K$ is a non-zero prime of $A \otimes K$, so it contains a. But $(P \otimes K) \cap A' = P$, so $a \in P$ and (0) is a G-ideal in A'. Applying 3.1, we find that (0) is a G-ideal in A as well.

We also need to show that G-ideals behave well in liberal extensions.

Proposition 3.3. *Let $A \subset B$ be a liberal extension of rings with B prime. If* (0) *is a G-ideal in B, then* (0) *is a G-ideal in A.*

Proof. By [8, 4.5 and 4.1], every non-zero ideal of B has non-zero intersection with A, and every prime of A is the intersection of a prime of B. Let I be the intersection of the non-zero primes of B. By assumption $I \neq (0)$, so $I \cap A \neq (0)$, and every non-zero prime of A contains $I \cap A$. Therefore (0) is a G-ideal in A.

We can now prove the main result of this section:

Theorem 3.4. *Let $F \subset K$ be fields of characteristic* 0 *with K algebraically closed, and let L be a finite-dimensional Lie algebra over F. If every primitive ideal is a G-ideal in $U(L) \otimes K$, then every primitive ideal is G-ideal in $U(L)$.*

Proof. Let P be a primitive ideal in $U(L)$ and set $A = U(L)/P$. The center Z of $Q(A)$ is algebraic over F, and it is of finite dimension over F, as the argument of 2.3 shows. Thus we may choose a finite extension F_1 of F in K which splits Z; that is, $Z \otimes_F F_1$ is a product of a finite number of copies of F_1. Then in $A \otimes F_1$, by [1, 3.4.2.], there is a primitive ideal P_1 with $P_1 \cap A = (0)$, and the heart of P_1 is F_1. The algebra $(A \otimes F_1)/P_1$ is a liberal extension of A, and by 3.3, if P_1 is a G-ideal, so is P.

We may therefore assume that the center of $Q(A)$ is F itself. Then $A \otimes K$ is prime, and by [1, 3.4.2] it is primitive, so the hypothesis implies that (0) is a G-ideal in $A \otimes K$. By 3.2, the property of being a G-ideal descends under purely transcendental field extensions, so we may as well assume that $K = \bar{F}$.

Let $a \in A \otimes \bar{F}$ be a non-zero element in every non-zero prime of $A \otimes \bar{F}$, and choose a finite field extension F_2 of F such that $a \in A \otimes F_2$. Since a prime $Q \neq 0$ in $A \otimes F_2$ is the intersection of $A \otimes F_2$ with a prime of $A \otimes \bar{F}$, the prime Q must contain a. Thus (0) is a G-ideal in $A \otimes F_2$. But this is a liberal extension of A, so (0) is a G-ideal in A by 3.3.

Corollary 3.5. *Let L be a finite-dimensional Lie algebra over a field F of characteristic* 0, *and suppose that for some algebraically closed field extension K of infinite transcendence degree over the rationals, the primitive ideals of $U(L)\otimes K$ are G-ideals. Then for the prime ideals of $U(L)$, conditions* (1), (2), *and* (3) *are equivalent.*

Proof. We need only show that (2) implies (1). By 2.3, we may assume that F is algebraically closed. Let P be a prime of $U(L)$ whose heart is F. Then $P\otimes K$ has heart K in $U(L)\otimes K$. Since K has infinite transcendence degree, by 2.4, the ideal $P\otimes K$ is primitive, so it is a G-ideal by assumption. Using the argument of 3.4, we find that P is a G-ideal, hence primitive.

Corollary 3.6. *Let L be an algebraic Lie algebra over a field F of characteristic* 0. *Then for the prime ideals of $U(L)$, conditions* (1), (2), *and* (3) *are equivalent.*

Proof. Moeglin has proved that primitive ideals are G-ideals provided F is uncountable and algebraically closed [6], so the result follows immediately from 3.5.

Remark. As noted in the introduction, the removal by Moeglin of the assumption that L must be algebraic implies that 3.5 can be applied to any finite-dimensional Lie algebra.

References

1. Dixmier, J.: Algèbres enveloppantes. Paris: Gauthier-Villars 1974
2. Dixmier, J.: Idéaux primitifs dans les algèbres enveloppantes. J. Algebra **48**, 96–112 (1977)
3. Duflo, M.: Certaines algèbres de type fini sont des algèbres de Jacobson. J. Algebra **27**, 358–365 (1973)
4. Formanek, E., Jategaonkar, A.V.: Subrings of noetherian rings. Proc. Amer. Math. Soc. **46**, 181–186 (1974)
5. Irving, R.: Primitive ideals of certain noetherian algebras. Math. Z. **169**, 77–92 (1979)
6. Moeglin, C.: Idéaux primitifs dans les algèbres enveloppantes. C. R. Acad. Sci. Paris Sér. A **288**, 709–712 (1979)
7. Quillen, D.: On the endomorphism ring of a simple module over an enveloping algebra. Proc. Amer. Math. Soc. **21**, 171–172 (1969)
8. Robson, J.C., Small, L.: Liberal extensions. Proc. London Math. Soc. (to appear)
9. Moeglin, C.: Idéaux bilatères dans les algèbres enveloppantes. Thèse de doctorat d'état, l'Université de Paris 6, 1980

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