Temperature around a nuclear waste rod using finite difference implicit method

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The governing equation in heat transfer is:

$$\frac{\partial T}{\partial t}(r,t)+\bar{u}*\bar{\nabla}T(r,t)=k\nabla^2 T(r,t)+S1(r,t)$$

K = thermal diffusivity m^2/s

For solids u is zero

$$\frac{}{k}\frac{\partial(T)}{\partial(t)}(r,t)-(\nabla^2 T(r,t))=S(r,t)$$

The source term due to radioactive decay of rod is defined as:

$$S(r,t)=\{Trod\,\mathrm{e}^{-t/tau}/a^2\, for\, r\le a\}$$

Where, a=25cm

Trod=1k

tau=100years

r_c=100cm

T_E= 300k

k= $2x10^7$ cm^2/year

0 < r < 100cm

0 < t <100years

Because the problem has circular symmetry, 2-D problem in (x,y) can be converted to 1-D problem in r

$$\nabla^2 T = \frac{\partial^2 T}{\partial^2 x} + \frac{\partial^2 T}{\partial^2 y} = \frac{\partial^2 T}{\partial^2 r} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial^2 \emptyset}$$

T(r,t) has circular symmetry, so

$$\nabla^2 T = \frac{\partial^2 T}{\partial^2 x} + \frac{\partial^2 T}{\partial^2 y} = \frac{\partial^2 T}{\partial^2 r} + \frac{1}{r}\frac{\partial T}{\partial r}$$

Hence 1-D heat equation in polar co-ordinates becomes:

$$\frac{1}{k}\frac{\partial T}{\partial t} - \frac{\partial^2 T}{\partial^2 r} - \frac{1}{r}\frac{\partial T}{\partial r} = S(r,t)$$

- We know that in the steady state solution, S(r, t) → 0 as t → ∞,
- Far away from the rod the temperature equals the ambient temperature, T (r = r c , t) =300K.
- T (r, t) → 300K once rod has finished radioactive decaying.

Finite Difference Method – Implicit method:

$$\frac{1}{k}\frac{\partial T}{\partial t}-\frac{\partial^2 T}{\partial^2 r}-\frac{1}{r}\frac{\partial T}{\partial r}=S(r,t)$$

Initial conditions : T(r,0) = 300k

There is singularity at r=0, thereby we take another boundary condition

Neumann boundary condition at r=0 as temperature cannot flow into r=0 region

$$\frac{\partial T}{\partial r}(r=0,t)=0$$

Dirichlet boundary condition at $r=r_c$

$$T(r=r_c, t) = 300k$$

Discretization of space and time:

$dr = r_c/(n+1)$ $dt=T_f/m$ $r_j=j(dr)$ $t_k=k(dt)$

$0<=j<=(n+1)$ $0<=k<=m$ $T(r_j,t_k) = T^k_j$ $S(r_j,t_k) = S^k_j$

Neumann boundary condition at r=0 becomes:

$$\frac{\partial T_j^k}{\partial t}(r=0,t)=\frac{\partial T_0^k}{\partial t}=0\approx\frac{T_1^k-T_0^k}{\Delta t}\Rightarrow T_1^k\approx T_0^k$$

Dirichlet boundary condition at $r=r_c$ becomes:

$$T_j^k(r=r_c,t)=T_{(n+1)}^k=300\,k$$

Writing spacial derivatives in r at future time step k+1 : implicit method

$$T_t(t_{k+1}, r_j) = \frac{T_j^{k+1} - T_j^k}{\Delta t}$$

$$T_{rr}(t_{k+1}, r_j) = \frac{T_{j+1}^{k+1} - 2T_j^{k+1} + T_{j-1}^{k+1}}{\Delta r^2}$$

$$T_r(t_{k+1}, r_j) = \frac{T_{j+1}^{k+1} - T_{j-1}^{k+1}}{2\Delta r}$$

Using r= j* dr, the discretized pde becomes:

$$\frac{1}{k}\frac{\partial T}{\partial t} - \frac{\partial^2 T}{\partial^2 r} - \frac{1}{r}\frac{\partial T}{\partial r} = S(r, t)$$

$$\frac{1}{k}\frac{T_j^{k+1} - T_j^k}{\Delta t} - \frac{T_{j+1}^{k+1} - 2T_j^{k+1} + T_{j-1}^{k+1}}{\Delta r^2} - \frac{1}{j\Delta r}\frac{T_{j+1}^{k+1} - T_{j-1}^{k+1}}{2\Delta r} = S_j^k$$

s = k*dt/ dr^2

$$T_{j+1}^{k+1}\left[-s - \frac{s}{2j}\right] + T_{j+1}^{k+1}\left[-s + \frac{s}{2j}\right] + T_j^{k+1}\left[1 + 2s\right] = T_j^k + S_j^k k\,\Delta t$$ Tridiagonal matrix for 1<=j<=n

Numerical solution for n=3

- $T^k_0 \approx T^k_{1,}$ Neumann boundary condition
- T^k_4 = 300k from T (r = r c , t) = 300
- The initial conditions are T^0_j = 300K.
- We solve the equation for T^k_1 , T^k_2 , T^k_3 at each time step (t^k):

$$\begin{pmatrix} 1+2s & (-s-\frac{s}{2j}) & 0 \\ (-s+\frac{s}{2j}) & 1+2s & (-s-\frac{s}{2j}) \\ 0 & (-s+\frac{s}{2j}) & 1+2s \end{pmatrix} \begin{pmatrix} T_1^{k+1} \\ T_2^{k+1} \\ T_3^{k+1} \end{pmatrix} + \begin{pmatrix} (-s+\frac{s}{2j})T_0^{k+1} \\ 0 \\ (-s-\frac{s}{2j})T_4^{k+1} \end{pmatrix} = \begin{pmatrix} T_1^k \\ T_2^k \\ T_3^k \end{pmatrix} + \kappa\Delta t \begin{pmatrix} S_1^k \\ S_2^k \\ S_3^k \end{pmatrix}$$

Using the boundary conditions: $T^{k+1}_0 \approx T^{k+1}_1$, T^{k+1}_4 = 300K

$$\begin{pmatrix} (1+s+\frac{s}{2j}) & (-s-\frac{s}{2j}) & 0 \\ (-s+\frac{s}{2j}) & (1+2s) & (-s-\frac{s}{2j}) \\ 0 & (-s+\frac{s}{2j}) & (1+2s) \end{pmatrix} \begin{pmatrix} T_1^{k+1} \\ T_2^{k+1} \\ T_3^{k+1} \end{pmatrix} = \begin{pmatrix} T_1^k \\ T_2^k \\ T_3^k \end{pmatrix} + \kappa\Delta t \begin{pmatrix} S_1^k \\ S_2^k \\ S_3^k \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ (-s-\frac{s}{2j})T_4^{k+1} \end{pmatrix}$$

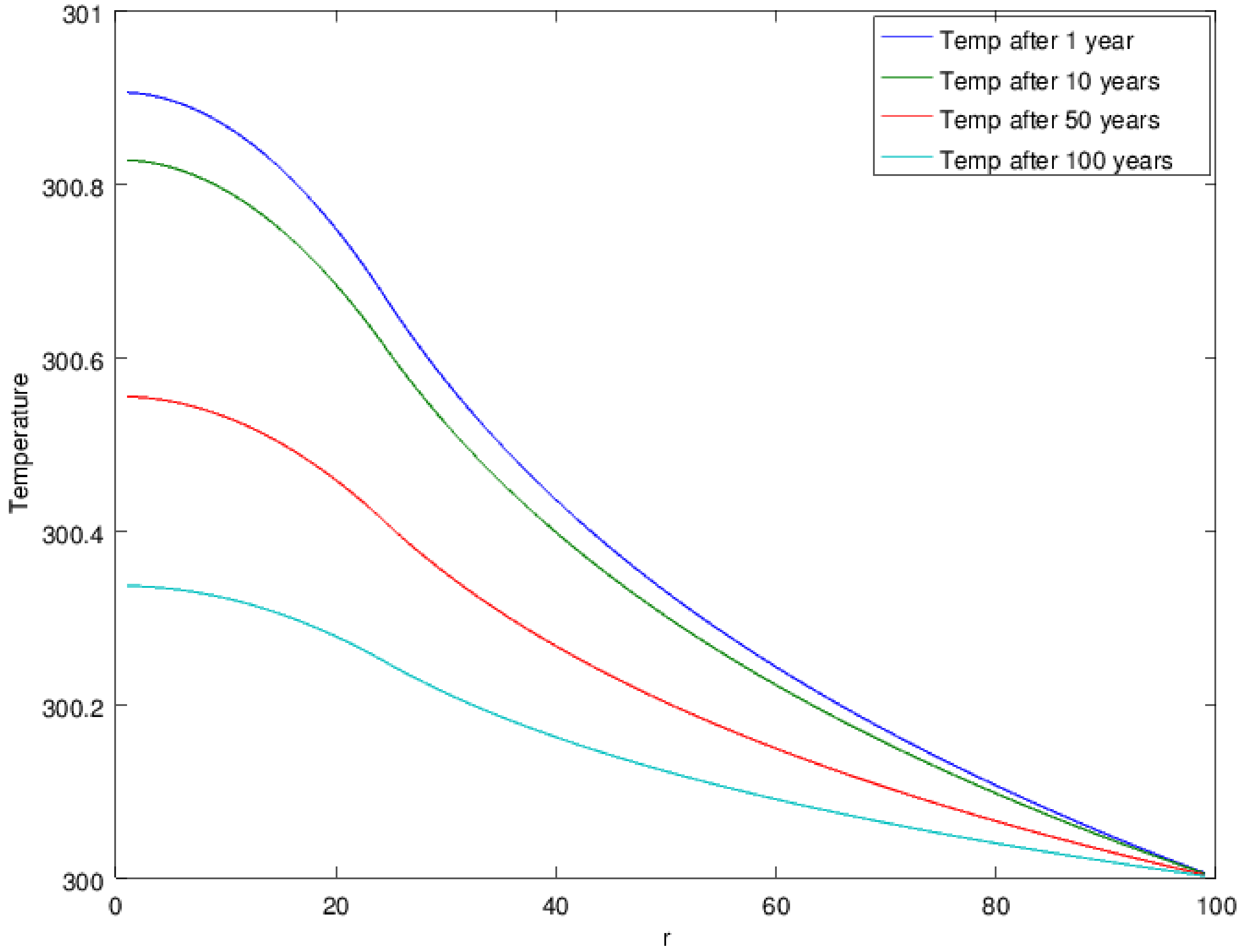
$$
\Rightarrow \begin{pmatrix} (1+s+\frac{s}{2}) & (-s-\frac{s}{2}) & 0 \\ (-s+\frac{s}{4}) & (1+2s) & (-s-\frac{s}{4}) \\ 0 & (-s+\frac{s}{6}) & (1+2s) \end{pmatrix} \begin{pmatrix} T_1^{k+1} \\ T_2^{k+1} \\ T_3^{k+1} \end{pmatrix}
$$

$$
= \begin{pmatrix} T_1^k \\ T_2^k \\ T_3^k \end{pmatrix} + \kappa\Delta t \begin{pmatrix} S_1^k \\ S_2^k \\ S_3^k \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ (-s-\frac{s}{6})300 \end{pmatrix}
$$

Fortran 90 code for the problem:

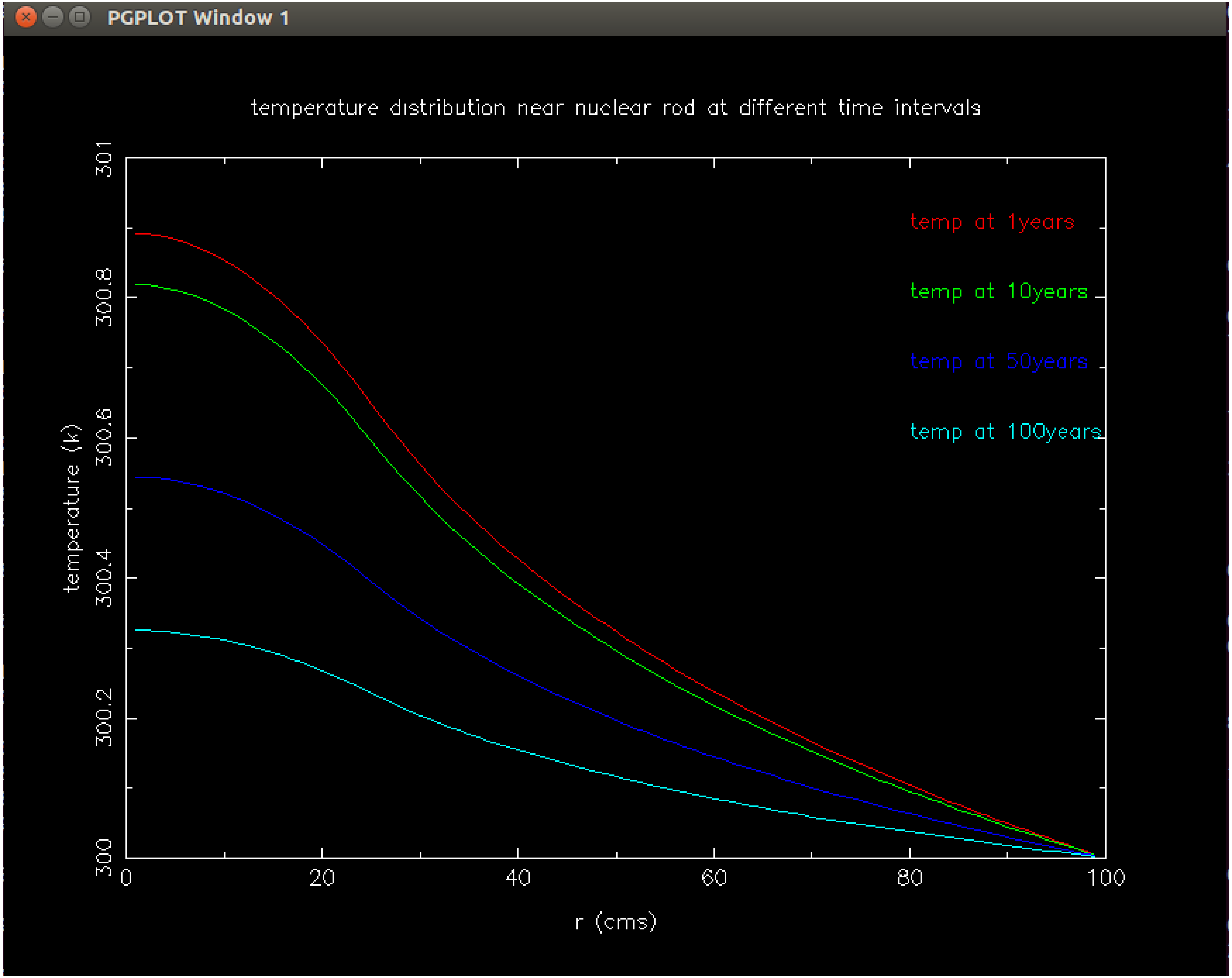
/home/chanakya/Fortran/sid/cylinder.f90

Documented output



http://espace.library.uq.edu.au/view/UQ:239427/Lectures_Book.pdf

Output:



PGPLOT Window 1
temperature distribution near nuclear rod at different time intervals
temp at 1years
temp at 10years
temp at 50years
temp at 100years
temperature (k)
300
300.2
300.4
300.6
300.8
301
0
20
40
60
80
100
r (cms)