Lecture 2: Matrix Algebra

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Contents

1 Matrices and Vectors

- A matrix is a rectangular array of numbers. For example, $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$.
  Its dimensions are (rows) $\times$ (columns). E.g., the dimensions of $\mathbf{A}$ are 2×3.

$$A_{m,n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

- a vector is a matrix with only one row or column. For example, $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

- Matrices are typically represented with bold capital letters (e.g., $\mathbf{A}$). Vectors are denoted by bold lower case letters (e.g., $\mathbf{x}$).

- An element of a matrix is denoted by a_{ij}, where i refers to the row and j to the column of the matrix.

In-class exercise: in R, Create a 4×4 matrix, and populate it with values drawn from a standard normal distribution. Save this matrix as $\mathbf{A}$.

1.1 Intuition: What is a matrix?

Why do we care about matrices? Because they are systems of vectors, and vectors store our data. For example:

$$Income = \text{some baseline} + \alpha Education + \alpha age + \text{random stuff}$$

$$\begin{pmatrix} Income_{John} \\ Income_{Barb} \end{pmatrix} = \begin{pmatrix} 1000 \\ 1000 \end{pmatrix} + \begin{pmatrix} BA & \text{34 years old} \\ PhD & \text{59 years old} \end{pmatrix} + \begin{pmatrix} \text{10000 because John is super lucky} \\ -\text{5000 because Barb's boss is a j\%\&k} \end{pmatrix} \quad (1)$$

Intuitively, a matrix can be represented in a Cartesian space by the coordinates of the vectors it defines. E.g.:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (2)$$

defines a vector (a,b) and a vector (c,d).

2 Matrix operations

2.1 Addition and Subtraction

$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$; $\mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}$

Both matrices must have the same dimensions. Here, both $\mathbf{A}$ and $\mathbf{B}$ are 2×3, so

$\mathbf{A} + \mathbf{B} = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & a_{13} + b_{13} \\ a_{21} + b_{21} & a_{22} + b_{22} & a_{23} + b_{23} \end{pmatrix}$

For example:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 10 & 12 \end{pmatrix} \tag{3}$$

Note that $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$

In-class exercise: in R, calculate $A + A$ (where A is the matrix you created in the previous exercise.

2.2 Multiplication

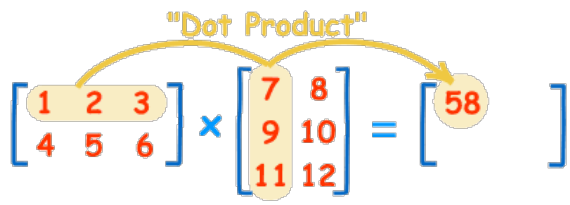


Figure 1: Matrix multiplication

$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$; $\mathbf{C} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{pmatrix}$

Inner dimensions must match: $(r \times c) \times (r \times c)$

$\mathbf{A}$ is 2×3 and $\mathbf{C}$ is 3×2, so the inner dimensions match, and we can calculate $\mathbf{D} = \mathbf{A} \times \mathbf{C}$ as:

$$\mathbf{D} = \begin{pmatrix} a_{11}c_{11} + a_{12}c_{21} + a_{13}c_{31} & a_{11}c_{12} + a_{12}c_{22} + a_{13}c_{32} \\ a_{21}c_{11} + a_{22}c_{21} + a_{23}c_{31} & a_{21}c_{12} + a_{22}c_{22} + a_{23}c_{32} \end{pmatrix}$$

The dimensions of the resulting matrix is 2×2, i.e., the outer dimensions: $(r \times c) \times (r \times c)$

CAREFUL: $\mathbf{A} \times \mathbf{C} \neq \mathbf{C} \times \mathbf{A}$

In-class exercises:

1. Calculate CA

2. What is $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \times \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$?

3. What is $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$?

2.3 Identity matrix

This is an important kind of a matrix. It is simply a square matrix with ones on the main diagonal and 0 everywhere else.

$$I = I_r = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

If the size of I is not specified, it is assumed to be conformable, i.e., as big as needed. Note that any matrix multiplied by I returns itself: $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$

In-class exercise: Rewrite $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ in matrix notation

2.4 Order of operations

This is important and hence worth repeating: matrix multiplication is non-commutative. i.e., the order matters: $\mathbf{AB} \neq \mathbf{BA}$ (at least not necessarily). However, $(AB)C = A(BC)$. I.e., matrix multiplication is associative. Also, $A(B+C) = AB + AC$ and $(A+B)C = AC + BC$.

In-class exercise: in R, create another 4×4 matrix B, also with values drawn from a standard normal distribution. Using R, calculate AB and BA. Note that the multiplication operator for matrices in R is %*%. E.g.,
>A%*%B

Because the order of operations matters, we have to be careful when we 'solve for x'. For example, say we want to solve

$$\begin{aligned}
\mathbf{Ax} &= \mathbf{c} \\
\mathbf{A}^{-1}\mathbf{Ax} &= \mathbf{A}^{-1}\mathbf{c} \quad \text{First pre-multiply by} \mathbf{A} \\
\mathbf{Ix} &= \mathbf{A}^{-1}\mathbf{c} \\
\mathbf{x} &= \mathbf{A}^{-1}\mathbf{c}
\end{aligned}$$

2.5 Transpose

The transpose of a matrix $\mathbf{A}$ is denoted $\mathbf{A}^T$ or $\mathbf{A}'$. To transpose, switch the row and column positions of the matrix, such that : a_{ij} becomes a_{ji}.

In-class exercise: Find the transpose of the following matrices: $x_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$; $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{pmatrix}$; $B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix}$.

In-class exercise: Let $x_2 = c(2,4,1,7)$. Using R, find $x'x$ and xx'. Hint: $\% * \%$ is the matrix multiplication operator, and t() is the function to transpose a matrix.

$$\text{If } \mathbf{A} = \begin{bmatrix} 1 & 5 & 9 \\ 2 & 3 & 8 \\ 6 & 8 & 7 \end{bmatrix}, \text{ then } \mathbf{A}^T = \begin{bmatrix} 1 & 2 & 6 \\ 5 & 3 & 8 \\ 9 & 8 & 7 \end{bmatrix}$$

Figure 2: Transpose of a matrix

This will be useful throughout. For example the sum of squares can be written:

$$\mathbf{x}'\mathbf{x} = \sum_{i=1}^{n} x_i^2.$$

Note that this is very different from $\mathbf{xx}'$. For example:

$$x_1' x_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = 30$$

But

$$x_1 x_1' = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \\ 4 & 8 & 12 & 16 \end{pmatrix}.$$

3 Determinant

The determinant of a matrix $\mathbf{A}$, denoted $|\mathbf{A}|$ or $\det(\mathbf{A})$, summarises the matrix in a way that is useful for systems of linear equations. It also helps us find the inverse of a matrix. The intuition for the determinant is that it is the size of the area (for a 2x2 matrix) or the volume (for a 3x3 matrix) spanned by the combination of the matrix's vectors.

The determinant of a 2×2 matrix is defined by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

In-class: Why is that the formula?

The determinant of a 3×3 matrix is defined by

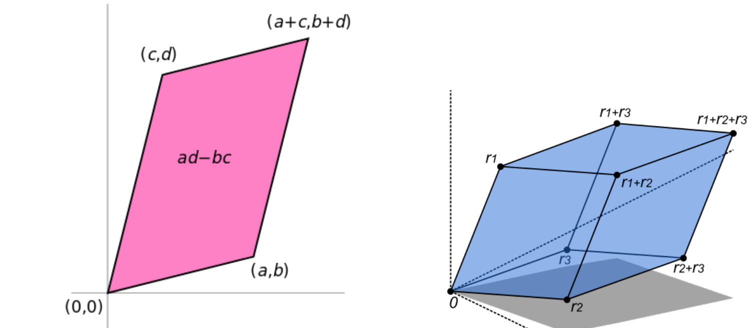
$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a(ei - fh) - b(di - fg) + c(dh - eg)$$

In-class exercise: Calculate

$$\begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{vmatrix}$$

It is not important that you remember how to calculate the determinant. You will never need to do it. But you should understand the intuition. Why should you care? Among other reasons:

- If the determinant of a matrix is 0, it has no inverse. This will be a problem when we need to calculate $(X'X)^{-1}X'y$, since $(X'X)^{-1}$ will be undefined.
- If the rows or columns of $\mathbf{A}$ are linearly dependent, then $|\mathbf{A}| = 0$. This will matter because we will assume that X, our matrix of independent variables, has full rank, i.e., its rows and columns are linearly independent. If they are not, the determinant will be 0 and the inverse will not exist.



(A) (B)

Figure 3: (A) Interpretation of determinant of a 2×2 matrix (A) and a 3×3 matrix (B).

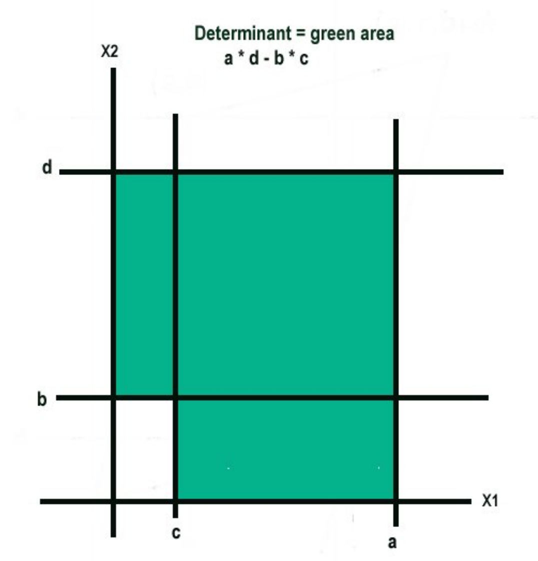


Figure 4: Calculating the determinant.

4 Inverse

The inverse of a matrix is (loosely) the equivalent of division in matrix-world. We say that a matrix is invertible or nonsingular if its inverse exists, i.e. if A^{-1} exists. Note that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$. Note that only square matrices have an inverse.

Finding the inverse of a matrix is not easy, but softwares do it for you. Inverses will be needed when we want to calculate $(X'X)^{-1}X'y$

5 Rank

The rank of a matrix $\mathbf{A}$ is the maximal number of linearly independent rows or columns of $\mathbf{A}$. To check independence, you can try to calculate the determinant (if 0, then independence is violated). The rank is the number of independent rows or columns (whichever is smaller).

When could this matter?

- Suppose that two of your variables are perfectly collinear. For example, you are using temperature in celsius and temperature in Fahrenheit. But C=aF, where a is a constant, and your software will complain (because it cannot calculate $X'X^{-1}$).
- Dummy variable trap. Suppose you want to measure the effect of being female on some variable y. So you write

$$\begin{aligned} y &= b_0 + b_1\text{male} + b_2\text{female} + e \\ &= \mathbf{Xb} + \mathbf{e} \qquad (4) \\ &= \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ \vdots & \ddots & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix} + e \qquad (5) \end{aligned}$$

But then quickly realize that $v_1 = v_2 + v_3$.

Example:

$$\begin{pmatrix} v_1 & v_2 & v_3 & v_4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 & 3 \\ 2 & 4 & 5 & 1 \\ 3 & 6 & 7 & 0 \\ 4 & 8 & 2 & 2 \end{pmatrix}$$

v_1, v_2 and v_3 are independent, but $v_4 = v_1 - v_2 - v_3$

For example, try this is R:

```
#--- v1 and v2 linearly dependent:
v1 = c(1,2,3,4)
v2 = c(2,4,6,8)
v3 = rnorm(4)
v4 = rnorm(4)
A <- matrix(c(v1,v2,v3,v4), nrow=4)
solve(A)

#--- dummy variable version:
v1 = c(1,1,1,1)
v2 = c(1,0,1,0)
v3 = c(0,1,0,1)
A <- matrix(c(v1,v2,v3), nrow=4)
#now try to calculate (A'A) inverse
solve(t(A)%*%A)
```

Listing 1: singularity.R