Lecture 4: Sampling and Probability Distributions

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PO7001: Quantitative Methods I

Probability Theory

Basic terminology

- Sample space: collection of all possible outcomes of an experiment
  - E.g., rolling a die: $S = \{1, 2, 3, 4, 5, 6\}$
- An event is a subset of the sample space
  - E.g., the event A that an even number is obtained is $A = \{2, 4, 6\}$.

Probability Axioms

- The probability of any event is non-negative

$$P(A) \geq 0 \; \forall A \in S, \text{ where S is the sample space}$$

- The probability of 'anything' occurring among all possible events is 1

$$P(S = 1)$$

- If two events are disjoint, the probability that one or the other will occur is the sum of their individual probabilities.

$$P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$$

Random Variables

- A variable: characteristics, quantity that can be measured or counted
- A random variable: variable that can take a set of values with some associated probability.
  - More formally: a random variable is a real-valued function that is defined on S, the sample space. I.e., a function that maps the sample space S to the real numbers.

Random Variables: examples

- E.g., number of heads in an experiment with three coin tosses:

$$X(\{H, T, H\}) = 2$$

$$X(\{H, H, H\}) = 3$$

$$X(\{T, T, H\}) = 1$$

- E.g., drawing a random person from a population and recording their height. This height is a random variable (in this case the sample space is the list of all possible people, and the function is a mapping of the form:

$$f(i) = \begin{cases} 178 & \text{if } i = \mathit{John} \\ 164 & \text{if } i = \mathit{Beth} \\ 192 & \text{if } i = \mathit{Ella} \\ \vdots & \vdots \end{cases}$$

Continuous and discrete R.V.

- If a variable can only take on a finite number of values, we call it discrete.
    - e.g., number of students in the classroom
- A variable that can take on any real value is called continuous
    - e.g., temperature

Distribution of Random Variables

- A random variable has a probability distribution, which specifies the probability that its value takes on a certain value or falls within any given interval.
- Random variables can be
  - discrete $\rightarrow$ probability *mass* function
  - continuous $\rightarrow$ probability density function

Probability Distributions

- Analogous to a frequency distribution, but derived from theory rather than observed data
- A probability distribution is a mathematical description of a random process in terms of the probabilities of its events.
- A probability distribution is like a frequency distribution where $N = \infty$.
- Note: terminology can be quite confusing, as many names refer to the same thing, with small (and inconsistent) variations. For practical purposes:
  - Probability density function = probability distribution function = probability function = probability distribution
  - However, be careful not to confuse “density function” with *cumulative* density function (more below)

Probability (mass) functions for continuous variables

- For a discrete variable, the probability function is

$$f(x) = P(X = x)$$

  - E.g., for a coin toss,

$$f(x) = \begin{cases} 0.5 & \text{if } \; x = \mathit{Heads} \\ 0.5 & \text{if } \; x = \mathit{Tails} \end{cases}$$

- Probability distribution for a single coin flip:

Event	Probability
Heads	0.5
Tails	0.5

- Simple example: the discrete uniform distribution has probability mass function $f(x) = \frac{1}{N}$

Probability (density) functions for continuous variables

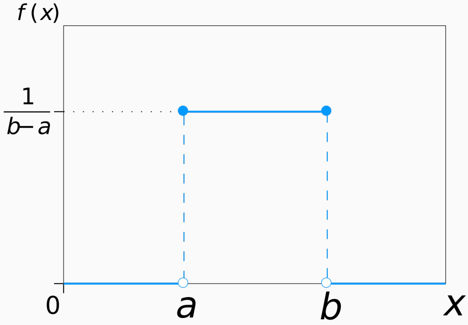
- Just like in the discrete case, the probability distribution function ('p.d.f.') assigns a probability to every possible outcome.
- More precisely, the pdf f is such that the probability that X takes a value in the interval is the integral of f over the interval:

$$P(a \leq X \leq b) = \int_a^b f(x)dx$$

E.g., the Uniform Distribution

For example, the uniform distribution has pdf:

$$f(x) = \begin{cases} \frac{1}{b-a} & \textit{for } a \leq x \leq b \\ 0 & \textit{otherwise} \end{cases}$$



E.g., the Uniform Distribution

- Note that the pdf can take values greater than 0
    - Exercise: in R, plot the density function of a uniform distribution over the interval $[0.2, 0.3]$

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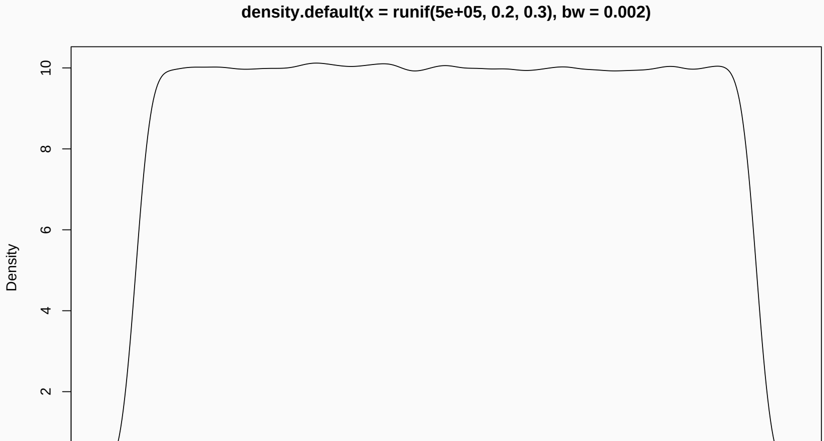
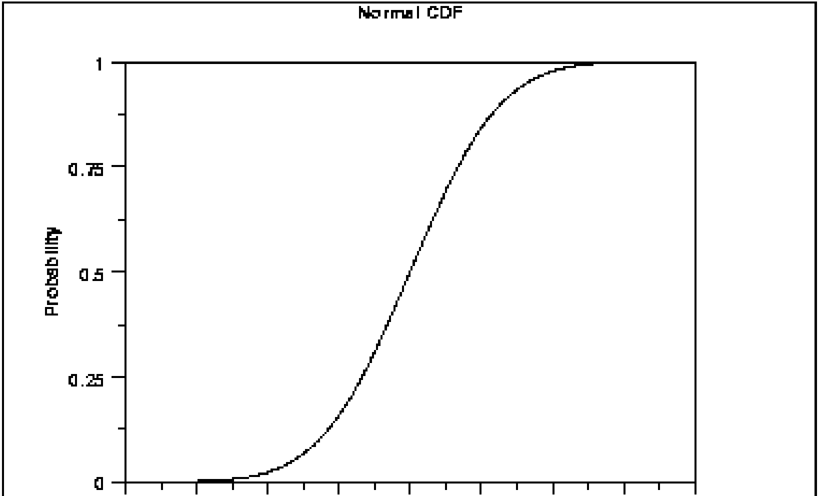
```
plot(density(runif(500000, 0.2,0.3), bw=0.002))
```



The *Cumulative* distribution function

The cumulative distribution function, $F(x)$, (note the use of a capital letter) is the function



Note: law of large numbers

- Law of large numbers: as N increases, the sample mean of a distribution approaches its theoretical mean
- e.g.:

```r
mean(rnorm(10))
```

```
## [1] -0.2983731
```

```r
mean(rnorm(1000))
```

```
## [1] -0.04338509
```

```r
mean(rnorm(100000))
```

```
## [1] -0.0008894246
```

```r
mean(rnorm(1000000))
```

```
## [1] 0.0006929246
```

Important aside: Notation

- Greek vs Roman letters
  - greek letters used for *population* statistics: μ is the mean, σ is the standard deviation, σ^2 is the variance, etc.
  - Roman letters for *sample* statistics: $\bar{x}$, s and s^2

Expected values

Let X be a random variable with a finite number of finite outcomes $x_1, x_2, ..., x_k$ occurring with probabilities $p_1, p_2, ..., p_k$, respectively. The expectation of X is defined as

$$E[X] = x_1 p_1 + x_2 p_2 + \cdots + x_k p_k$$

Expected values: example

Let X represent the outcome of a roll of a fair six-sided die. More specifically, X will be the number of pips showing on the top face of the die after the toss. The possible values for X are $1, 2, 3, 4, 5, 6$, all equally likely (each having the probability of $1/6$).

The expectation of X is

$$\mathrm{E}[X] = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = 3.5.$$

Important Distributions

The Bernoulli distribution

- A Bernoulli random variable has only two possible values: 0 and 1.
- A Bernoulli(p) random variable is defined by $P(X = 1) = p$
  - For example, the toss of one fair coin follows a Bernoulli(.5) distribution.
- The Bernoulli(p) random variable has probability mass function over possible outcomes k:

$$f(k,p) = \begin{cases} p & \textit{if } \text{k=1} \\ 1 - p & \textit{if } \text{k=0} \end{cases}$$

The Bernoulli distribution (cont'd)

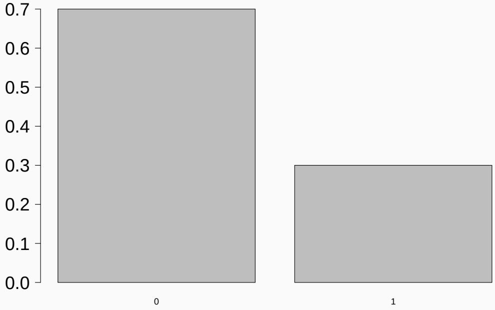
- For Bernoulli(p),

$$\mu = p, \sigma^2 = p(1-p)$$

  - Exercise: prove that for Bernoulli(2p), $\sigma^2 = 2p(1-2p)$
- The Bernoulli distribution is a special case of the binomial distribution, i.e., the case where $n = 1$ (see below)
- Exercise: graph the probability mass function of a Bernoulli(.3) variable.

PMF of A Bernoulli(.3) variable

```
x <- c(rep(0,7), rep(1,3))
b1 <- barplot(c(0.7, 0.3), las=1,
              cex.axis=2, cex.lab=2, names.arg=c(0,1))
```



Binomial Distribution

- Counts the number of successes in Bernoulli trials
- E.g., let X = number of Heads from flipping a (fair) coin 5 times
    - Note: like all variables, this takes a particular outcome and converts it into a number
    - This variable could take any number in [0,5]. i.e., we could get 0 Heads, 1 Heads, . . . 5 Heads

Binomial Distribution

- Question: What is the probability that we get 2 Heads, 3 Heads, etc?
- Possible outcomes: $\{H, T, H, T, T\}$, $\{T, T, T, H, T\}$, etc.
  - How many possible outcomes? 2 for the first flip, 2 for the second, etc. So $2^5 = 32$ equally likely possibilities.
- First, let's think about the probability that we get 0 Heads. There is only one way to get this: $\{T, T, T, T, T\}$, which is one of the 32 possible outcomes. So $P(X = 0) = \frac{1}{32}$
- How about $P(X = 1)$? There are 5 ways to get that, so $P(X = 1) = \frac{5}{32}$

Binomial Distribution

- The binomial distribution counts the number of successes in Bernoulli trials, and there is a formula for it, though we won't go into details

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k},$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

is called the binomial coefficient, n is the number of trials, k the total number of "successes" and p the probability of success in a given trial.

Binomial Distribution: an example

- E.g., what is the probability that we get exactly one Heads if we toss a coin twice?

$$P(X=k)=\binom{2}{1}.5^1(1-.5)^{2-1}=\frac{2\times 1}{1(2-1)!}\times .25=0.5$$

Check in R:

```r
dbinom(1, 2, .5)
```

```
## [1] 0.5
```

Binomial Distribution: another example

- E.g., what is the probability that we get 12 Heads if we toss a coin 20 times?

$$
\begin{aligned}
P(X = k) &= \binom{20}{12} .5^12(1 - .5)^{20-12} && (1)\\
&= \frac{20 \times 19 \times \ldots \times 2 \times 1}{12!(20-12)!} \times 0.0000009536742 && (2)\\
&\approx 0.12 && (3)
\end{aligned}
$$

```r
dbinom(12, 20, .5)
```

```
## [1] 0.1201344
```

Binomial Distribution

When n is large, a reasonable approximation to $B(n, p)$ is given by the normal distribution

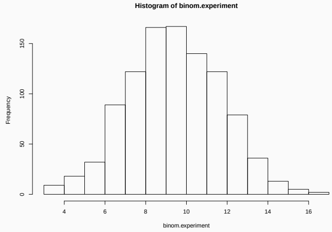
$$\mathcal{N}(np,\ np(1-p))$$

Binomial Distribution: why should I care?

E.g., In the logit, the errors will be drawn from a binomial distribution

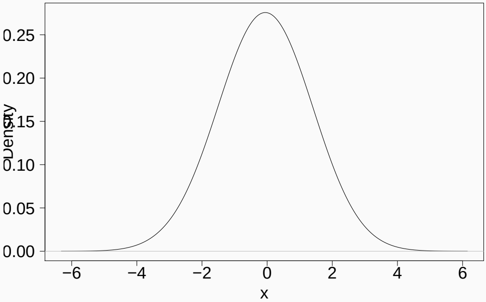
Binomial Distribution: How Generate a random draw from one?

```r
# Let our experiment be the flipping of a fair coin 20 time
binom.experiment <- rbinom(1000, 20, .5)
hist(binom.experiment)
```



The normal distribution

- A variable X is normally distributed if, loosely, it has a ‘bell’ like curve.



The normal distribution

- The most important (and common) distribution in statistics
- It is:
  - Symmetric
  - continuous
  - unimodal
  - follows a specific mathematical form involving two parameters: the mean and the variance. We write for example $X \tilde{} N(\mu, \sigma^2)$ to say that X is normally distributed with mean μ and variance σ^2.

Normal curve: p.d.f.

- It has the pdf

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

- NB: With this formula, we can reconstruct the curve. For example, let $X\tilde{}N(0,1)$. Then:

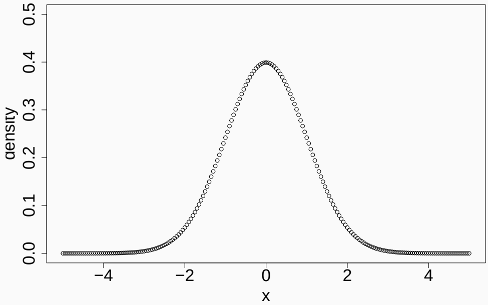
$$f(0) = \frac{1}{1\sqrt{2\pi}} e^{0} = 0.398$$

```
# Check with R
dnorm(0)
```

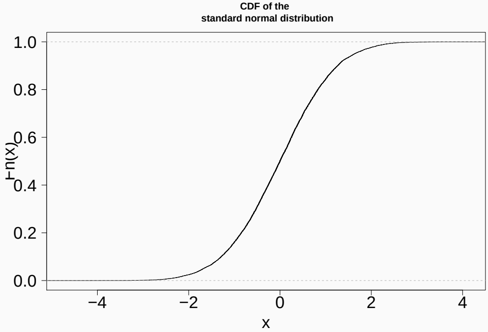
```
## [1] 0.3989423
```

Normal curve: p.d.f.

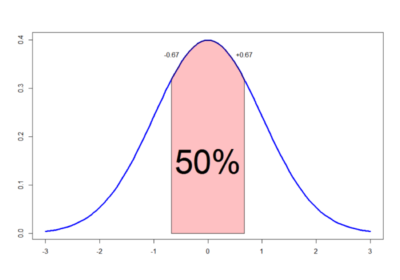
```
plot(c(-5,5), c(0,0.5), type='n', xlab='x', ylab='density'
for(i in seq(-5,5, 0.05)){
   points(i, dnorm(i))
}
```

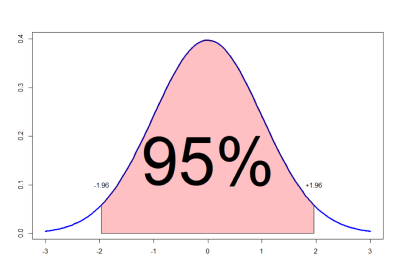


Normal curve: p.d.f.



The area under the normal curve





Standardized scores (z-scores)

You often need to standardize a variable that follows a normal distribution, in particular to transform it into a *standard* normal distribution.

$$z = \frac{x - \mu}{\sigma}$$

Many other distributions that we will come across

- Exponential distribution
- lognormal distribution
- t-distribution
- F-distribution
- χ^2 (chi-squared) distribution

Probability distributions in R

- d, p, q, and r functions
  - d: You know x and want the *density* at this point. I.e., f(x)

```r
dnorm(0)
```

```
## [1] 0.3989423
```

  - p: you know x and want the *area* up to this point. I.e., it gives you F(x)

```r
pnorm(0)
```

```
## [1] 0.5
```

  - q: you have the area and want to know the corresponding x. I.e., it gives you the inverse of the cdf. For example, you want to know which point splits the area in two:

```r
qnorm(0.5)
```

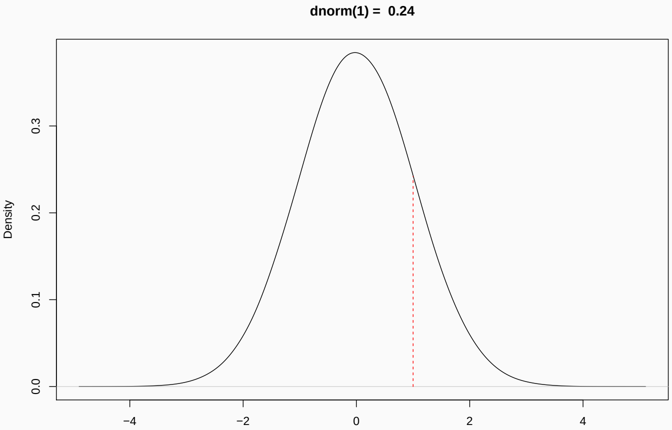
```
## [1] 0
```

  - r: you want to generate random numbers from that distribution

Probability distributions in R

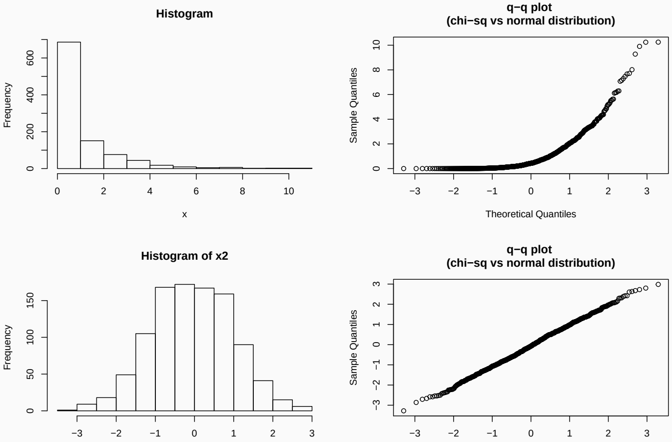
These functions work on many distributions: - rnorm() for the normal distribution - rchisq() for the chi-squared distribution - rf() for the F-distribution - rbinom() for the binomial - rt() for the t distribution

Examples



Quantile plots

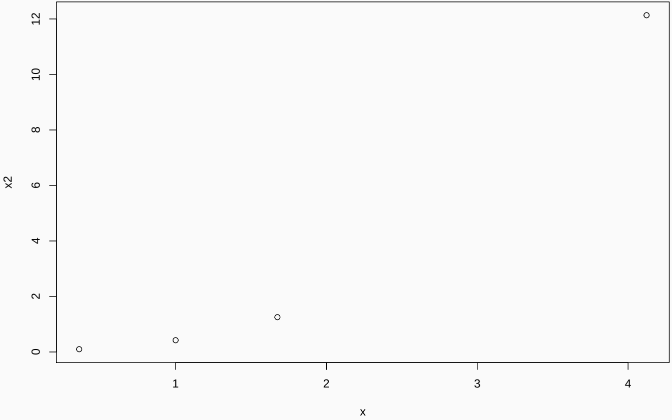
A graphical method for comparing two probability distributions. Of particular interest will be whether a given variable is normally distributed, for example. A Q-Q plot (quantile-quantile) achieves this by plotting the quantiles of two distributions against each other.



qq-plot: a rough example

A very rough example, for intuition purposes, using only 4 quantiles:

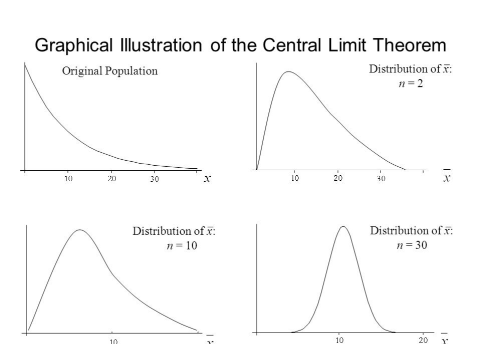
```
x <- quantile(rnorm(1000, 1), probs=c(0.25,0.5,0.75,1))
x2 <- quantile(rchisq(1000, 1), probs=c(0.25,0.5,0.75,1))
plot(x, x2)
```



The central limit theorem (Important)

The central limit theorem is one of the reasons why the normal distribution is so important.

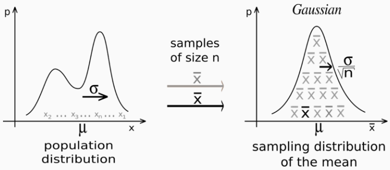
- For a large number of samples, the distribution *of the sample means* will be *normally* distributed, *regardless of the shape of the original distribution*. NB there are exception (e.g., highly skewed distribution, etc.)



The central limit theorem (Important)

http://onlinestatbook.com/stat_sim/sampling_dist/

The central limit theorem



The Central Limit Theorem (formally)

Suppose $\{X_1, \ldots, X_n\}$ is a sequence of i.i.d. random variables with $E[X_i = \mu]$ and $Var[X_i] = \sigma^2 < \infty$.

Then as n approaches infinity, the random variables $(\bar{X}_n - \mu)$ converge in distribution to a normal $N(0, \frac{\sigma^2}{\sqrt{n}})$:

$$\left(\bar{X}_n - \mu\right) \xrightarrow{d} N\left(0, \frac{\sigma^2}{n}\right).$$

Sampling distribution of the mean (optional)

Why σ^2/n?

Let $X = \{x_1, x_2, \ldots, x_n\}$ and $m(X)$ be the mean of X, i.e., $m(X) = \frac{1}{n}(x_1 + x_2 + \ldots + x_n)$

then:

$$
\begin{aligned}
Var[m(X)] &= var[m(x_1, x2, \ldots, x_n)] && (4)\\
&= Var[\frac{1}{n}(x_1 + x2 + \ldots + x_n)] && (5)\\
&= (\frac{1}{n})^2(Var(x_1) + Var(x_2) + \ldots + Var(x_3)) && (6)\\
&= (\frac{1}{n})^2(\sigma^2 + \sigma^2 + \ldots + \sigma^2) && (7)\\
&= \frac{1}{n^2}(n\sigma^2) && (8)\\
&= \frac{\sigma^2}{n} && (9)
\end{aligned}
$$