Inference at the data's edge: Characterizing and including counterfactual uncertainty with Gaussian Processes

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The problem

We rely on models to fit things like $\mathbb{E}[Y(d)|X]$, but use their predictions at values of X that may be at or near the edge of the data.

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We would like an approach that can, under reasonable assumptions, characterize (counterfactual) uncertainty at or beyond edge of data and include it in our inferences.

GP intuition: Ignorance and neighbors

If I tell you Y^* has mean 0 and variance σ^2, all you can believe is.

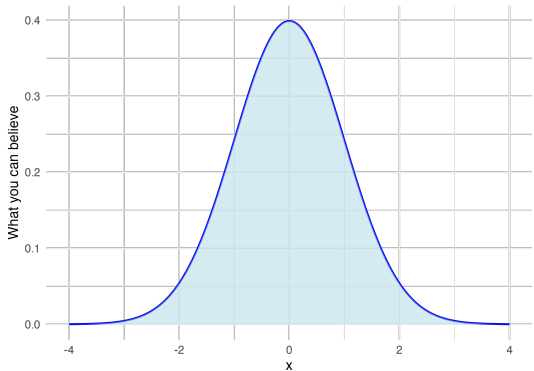

Figure: Total ignorance

GP intuition: Ignorance and neighbors

Now I tell you we see $Y_{obs} = -1$, and $cor(Y^*, Y_{obs}) = .8$. Then you believe:

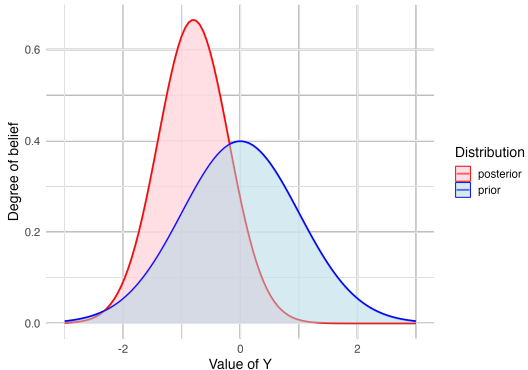

Figure: Informed ignorance

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Big idea: observations nearer each other in X should have similar Y, whatever the values:

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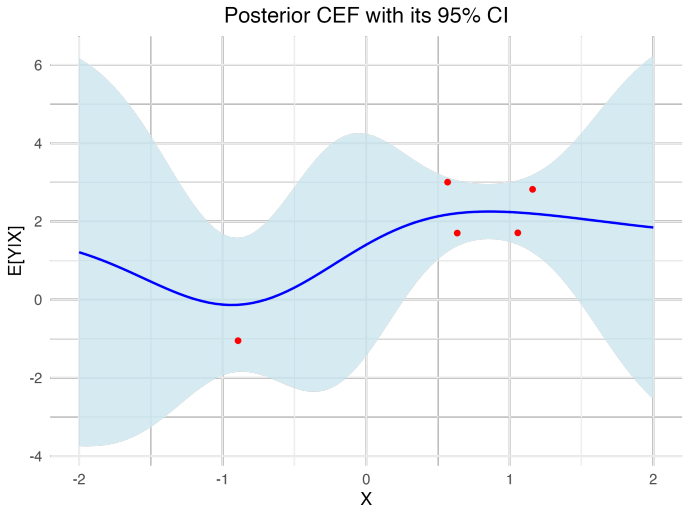
At values of x^* far from observed points, low covariance of observed outcomes with Y^* leaves us more uncertain about $p(Y^*)$.

Inferences given a few observations

Given GP assumptions and 5 observations of $\{X_i, Y_i\}$, ask for predictions at any X:



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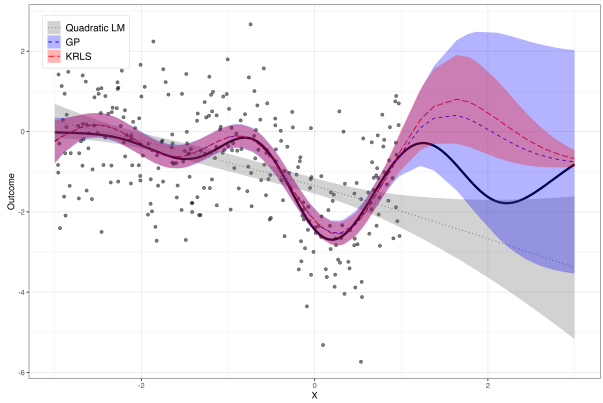
7. "Conditioning on the observations", beliefs about the distribution of test points (Y^*) is
$$Y^* \sim \mathcal{N}(K_* \underbrace{(K + \sigma^2 I)^{-1} Y}_{\text{KRLS coefs.}}, \underbrace{K_{*,*} + \sigma^2 I}_{\text{prior var}} - \underbrace{K_*^\top (K + \sigma^2 I)^{-1} K_*}_{\text{reduction due to data}}) \quad (1)$$

Uncertainty and extrapolation in 3 models:



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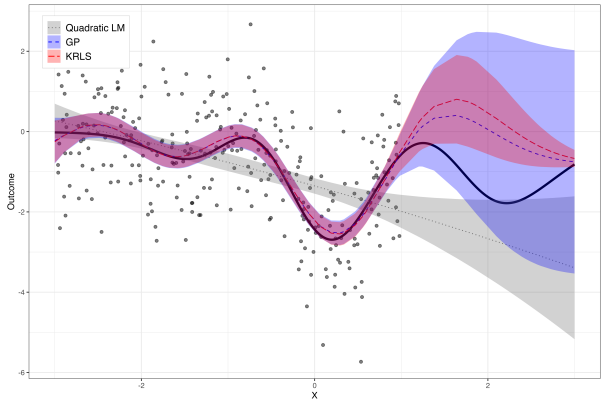

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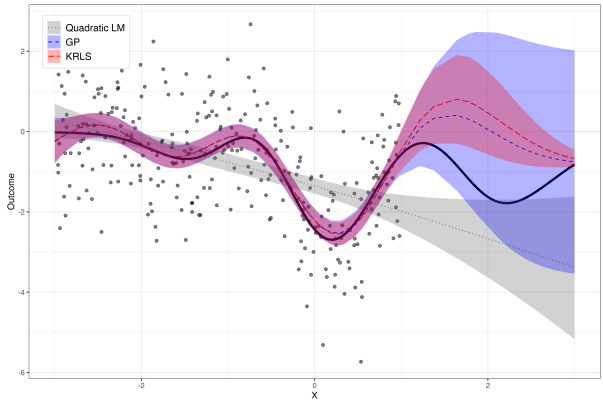

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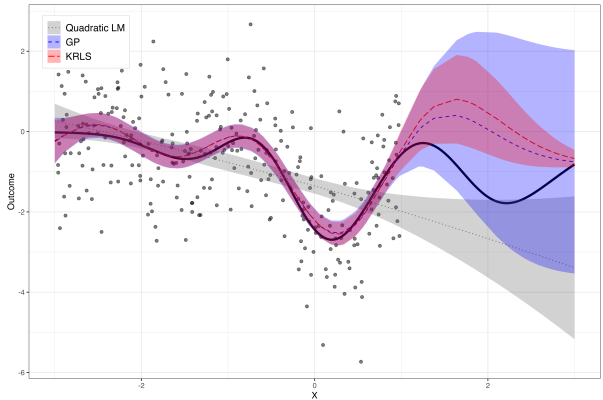

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- GP and KRLS have same CEF/*maximum a posteriori*, including the "return to mean"
- But with GP the uncertainty shows our ignorance farther from the data

Additional details

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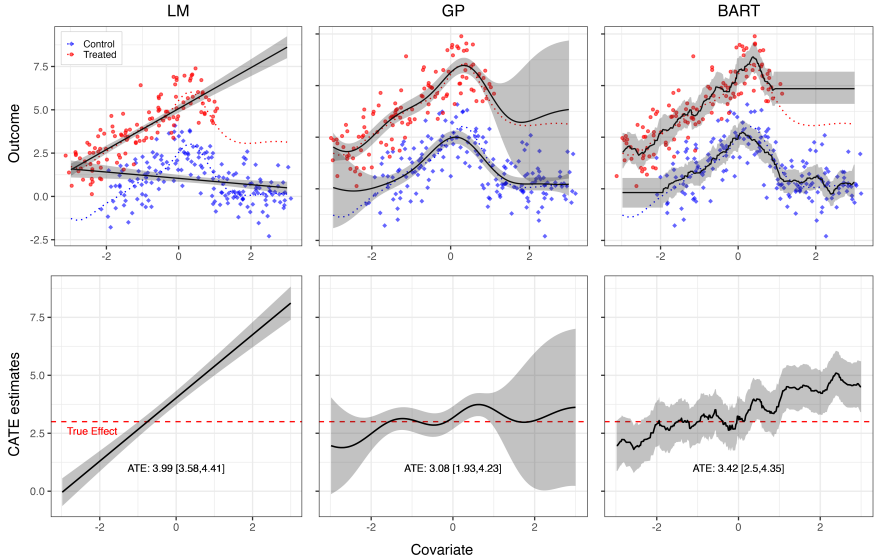
- in case of the Gaussian with bandwidth b, choose $\underset{b}{argmax}\ Var(\mathbf{K})$.
- The resulting procedure avoids all user-chosen hyperparameters and works very well empirically.

Case 1: Comparing groups with poor overlap

Simulation setting:

- $-3 < X < 3$, but no controls below X=-2, no treated above X=1.
- CEF drawn from a space of random functions on each iteration
- Truth: constant treatment effect of 3
- Compare approaches on ATE, CATE, coverage of both.

Case 1: Checking one draw



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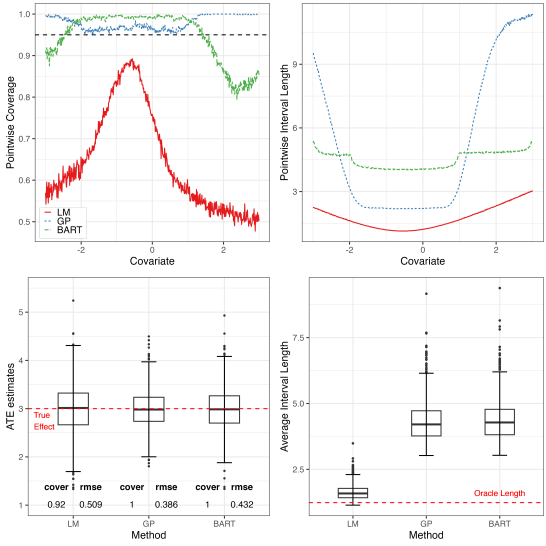

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- That is, interpret modestly! e.g.,

  "This is our best estimate of how the outcome might look absent the treatment, given the prior trend information, and if nothing else of importance to the outcome were to occur over the post-treatment window"

Case 2: Heller effect (application)

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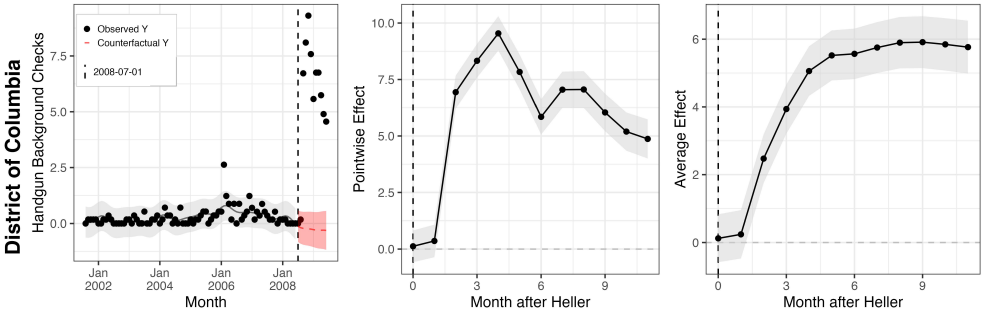
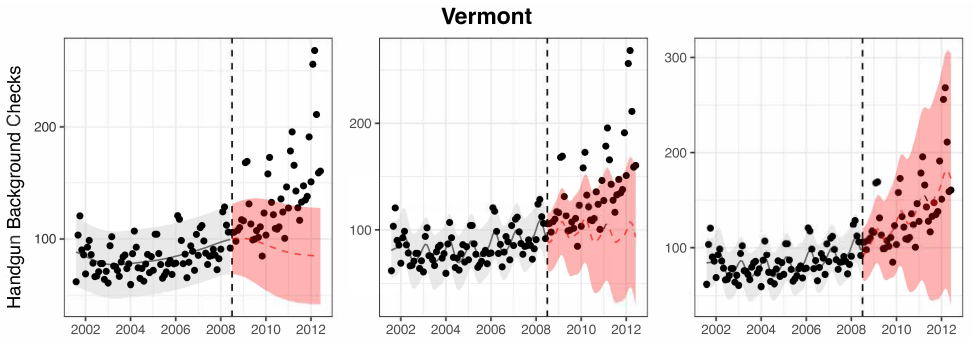
- Kernel: Gaussian + periodic + polynomial.

Case 2: Heller and firearms in DC



Case 2: Not so clear in Vermont

Gaussian (*left*), Gaussian+periodic+linear (*center*), Gaussian+periodic+quadratic (*right*)



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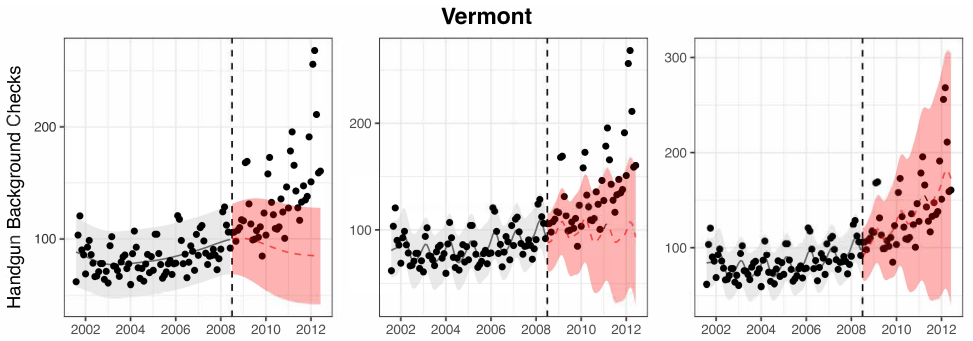

Defensible claims (still) depend on assumptions from outside the data,
here involving the space of functions you cannot rule out for extrapolation.

Case 3: Regression Discontinuity

RD always involves estimating to the edge of the data, or slightly past it:

- $\widehat{Y(0)}_{Z=c}$: Train $\mathbb{E}[Y(0)|Z]$ on $Z < c$ and predict at $Z = c$
- $\widehat{Y(1)}_{Z=c}$: Train $\mathbb{E}[Y(1)|Z]$ on $Z > c$ and predict at $Z = c$

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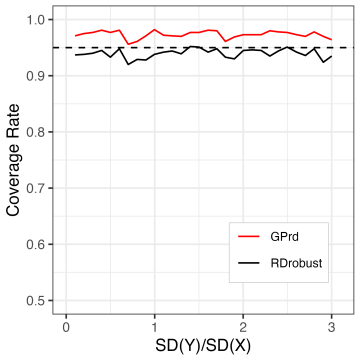
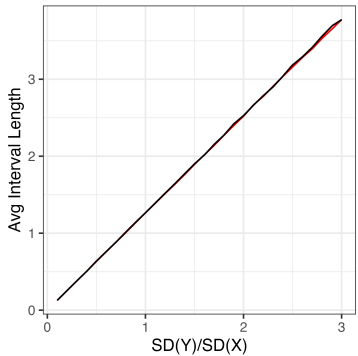
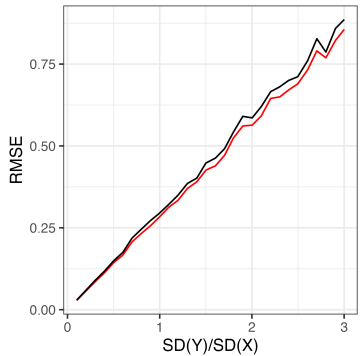
For investigators with small sample RDs, can GP help?

RD: random Z and Y, large sample

$\tau = 3$ (or $\tau = 0$), $N = 500$, `rdrobust` at defaults (robust estimate), `GPrd` at defaults







RD: random Z and Y, small sample

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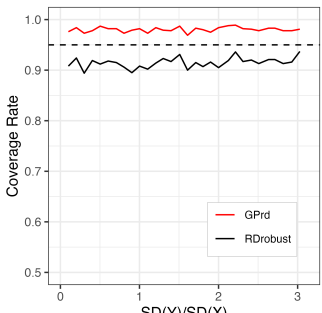

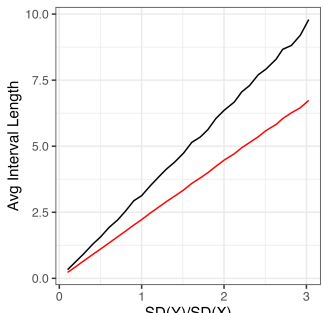

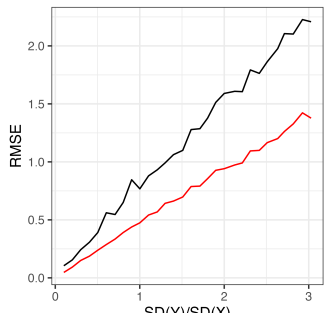

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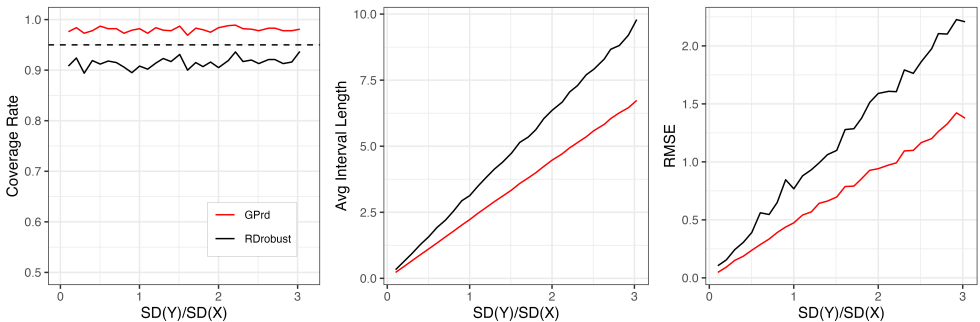

(Similar story if you posit a latent variable confounding running variable and outcome)

Placebo cutoffs in close elections (application)

Lee (2004) close election setting:

- Forcing variable: Democrats' vote share in US House election (1948 - 1990)
- Outcome variable: ADA score (liberal vote score of each representative)

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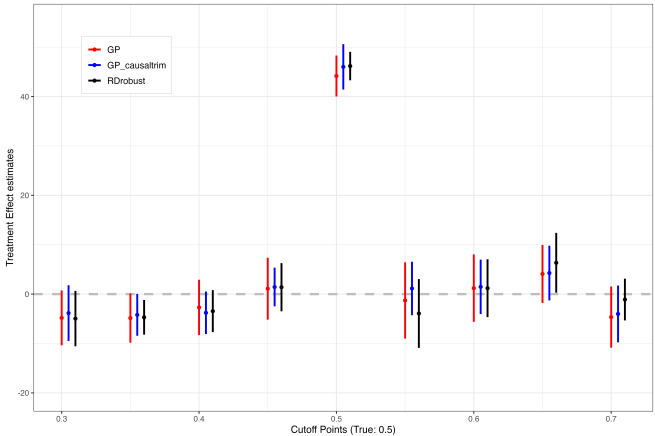
Another twist: Should our RD estimates be informed by some *causal assumptions* regarding comparability?

- Should we worry if an automated RD procedure chooses a very wide bandwidth?
- In the `gp_causaltrim` approach we:
  1. use small bandwidth to limit comparison range: e.g. within 2% vote share, cov=cor=0.9 $\rightarrow$ b=0.005
  2. trim: remove data outside useful range ($\pm$10%, so scaling and σ^2 choice are focused)

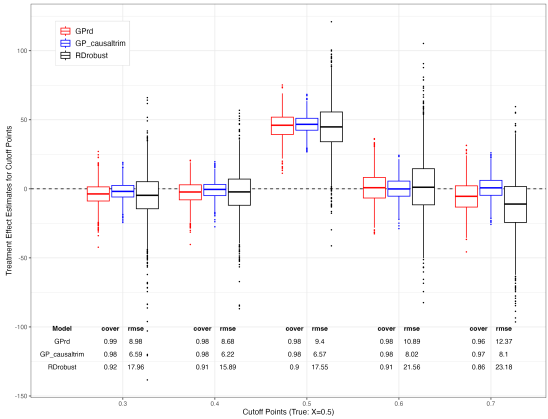
Placebo cutoffs in close elections: Large sample

Full sample: $N = 13,577$ (though each cutoff analysis sees only part of this)



Placebo cutoffs in close elections: Small samples

Repeated sampling of sub-samples: $N = 200$ in each segment



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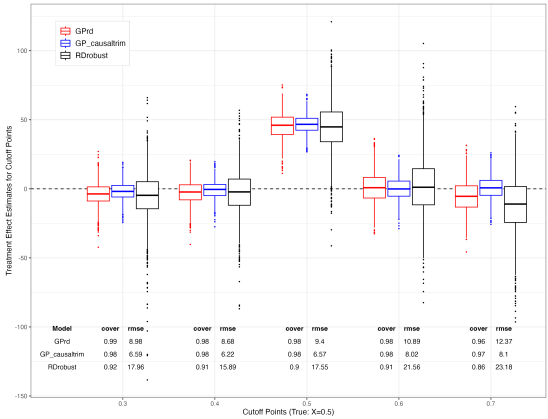

`GP_causaltrim`: conveniently does best while being a bit more transparent, restrictive.

Wrap-up: Entailing counterfactual uncertainty

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- limitations: violation of $Y \sim \mathcal{N}(0, K)$; uncertainty ultimately limited to σ^2

We also hope to improve accessibility through our explanations and simplified implementation...

Give it a go!

https://doeunkim.org/gpss/