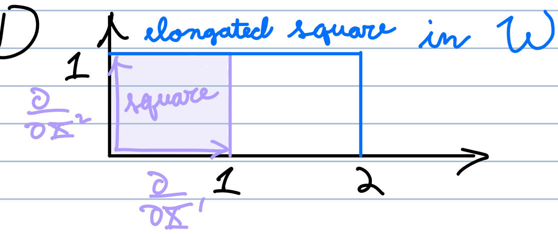
# 12. HW 3

## Continuous Theory

① 2D Square Elongation
( youtube example )

- Calculate $F^i_a$

① 




$$\mathcal{M} \xrightarrow{\ \phi\ } \mathcal{W}$$

$$\frac{\partial}{\partial X^a} \qquad\qquad \frac{\partial}{\partial x^i}$$

$$\vec{x} = \phi(\vec{X}) = \begin{array}{c} \\ \\ \end{array}\begin{bmatrix} 2X_1 \\ X_2 \end{bmatrix} \begin{array}{l} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{array} \ \downarrow i$$

$$F^i_a = \frac{\partial \phi^i}{\partial X^a} = i\downarrow \overset{a\ \longrightarrow}{\begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} \end{bmatrix}}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

② 3D Abstract Time-Dependent Motion (youtube ex.)

- Calculate $F^i_a$

② Assume :

$$\phi(X,t) = \begin{bmatrix} 3X_1^2 X_2 + tX_3^3 \\ X_3^2 - tX_1 \\ 5X_2X_3 \end{bmatrix} \downarrow i$$

$$F^i_a = \frac{\partial \phi^i}{\partial X^a} = i\downarrow \overset{a\ \longrightarrow}{\begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix}}$$

$$= \begin{bmatrix} 6X_1X_2 & 3X_1^2 & 3tX_3^2 \\ -t & 0 & 2X_3 \\ 0 & 5X_3 & 5X_2 \end{bmatrix}$$

- F may not be symmetric.
- F is a fn of X and t.

# Discrete Theory

① What are the steps involved in doing an elastic body simulation?

**Note**: the subscript "c" means "per cell" and the subscript "v" means "per vertex".

Equivalent approach:

$C_c \rightarrow \mathcal{U}_c \rightarrow \mathcal{U}_{tot} = \sum_c \mathcal{U}_c$

$f = \frac{\partial \mathcal{U}_{tot}}{\partial x_i}$

① (i) $\vec{x}_i$ (flow map per vertex)

(ii) $F_c$ (deform. grad.)

(iii) $C_c = F_c^T F_c$ (induced metric)

(iv) $E_c = \frac{1}{2}(C_c - I)$ (strain tensor)

(v) Find appropriate stress-strain relation. One possible choice:

$S_c = 2\mu E_c + \lambda \, tr(E_c) I$ ($2^{nd}$ Piola stress)

(vi) $P_c = F S_c$ ($1^{st}$ Piola stress)

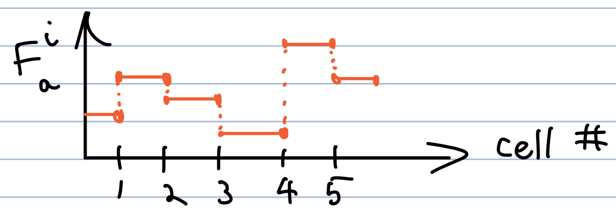
(vii) $\vec{f}_v = \frac{1}{n} \sum_{c \succ v} P_c \hat{n}_{c,v} A_{c,v}$

( = div($P_c$) = discrete divergence)

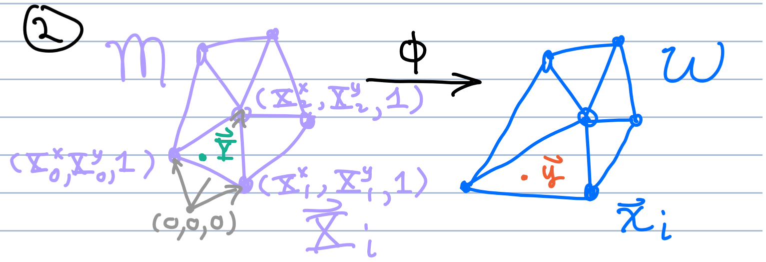
(ix) $\dot{\vec{v}}_v = \vec{f}_v / m_v$, $\dot{\vec{x}}_v = \vec{v}_v$

② Calculate $F_c$.

**Note**: We have a different $F_c$ for each cell. Thus, $F^i_a$ is piecewise constant.




②




$\vec{x}_i = \phi(\vec{X}_i)$ is the discrete (finite-dimensional) flow map.

$\vec{y} = \phi(\vec{Y})$ is the continuous (infinite-dimensional) flow map.

$F^i_a = \frac{\partial \phi^i}{\partial Y^a}$ ← $i$ increment over $\dim(\mathcal{W}) = 2$; ← $a$ increment over $\dim(\mathcal{M}) = 2$

Given $\begin{bmatrix} y^x \\ y^y \\ 1 \end{bmatrix} = \phi_c(\vec{Y}) = AB \begin{bmatrix} Y^x \\ Y^y \\ 1 \end{bmatrix}$

$$A = \begin{bmatrix} x^x_0 & x^x_1 & x^x_2 \\ x^y_0 & x^y_1 & x^y_2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} X^x_0 & X^x_1 & X^x_2 \\ X^y_0 & X^y_1 & X^y_2 \\ 1 & 1 & 1 \end{bmatrix}^{-1}$$

$$Q = AB = \begin{bmatrix} Q^x_a & Q^x_b & Q^x_c \\ Q^y_a & Q^y_b & Q^y_c \\ Q^z_a & Q^z_b & Q^z_c \end{bmatrix}$$

$$\therefore \quad \phi_c(\vec{\Upsilon}) = \begin{bmatrix} Q_a^x \Upsilon^x + Q_b^x \Upsilon^y + Q_c^x \\ Q_a^y \Upsilon^x + Q_b^y \Upsilon^y + Q_c^y \\ Q_a^z \Upsilon^x + Q_b^z \Upsilon^y + Q_c^z \end{bmatrix} \Big\downarrow i$$

$$F_c = \frac{\partial \phi_c^i}{\partial \Upsilon^a} = \begin{bmatrix} Q_a^x & Q_b^x \\ Q_a^y & Q_b^y \end{bmatrix} \qquad \begin{matrix} i = 0, 1 \\ a = 0, 1 \end{matrix}$$

(per cell)

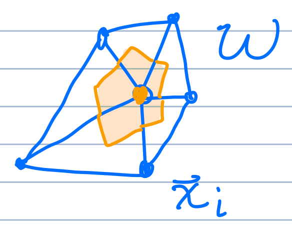
Alternative approach for getting $F_c$ (produces equivalent result):

$$F_c = -\frac{1}{nV_c} \sum_{j=0}^{n} \begin{bmatrix} | \\ x_j \\ | \end{bmatrix} \begin{bmatrix} - & A_{c,j} n_{c,j}^T & - \end{bmatrix}$$

We use this to derive our discrete divergence operator in $\vec{f}_v = \mathrm{div}(P)$.

③ Calculate $m_v$.

③ $$m_v \approx \rho \sum_{c \ni v} \frac{1}{n+1} V_c$$




$m_v$ is a diagonal matrix (which makes matrix inversion easier).