Chapter 1

Modal Synthesis of Strings

Modal synthesis is a useful physics-based modeling technique for achieving real-time simulation of systems with complex damping laws [1]. Based on the frequency-domain decomposition of acoustic bodies into normal modes of vibration, modal synthesis is well-suited to many systems of interest in sound synthesis. In this chapter, a description of real-time modal synthesis of vibrating strings is provided. Forcing was modeled by applying an input signal at a discrete point along the string. Following a brief overview of modal synthesis, we review the equation describing the motion of strings, explain the modal synthesis algorithm for strings, then discuss the MATLAB code used to implement the algorithm.

1.1 Overview of Modal Synthesis

To illustrate how modal synthesis works, let's consider a string of finite length $L = 1$ excited into motion by the triangular shape $g(x)$ defined by

$$g(x) \;=\; \begin{cases} \frac{3}{10}x & 0 \leq x \leq \frac{1}{3}, \\ \frac{3(1-x)}{20} & \frac{1}{3} \leq x \leq 1. \end{cases}$$

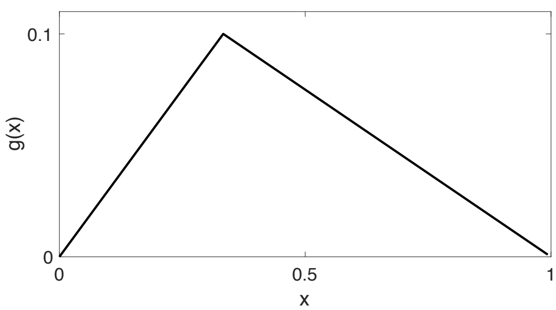


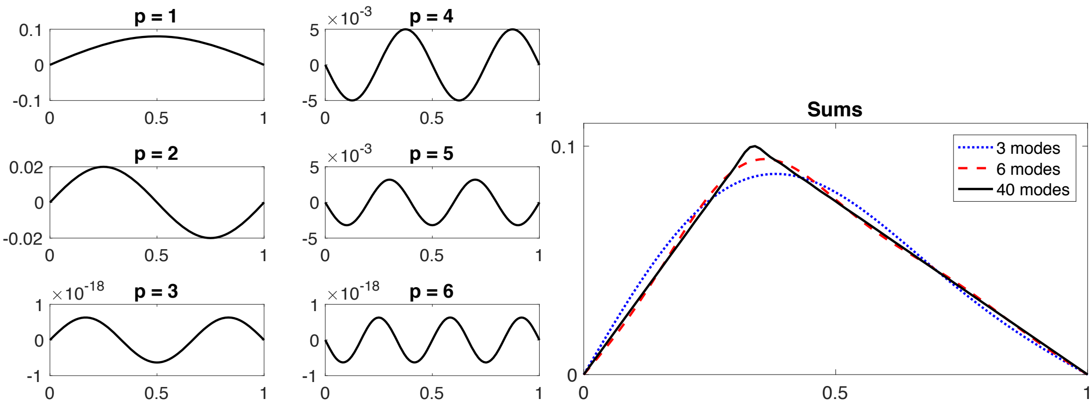
Figure 1.1: Plot of the triangular excitation shape $g(x)$.

A plot of $g(x)$ is shown in Fig. 1.1. In the modal synthesis framework, the formula used to model the displacement $u(x,t)$ of the vibrating string takes the form of an N-level series expansion given by

$$u(x,t) \;=\; \sum_{p=1}^{N} U_p(x)\Phi_p(t) \tag{1.1}$$

where $U_p(x)$ are the shapes of each normal mode, $\Phi_p(t)$ are the time-dependent amplitudes of the mode shapes, and N is the number of modes included in the series expansion. In other words, the moving string is modeled by summing appropriately weighted amounts of each normal mode of vibration at each time step of the acoustic simulation. Figure 1.2a shows the first six weighted mode shapes of the string at time $t = 0$ seconds. As depicted in Fig. 1.2b, when more weighted modes are added to the output sum $u(x,0)$, the composite shape approaches the exact shape of the string when it is initially excited by the function $g(x)$. We may continue to add more modes as long as the value of the highest modal frequency is less than or equal to the Nyquist frequency of the simulation.

As the sound simulation proceeds, different amounts of each normal mode are summed in order to produce the correctly evolving shape of the moving string. Specifically, the amplitude of each normal mode is modulated in time by the values of $\Phi_p(t)$. Figure 1.3 graphs the modal solution to the vibrating string at six separate moments in time using $N = 40$ normal modes.



(a) Weighted mode shapes **(b)** Sums of mode shapes

Figure 1.2: Plots of (a) the first six weighted mode shapes (i.e. $U_1(x)\Phi_1(0)$, $U_2(x)\Phi_2(0)$, etc.) and (b) the sums $u(x,0)$ with $N = 3$, $N = 6$, and $N = 40$ weighted mode shapes of a string excited by the triangular shape $g(x)$ at time $t = 0$ seconds. The y-axis of each plot represents the transverse displacement, and the x-axis represents the length along the string.

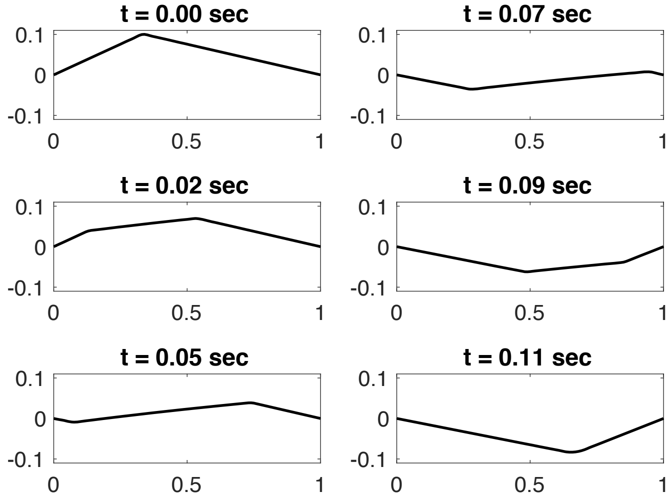


Figure 1.3: Modal solution of a vibrating string at six moments in time. The string was excited by the triangular shape $g(x)$ at time $t = 0$ seconds. The modal solution includes $N = 40$ modes of vibration. The speed of sound in the string was set to 9 meters per second.

1.2 Physical Model of a String

The displacement $u = u(x,t)$ of a lossy forced string of finite length L and fixed boundary conditions (i.e. $u(0,t) = u(L,t) = 0$) is described by the 1D wave equation with loss and forcing [2]:

$$u_{tt} = c^2 u_{xx} - 2\sigma_0 u_t + \delta F \tag{1.2}$$

Here, c is the speed of sound in the string, σ_0 is a non-negative damping constant, $\delta = \delta(x - x_i)$ is the 1D delta function centered at the excitation location x_i, and F is the time-dependent forcing signal. Note that $u_t = \frac{\partial u}{\partial t}$, $u_{tt} = \frac{\partial^2 u}{\partial t^2}$, and $u_{xx} = \frac{\partial^2 u}{\partial x^2}$. The real-valued modal solution to Eq. (2.1) takes the form of a semi-infinite series expansion:

$$u(x,t) = \sum_{p=1}^{\infty} U_p(x)\Phi_p(t) \tag{1.3}$$

The spatial mode shapes $U_p(x)$ are modulated in time by the amplitudes $\Phi_p(t)$. Each term is defined by [3]

$$U_p(x) = \sin\left(\frac{p\pi x}{L}\right) \tag{1.4a}$$

$$\Phi_p(t) = a_p\cos(\omega_p t) + b_p\sin(\omega_p t) \tag{1.4b}$$

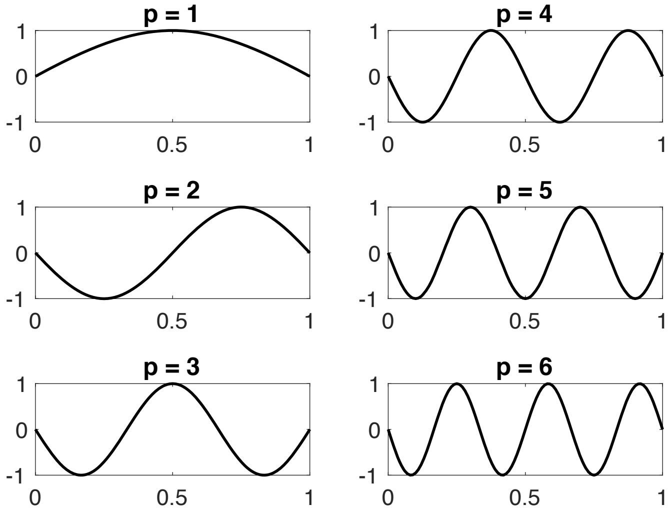


Figure 1.4: Plots of the first six normal mode shapes $U_{1-6}(x)$ of a vibrating string fixed at both ends. The y-axis of each plot represents the transverse displacement, and the x-axis represents the length along the string.

where p is the integer mode index, ω_p are the angular modal frequencies, and a_p and b_p are coefficients determined by the initial conditions of the string vibration. The angular modal frequencies ω_p are given by [3]

$$\omega_p = \frac{p\pi c}{L} \tag{1.5}$$

The first six mode shapes $U_{1-6}(x)$ are plotted in Fig. 2.1.

1.3 Modal Synthesis of a String

To compute sound from the physical model of a vibrating string, we must solve for the displacement $u = u(x,t)$ at each time step. To do this, we first insert the modal solution given by Eq. (1.3) into the 1D wave equation defined by Eq. (1.2) as follows

$$\left[\sum_{p=1}^{\infty} U_p(x)\Phi_p(t)\right]_{tt} = c^2\left[\sum_{p=1}^{\infty} U_p(x)\Phi_p(t)\right]_{xx} - 2\sigma_0\left[\sum_{p=1}^{\infty} U_p(x)\Phi_p(t)\right]_{t} + \delta F$$

where once again the subscripts tt, xx, and t denote differentiation. After differentiating the terms, we get

$$\sum_{p=1}^{\infty} U_p(x)\ddot{\Phi}_p(t) = c^2\sum_{p=1}^{\infty}\left(\frac{-p^2\pi^2}{L^2}\right)U_p(x)\Phi_p(t) - 2\sigma_0\sum_{m=1}^{\infty} U_p(x)\dot{\Phi}_p(t) + \delta F$$

Next, we multiply both sides by $U_m(x) = \sum_{m=1}^{\infty} \sin\left(\frac{m\pi x}{L}\right)$ and integrate over the length of the string as follows

$$\begin{aligned}
\int_0^L \left[\sum_{m=1}^{\infty}\sum_{p=1}^{\infty} U_m(x)U_p(x)\ddot{\Phi}_p(t)\right]dx \;&=\; \int_0^L \left[c^2\sum_{m=1}^{\infty}\sum_{p=1}^{\infty}\left(\frac{-p^2\pi^2}{L^2}\right)U_m(x)U_p(x)\Phi_p(t)\right]dx \\
&- \int_0^L \left[2\sigma_0\sum_{m=1}^{\infty}\sum_{p=1}^{\infty} U_m(x)U_p(x)\dot{\Phi}_p(t)\right]dx \\
&+ \int_0^L \left[\sum_{m=1}^{\infty} U_m(x)\delta F\right]dx
\end{aligned}$$

As discussed in [3], if two functions f and g are orthogonal on the interval (a, b), then $\int_a^b f(x)g(x)dx = 0$. Therefore, since the functions $U_m(x)$ and $U_p(x)$ are orthogonal whenever $p \neq m$, the integrals reduce to

$$\begin{aligned}
\int_0^L \left[\sum_{p=1}^{\infty} U_p(x)^2\ddot{\Phi}_p(t)\right]dx \;&=\; \int_0^L \left[c^2\sum_{p=1}^{\infty}\left(\frac{-p^2\pi^2}{L^2}\right)U_p(x)^2\Phi_p(t)\right]dx \\
&- \int_0^L \left[2\sigma_0\sum_{p=1}^{\infty} U_p(x)^2\dot{\Phi}_p(t)\right]dx \\
&+ \int_0^L \left[\sum_{p=1}^{\infty} U_p(x)\delta F\right]dx
\end{aligned}$$

Given that $\int_a^b f(x)\delta(x - x_i)dx = f(x_i)$ according to [3] and $\int_0^L \sin^2\left(\frac{p\pi x}{L}\right)dx = \frac{L}{2}$, we have

$$\frac{L}{2}\sum_{p=1}^{\infty}\ddot{\Phi}_p(t) \;=\; \frac{L}{2}\sum_{p=1}^{\infty}\left(\frac{-p^2\pi^2c^2}{L^2}\right)\Phi_p(t) - \;\sigma_0 L\sum_{p=1}^{\infty}\dot{\Phi}_p(t) + \;\sum_{p=1}^{\infty} U_p(x_i)F$$

Finally, substituting $\omega_p^2 = \frac{p^2\pi^2c^2}{L^2}$ and removing the sums for convenience, we see that the time-dependent amplitudes Φ_p take the form of a parallel combination of lossy forced harmonic oscillators:

$$\ddot{\Phi}_p \;=\; -\omega_p^2\Phi_p - 2\sigma_0\dot{\Phi}_p + \frac{2}{L}\sin\left(\frac{p\pi x_i}{L}\right)F \tag{1.6}$$

Frequency-dependent loss is modeled by assigning unique damping coefficients σ_p to each mode of vibration. That is, the vector of damping coefficients σ_p takes the place of the single damping coefficient σ_0. The finite-difference scheme for Eq. (1.6) then takes the form [2]:

$$\delta_{tt}\Phi_p = -\omega_p^2\Phi_p - 2\sigma_p\delta_{t.}\Phi_p + \frac{2}{L}\sin\left(\frac{p\pi x_i}{L}\right)F^n \tag{1.7}$$

where F^n is a time vector of sampled values of the force F, and n is the time step. The difference operators δ_{tt} and $\delta_{t.}$ are defined by [2]

$$\delta_{t.} = \frac{1}{2k}(e_{t+} - e_{t-}) \tag{1.8a}$$

$$\delta_{tt} = \frac{1}{k^2}(e_{t+} - 2 + e_{t-}) \tag{1.8b}$$

where k is the duration between time samples, and e_{t+} and e_{t-} are operators that shift the operand forward and backward in time, respectively. The numerical solution to Eq. (1.7) is given by

$$\Phi_p^{n+1} = \frac{1}{1+\sigma_p k}\left[(2-\omega_p^2 k^2)\Phi_p^n + (\sigma_p k - 1)\Phi_p^{n-1} + \frac{2k^2}{L}\sin\left(\frac{p\pi x_i}{L}\right)F^n\right] \tag{1.9}$$

The output sound signal u^{n+1} is computed using the following N-level numerical approximation to Eqn. (1.3) at a read-out position[1] $x = x_0$ along the string:

$$u^{n+1}(x_0, t) = \sum_{p=1}^{N} U_p(x_0)\Phi_p^{n+1} \tag{1.10}$$

Here, N is the total number of modes included in the truncated series expansion. The frequency f_N corresponding to the highest mode N must be less than or equal to the Nyquist frequency of the simulation. Since Φ_p^{n+1} was computed using a finite-difference approach, the Nyquist frequency of the simulation is $\frac{f_s}{\pi}$, where f_s is the sampling frequency. That is,

$$f_N \leq \frac{f_s}{\pi} \tag{1.11}$$

The highest modal frequency f_N can be expressed in terms of the mode number N using Eq. (1.5), where $f_p = \frac{\omega_p}{2\pi} = \frac{pc}{2L}$:

[1]The point x_0 must lie in the range $0 < x \leq L$.

$$f_N = \frac{Nc}{2L} \tag{1.12}$$

Substituting Eq. (1.12) into Eq. (1.11) and solving for N, we find the limit on N to be

$$N \leq \left\lfloor \frac{2Lf_s}{c\pi} \right\rfloor \tag{1.13}$$

1.4 MATLAB Implementation

The modal synthesis algorithm described in the previous section was implemented using the MATLAB programming language. First, the force vector F^n was defined to be an impulse signal. The global string parameters listed in Table 1.1 were then set. Estimated values for an acoustic nylon guitar string tuned to 110 Hz are given in the table. The parameter ranges for `L`, `T`, `r`, and `rho` were chosen by calculating a margin of approximately ±50% of the values estimated for the string. For example, ±50% of 0.6 meters produced a range of 0.3 – 0.9 meters for the string length. More variability was added to the decay scale and decay rate in order to reflect the fact that materials

Table 1.1: Global string parameters. Estimated values for an acoustic nylon string tuned to 110 Hz are listed. Corresponding ranges recommended for the synthesized string are given.

String Parameter	MATLAB Symbol	Nylon String	Range	Units
Length	`L`	0.6	0.3 – 0.9	m
Tension	`T`	11	5 – 15	N
Radius	`r`	4×10^{-4}	$2 - 6\times10^{-4}$	m
Density	`rho`	1150	500 – 1600	$\frac{\text{kg}}{\text{m}^3}$
T60 Decay Rate	`dR`	0.09	0.001 – 2.0	—
T60 Decay Scale	`dS`	15	0.1 – 20.0	s
Excitation Location	`xi`	0.2	0.01*`L` – `L`	m
Read-Out Location	`xo`	0.3	0.01*`L` – `L`	m

display different frequency-dependent damping behavior. For example, rubber strings have more high-frequency damping compared to low-frequency damping, while metal strings exhibit the opposite behavior. The value of the excitation and read-out locations spanned the length of the string.

In the main update of the script, the time-dependent amplitudes Φ_p^{n+1} and the output sound signal u^{n+1} were computed for each time step n as follows:

```
for n=1:Nf
        % Compute time-dependent amplitudes
        phi = a1.*phi1 + a2.*phi2 + a3.*F(n);

        % Compute output signal
        out(n) = U*phi;

        % Move past amplitude vectors forward one time step
        phi2 = phi1;
        phi1 = phi;
end
```

Here, `Nf` is the total duration of the simulation measured in time samples, `a1`, `a2`, and `a3` are the coefficients in Eq. (1.9), `out` is the output signal u^{n+1}, `Nf` are the spatial modes defined by Eq. (1.4a), and `phi`, `phi1`, and `phi2` are the time-dependent amplitudes Φ_p^{n+1}, Φ_p^n, and Φ_p^{n-1} in Eq. (1.9), respectively. In the absence of initial conditions, the value of the coefficient `a3` can be arbitrarily set since it only scales the output; that is, the underlying physics of the simulation is unchanged.

Bibliography

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[3] N. H. Asmar, *Partial differential equations with Fourier series and boundary value problems.* Courier Dover Publications, 2005.

