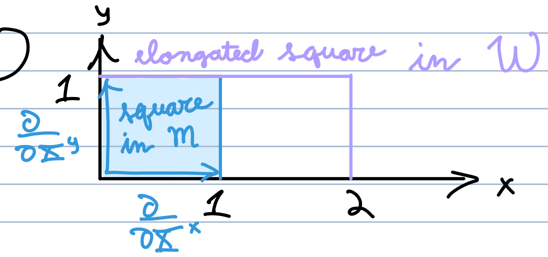
13. HW 3 : Elasticity

Continuous Theory

(1) 2D Square Elongation (youtube example)

- Calculate the deformation gradient F of a 2D square that has been elongated by twice its length along one of its dimensions.

(1)



$$\mathcal{M} \xrightarrow{\phi} \mathcal{W}$$

$$\frac{\partial}{\partial X^a} \qquad \frac{\partial}{\partial x^i}$$

$$\vec{x} = \phi(\vec{X}) = \begin{bmatrix} 2X^x \\ X^y \end{bmatrix}$$

$$F = \frac{\partial \phi^i}{\partial X^a} = \begin{bmatrix} \frac{\partial x^x}{\partial X^x} & \frac{\partial x^x}{\partial X^y} \\ \frac{\partial x^y}{\partial X^x} & \frac{\partial x^y}{\partial X^y} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

(2) 3D Abstract Time-Dependent Motion (youtube ex.)

- Calculate F from the following flow map ϕ.

$$\phi(\vec{X},t) = \begin{bmatrix} 3{X^x}^2 X^y + t{X^z}^3 \\ {X^z}^2 - tX^x \\ 5X^y X^z \end{bmatrix}$$

(2) Assume :

$$\phi(\vec{X},t) = \begin{bmatrix} 3{X^x}^2 X^y + t{X^z}^3 \\ {X^z}^2 - tX^x \\ 5X^y X^z \end{bmatrix}$$

$$F = \frac{\partial \phi^i}{\partial X^a} = \begin{bmatrix} \frac{\partial x^x}{\partial X^x} & \frac{\partial x^x}{\partial X^y} & \frac{\partial x^x}{\partial X^z} \\ \frac{\partial x^y}{\partial X^x} & \frac{\partial x^y}{\partial X^y} & \frac{\partial x^y}{\partial X^z} \\ \frac{\partial x^z}{\partial X^x} & \frac{\partial x^z}{\partial X^y} & \frac{\partial x^z}{\partial X^z} \end{bmatrix}$$

$$= \begin{bmatrix} 6X^x X^y & 3{X^x}^2 & 3t{X^z}^2 \\ -t & 0 & 2X^z \\ 0 & 5X^z & 5X^y \end{bmatrix}$$

- F may not be symmetric.
- F is a f^n of X and t.

Discrete Theory

(1) What are the steps involved in doing an elastic body simulation based on the Piola stress tensor framework?

Note: the subscript "c" means "per cell" and the subscript "v" means "per vertex".

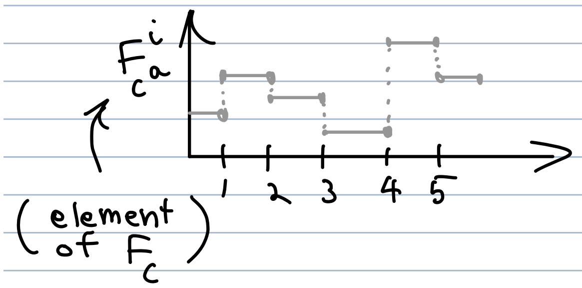
Equivalent approach:

$C_c \to U_c \to U_{tot} = \sum_c U_c$

$f = \frac{\partial U_{tot}}{\partial x_i}$

(2) Calculate F_c for a 2D mesh.

Note: We have a different F_c for each cell. Thus, $F^i_{c\,a}$ is piecewise constant.



(1) (i) $\phi(\vec{X}) = \vec{x}_v$ (flow map)

(ii) F_c (deformation grad.)

(iii) $C_c = F_c^T F_c$ (induced metric)

(iv) $E_c = \frac{1}{2}(C_c - I)$ (strain tensor)

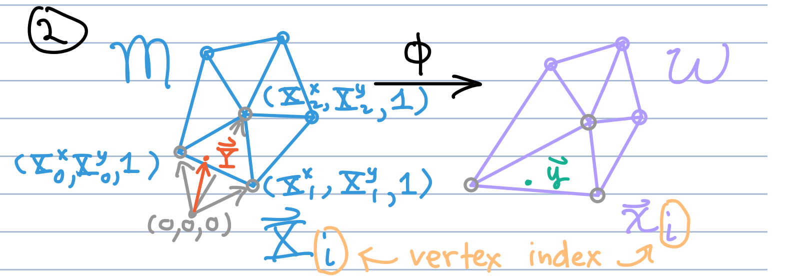
(v) Find appropriate stress-strain relation. One possible choice:
$S_c = 2\mu E_c + \lambda \, tr(E_c) I$ (2nd Piola stress)

(vi) $P_c = F S_c$ (1st Piola stress $= \frac{\partial U}{\partial F}$)

(vii) $\vec{f}_v = \frac{1}{n} \sum_{c \ni v} P_c \hat{n}_{c,v} A_{c,v}$
(= div(P_c) = discrete divergence)

(ix) $\dot{\vec{v}}_v = \vec{f}_v / m_v$, $\dot{\vec{X}}_v = \vec{v}_v$

(2)



$\vec{x}_i = \phi(\vec{X}_i)$ is the discrete (finite-dimensional) flow map.

$\vec{y} = \phi(\vec{Y})$ is the continuous (infinite-dimensional) flow map.

$F_c = \frac{\partial \phi^i}{\partial Y^a}$ (i ← increment over $\dim(\mathcal{W}) = 2$; a ← increment over $\dim(\mathcal{M}) = 2$)

Given $\begin{bmatrix} y^x \\ y^y \\ 1 \end{bmatrix} = \phi_c(\vec{Y}) = AB \begin{bmatrix} Y^x \\ Y^y \\ 1 \end{bmatrix}$

$$A = \begin{bmatrix} x_0^x & x_1^x & x_2^x \\ x_0^y & x_1^y & x_2^y \\ 1 & 1 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} X_0^x & X_1^x & X_2^x \\ X_0^y & X_1^y & X_2^y \\ 1 & 1 & 1 \end{bmatrix}^{-1}$$

$$Q = AB = \begin{bmatrix} Q_a^x & Q_b^x & Q_c^x \\ Q_a^y & Q_b^y & Q_c^y \\ Q_a^z & Q_b^z & Q_c^z \end{bmatrix}$$

③ Calculate m_v.

Note: ρ is the mass density.

we care about this part:

$$\therefore \phi_c(\vec{Y}) = \begin{bmatrix} \boxed{\begin{matrix} Q_a^x Y^x + Q_b^x Y^y \\ Q_a^y Y^x + Q_b^y Y^y \end{matrix}} & \begin{matrix} + Q_c^x \\ + Q_c^y \end{matrix} \\ Q_a^z Y^x + Q_b^z Y^y & + Q_c^z \end{bmatrix} \downarrow i$$

$$F_c = \frac{\partial \phi_c^i}{\partial Y^a} = \begin{bmatrix} Q_a^x & Q_b^x \\ Q_a^y & Q_b^y \end{bmatrix} \qquad \begin{matrix} i = 0, 1 \\ a = 0, 1 \end{matrix}$$

(per cell)

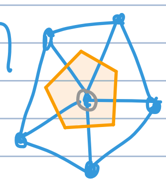
Alternative approach for getting F_c (produces equivalent result):

$$F_c = -\frac{1}{n V_c} \sum_{j=0}^{n} \begin{bmatrix} | \\ x_j \\ | \end{bmatrix} \begin{bmatrix} - A_{c,j} n_{c,j}^T - \end{bmatrix}$$

We use this to derive our discrete divergence operator in $\vec{f}_v = \mathrm{div}(P_c)$.

③ $$m_v = \rho \sum_{c \ni v} \frac{1}{n+1} V_c$$

m



m_v is a diagonal matrix (which makes matrix inversion easier).