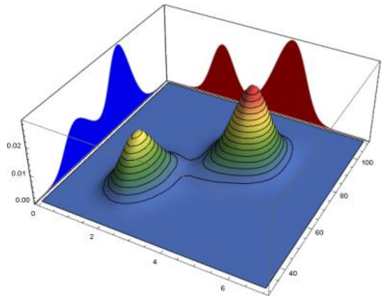
1. 확률, 확률분포

<table>
<tr><td rowspan="1">++조건부
확률

독립성</td><td>$C_1 \subset S$ 에서 C_1을 새로운 표본 공간으로 설정 ➔ $C_2 \subset S$에 대해
1. $P(C_2|C_1) = P(C_1 \cap C_2|C_1) = {P(C_1 \cap C_2)}/{P(C_1)}$ ⇔ $P(C_1 \cap C_2) = P(C_2|C_1)\, P(C_1)$
2. Bayes (C_i는 상호 배반=disjoint, S의 partition)
1) Law of total prob: $P(A) = \sum P(A \cap C_i) = \sum P(A \mid C_i)\, P(C_i)$
2) Bayes' theorem: $P(C_i \mid A) = {P(A \cap C_i)}/{P(A)} = {P(A \cap C_i)}/{\sum P(A \cap C_i)} = {P(A \mid C_i)\, P(C_i)}/{\sum P(A \mid C_i)\, P(C_i)}$
① $P(C_i)$: C_i 사전확률 (prior)
② $P(C_i \mid A)$: C_i 사후확률 (posterior) ← 표본 A에서 관찰된 C_i
3) 독립성: $P(A \cap B \cap C) = P(A)P(B)P(C)$ 이면 A,B,C는 statistically independent</td></tr>
<tr><td rowspan="3">확률
변수</td><td>1. Prob mass function; PMF (discrete) ➔ CDF of PMF: $F(x) = P\big((-\infty, x]\big) = \sum_{(-\infty,x]} p(x)$
*변환: $p_y(y) = p_X(w(y))$; for 일대일 함수 $x = w(y)$</td></tr>
<tr><td>2. Prob density function; PDF (continuous) >0
1) $F(x) = \int_{-\infty}^{x} f(t)dt$ ⇔ 2) $\frac{d}{dx}F(x) = f(x)$ (F는 f의 CDF)
3) $P[(a,b)] = \int_a^b f(x)dx = F(b) - F(a)$
*변환: X가 pdf f_X on S_X & Y가 pdf f_Y on S_Y ; 1-on-1 $\boldsymbol{w(y) = x}$
➔ $f_Y(y) = f_x\big(w(y)\big)|dx/dy|$ ⇔ $f_Y(y) = f_x\big(w(y)\big)\, \mathbf{abs}(\boldsymbol{J})$ (Jacobian: J=|w'(y)| / J=$\begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix}$) *J는 행렬식</td></tr>
<tr><td>* Support(받침): PDF $\neq 0$인 space // *CDF는 유일 for PDF, PMF</td></tr>
<tr><td rowspan="3">기대값</td><td>1. 조건: $\boldsymbol{E(|X|)}$ 존재 ⇔ ① 연속 pdf 존재 ② $\int_{-\infty}^{\infty} \boldsymbol{|x| f(x) dx} < \infty$ (이산 pmf 존재 ➔ $\sum |x_i|\, p_i(x) < \infty$)
2. 기대값: 1) $E(X) = \int_{-\infty}^{\infty} x\, f(x)dx$ 2) $E(X) = \sum x_i\, p(x_i)$
3. $y = g(x)$: 1) $E\big(g(x)\big) = \int_{-\infty}^{\infty} g(x)f(x)dx$ & $E\big(g(x)\big) = \sum g(x_i)p(x_i)$
2) $E\big(k_1 g_1(x) + k_2 g_2(x)\big) = k_1 E\big(g_1(x)\big) + k_2 E\big(g_2(x)\big)$</td></tr>
<tr><td>1. 평균: $\mu = E(X)$
2. 분산: $\sigma^2 = Var(X) = E[(X-\mu)^2] = E(X^2) - \mu^2 = M''(0) - M'(0)^2$ *$\mathrm{Var}(aX+b) = a^2\mathrm{Var}(X)$</td></tr>
<tr><td>3. 적률생성함수 (MGF) *조건: $t \in (-h, h)$ for $\forall h > 0$ ← 0을 포함하는 개구간에서 mgf 존재
1) $\boldsymbol{M(t) = E(e^{tX})}$ ➔ $M_X(0)^{(r)} = E(X^r)$ *분포의 r차 moment
① $M(0)^{(r)} = \frac{d^r}{dt^r}\int_{-\infty}^{\infty} e^{tx} f(x)dx\,|_{t=0} = \int_{-\infty}^{\infty} \frac{d^r}{dt^r} e^{tx} f(x)dx\,|_{t=0} = \int_{-\infty}^{\infty} x^r e^{tx} f(x)dx\,|_{t=0} = \int_{-\infty}^{\infty} x^r\, f(x)dx = E(X^r)$
② $M(0)^{(r)} = \frac{d^r}{dt^r}\sum e^{tx_i}\, p(x_i)|_{t=0} = \sum \frac{d^r}{dt^r} e^{tx_i}\, p(x_i) = \sum x_i^r e^{tx_i}\, p(x_i)|_{t=0} = \sum x_i^r\, p(x_i) = E(X^r)$
2) 성질 ① MGF의 유일성: $\boldsymbol{M_x(t) = M_y(t)}$ ⇔ $\boldsymbol{X = Y}$ (pdf 동일)
② $M_{X+\alpha}(t) = e^{\alpha t}\, M_X(t)$ $\because M_{X+\alpha}(t) = E\big(e^{t(x+\alpha)}\big) = e^{\alpha t}\, E(e^{tx}) = e^{\alpha t}\, M_X(t)$
③ $M_{\alpha X}(t) = M_X(\alpha t)$ $\because M_{\alpha X}(t) = E\big(e^{t(\alpha X)}\big) = E\big(e^{(\alpha t)X}\big) = M_X(\alpha t)$
④ $M_{aX+b}(t) = e^{bt} M_X(at)$
⑤ $M_Y(t) = \prod M_{X_i}(k_i t)$, $t < |\min(h_i)|$ (for $Y = \sum k_i X_i$, X_i는 모두 <u>독립</u>)
$\because M_Y(t) = E(\exp(tY)) = E(\exp(t\sum k_i X_i)) = E(\prod \exp(tk_i X_i)) = \prod E(\exp(tk_i X_i))$
⑥ $M_Y(t) = [M(t)^n]$ (for $Y = \sum X_i$; X_i는 iid 확률변수)</td></tr>
<tr><td>중요한
부등식</td><td>1. $E(X^m)$이 존재하면 ➔ $E(X^k)$ 존재 for $k \le m$
*증명: $E(X^k) = \int_{-\infty}^{\infty} |x|^k f(x)dx = \int_{|x|\le 1} |x|^k f(x)dx + \int_{|x|\ge 1} |x|^k f(x)dx \le \int_{|x|\le 1} f(x)dx + \int_{|x|\ge 1} |x|^m f(x)dx$
$\le \int_{-\infty}^{\infty} f(x)dx + \int_{-\infty}^{\infty} |x|^m f(x)dx \le 1 + E(X^m)$ ∴ 유한함
2. Markov: $\boldsymbol{P[u(X) \ge c] \le E[u(X)]/c}$ (for u(X)≥0, c>0; E[u(X)]존재)
*증명: $E[u(x)] = \int_{-\infty}^{\infty} u(x)f(x)dx \ge \int_{u(x)\ge c} u(x)f(x)dx \ge c\int_{u(x)\ge c} f(x)dx = c\, P[u(x) \ge c]$
*직관: 평균 나이의 5배 보다 나이 많은 사람의 확률한계 → $P[X \ge 5\mu] \le \frac{1}{5}$
3. Chevyshev: $\boldsymbol{P(|X-\mu| \ge k\sigma) \le 1/k^2}$ (for k>0; X가 μ,σ^2(유한) 가짐)
*증명: Markov에서 $u(X) = (X-\mu)^2$, $c = k^2\sigma^2$</td></tr>
</table>

2-1. 이변량분포

이변수	1) Joint CDF: $F(x,y) = P[\{X \le x\} \cap \{Y \le y\}]$ * $\mathbf{X} = (X,Y)^T \in$ D ; **Random vector X** $*P\big((a_1,a_2] \times (b_1,b_2]\big) = F(a_2,b_2) - F(a_1,b_2) - F(a_2,b_1) + F(a_1,b_1) = \int_{b_1}^{b_2}\int_{a_1}^{a_2} f(x,y)\,dxdy$ 2) Joint PMF: $\sum_y \sum_x p(x,y) = 1$ 3) Joint PDF: $F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(x,y)\,dydx$ $(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dxdy = 1)$ $\Leftrightarrow \partial^2(F)/\partial x \partial y = f(x,y)$ 4) Marginal dist: 한 변수의 효과만 봄; 다른 변수는 (-∞, ∞) 전부 포괄 * $F_X(x) = P(\{X \le x\}) = P(\{X \le x\} \cap \{-\infty < Y < \infty\})$ ①PMF of x: $F_X(x) = \sum_{(-\infty,x]}\{\sum_{y\in(-\infty,\infty)} p(x,y)\}$ ➔ $p_X(x) = \sum_{y\in(-\infty,\infty)} p(x,y)$ ②PDF of x: $F_x(x) = \int_{-\infty}^{x}\{\int_{-\infty}^{\infty} \boldsymbol{f(x,y)dy}\}\,dx$ ➔ $f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy$ 
이변수 기대값	$*E(g(X,Y))$ 존재 조건 ⇔ $E(\|g(X,Y)\|) < \infty$ 기대값: $E\big(g(X,Y)\big) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\,f(x,y)\,dxdy$ (이산형: $E\big(g(X,Y)\big) = \sum\sum g(x,y)p(x,y)$) 1) $E(k_1g_1 + k_2g_2) = k_1E(g_1) + k_2E(g_2)$ 2) $E(\mathbf{X}) = [E(X)\ E(Y)]^{\mathrm{T}} = \big[\int_{-\infty}^{\infty} x\,f(x,y)dxdy\ \ \int_{-\infty}^{\infty} y\,f(x,y)dxdy\big]^{\mathrm{T}}$ 3) $M(t_1,t_2) = E(\exp(t_1X + t_2Y))$ ➔ $\mathbf{t} = (t_1,t_2)^T$에 대해 $M(\mathbf{t}) = E(\exp(\mathbf{t}^T\mathbf{X}))$ $E(X^kY^m) = \frac{\partial^{k+m}}{\partial t_1^k \partial t_2^m} M(0,0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^k y^m \exp(t_1x + t_2y)\,f(x,y)\,dxdy$
이변수 변환	*변환 조건: 1) $\mathbf{X} = (X_1,X_2)$의 받침 S 2) S→T 사상하는 일대일대응: $y_1 = u_1(x_1,x_2)$ & $y_2 = u_2(x_1,x_2)$ 3) T→S 사상하는 위 대응 역: $x_1 = w_1(y_1,y_2)$ & $x_2 = w_2(y_1,y_2)$
	1. 이산형 변환: $p_{\mathbf{Y}}(y_1,y_2) = p_{\mathbf{X}}[w_1(y_1,y_2), w_2(y_1,y_2)]$ for $(y_1,y_2) \in T$ & 나머지 pmf 0 * X_1, X_2 ➔ Y로만 변환 시, dummy 변수를 하나 더 만들어 Y_2로 지정해주고 marginal Y dist를 구 함
	2. 연속형 변환: $f_{\mathbf{Y}}(y_1,y_2) = f_{\mathbf{X}}[w_1(y_1,y_2), w_2(y_1,y_2)]\|\boldsymbol{J}\|$ for $(y_1,y_2) \in T$ & 나머지 pdf 0
	* MGF 이용 변환: $E\big(\exp(tY)\big) = E(\exp(t(X_1 + X_2))$ ➔ MGF 유일성으로 Y의 PMF/PDF 구함
조건부	1. 조건부 PMF: $p_{2\|1}(x_2\|x_1) = p(x_1,x_2)/p_1(x_1)$
	2. 조건부 PDF: $f_{2\|1}(x_2\|x_1) = f(x_1,x_2)/f_1(x_1)$ (f_1는 $f_{1,2}$의 marginal 분포, 직관적으로는 $f_1(x_1)$는 스케일러) 1) $P(a < Y < b\ \|X = x) = \int_a^b f_{Y\|X}(y\|x)\,dy$ & $P(c < X < d\|Y = y) = \int_c^d f_{X\|Y}(x\|y)\,dx$ 2) $P(-\infty < Y < \infty\ \|X = x) = \int_{-\infty}^{\infty} f_{Y\|X}(y\|x)\,dy = \int_{-\infty}^{\infty} \frac{f(x,y)}{f_X(x)}dy = \frac{1}{f_X(x)}\int_{-\infty}^{\infty} f(x,y)\,dy=1$ 3) 조건부 기대값: $E[\boldsymbol{u}(\boldsymbol{Y})\|x] = \int_{-\infty}^{\infty} u(y)\,f_{Y\|X}(y\|x)\,dy$ ➔ **x의 함수** ① 조건부 평균: $E(Y\|x) = \int_{-\infty}^{\infty} y\,f_{Y\|X}(y\|x)\,dy$ ② 조건부 분산: $\mathrm{Var}(Y\|x) = E(Y^2\|x) - [E(Y\|x)]^2$
	* 정리: μ_Y 추정 ← $E(Y\|X)$이 Y보다 더 신뢰도 높음 (Rao & Blackwell) 1) $E[E(Y\|X)] = E(Y)$ 2) $\mathrm{Var}\big(E(Y\|X)\big) \le \mathrm{Var}(Y)$ *Y 분산 유한 * 증명: $E(Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yf(x,y)\,dydx = \int_{-\infty}^{\infty}[\int_{-\infty}^{\infty} yf_{Y\|X}(y\|x)dy]\,f_X(x)dx = \int_{-\infty}^{\infty} E(Y\|X)\,f_X(x)dx = E(E(Y\|X))$
공분산 / 상관 계수	1.공분산: $\mathbf{Cov}(\boldsymbol{X},\boldsymbol{Y}) = \boldsymbol{E}[(\boldsymbol{X} - \boldsymbol{\mu_X})(\boldsymbol{Y} - \boldsymbol{\mu_Y})] = \boldsymbol{E}(\boldsymbol{XY}) - \boldsymbol{E}(\boldsymbol{X})\boldsymbol{E}(\boldsymbol{Y})$ *독립이면 $\mathrm{Cov}(X,Y) = 0 \Leftrightarrow E(XY) = E(X)E(Y)$
	2.상관계수: $\boldsymbol{\rho} = \mathbf{Cov}(\boldsymbol{X},\boldsymbol{Y})/\boldsymbol{\sigma_X\sigma_Y} = \boldsymbol{\sigma_{XY}}/(\boldsymbol{\sigma_X\sigma_Y})$ $(-1 \le \rho \le 1)$➔$y = a + bx\ (b > 0)$에 ρ의 강도로 집중 $(0 < \rho \le 1)$
	3.선형조건부평균: $E(Y\|X) = a + bx$ ➔ $E(Y\|X) = \mu_y + \rho\frac{\sigma_y}{\sigma_x}(X - \mu_x)$ & $E\big(\mathrm{Var}(Y\|X)\big) = \sigma_y^2(1 - \rho^2)$ *회귀분석 모회귀계수 $\beta = \rho(\sigma_y/\sigma_x) = \mathrm{Cov}(X,Y)/Var(X)$; *X,Y 분산 유한
독립	*정의: $\boldsymbol{f(x,y) = f_x(x)f_y(y)}$ ⇔ **X,Y는 독립** $[x \in (a,b)\ \&\ y \in (c,d)]$ (받침이 수평/수직선 box에 존재해야 함) 1. 조건부 증명: $f_y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_{-\infty}^{\infty} f_{y\|x}(y\|x)\,f_x(x)\,dx = f_{y\|x}(y\|x)\int_{-\infty}^{\infty} f_x(x)dx = f_{y\|x}(y\|x)$ 2. 동치류 1) $\boldsymbol{f(x,y) = f_x(x)\,f_y(y)}$ 2) $\boldsymbol{F(x,y) = F_x(x)\,F_y(y)}$ *증명: $\partial^2F/\partial x\partial y = f_x(x)f_y(y)$ 3) $\boldsymbol{P(a < X < b,\ c < Y < d) = P(a < X < b)\,P(c < Y < d)}$ *증명: $P(a < X < b, c < Y < d) = F(b,d) - F(a,d) - F(b,c) + F(a,c) = [F_x(b) - F_x(a)][F_y(d) - F_y(c)]$ 4) $\boldsymbol{E[u(X)v(Y)] = E[u(X)]\,E[v(Y)]}$ $\Rightarrow E(XY) = E(X)E(Y) \Leftrightarrow \mathrm{Cov}(X,Y) = 0$ 5) $\boldsymbol{M(t_1,t_2) = M(t_1,0)\,M(0,t_2)}$ *증명: $M(t_1,t_2) = E(e^{t_1x+t_2y}) = E(e^{t_1x})E(e^{t_2y}) = M(t_1,0)\,M(0,t_2)$ * $\boldsymbol{M(t_1,0)}$는 $\boldsymbol{f_x}$의 **MGF** $(\because M(t_1,0) = E(e^{t_1x}) = \int_{-\infty}^{\infty} e^{t_1x}[\int_{-\infty}^{\infty} f(x,y)\,dy]dx = \int_{-\infty}^{\infty} e^{t_1x} f_x(x)\,dx$

2-2. 다변량분포

<table>
<tr><td rowspan="5">다변수</td><td>* $\mathbf{x} = (x_1, x_2, \cdots, x_p)^T = \left(X_1(c), X_2(c), \cdots, X_p(c)\right)^T$ for 확률 실험 $c \in C$
1. 결합 확률 함수들
1) Joint CDF: $F(\mathbf{x}) = P[\{X_1 \le x_1\} \cap \{X_2 \le x_2\} \cap \cdots \cap \{X_p \le x_p\}]$
2) Joint PMF $F(\mathbf{x}) = \sum_{w_1 \le x_1} \cdots \sum_{w_p \le x_p} p(w_1, \cdots, w_p)$
3) Joint PDF: $F(\mathbf{x}) = \int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\cdots\int_{-\infty}^{x_p} f(x_1,\cdots,x_p)\,dx_p\cdots dx_1$ $\left(\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(x_1,\cdots,x_p)\,dx_p\cdots dx_1 = 1\right)$
⇔ $\partial^p\{F(\mathbf{x})\} \big/ \partial x_1 \cdots \partial x_p = f(\mathbf{x})$</td></tr>
<tr><td>2. Marginal/Conditional
1) $f_1(x_1) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(x_1,\cdots,x_p)\,dx_2\cdots dx_p$
➔ $f_{2,\cdots,p|1}(x_2,\cdots,x_p \mid x_1) = f(\mathbf{x}) \big/ f_1(x_1)$
2) $f_{2,4,5}(x_2,x_4,x_5) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\mathbf{x})\,dx_1 dx_3 dx_6$ ← $\mathbf{x} = (x_1,x_2,x_3,x_4,x_5,x_6)^T$
➔ $f_{1,3,6|2.4.5}(x_1,x_3,x_6 \mid x_2,x_4,x_5) = f(\mathbf{x}) \big/ f_{2,4,5}(x_2,x_4,x_5)$</td></tr>
<tr><td>3. 기대값
1) $E(u(\mathbf{x})) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} u(x_1,\cdots,x_p)\,dx_1\cdots dx_p$ (존재성: $\exists E(|u(\mathbf{x})|)$) *이산: $E(u(\mathbf{x})) = \sum_{x_1}\cdots\sum_{x_p} u(x_1,\cdots,x_p)$
2) $E(\sum k_i Y_i) = \sum k_i E(Y_i)$
3) $E[u(X_2,\cdots,X_p) \mid x_1] = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \boldsymbol{u}(\boldsymbol{x_2},\cdots,\boldsymbol{x_p})\, f_{2,\cdots,p|1}(x_2,\cdots,x_p \mid x_1)\,dx_2\cdots dx_p$</td></tr>
<tr><td>4. 독립: $E(\prod u_i(X_i)) = \prod E(u_i(X_i))$ 등 동치류 (f, F, P, E, M)
***iid** (independent and identically distributed): 여러 확률 변수가 서로 독립 & 동일한 분포</td></tr>
<tr><td>5. 변환: **X**의 받침 S에 대해, X⇔Y가 일대일이 되는 $S_1,\cdots,S_k$의 부분 공간 상 각각의 야코비안 J_i 정의
$$g(\mathbf{y}) = \sum_{i=1}^{k} |\boldsymbol{J_i}|\, f[w_{1i}(\mathbf{y}),\cdots,w_{pi}(\mathbf{y})]$$</td></tr>
<tr><td rowspan="2">Random matrix</td><td>* **Random matrix** $\mathbf{W} = [W_{ij}]$, $W_{ij}\ (1 \le i \le m,\ 1 \le j \le n)$
1. $E(\mathbf{W}) = [E(W_{ij})]$ (일렬로 배열하여 mn x 1의 벡터로 생각)
1) $E[\mathbf{AW} + \mathbf{BV}] = \mathbf{A}\,E[\mathbf{W}] + \mathbf{B}\,E[\mathbf{V}]$ (A,B: k x m 상수 행렬, W,V: m x n 확률 행렬)
2) $E[\mathbf{AWB}] = \mathbf{A}\,E(\mathbf{W})\,\mathbf{B}$ (A: k x m, W: m x n, B: n x l)
2. 분산-공분산 행렬 (Variance-Covariance matrix) $*\mathbf{X} = (X_1, X_2, \cdots, X_p)^{\mathrm{T}}$; 모든 VCM는 양의 반정부호(psd)
1) 정의: $\mathrm{Cov}(\mathbf{X}) = E[(\mathbf{X}-\boldsymbol{\mu})(\mathbf{X}-\boldsymbol{\mu})^{\mathrm{T}}] = [\sigma_{ij}]$ $(\boldsymbol{\mu} = E(\mathbf{X}))$
➔ $\sigma_i^2 = \mathrm{Var}(X_i)$ & $\sigma_{ij} = \mathrm{Cov}(X_i, X_j)$
2) 정리 ① $\mathrm{Cov}(\mathbf{X}) = \mathrm{E}(\mathbf{XX}^{\mathrm{T}}) - \boldsymbol{\mu\mu}^{\mathrm{T}}$ $(\sigma_i^2 \lt \infty)$
② $\mathrm{Cov}(\mathbf{AX}) = \mathbf{A}\mathrm{Cov}(\mathbf{X})\mathbf{A}^{\mathrm{T}}$ $(\sigma_i^2 \lt \infty,\ \mathrm{A}: \mathrm{m} \times \mathrm{p})$ $\because \mathrm{Cov}(\mathbf{AX}) = \mathbf{A}\mathrm{E}(\mathbf{XX}^{\mathrm{T}})\mathbf{A}^{\mathrm{T}} - \mathbf{A}\mathrm{E}(\mathbf{X})\mathrm{E}(\mathbf{X})^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$
3. MGF: $M(\mathbf{t}) = E[\exp(\mathbf{t}^T\mathbf{X})] = E[\prod_{i=1}^{p} \exp(t_i X_i)]$ (*X_i 독립➔ $M(\mathbf{t}) = M(t_1,\cdots,0)\cdots M(0,\cdots,t_p) = \prod_{i=1}^{p} E[\exp(t_i X_i)]$)
1) $M_{\mathbf{Y}}(\mathbf{t}) = \prod M_{\mathbf{X}_i}(\mathbf{t})$ $(\mathbf{Y} = \sum \mathbf{X}_i$, 각 $\mathbf{X}_i \in \mathbb{R}^n$은 독립) *다변량 MGF는 스칼라
2) $M_{\mathbf{Y}}(\mathbf{t}) = \mathrm{e}^{\mathbf{b}^T\mathbf{t}} M_{\mathbf{X}}(\mathbf{A}^T\mathbf{t})$ $(\mathbf{Y} = \mathbf{AX} + \mathbf{b};\ \mathbf{A}: m \times p;\ \mathbf{t} \in \mathbb{R}^m;\ \mathbf{b} \in \mathbb{R}^m)$</td></tr>
<tr><td>3. 선형결합: $T = \sum_{i=1}^{n} a_i X_i$, $W = \sum_{i=1}^{m} b_i Y_i$
1) $E(T) = \sum_{i=1}^{n} a_i E(X_i)$ $(*E[|X_i|] \lt \infty)$ $\because E(X_1 + X_2) = \int\int (x_1 + x_2) f(x_1, x_2)\,dx_1 dx_2 = E(X_1) + E(X_2)$
2) $\mathrm{Cov}(T, W) = \sum\sum a_i b_j\, \mathrm{Cov}(X_i, Y_j)$ $(*E[X_i^2] \lt \infty,\ E[Y_{ij}^2] \lt \infty)$
① $Var(T) = \mathrm{Cov}(T,T) = \sum_{i=1}^{n} a_i^2\, \mathrm{Var}(X_i) + 2\sum_{i \lt j} a_i a_j\, \mathrm{Cov}(X_i, X_j)$ $(*E[X_i^2] \lt \infty)$
② $Var(T) = \mathrm{Cov}(T,T) = \sum_{i=1}^{n} a_i^2\, \mathrm{Var}(X_i)$ (*$X_1,\cdots,X_n$이 유한 분산, **독립**)
3) 표본 추정량: $X_1,\cdots,X_n$이 μ, σ^2 가지는 **iid** 확률변수
① 표본평균: $\boldsymbol{\bar{X}} = \sum_{i=1}^{n} \boldsymbol{X_i} \big/ \boldsymbol{n}$ & 표본분산: $\boldsymbol{S^2} = \sum_{i=1}^{n} (\boldsymbol{X_i} - \boldsymbol{\bar{X}})^2 \big/ \boldsymbol{n-1} = \sum X_i^2 - n\bar{X}^2 \big/ n-1$
② $\boldsymbol{E(\bar{X})} = \sum_{i=1}^{n} E(X_i)/n = n\mu/n = \boldsymbol{\mu}$ & $\mathbf{Var}(\boldsymbol{\bar{X}}) = \sum_{i=1}^{n} \left(\frac{1}{n}\right)^2 \mathrm{Var}(X_i) = n\left(\frac{1}{n}\right)^2 \sigma^2 = \boldsymbol{\sigma^2/n}$
③ $\boldsymbol{E(S^2)} = \sum E(X_i^2) - nE(\bar{X}^2) \big/ n-1 = n(\mu^2 + \sigma^2) - n(\mu^2 + \frac{\sigma^2}{n}) \big/ n-1 = \boldsymbol{\sigma^2}$
($\bar{X}$, S^2는 독립 by Student's정리)</td></tr>
</table>

3-1. 주요 분포: 이산

이항 분포	* Bernoulli experiment: 성공/실패로 서로 배반인 확률 실험 * Bernoulli trial: 베르누이 실험을 독립적으로 반복 (성공 확률 p 동일) * Bernoulli distribution의 유도: X(성공)=1, X(실패)=0 ➔ PMF: $p(x)=p^x(1-p)^{1-x}$ *$\mu=p,\ \sigma^2=p(1-p)$
	* **Binomial distribution (이항분포)**: n회 반복한 베르누이 시행에서 **성공한 총 횟수** 분포 1. PMF: $\boldsymbol{p(x)=\binom{n}{x}p^x(1-p)^{n-x}\ \sim b(n,p)}$ $(x=0,1,\cdots,n)$ 2. MGF: $M(t)=\sum e^{tx}p(x)=[(1-p)+(pe^t)]^n \quad (t\in\mathbb{R})$ 3. 기대값 1) $\boldsymbol{\mu=np}$ *$\mu=M'(0)=n[(1-p)+pe^t]^{n-1}(pe^t)\vert_{t=0}=np$ 2) $\boldsymbol{\sigma^2=np(1-p)}$ *$\sigma^2=M''(0)-\left(M'(0)\right)^2=n(n-1)p^2+np-(np)^2=np(1-p)$ 4. 가법성: $Y=\sum X_i,\ X_i\sim B(n_i,\boldsymbol{p})$ ➔ $Y\sim B(\sum n_i,\boldsymbol{p})$ (증명) $M_Y(t)=\prod M_{X_i}(t)=\prod[(1-p)+(pe^t)]^{n_i}=[(1-p)+(pe^t)]^{\sum n_i}$ $*\,p(x)$가 성공확률(= 평균) p인 Bernoulli분포 $\leftrightarrow X\sim B(1,p)$ $\rightarrow\ Y=\sum_{i=1}^{20}X_i\ (iid)$ 에 대해 $p(y)$는 20회 시행 중 평균 $20p$회 성공하는 Bernoulli $\leftrightarrow Y\sim B(20,p)$
	* **Multinomial distribution (다항분포)** 1. PMF: $p(x_1,\cdots,x_{k-1})=\frac{n!}{(x_1)!\cdots(x_k)!}(p_1)^{x_1}\cdots(p_k)^{x_k}$ ➔ $p_k=1-\sum_{i=1}^{k-1}p_i$ & $x_k=n-\sum_{i=1}^{k-1}x_i$ 2. MGF: $M(t_1,\cdots,t_{k-1})=(p_1e^{t_1}+\cdots+p_{k-1}e^{t_{k-1}}+p_k)^n$
	* R codes 1) dbinom (k,n,p): P(X=k) 2) pbinom (k,n,p): P(X≤k)
	* **Negative binomial distribution (음이항분포)**: X번 실패 후 r번 성공 (베르누이 시행) *r번 성공시 나감 1. PMF: $p(x)=\binom{x+r-1}{r-1}p^r(1-p)^x$ 2. MGF: $M(t)=p^r[1-(1-p)e^t]^{-r}$ ($e^t<1/(1-p)$) ⇔ [이항:x+(r-1)번 중 (r-1)번 성공] x [p] *$\binom{-n}{k}=(-n)(-n-1)\cdots(-n-k+1)/k!=(-1)^k\binom{n+k-1}{k}$ * **Geometric distribution (기하 분포)**: X번 실패 후 처음 성공 (베르누이 시행) ⇔ r=1인 음이항분포 1. PMF: $p(x)=p(1-p)^x$ 2. MGF: $M(t)=p[1-(1-p)e^t]^{-1}$
	* **Hypergeometric distribution (초기하분포)** 1. PMF: $p(x)=\frac{\binom{D}{x}\binom{N-D}{n-x}}{\binom{N}{n}}$ *N개 중 D개가 성공 & **비복원추출:** n번 시행 ➔ x번 성공 확률 2. 기대값: 1) $\mu=n\left(\frac{D}{N}\right)$ 2) $\sigma^2=n\left(\frac{D}{N}\right)\left(1-\frac{D}{N}\right)\left(\frac{N-n}{N-1}\right)$ N>>n이면 이항분포로 근사 가능
Poisson 분포	* Poisson process: 일정한 구간 (시간, 공간)에서 독립적으로 발생하는 event를 생성하는 과정 (**비기억성**) * Poisson postulate: 짧은 구간 h (h->0)에 대해 1) $g(1,h)=\lambda h+o(h)$ $*\,g(x,w)$는 구간 길이 w 내에 x회 발생 확률 2) $\sum_{x=2}^{\infty}g(x,h)=o(h)$ (≒미소 구간 h에 둘 이상은 본질적 불가) * $\lim_{h\to0}o(h)/h=0$ (little-o) 3) 겹치지 않는 구간 ➔ 확률적으로 독립 $\therefore\ g(x,w)=\frac{e^{-\lambda w}(\lambda w)^x}{x!}$
	1. PMF: $\boldsymbol{p(x)=\frac{e^{-\mu}\mu^x}{x!}}$ $(x=0,1,2,\cdots)$ (*$\sum p(x)=e^{-\mu}\sum\left(\frac{\mu^x}{x!}\right)=e^{-\mu}e^{\mu}=1$) 🡐 **주어진 시간**에 **x회 발생** 분포 2. 기댓값: $\boldsymbol{\mu=\sigma^2=\lambda w}$ (λ: 단위 길이당 발생률, w: 주어진 영역 크기) 3. MGF: $M(t)=e^{\mu(e^t-1)}$ $(t\in\mathbb{R})$ 4. 가법성: $Y=\sum X_i,\ X_i\sim\mathrm{Poi}(m_i)$ ➔ $Y\sim\mathrm{Poi}(\sum m_i)$ (증명) $M_Y(t)=\prod M_{X_i}(t)=\prod e^{m_i(e^t-1)}=e^{(\sum m_i)(e^t-1)}$) $*\,p(x)$가 주어진 100초당 평균 μ회 발생 Poisson $\leftrightarrow X\sim\mathrm{Poi}(\mu)$ $\rightarrow\ Y=\sum_{i=1}^{20}X_i\ (iid)$ 에 대해 $p(y)$는 $Y\sim\mathrm{Poi}(20\mu)$ (* 두 가지 해석: 20x100초로 확장 or 20λ 발생률로 중첩) * **이항분포** $\boldsymbol{b(n,p)\xrightarrow{D}}$ **푸아송분포** $(\boldsymbol{\mu=np})$ (MGF의 극한으로 분포수렴 증명 / 짧은 간격의 n회 베르누이)
	* R codes 1) dpois (k,m): P(X=k) 2) ppois (k,m): P(X≤k)

3-2. 주요 분포: 감마 연관 분포

Γ 분포	* 감마함수: $\boldsymbol{\Gamma(\alpha) = \int_0^\infty y^{\alpha-1}e^{-y}dy}$ $(\alpha > 0)$ * $\mathbf{\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)}$ ➔ $\mathbf{\Gamma(n) = (n-1)!}$ for 자연수 n *$\Gamma(1) = 1,\ \Gamma(1/2) = \sqrt{\pi}$ $*$ 스털링 근사: $\Gamma(k+1) \approx \sqrt{2\pi k}\left(\frac{k}{e}\right)^k$
	***Gamma distribution (감마분포): α** (∈ ℝ) **번째** Poisson event 발생까지 걸리는 **대기 시간** 1. PDF**:** $\boldsymbol{f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}e^{-x/\beta}}$ $\sim \boldsymbol{\Gamma(\alpha,\beta)}$ $(0 \le x < \infty)$ (감마함수 식에 $y = x/\beta$ 대입; α>0 & β>0) 2. 기댓값: 1) $\boldsymbol{\mu = \alpha\beta}$, 2) $\boldsymbol{\sigma^2 = \alpha\beta^2}$ 3. MGF: $M(t) = 1/(1-\beta t)^\alpha$ (t < 1/β) 4. 가법성: $Y = \sum X_i,\ X_i \sim \Gamma(\alpha_i, \beta)$ ➔ $Y \sim \Gamma(\sum\alpha_i, \beta)$ (증명) $M_Y(t) = \prod M_{X_i}(t) = \prod(1-\beta t)^{-\alpha_i} = (1-\beta t)^{-\sum\alpha_i}$ 5. 스칼라배: $X \sim \Gamma(\alpha,\beta) \Rightarrow kX \sim \Gamma(\alpha, k\beta)$ (*증명: 야코비안 변수변환) 6. 유도: k번 Poisson event 발생까지 시간을 T_i로 분할 ➔ 각 $T_i \sim \Gamma(1, \frac{1}{\lambda})$ ➔ $Y = \sum_{i=1}^k T_i \sim \Gamma(k, \frac{1}{\lambda})$ (*Erlang 분포: 자연수 k인 감마 분포)
	* R codes 1) dgamma (x,shape=a,scale=b): f(X=x) 2) pgamma (x, shape=a, scale=b): P(X≤x)
지수 분포	***Exponential distribution (지수분포): 1번째** Poisson event 발생까지 대기 시간 $= \Gamma(1,\beta)$ 1. PDF**:** $\boldsymbol{f(x) = \frac{1}{\beta}e^{-x/\beta}}$ 2. 기댓값: 1) $\boldsymbol{\mu = \beta}$, 2) $\boldsymbol{\sigma^2 = \beta^2}$ 3. 유도: W가 첫 번째 Poisson event 까지 걸린 시간 ➔ w시간 내 푸아송 사건 없을 확률: $P(W > w) = \frac{e^{-\lambda w}(\lambda w)^0}{0!} = e^{-\lambda w}$ ⇔ $P(0 < W < w) = 1 - e^{-\lambda w}$ $\therefore f(w) = \lambda e^{-\lambda w}$ (β=1/λ)
χ² 분포	***Chi-square distribution (카이제곱 분포)**: 자유도 r에 대해, $\boldsymbol{\chi^2(r) = \Gamma(\frac{r}{2}, 2)}$ 1. PDF**:** $\boldsymbol{f(x) = \frac{1}{\Gamma(\frac{r}{2})2^{(\frac{r}{2})}}x^{(\frac{r}{2})-1}e^{-\frac{x}{2}}}$ $\sim \boldsymbol{\chi^2(r)}$ $(0 \le x < \infty)$ 2. 기댓값: 1) $\boldsymbol{\mu = r}$, 2) $\boldsymbol{\sigma^2 = 2r}$ 3. MGF: $M(t) = 1/(1-2t)^{r/2}$ (t < 1/2) 4. $E(X^k) = 2^k\frac{\Gamma(\frac{r}{2}+k)}{\Gamma(\frac{r}{2})}$, $k > -\frac{r}{2}$ 5. 가법성 (corollary): $Y = \sum X_i,\ X_i \sim \chi^2(r_i)$ ➔ $Y \sim \chi^2(\sum r_i)$
	* R codes 1) dchisq (x,r): f(X=x) 2) pchisq (x,r): P(X≤x)
β 분포	*베타함수: $\mathrm{B}(\alpha,\beta) = \int_0^1 y^{\alpha-1}(1-y)^{\beta-1}dy$ $(\alpha > 0, \beta > 0)$ ①$\mathrm{B}(\alpha,\beta) = \mathrm{B}(\beta,\alpha)$, ②$\mathrm{B}(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$
	*결합 PDF: $h(x_1,x_2) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}x_1^{\alpha-1}x_2^{\beta-1}e^{-(x_1+x_2)}$; $0 \le x_1 < \infty, 0 \le x_2 < \infty$ (X_1, X_2독립) $*$ $Y_1 = X_1/(X_1+X_2)$ & $Y_2 = X_1 + X_2$ ➔ Y_1에 대한 marginal distribution이 beta(α,β) 1. PDF**:** $\boldsymbol{f(x) = \frac{1}{\mathrm{B}(\alpha,\beta)}x^{\alpha-1}(1-x)^{\beta-1}}$ $(0 < x < 1)$
	* R codes 1) dbeta(x,a,b): f(X=x) 2) pbeta (x,a,b): P(X≤x)
Dirichlet 분포 (β 확장)	*결합 PDF: $h(x_1,\cdots,x_{k+1}) = \prod_{i=1}^{k+1}\frac{1}{\Gamma(\alpha_i)}x_i^{\alpha_i-1}e^{-x_i}$; $0 \le x_i < \infty$ (X_1, … ,X_k 독립) $*$ $Y_i = \frac{X_i}{\sum_{i=1}^{k+1}X_i}(i = 1,2,\cdots,k)$ & $Y_{k+1} = \sum_{i=1}^{k+1}X_i$ ➔$Y_1,\cdots,Y_k$에 대한 marginal dist이 Dirichlet(α_1,…α_{k+1}) 1. PDF: $g(\boldsymbol{y_1},\cdots,\boldsymbol{y_{k+1}}) = \frac{\boldsymbol{\Gamma(\alpha_1+\cdots+\alpha_{k+1})}}{\boldsymbol{\Gamma(\alpha_1)\cdots\Gamma(\alpha_{k+1})}}\boldsymbol{y_i^{\alpha_i-1}\cdots y_k^{\alpha_k-1}[1-(y_1+\cdots+y_k)]^{\alpha_{k+1}-1}}$ ($0 \le y_k$, $\sum_{i=1}^k y_i < 1$)

3-3. 주요 분포: 정규 분포

정규 분포

*표준정규분포: $I=\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{2}\right)dz$ → $0<\exp\left(-\frac{z^2}{2}\right)<\exp(-|z|+1)$ 유계 ($\int_{-\infty}^{\infty}e^{-|z|+1}dz=2e$)

→ $I^2=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(-\frac{x^2+y^2}{2}\right)dxdy=\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty}\mathrm{e}^{-\frac{r^2}{2}}r\,drd\theta=1$ ∴ $f(z)=\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{2}\right)$

*정규분포: $X=\sigma Z+\mu$ 로 변수 변환 $f(x)=\frac{1}{\sigma\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}$

*Bell shape 분포: location 모수 (μ), scale 모수 (σ^2) vs. 감마분포 등: shape 모수 (α), scale 모수 (β)

***표준 정규 분포 $N(0, 1^2)$**

1. PDF: $\phi(z)=\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}z^2\right)$ $(-\infty<z<\infty)$
2. MGF: $M(t)=\exp\left(\frac{1}{2}t^2\right)$, $t\in\mathbb{R}$ 3. 기대값: $E(Z)=0$, $\mathrm{Var}(Z)=1$
4. $E(Z^k)={k!}\Big/{2^{\frac{k}{2}}\left(\frac{k}{2}\right)!}$ (k가 짝수), $E(Z^k)=0$ (k가 홀수) $*\,M(t)=\exp\left(\frac{1}{2}t^2\right)=\sum_{m=0}^{\infty}\left(\frac{t^2}{2}\right)^m/m!$

***정규 분포 $N(\mu, \sigma^2)$**

1. PDF: $f(x)=\frac{1}{\sigma\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}$ $(-\infty<x<\infty\ ,\ \sigma>0)$
2. MGF: $M(t)=\exp\left(\mu t+\frac{1}{2}\sigma^2t^2\right)$, $t\in\mathbb{R}$ ($\because M_X(t)=M_{\sigma Z+\mu}(t)=e^{\mu t}M(\sigma t)=e^{\mu t+\frac{1}{2}\sigma^2t^2}$)
3. 기대값: $E(Z)=\mu$, $\mathrm{Var}(Z)=\sigma^2$
4. $E(X^k)=E\left[(\sigma Z+\mu)^k\right]=\sum_{j=0}^{k}\binom{k}{j}\sigma^jE(Z^j)\mu^{k-j}$
5. 가법성: $Y=\sum a_iX_i$, $X_i\sim N(\mu_i,\sigma_i^2)$ → $Y\sim N[\sum(a_i\mu_i),\sum(a_i\sigma_i)^2]$ *두 모수에 대한 가법성

(증명) $M_Y(t)=\prod M_{a_iX_i}(t)=\prod M_{X_i}(a_it)=\prod\exp\left(\mu_i(a_it)+\frac{1}{2}\sigma_i^2(a_it)^2\right)=\exp\left((\sum a_i\mu_i)t+\frac{1}{2}(\sum a_i^2\sigma_i^2)t^2\right)$

6. Corollary: $\bar{X}={\sum_{i=1}^{n}X_i}/{n}$, $X_i\sim N(\mu,\sigma^2)$ (iid) → $\bar{X}\sim N(\mu,\ \sigma^2/n)$ *X_i가 iid 정규분포 → $\bar{X}$ 무조건 정규분포

*** 정리: $Z^2\sim\chi^2(1)$**

pf) $W=Z^2$일 때, $F(x)=P(W\le x)=P(Z^2\le x)=P(-\sqrt{x}\le Z\le\sqrt{x})$, $x\ge0$

→ $z=\sqrt{w}$ 변환 시, $F(x)=2\int_0^{\sqrt{x}}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{2}\right)dz=\int_0^x\frac{1}{\sqrt{2\pi}\sqrt{w}}\exp\left(-\frac{w}{2}\right)dw$

→ $f(x)=\frac{1}{\sqrt{2\pi}}x^{-\frac{1}{2}}e^{-\frac{x}{2}}=\frac{1}{\Gamma\left(\frac{1}{2}\right)2^{1/2}}x^{\frac{1}{2}-1}e^{-\frac{x}{2}}$ $\sim\chi^2(1)$ $(0\le x<\infty)$

* **따름 정리**: $Y=\sum_{i=1}^{n}Z_i^2\sim\chi^2(n)$ (가법성 of χ^2 using MGF; for iid $Z\sim N(0,1^2)$)

*** Contaminated normal distribution**: 대부분 $Z\sim N(0,1^2)$, 일부 outlier $\sim N(0,\sigma_c^2)$ (오염 비율: ε)

1) $W=KZ+(1-K)\,\sigma_cZ$ for $K=\begin{cases}1 & \text{확률 } 1-\varepsilon\\ 0 & \text{확률 } \varepsilon\end{cases}$ (Z, K는 독립)

$F_W(w)=P(W\le w)=P(W\le w, I=1)+P(W\le w, I=0)=P(Z\le w)(1-\varepsilon)+P\left(Z\le\frac{w}{\varepsilon_c}\right)\varepsilon=(1-\varepsilon)\Phi(w)+\varepsilon\,\Phi\left(\frac{w}{\sigma_c}\right)$

① PDF: $f_W(w)=(1-\varepsilon)\phi(w)+\frac{\varepsilon}{\sigma_c}\phi\left(\frac{w}{\sigma_c}\right)$ ② $E(W)=0$, $\mathrm{Var}(W)=1+\varepsilon(\sigma_c^2-1)$

2) $X=a+bW$ $(b>0)$

① PDF: $f_X(x)=(1-\varepsilon)\phi\left(\frac{X-a}{b}\right)+\frac{\varepsilon}{\sigma_c}\phi\left(\frac{X-a}{b\sigma_c}\right)$ ② $E(W)=a$, $\mathrm{Var}(W)=b^2[1+\varepsilon(\sigma_c^2-1)]$

* R codes 1) dnorm(x,a,b): f(X=x) 2) pnorm (x,a,b): P(X≤x) (평균 a, 표준편차 b)

3-3. 주요 분포: 정규 분포

다변량 정규 분포

* $\mathbf{Z} \sim N_p(\mathbf{0}, \mathbf{I}_p)$ / $\mathbf{z} = (Z_1, \cdots, Z_p)^T \in \mathbb{R}^p \sim \text{iid } N(0,1)$

1) PDF: $f_{\mathbf{Z}}(\mathbf{z}) = \left(\frac{1}{2\pi}\right)^{p/2} \exp\left(-\frac{1}{2}\mathbf{z}^T\mathbf{z}\right)$ pf) $f_{\mathbf{Z}}(\mathbf{z}) = \prod \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}z_i^2\right) = \left(\frac{1}{2\pi}\right)^{p/2} \exp\left(-\frac{1}{2}\sum z_i^2\right)$ $\because$ 각 Z_i는 독립

2)MGF: $M_{\mathbf{Z}}(\mathbf{t}) = \exp\left(\frac{1}{2}\mathbf{t}^T\mathbf{t}\right)$ $(\mathbf{t} \in \mathbb{R}^{\boldsymbol{p}})$ pf) $M_{\mathbf{Z}}(\mathbf{t}) = E\{\exp(\mathbf{t}^T\mathbf{Z})\} = E\{\prod \exp(t_i Z_i)\} = \prod E\{\exp(t_i Z_i)\} = \exp\left(\frac{1}{2}\sum t_i^2\right)$

3) 기대값: $E[\mathbf{Z}] = \mathbf{0}$, $\text{Cov}[\mathbf{Z}] = \mathbf{I}_{\boldsymbol{p}}$

* $\mathbf{X} \sim N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ / $\text{Cov}[\mathbf{X}] = \boldsymbol{\Sigma}$가 psd (양반정치)

⇔ p개의 **<u>의존관계</u>**인 정규분포 확률변수의 **결합 분포**

0) 변환: $\mathbf{X} = \boldsymbol{\Sigma}^{1/2}\mathbf{Z} + \boldsymbol{\mu}$ & $\mathbf{Z} = \boldsymbol{\Sigma}^{-1/2}(\mathbf{X} - \boldsymbol{\mu})$

1) PDF: $f_{\mathbf{X}}(\mathbf{x}) = \dfrac{1}{(2\pi)^{\frac{p}{2}}|\boldsymbol{\Sigma}|^{\frac{1}{2}}} \exp\left\{-\dfrac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T(\boldsymbol{\Sigma}^{-1})(\mathbf{x} - \boldsymbol{\mu})\right\}$

2) MGF: $M_{\mathbf{X}}(\mathbf{t}) = \exp\left\{\mathbf{t}^T\boldsymbol{\mu} + \dfrac{1}{2}\mathbf{t}^T(\boldsymbol{\Sigma})\mathbf{t}\right\}$, $(\mathbf{t} \in \mathbb{R}^{\boldsymbol{p}})$

3) 기대값: $E[\mathbf{X}] = \boldsymbol{\mu}$, $\text{Cov}[\mathbf{X}] = \boldsymbol{\Sigma}$

<유도> $\boldsymbol{\Sigma}$가 psd & 대칭 ➔ EVD 가능

$\boldsymbol{\Sigma} = \boldsymbol{\Gamma}^T\boldsymbol{\Lambda}\boldsymbol{\Gamma}$ $(\boldsymbol{\Lambda} = \text{diag}(\lambda_1, \cdots, \lambda_p);\ \lambda_1 \geq \cdots \geq \lambda_p)$

$\boldsymbol{\Sigma}^{1/2} = \boldsymbol{\Gamma}^T\boldsymbol{\Lambda}^{1/2}\boldsymbol{\Gamma}$, $\boldsymbol{\Sigma}^{-1/2} = \boldsymbol{\Gamma}^T\boldsymbol{\Lambda}^{-1/2}\boldsymbol{\Gamma}$ (if $\boldsymbol{\Sigma}$ is pd)

$E[\mathbf{X}] = E[\boldsymbol{\Sigma}^{1/2}\mathbf{Z}] + \boldsymbol{\mu} = \boldsymbol{\Sigma}^{1/2}E[\mathbf{Z}] + \boldsymbol{\mu} = \boldsymbol{\mu}$

$\text{Cov}[\mathbf{X}] = E[(\mathbf{X} - \boldsymbol{\mu})(\mathbf{X} - \boldsymbol{\mu})^T] = E[(\boldsymbol{\Sigma}^{1/2}\boldsymbol{Z})(\boldsymbol{\Sigma}^{1/2}\boldsymbol{Z})^{\boldsymbol{T}}]$

$= \left(\boldsymbol{\Sigma}^{\frac{1}{2}}\right)E(\mathbf{Z}\mathbf{Z}^T)\left(\boldsymbol{\Sigma}^{\frac{1}{2}}\right) = \boldsymbol{\Sigma}$ $\quad * E[\mathbf{Z}\mathbf{Z}^T] = \text{Cov}(\mathbf{Z}) + \mathbf{0} = \mathbf{I}_p$

1-1. Theorem $\quad * \mathbf{X} \sim N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$; $\mathbf{A}: \mathrm{m} \times \mathrm{p}, \mathbf{b} \in \mathbb{R}^{\mathrm{m}}$

$\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b} \rightarrow \mathbf{Y} \sim N_{\mathrm{m}}(\mathbf{A}\boldsymbol{\mu} + \mathbf{b}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^T)$ (MGF로 증명)

1-2. Corollary (m개 변수에 대한 **<u>주변 분포</u>**)

$*\mathbf{X} \rightarrow \mathbf{X_1} \in \mathbb{R}^m$, $\mathbf{X_2} \in \mathbb{R}^q$ $(\mathrm{p} = \mathrm{m} + \mathrm{q})$ 분할

- $\mathbf{X} = \begin{bmatrix}\mathbf{X_1} \\ \mathbf{X_2}\end{bmatrix}$, $\boldsymbol{\mu} = \begin{bmatrix}\boldsymbol{\mu_1} \\ \boldsymbol{\mu_2}\end{bmatrix}$, $\boldsymbol{\Sigma} = \begin{bmatrix}\boldsymbol{\Sigma_{11}} & \boldsymbol{\Sigma_{12}} \\ \boldsymbol{\Sigma_{21}} & \boldsymbol{\Sigma_{22}}\end{bmatrix}$
- $\mathbf{A} = [\mathbf{I}_m \quad \mathbf{O}_{mq}] \rightarrow \mathbf{X_1} = \mathbf{A}\mathbf{X}$

➔ $\mathbf{X} \sim N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \rightarrow \mathbf{X_1} \sim N_m(\boldsymbol{\mu_1}, \boldsymbol{\Sigma_{11}})$

$(\because A\mu = \mu_1,\ \mathrm{A}\Sigma\mathrm{A}^T = [\mathrm{I}_m \quad \mathrm{O}_{mq}]\begin{bmatrix}\Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22}\end{bmatrix}\begin{bmatrix}\mathrm{I}_m \\ \mathrm{O}_{mq}\end{bmatrix} = \Sigma_{11})$

<MGF 유도>

$M_{\mathbf{X}}(t) = \exp(\mathbf{t}^T\boldsymbol{\mu})\, M_{\mathbf{Z}}\{(\boldsymbol{\Sigma}^{\frac{1}{2}})^T\,\mathbf{t}\}$

$= \exp(\mathbf{t}^T\boldsymbol{\mu})\exp\{(1/2)[(\boldsymbol{\Sigma}^{1/2})^T\mathbf{t}]^{\boldsymbol{T}}[(\boldsymbol{\Sigma}^{1/2})^T\mathbf{t}]\}$

$= \exp(\mathbf{t}^T\boldsymbol{\mu})\exp[(1/2)\mathbf{t}^T(\boldsymbol{\Sigma}^{1/2})^T(\boldsymbol{\Sigma}^{1/2})\,\mathbf{t}] = e^{\mathbf{t}^T\boldsymbol{\mu} + \frac{1}{2}\mathbf{t}^T(\boldsymbol{\Sigma})\mathbf{t}}$

2. 주변분포 독립성: $\mathbf{X_1}, \mathbf{X_2}$ 독립 ⇔ $\boldsymbol{\Sigma_{12}} = \boldsymbol{\Sigma_{21}} = \mathbf{0}$

pf) $M_{\mathbf{X_1},\mathbf{X_2}}(\mathbf{t_1}, \mathbf{t_2}) = \exp\left\{[\mathbf{t_1} \quad \mathbf{t_2}]\begin{bmatrix}\boldsymbol{\mu_1} \\ \boldsymbol{\mu_2}\end{bmatrix} + \frac{1}{2}[\mathbf{t_1} \quad \mathbf{t_2}]\begin{bmatrix}\boldsymbol{\Sigma_{11}} & \boldsymbol{\Sigma_{12}} \\ \boldsymbol{\Sigma_{21}} & \boldsymbol{\Sigma_{22}}\end{bmatrix}\begin{bmatrix}\mathbf{t_1} \\ \mathbf{t_2}\end{bmatrix}\right\}$

$M_{\mathbf{X_1}}(\mathbf{t_1})M_{\mathbf{X_2}}(\mathbf{t_2}) = \exp\{\mathbf{t_1}\boldsymbol{\mu_1} + \mathbf{t_2}\boldsymbol{\mu_2} + \dfrac{1}{2}(\mathbf{t_1^T}\boldsymbol{\Sigma_{11}}\mathbf{t_1} + \mathbf{t_2^T}\boldsymbol{\Sigma_{22}}\mathbf{t_2})\}$ $\therefore M_{\mathbf{X_1},\mathbf{X_2}}(\mathbf{t_1}, \mathbf{t_2}) = M_{\mathbf{X_1}}(\mathbf{t_1})M_{\mathbf{X_2}}(\mathbf{t_2}) \leftrightarrow \boldsymbol{\Sigma_{12}} = \boldsymbol{\Sigma_{21}} = \mathbf{0}$

3. 조건부 분포: $\mathrm{X_1}|\mathrm{X_2} \sim N_m(\boldsymbol{\mu_1} + \boldsymbol{\Sigma_{12}}\boldsymbol{\Sigma_{22}^{-1}}(\mathbf{X_2} - \boldsymbol{\mu_2}),\ \boldsymbol{\Sigma_{11}} - \boldsymbol{\Sigma_{12}}\boldsymbol{\Sigma_{22}^{-1}}\boldsymbol{\Sigma_{21}})$ ($\boldsymbol{\Sigma}$는 양정치)

pf) $\mathbf{W} = \mathbf{X_1} - \boldsymbol{\Sigma_{12}}\boldsymbol{\Sigma_{22}^{-1}}\mathbf{X_2} \rightarrow \begin{bmatrix}\mathbf{W} \\ \mathbf{X_2}\end{bmatrix} = \begin{bmatrix}\mathbf{I_m} & -\boldsymbol{\Sigma_{12}}\boldsymbol{\Sigma_{22}^{-1}} \\ \mathbf{O_{qm}} & \mathbf{I}_{\boldsymbol{q}}\end{bmatrix}\begin{bmatrix}\mathbf{X_1} \\ \mathbf{X_2}\end{bmatrix}$ $\left(\mathbf{A} = \begin{bmatrix}\mathbf{I_m} & -\boldsymbol{\Sigma_{12}}\boldsymbol{\Sigma_{22}^{-1}} \\ \mathbf{O_{qm}} & \mathbf{I}_{\boldsymbol{q}}\end{bmatrix}\right)$

$\begin{bmatrix}\mathbf{W} \\ \mathbf{X_2}\end{bmatrix} \sim N_p(\mathbf{A}\boldsymbol{\mu}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A^T})$; $\mathbf{A}\boldsymbol{\Sigma}\mathbf{A^T} = \begin{bmatrix}\boldsymbol{\Sigma_{11}} - \boldsymbol{\Sigma_{12}}\boldsymbol{\Sigma_{22}^{-1}}\boldsymbol{\Sigma_{21}} & \mathbf{O_{mq}} \\ \mathbf{O_{qm}} & \boldsymbol{\Sigma_{22}}\end{bmatrix}$ ➔ **W, X₂ 독립**

$\mathbf{W}|\mathbf{X_2} = \mathbf{W} \sim N_m(\boldsymbol{\mu_1} - \boldsymbol{\Sigma_{12}}\boldsymbol{\Sigma_{22}^{-1}}\boldsymbol{\mu_1}, \boldsymbol{\Sigma_{11}} - \boldsymbol{\Sigma_{12}}\boldsymbol{\Sigma_{22}^{-1}}\boldsymbol{\Sigma_{21}}) \rightarrow \therefore \boldsymbol{X_1}|\boldsymbol{X_2} \sim N_m(\boldsymbol{\mu_1} + \boldsymbol{\Sigma_{12}}\boldsymbol{\Sigma_{22}^{-1}}(\mathbf{X_2} - \boldsymbol{\mu_2}),\ \boldsymbol{\Sigma_{11}} - \boldsymbol{\Sigma_{12}}\boldsymbol{\Sigma_{22}^{-1}}\boldsymbol{\Sigma_{21}})$

4. 카이 제곱: $W = (\mathbf{X} - \boldsymbol{\mu})^T(\boldsymbol{\Sigma}^{-1})(\mathbf{X} - \boldsymbol{\mu}) = \mathbf{Z}^T\mathbf{Z} \sim \chi^2(p)$ ($\boldsymbol{\Sigma}$는 양정치)

pf) $W = \mathbf{Z}^T\mathbf{Z} = \sum_{i=1}^p Z_i^2 \sim \chi^2(p)$ $\quad *$ 가법성 of $\chi 2$ using MGF; for iid $\mathrm{Z} \sim \mathrm{N}(0, 1^2)$ ➔ $\sum_{i=1}^p [(X_i - \mu_i)/\sigma_i]^2 \sim \chi^2(p)$

*** Bivariate normal distribution (이변량 정규 분포)**

1) **기댓값**: $\boldsymbol{\mu} = \begin{bmatrix}\boldsymbol{\mu_1} \\ \boldsymbol{\mu_2}\end{bmatrix}$, $\boldsymbol{\Sigma} = \begin{bmatrix}\boldsymbol{\sigma_1^2} & \boldsymbol{\sigma_{12}} \\ \boldsymbol{\sigma_{12}} & \boldsymbol{\sigma_2^2}\end{bmatrix}$, $\boldsymbol{\sigma_{12}} = \boldsymbol{\rho}\,\boldsymbol{\sigma_1}\boldsymbol{\sigma_2}$

2) PDF: $f(x, y) = \dfrac{1}{2\pi\sigma_1\sigma_2\sqrt{1 - \rho^2}} \exp\left\{-\dfrac{1}{2(1 - \rho^2)}\left[\left(\dfrac{x - \mu_1}{\sigma_1}\right)^2 + \left(\dfrac{y - \mu_2}{\sigma_2}\right)^2 - 2\rho\left(\dfrac{x - \mu_1}{\sigma_1}\right)\left(\dfrac{y - \mu_2}{\sigma_2}\right)\right]\right\}$

3) 조건부 분포: $Y|X \sim N[\mu_2 + \rho\dfrac{\sigma_2}{\sigma_1}(x - \mu_1), \sigma_2^2(1 - \rho^2)]$

PCA 기본

$\mathbf{Y} = \boldsymbol{\Gamma}\mathbf{X} = (\mathbf{PC_1}, \mathbf{PC_2}, \cdots, \mathbf{PC}_{\boldsymbol{p}})^T$ ➔ $\mathbf{PC_1} = \mathbf{v_1^T}\mathbf{X}$ ($\mathbf{v_1}$: $\text{Cov}(\mathbf{X}) = \boldsymbol{\Sigma}$의 λ_1 대응 고유벡터)

pf)$\mathbf{Y} \sim N_p(\boldsymbol{\Gamma}\boldsymbol{\mu}, \boldsymbol{\Gamma}\boldsymbol{\Sigma}\boldsymbol{\Gamma}^T) = N_p(\boldsymbol{\Gamma}\boldsymbol{\mu}, \boldsymbol{\Lambda}) \rightarrow \text{TV}(\mathbf{X}) = \sum \sigma_i^2 = tr(\boldsymbol{\Sigma}) = tr(\boldsymbol{\Lambda}) = \sum\lambda_i = \text{TV}(\mathbf{Y})$

어떤 $\|\mathbf{a}\|^2 = 1$, $\mathbf{a} = \sum_{j=1}^p a_j\mathbf{v}_j$ *에 대해* $\mathbf{a}^T\mathbf{v}_1 = a_i$

$\rightarrow \text{Var}(\mathbf{a}^T\mathbf{X}) = \text{Cov}(\mathbf{a}^T\mathbf{X}) = \mathbf{a}^T\Sigma\mathbf{a} = (\boldsymbol{\Gamma}\mathbf{a})^T\boldsymbol{\Lambda}(\boldsymbol{\Gamma}\mathbf{a}) = \sum_i^p \lambda_i(\mathbf{a}^T\mathbf{v}_i)^2 = \sum_i^p \lambda_i a_i^2 \leq \lambda_1 = \text{Var}(Y_1)$, 등호조건: $\mathbf{a} = \mathbf{v}_1$

$\therefore Y_1 = \mathbf{v_1^T}\mathbf{X}$ (고유벡터 v_1으로 총 데이터 X 사영): 총분산 $\sum\lambda_i$ 중 최대 분산 λ_1 을 설명하는 **PC₁**

$\rightarrow \mathbf{X} = \boldsymbol{\Gamma}^{\mathbf{T}}\mathbf{Y}$ 에서 $X_k = (v_{1k})\mathbf{PC_1} + (v_{2k})\mathbf{PC_2} + \cdots$ (각 v_{ik}는 X_k의 PC_i에 대한 PC score)

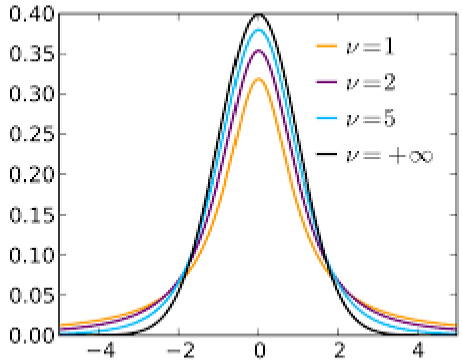
3-4. 주요 분포: t-분포, F-분포

t-분포

Joint PDF: $h(w,v)=\left\{\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}w^2\right)\right\}\left\{\frac{1}{\Gamma\left(\frac{r}{2}\right)2^{\left(\frac{r}{2}\right)}}v^{\left(\frac{r}{2}\right)-1}e^{-\frac{v}{2}}\right\}$ $w\in(-\infty,\infty), v\in(0,\infty)$; 서로 독립

$T=\frac{W}{\sqrt{V/r}}$ 일때 $t=\frac{w}{\sqrt{v/r}}, u=v$로 변환 시 $|J|=\frac{\sqrt{u}}{\sqrt{r}}$ $\rightarrow g(t,u)$ 에서 T의 marginal PDF $g_1(w)=\int_0^\infty g(t,u)du$

$T=\frac{\boldsymbol{W}}{\sqrt{\boldsymbol{V}/\boldsymbol{r}}}$ $\leftarrow$ $W\sim N(0,1^2), V\sim\chi^2(r)$; 서로 독립



1. PDF: $f(x)=\frac{\Gamma\left(\frac{r+1}{2}\right)}{\sqrt{r\pi}\,\Gamma\left(\frac{r}{2}\right)}\left(1+\frac{x^2}{r}\right)^{-\left(\frac{r+1}{2}\right)}$, $x\in\mathbb{R}$

2. 기대값: $E(T)=0\ (r>1)$, $\mathrm{Var}(T)=\frac{r}{r-2}\ (r>2)$

$*\ E(T^k)=E\left[W^k\left(\frac{V}{r}\right)^{-k/2}\right]=E(W^k)E\left[\left(\frac{V}{r}\right)^{-k/2}\right]=\frac{1}{r^{-\frac{k}{2}}}E(W^k)E(V^{-k/2})$

$=E(W^k)\left(\frac{r}{2}\right)^{\frac{k}{2}}\frac{\Gamma\left(\frac{r}{2}-\frac{k}{2}\right)}{\Gamma\left(\frac{r}{2}\right)},\quad r>k$

(Cauchy 분포: df = 1인 t 분포, $f(x)=\frac{1}{\pi(1+x^2)}$)

3. Student's theorem *iid $X_i\sim N(\mu,\sigma^2)$ $\boldsymbol{n}$개 **[직관: $\boldsymbol{\sigma^2}$를 모르나, $\boldsymbol{X_i\sim N(\mu,\sigma^2)}$ ➔ $\boldsymbol{\bar{X}}$, $\boldsymbol{S}$로 $\boldsymbol{\mu}$ 정확 추정]**

1) $\frac{(n-1)S^2}{\sigma^2}\sim\chi^2(n-1)$, 2) $T=\frac{\bar{X}-\mu}{S/\sqrt{n}}\sim t\,(n-1)$ $\leftarrow$ $T=\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right)\Big/\sqrt{\frac{(n-1)S^2}{\sigma^2}/(n-1)}$

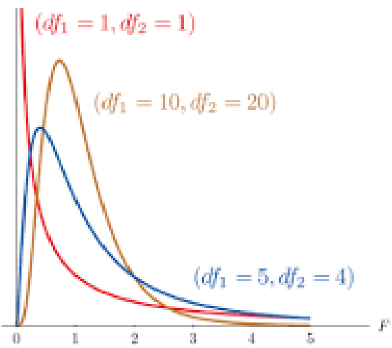
pf) $V=\sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2=\sum_{i=1}^{n}\left(\frac{(X_i-\bar{X})+(\bar{X}-\mu)}{\sigma}\right)^2=\sum_{i=1}^{n}\left(\frac{X_i-\bar{X}}{\sigma}\right)^2+\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right)^2=\frac{(n-1)S^2}{\sigma^2}+\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right)^2$

$\sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2\sim\chi^2(n)$, $\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right)^2\sim\chi^2(1)$ $\rightarrow\bar{X},S^2$ 독립; 양변 mgf 취하면 $(1-2t)^{-\frac{n-1}{2}}=E\left(\exp\left\{\frac{(n-1)S^2}{\sigma^2}\right\}t\right)$

* R codes 1) dt(x,r): f(X=x) 2) pt(x,r): P(X≤x) (자유도 r) // 3) qt(0.975,r): 97.5%인 t값

F-분포

$F=\frac{U/r_1}{V/r_2}$ $\leftarrow$ $U\sim\chi^2(r_1), V\sim\chi^2(r_2)$; 서로 독립



1. PDF: $f(x)=\frac{1}{B\left(\frac{r_1}{2},\frac{r_2}{2}\right)}\left(\frac{r_1}{r_2}\right)^{\frac{r_1}{2}}\left(1+\frac{r_1}{r_2}x\right)^{-\frac{r_1+r_2}{2}}x^{\frac{r_1}{2}-1}$ $0<x<\infty$

2. 기댓값: $E(F)=\frac{r_2}{r_2-2}$ $(r_2>2)$

$*\ E(F^k)=E\left[\left(\frac{U/r_1}{V/r_2}\right)^k\right]=\left(\frac{r_2}{r_1}\right)^kE(U^k)E(V^{-k})$

$=\left(\frac{r_2}{r_1}\right)^k\frac{\Gamma\left(\frac{r_1}{2}+k\right)}{\Gamma\left(\frac{r_1}{2}\right)}\frac{\Gamma\left(\frac{r_2}{2}-k\right)}{\Gamma\left(\frac{r_2}{2}\right)}$ $(r_2>2k)$

* R codes 1) df(x,a,b): f(X=x) 2) pf(x,a,b): P(X≤x) (자유도 a,b) // 3) qf(0.975,a,b): 97.5%인 F값

3-5. 혼합 분포

혼합 분포

1. 혼합분포: $f(x)=\sum_{i=1}^{k}p_if_i(x)$ (각각 받침 S_i, 평균 μ_i, 분산 σ_i^2 , $\sum_{j=1}^{k}p_j=1$)

1) $E(X)=\sum_{i=1}^{k}p_i\int_{-\infty}^{\infty}xf_i(x)dx=\sum_{i=1}^{k}p_i\mu_i=\bar{\mu}$

2) $Var(X)=\sum_{i=1}^{k}p_i\sigma_i^2+\sum_{i=1}^{k}p_i(\mu_i-\bar{\mu})^2$

* $Var(X)=\sum_{i=1}^{k}p_i\int_{-\infty}^{\infty}(x-\bar{\mu})^2f_i(x)dx=\sum_{i=1}^{k}p_i\int_{-\infty}^{\infty}(x-\mu_i)^2f_i(x)dx+\sum_{i=1}^{k}p_i(\mu_i-\bar{\mu})^2\int_{-\infty}^{\infty}f_i(x)dx$

2. 응용: 유한 개로 제한할 필요 X

1) **감마(θ,α,β)+푸아송(x,θ) ➔ 음이항 분포**

$p(x)=\int_0^\infty\left[\frac{1}{\Gamma(\alpha)\beta^\alpha}\theta^{\alpha-1}e^{-\theta/\beta}\right]\left[\frac{e^{-\theta}\theta^x}{x!}\right]d\theta=\frac{1}{\Gamma(\alpha)\beta^\alpha x!}\int_0^\infty\theta^{\alpha+x-1}e^{-\theta(\alpha+\beta)/\beta}d\theta=\frac{\Gamma(\alpha+x)}{\Gamma(\alpha)x!}\frac{\beta^x}{(1+\beta)^{\alpha+x}}$

$\rightarrow p(x)=\frac{(r+x-1)!}{(r-1)!}\frac{p^r(1-p)^x}{x!}$, $\alpha=r\in\mathbb{Z}^+$, $\beta=\frac{1-p}{p}$

2) **베이지안 추론**: $h(x)=\int_\theta g(\theta)f(x|\theta)d\theta$; $g(\theta)$: Conjugate prior, $h(x)$: 무조건부

① $X|\theta\sim$N(0, 1/θ), θ~Γ(r/2,2/r) → $X\sim t(r)$ ② 이항분포 (p모름)→베타분포 β(p)로 추출 $\int_0^1p(x|p)g(p)dp$

4. 통계적 추론 – MLE / 신뢰구간 / 가설검정

표본 / 통계량

* 표본 → 1) 분포 f(x), p(x)의 추론 // 2) θ 추론 ← f(x), p(x)는 알고 있음 (X_i: 확률변수, x_i: 실현값)

1. 확률 표본 (Random sample): iid $[X_1,\cdots,X_n]$
2. 통계량 (Statistic): $T = T(X_1,\cdots,X_n)$ (표본에 대한 함수)

   ➔ **$\boldsymbol{\theta} \in \Omega$에 대한 추정량**이면 T: 점추정량 (point estimator), 실현값 t: 점추정값 (point estimate)
3. **불편추정량 (Unbiased estimator)**: $E(T) = \theta$ $[E(\bar{X}) = \mu,\ \ E(S^2) = \sigma^2]$
4. **Maximum likelihood estimator (mle)**

 1) **가능도 함수:** $L(\theta) = \prod_{i=1}^{n} f(x_i;\theta)$ ← **MLE:** $\hat{\theta} = \mathrm{Argmax}[L(\theta)]$

 2) **로그우도 함수:** $l(\theta) = \sum_{i=1}^{n} \ln f(x_i,\theta)$ ← **MLE:** $\hat{\theta} = \mathrm{Argmax}[l(\theta)]$ $\leftarrow \partial l/\partial\theta = 0$

지수	$l(\beta) = \sum_{i=1}^{n} \ln \frac{1}{\beta} e^{-\frac{x_i}{\beta}} = -\frac{1}{\beta}\sum x_i - n\ln\beta = -\mathrm{n}\left(\frac{1}{\beta}\bar{X} + \ln\beta\right) \to \frac{\partial l}{\partial\beta} = \mathrm{n}\left(\frac{\bar{\mathrm{X}}}{\beta^2} - \frac{1}{\beta}\right) \to \hat{\beta} = \bar{X}$ (also 불편)
이항	$l(p) = \sum_{i=1}^{n} \ln p^{x_i}(1-p)^{1-x_i} = n\bar{X}\ln p + (n - n\bar{X})\ln(1-p) \to \frac{\partial l}{\partial p} = n\left(\frac{\bar{\mathrm{X}}}{p} - \frac{1-\bar{X}}{1-p}\right) \to \hat{p} = \bar{X}$ (also 불편)
정규	$l(\mu,\sigma) = -\frac{n}{2}\ln 2\pi - n\ln\sigma - \frac{1}{2}\sum\left(\frac{x_i-\mu}{\sigma}\right)^2 \to \nabla l(\mu,\sigma) = \left[\frac{1}{\sigma}\sum\left(\frac{x_i-\mu}{\sigma}\right), -\frac{n}{\sigma} + \frac{1}{\sigma^3}\sum(x_i-\mu)^2\right]^T$ $\to \hat{\mu} = \bar{X},\ \ \hat{\sigma}^2 = \frac{1}{n}\sum_{\mathrm{i=1}}^{\mathrm{n}}(X_i - \bar{X})^2 = \frac{n-1}{n}S^2$

CLT

1) Pivot 확률변수: (추정량-모수)/표준오차
2) 중심극한정리: $Z_n = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \xrightarrow{D} N(0,1)$ ↔ 근사적으로 $N(0,1)$에 수렴 cf) $X \sim N(\mu,\sigma^2)$이면 정확히 정규분포

신뢰 구간

*신뢰구간: **모수 $\boldsymbol{\theta}$가 추정량 $\hat{\boldsymbol{\theta}}$에서** 얼마나 벗어났는가?

1. 신뢰구간: $1-\alpha = P_\theta[\theta \in (\hat{\theta}_L, \hat{\theta}_U)]$ → 100(1-α)% 신뢰구간 (같은 신뢰계수 → 구간 길이 최소화)

 *해석: **모수 $\boldsymbol{\theta}$가 추정량 $(\hat{\boldsymbol{\theta}}_L, \hat{\boldsymbol{\theta}}_U)$ 구간에 있는 사건 $\sim B(1, 1-\alpha)$ (95% CI: $\boldsymbol{\theta}$가 $(\hat{\boldsymbol{\theta}}_L, \hat{\boldsymbol{\theta}}_U)$에 평균 19회/20회)**

2. 평균 신뢰 구간 ($\boldsymbol{z}_{\alpha/2}$: 상위 $\alpha/2$에서 z값) ⇔ $(z_{\alpha/2} = \xi_{1-\alpha/2} \leftrightarrow F(z_{\alpha/2}) = F(\xi_{1-\alpha/2}) = 1-\alpha/2)$

상황	가정	Pivot statistic	$\boldsymbol{\mu}$의 100(1-α)% CI
대표본 (근사;CLT)	평균 μ 분산 σ^2	$Z = \frac{\bar{X}-\mu}{S/\sqrt{n}} \sim N(0,1)$ 중심극한정리	$1-\alpha \approx P\left(-\boldsymbol{z}_{\alpha/2} < \frac{\bar{X}-\mu}{S/\sqrt{n}} < \boldsymbol{z}_{\alpha/2}\right)$
t-구간 정규성 (정확)	$X_i \sim N(\mu,\sigma^2)$	$T = \frac{\bar{X}-\mu}{S/\sqrt{n}}$ $(df = n-1)$	$1-\alpha = P\left(-\boldsymbol{t}_{\alpha/2,n-1} < \frac{\bar{X}-\mu}{S/\sqrt{n}} < \boldsymbol{t}_{\alpha/2,n-1}\right)$

3. 평균 차이 $(\bar{\boldsymbol{X}} - \bar{\boldsymbol{Y}})$ 신뢰 구간 $*\,E(\bar{X}-\bar{Y}) = \mu_1 - \mu_2,\ \ \mathrm{Var}(\bar{X}-\bar{Y}) = (\sigma_1^2/n_1) + (\sigma_2^2/n_2)$

상황	가정	Pivot statistic	유도/비교
대표본 (근사;CLT)	평균 $\mu_1 - \mu_2$ 분산 $(\sigma_1^2/n_1) + (\sigma_2^2/n_2)$	$\boldsymbol{Z} = \frac{(\bar{\boldsymbol{X}}-\bar{\boldsymbol{Y}}) - (\boldsymbol{\mu_1}-\boldsymbol{\mu_2})}{\sqrt{(\boldsymbol{S_1^2/n_1}) + (\boldsymbol{S_2^2/n_2})}} \sim N(0,1)$	
t-통계량 정규성 (등분산)	$X_i \sim N(\mu_1, \boldsymbol{\sigma^2})$ $Y_i \sim N(\mu_2, \boldsymbol{\sigma^2})$	$\boldsymbol{T} = \frac{(\bar{\boldsymbol{X}}-\bar{\boldsymbol{Y}}) - (\boldsymbol{\mu_1}-\boldsymbol{\mu_2})}{\boldsymbol{S_p}\sqrt{(\boldsymbol{1/n_1}) + (\boldsymbol{1/n_2})}}$ $(df = n_1 + n_2 - 2)$ $S_p^2 = \frac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{(n_1-1)+(n_2-1)}$	1) $\mathrm{E}(S_p^2) = \sigma^2$ 2) $(\mathrm{n}_1 - 1)S_1^2/\sigma^2 \sim \chi^2(n_1-1)$ $\to (\mathrm{n}_1 + \mathrm{n}_2 - 2)S_p^2 \sim \chi^2(\mathrm{n}_1 + \mathrm{n}_2 - 2)$ 3) $S_p^2 \leftrightarrow (\bar{X}-\bar{Y})$ 독립
t-통계량 정규성 (이분산)	$X_i \sim N(\mu_1, \boldsymbol{\sigma_1^2})$ $Y_i \sim N(\mu_2, \boldsymbol{\sigma_2^2})$	$\boldsymbol{T} = \frac{(\bar{\boldsymbol{X}}-\bar{\boldsymbol{Y}}) - (\boldsymbol{\mu_1}-\boldsymbol{\mu_2})}{\sqrt{(\boldsymbol{S_1^2/n_1}) + (\boldsymbol{S_2^2/n_2})}}$ **(Welch's, df<-round(df))**	$df = \frac{\left(\frac{s_x^2}{n_x} + \frac{s_y^2}{n_y}\right)^2}{\frac{\left(\frac{s_x^2}{n_x}\right)^2}{n_x-1} + \frac{\left(\frac{s_y^2}{n_y}\right)^2}{n_y-1}}$

4. 통계적 추론 – MLE / 신뢰구간 / 가설검정

신뢰 구간

4. 비율 차이 (극한 표준정규분포; CLT)

1) 가정: $X\sim b(1,p_1),\ Y\sim b(1,p_2) \rightarrow\ \hat{p}_1=\bar{X},\ \hat{p}_2=\bar{Y}$ $\qquad\therefore E(\hat{p}_1)=p_1,\ \mathrm{Var}(\hat{p}_1)=p_1(1-p_1)/n_1$

상황	가정	Pivot statistic
대표본 (근사;CLT)	평균 p_1-p_2 분산 $\dfrac{p_1(1-p_1)}{n_1}+\dfrac{p_2(1-p_2)}{n_2}$	$\boldsymbol{Z}=\dfrac{(\hat{\boldsymbol{p}}_1-\hat{\boldsymbol{p}}_2)-(\boldsymbol{p}_1-\boldsymbol{p}_2)}{\sqrt{\dfrac{\hat{\boldsymbol{p}}_1(1-\hat{\boldsymbol{p}}_1)}{\boldsymbol{n}_1}+\dfrac{\hat{\boldsymbol{p}}_2(1-\hat{\boldsymbol{p}}_2)}{\boldsymbol{n}_2}}}\sim N(0,1)$

5. 이산형 모수

1) $F_T(T;\theta)$: 통계량 T의 cdf; θ에 대해 단조 감소 → 신뢰 구간: $F_T(T_{n-1};\ \underline{\theta})=1-\alpha_2,\ F_T(T_n;\bar{\theta})=\alpha_1$

2) **Bisection algorithm**: 순감소 $g(x)=d\in g([a,\ b])\ \rightarrow\ $ 1) if $g\{(a+b)/2\}>d\ \rightarrow$ 구간 $[(a+b)/2,\ b]$ 재설정

$\rightarrow$ 2) if $g\{(a+b)/2\}<d\ \rightarrow$ 구간 $[a,\ (a+b)/2]$ 재설정

상황	조건	유도
Binomial	$X\sim b(1,p)$ $n=30,\bar{x}=0.60$ $T=n\bar{X}\sim b(30,p)$ $(T_{n-1}=17,\quad T_n=18)$	① **하한:** pbinom(17, 30, 0.4)=0.9787, pbinom(17,30,0.45)=0.9286 ➔ pbinom(17, 30, **0.434**) ≈ 0.95 ② **상한:** pbinom(18, 30, 0.7)=0.1593, pbinom(18,30,0.8)=0.0094 ➔ pbinom(18, 30, **0.747**) ≈ 0.05 $\therefore\ \boldsymbol{p}$의 90% CI: $[\mathbf{0.434},\mathbf{0.747}]$
Poisson	$X\sim\mathrm{Poi}(\mu)$ $\mathrm{n}=25,\bar{x}=5$ $T=n\bar{X}\sim\mathrm{Poi}(25\mu)$ $(T_{n-1}=124,\quad T_n=125)$	① **하한:** ppois(124, 25 x 4)=0.9912, ppois(124, 25 x 4.4)=0.9145 ➔ ppois(124, 25 x **4.287**) ≈ 0.95 ② **상한:** ppois(125, 25 x 5.5)=0.1330, ppois(125, 25 x 6)=0.0204 ➔ ppois(125, 25 x **5.8**) ≈ 0.05 $\therefore\ \boldsymbol{\mu}$의 90% CI: $[\mathbf{4.287},\mathbf{5.8}]$

순서 통계량

*정의: $(Y_1<\cdots<Y_n)\leftarrow[X_1,\cdots,X_n]$ 재배열 $\qquad$ *강점: 분포에 종속되지 않음.

1. **PDF**: $g(y_1,\cdots,y_n)=\boldsymbol{n}!\,f(y_1)\cdots f(y_n)\quad(\text{on } a<y_1<\cdots<y_n<b)$ $\qquad$ pf) $g(y_1,\cdots,y_n)=\sum_{i=1}^{n!}|J_i|f(y_1)\cdots f(y_n)$

2. **Marginal PDF** $\quad$ 1) $\boldsymbol{g_k(y_k)}=\dfrac{\boldsymbol{n}!}{(\boldsymbol{k}-\mathbf{1})!\,(\boldsymbol{n}-\boldsymbol{k})!\,(\mathbf{1})!}[\boldsymbol{F(y_k)}]^{\boldsymbol{k}-\mathbf{1}}[\mathbf{1}-\boldsymbol{F(y_k)}]^{\boldsymbol{n}-\boldsymbol{k}}\boldsymbol{f(y_k)}$

pf) $g_k(y_k)=\displaystyle\int_a^{y_k}\cdots\int_a^{y_2}\int_{y_k}^{b}\cdots\int_{y_{n-1}}^{b}\boldsymbol{n}!\,f(y_1)\cdots f(y_n)dy_n\cdots dy_{k+1}dy_1\cdots dy_{k-1}\quad(y_n\rightarrow y_{k+1}\,;y_1\rightarrow y_{k-1})$

2) $\boldsymbol{g_{ij}(y_i,y_j)}=\dfrac{\boldsymbol{n}!}{(\boldsymbol{i}-\mathbf{1})!\,(\boldsymbol{j}-\boldsymbol{i}-\mathbf{1})!\,(\boldsymbol{n}-\boldsymbol{j})!\,(\mathbf{1})!\,(\mathbf{1})!}[\boldsymbol{F(y_i)}]^{\boldsymbol{i}-\mathbf{1}}[\boldsymbol{F(y_j)}-\boldsymbol{F(y_i)}]^{\boldsymbol{j}-\boldsymbol{i}-\mathbf{1}}[\mathbf{1}-\boldsymbol{F(y_j)}]^{\boldsymbol{n}-\boldsymbol{j}}\boldsymbol{f(y_i)f(y_j)}$

3. **Quantile (분위수)**: cdf $F(\xi_p)=p\ \leftrightarrow\ \boldsymbol{\xi_p}=\boldsymbol{F^{-1}(p)},\qquad \boldsymbol{k}=\mathbf{floor}[\boldsymbol{p}(\boldsymbol{n}+\mathbf{1})]$

1) $\boldsymbol{F(Y_k)}$는 $\dfrac{\boldsymbol{k}}{\boldsymbol{n}+\mathbf{1}}$ 의 불편추정량: $E(F(Y_k))=\displaystyle\int_a^b F(y_k)g_k(y_k)dy_k=\int_0^1\frac{n!}{(k-1)!\,(n-k)!}z^k(1-z)^{n-k}dz=\frac{k}{n+1}$

2) Quartile: 1분위수 $(\boldsymbol{Q_1}=Y_{[0.25(n+1)]})$ ⬄ 중위수 $(\boldsymbol{Q_2}=Y_{[0.5(n+1)]})$ ⬄ 3분위수 $(\boldsymbol{Q_3}=Y_{[0.75(n+1)]})$

*중위수: 홀수➔중간값 $Y_{(n+1)/2}$ / 짝수➔ $(Y_{(n/2)}+Y_{(n/2)+1})/2$

➔ **Box plot**: $h=1.5(Q_3-Q_1),\ LF=Q_1-h,\ UF=Q_3+h$ $\quad$ (LF, UF 바깥: 이상값; 정규분포상 P≤0.007)

3) **Q-Q plot**: 표본의 순서통계량 $(Y_1,Y_2,\cdots,Y_{50})$ ⬄ 이론적 분위수 $(Z_{0.02},Z_{0.04},\cdots,Z_{1.00})$ $\quad$ ← any 분포

4) 신뢰구간: $1-\alpha=P(Y_i<\xi_p<Y_j)=\sum_{w=i}^{j-1}\binom{n}{w}p^w(1-p)^{n-w}\quad\leftarrow\ \boldsymbol{p}=\boldsymbol{F(\xi_p)}$ $\quad$ (중위수: $p=1/2$)

4. 통계적 추론 – MLE / 신뢰구간 / 가설검정

<table>
<tr><td>가설 검정</td><td>

1) 가설 정의: $H_0:\theta\in\boldsymbol{\omega_0}$ (Null) vs. $H_1:\theta\in\boldsymbol{\omega_1}$ (alternative) ← $X\sim f(x;\theta)$에 대해 $\theta\in\Omega=(\omega_0\cup\omega_1)$, 분할
2) 가설 검정: 표본 $(X_1,\cdots,X_n)\in C$ → H_1채택 (기각역 $C\subset D=\text{span}\{(X_1,\cdots,X_n)\}$)
표본 $(X_1,\cdots,X_n)\notin C$ → H_0유지
3) 유의 수준: $\alpha=\max_{\theta\in\omega_0}P_\theta[(X_1,\cdots,X_n)\in C]$ (복합귀무가설에 대해 모든 null 모수 ➔ 기각역에 속할 확률 최대)
* 1종 오류: H_0 참, but 기각 ➔ H_1 채택 (=FP) ∴ 유의수준(α): 1종 오류 범할 최대 확률
4) 검정력: $\gamma(\theta)=P_\theta[(X_1,\cdots,X_n)\in C],\ \theta\in\omega_1$ ➔ 유의 수준과 다르게 대립가설 모수에 따라 달라짐
① 2종 오류: H_0 거짓, but 유지 ➔ H_0 유지 (=FN) ∴ β: 2종 오류 범할 확률 (under given $\theta\in\boldsymbol{\omega_1}$)
② 검정력: H_0 거짓 ➔ 알맞게 H_1 채택 (TP)
5) P-값: 1) Upper tail: P-값$=\boldsymbol{P_{H_0}(X\ge x_{obs})=1-F_{H_0}(x_{obs})}$
2) Lower tail: P-값$=\boldsymbol{P_{H_0}(X\le x_{obs})=F_{H_0}(x_{obs})}$
3) 2-sided: P-값$=\boldsymbol{2\times P_{H_0}(X\ge|x_{obs}|)=2[1-F_{H_0}(|x_{obs}|)]}$ (X=0 좌우 대칭)
➔ $X=F^{-1}(U)$ (단조 증가) 정리의 역에 의해 P-값 ~ unif(0,1) under 귀무가설 H_0

<table>
<tr><th>예시</th><th>분포</th><th>가설</th><th>유도</th></tr>
<tr><td rowspan="2">단일 이항 단측</td><td>$X_i\sim B(1,p)$</td><td>$H_0:p=p_0$
$H_1:p<p_0$</td><td>*표본통계량: $\boldsymbol{S=\sum_{i=1}^{n}X_i\sim B(n,p)}$
1) 기각역 설정: 귀무가설 하에서 $S\sim B(n,p_0)\to\alpha=P_{p_0}[S\le k]$
➔ $0.11=P_{p_0}[S\le 11]$ $(n=20,\ p_0=0.7)$
2) 검정력 함수: $\gamma(p)=P_p[S\le 11]$ (단조 감소 of p)
∴ <u>$H_0:p\ge p_0$로 확장</u> ⬅ $\max_{p\ge p_0}P_p[S\le k]=P_{p_0}[S\le k]$ (단조성)</td></tr>
<tr><td colspan="3">대표본에서 $\boldsymbol{\frac{\hat{p}-p_0}{\sqrt{\hat{p}(1-\hat{p})/n}}\approx\frac{\hat{p}-p_0}{\sqrt{p(1-p)/n}}\xrightarrow{D}N(0,1)}$</td></tr>
<tr><td colspan="3">대표본</td><td>* 표본통계량: $\boldsymbol{\frac{\bar{X}-\mu}{S/\sqrt{n}}\approx\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\xrightarrow{D}N(0,1)}$</td></tr>
<tr><td>대표본 단측 (Upper)</td><td rowspan="3">X_i ~ 미지 분포
1) 평균: μ
2) 분산: σ^2</td><td>$H_0:\mu=\mu_0$
$H_1:\mu>\mu_0$</td><td>1) 기각역 설정: $\alpha=P_{\mu_0}\left[\boldsymbol{\frac{\bar{X}-\mu_0}{S/\sqrt{n}}}\ge z_\alpha\right]\approx 1-\Phi(z_\alpha)$
2) 검정력 함수: $\gamma(\mu)=P_\mu\left[\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}}\ge z_\alpha\right]=P_\mu\left[\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\ge\frac{\mu_0-\mu}{\sigma/\sqrt{n}}+z_\alpha\right]$
$\approx 1-\Phi\left(\frac{\sqrt{n}(\mu_0-\mu)}{\sigma}+z_\alpha\right)=\boldsymbol{\Phi\left(\frac{\sqrt{n}(\mu-\mu_0)}{\sigma}-z_\alpha\right)}$ (단조 증가 of μ)
* Power 증가: n↑, 효과크기 (μ-μ$_0$)↑, α↑ & σ↓</td></tr>
<tr><td>대표본 단측 (Lower)</td><td>$H_0:\mu=\mu_0$
$H_1:\mu<\mu_0$</td><td>1) 기각역 설정: $\alpha=P_{\mu_0}\left[\frac{\bar{X}-\mu_0}{S/\sqrt{n}}\le -z_\alpha\right]\approx\Phi(z_\alpha)$
2) 검정력 함수: $\gamma(\mu)=P_\mu\left[\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}}\le -z_\alpha\right]=P_\mu\left[\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\le\frac{\mu_0-\mu}{\sigma/\sqrt{n}}-z_\alpha\right]$
$\approx\boldsymbol{\Phi\left(\frac{\sqrt{n}(\mu_0-\mu)}{\sigma}-z_\alpha\right)=\Phi\left(-\frac{\sqrt{n}(\mu-\mu_0)}{\sigma}-z_\alpha\right)}$ (단조 감소 of μ)
* Power 증가: n↑, 효과크기 (μ-μ_0)↑, α↑ & S↓</td></tr>
<tr><td>대표본 양측</td><td>$H_0:\mu=\mu_0$
$H_1:\mu\ne\mu_0$</td><td>1) 기각역 설정: $\boldsymbol{\alpha}=P_{\mu_0}\left[\left|\frac{\bar{X}-\mu_0}{S/\sqrt{n}}\right|\ge z_{\alpha/2}\right]$ ⇐ (양측 동일 배분)
2) 검정력 함수: $\gamma(\mu)=P_\mu\left[\left|\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}}\right|\ge z_{\alpha/2}\right]$
$\approx\boldsymbol{\Phi\left(\frac{\sqrt{n}(\mu-\mu_0)}{\sigma}-z_{\alpha/2}\right)+\Phi\left(-\frac{\sqrt{n}(\mu-\mu_0)}{\sigma}-z_{\alpha/2}\right)}$ (U자 함수 of μ)
➔ (μ_0에서 최소값)</td></tr>
<tr><td>t-검정 정규성</td><td>$X_i\sim N(\mu,\sigma^2)$</td><td>$H_0:\mu=\mu_0$
$H_1:\mu\ne\mu_0$</td><td>* 표본통계량: $\boldsymbol{T=\frac{\bar{X}-\mu_0}{S/\sqrt{n}}\sim t(n-1)}$
* t-분포는 N(0,1) 보다 누워 있음 ➔ "보수적" // 정규성 하 "정확"</td></tr>
<tr><td>2-표본 t-검정</td><td>$X_i\sim N(\mu_1,\boldsymbol{\sigma^2})$
$Y_i\sim N(\mu_2,\boldsymbol{\sigma^2})$
(정규,등분산)</td><td>$H_0:\mu_1=\mu_2$
$H_1:\mu_1\ne\mu_2$</td><td>* 표본통계량: $\boldsymbol{T=\frac{(\bar{X}-\bar{Y})-0}{S_p\sqrt{(1/n_1)+(1/n_2)}}\sim t(n_1+n_2-2)}$
* $\boldsymbol{|T|\ge t_{0.025,n_1+n_2-2}}$ 이면 $\boldsymbol{H_0}$ 기각</td></tr>
</table>

</td></tr>
</table>

4. 통계적 추론 – MLE / 신뢰구간 / 가설검정

Pearson χ^2 검정	자유도: n(확률표본)-n(미지수 or 제약)		
	2 cells		1. 상황: $\boldsymbol{X_1\sim b(n,p_1)}$, $X_2=n-X_1$, $p_2=1-p_1$ ⇒ $Y=\frac{X_1-np_1}{\sqrt{np_1(1-p_1)}}\xrightarrow{D}N(0,1)$; $Q_1=Y^2\xrightarrow{D}\chi^2(1)$ 2. 검정통계량: $\boldsymbol{Q_1}=\frac{(X_1-np_1)^2}{np_1(1-p_1)}=\frac{(X_1-np_1)^2}{np_1}+\frac{(X_1-np_1)^2}{n(1-p_1)}=\boldsymbol{\frac{(X_1-np_1)^2}{np_1}+\frac{(X_2-np_2)^2}{np_2}\xrightarrow{D}\chi^2(1)}$
	k cells		1. 상황: k항;n회 다항분포 ($p_k=1-\sum_{i=1}^{k-1}p_i$ & $x_k=n-\sum_{i=1}^{k-1}x_i$) 2. 검정통계량: $\boldsymbol{Q_{k-1}=\sum_{i=1}^{k}\frac{(X_i-np_i)^2}{np_i}\xrightarrow{D}\chi^2(k-1)}$ ⇐ $(k-1)$개 알면 나머지 1개 앎
		적합도 검정	1) 귀무가설: $H_0: p_1=p_{1,0}$, $p_2=p_{2,0}$, $\cdots$, $p_k=p_{k,0}$ 2) 검정통계량: $Q_{k-1}=\sum_{i=1}^{k}\frac{(O_i-E_i)^2}{E_i}=\sum_{i=1}^{k}\frac{(X_i-np_{i,0})^2}{np_{i,0}}\xrightarrow{D}\boldsymbol{\chi^2(k-1)}$ (귀무가설 하)
		최소 χ^2 추정량	<예시> 정규분포 모수 추정 $N(\mu,\sigma^2)$ 1) 상황: 실수구간 → k등분 $(A_1,\cdots,A_k)$; $\boldsymbol{p_i}=\int_{A_i}\frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{1}{2}(y-\boldsymbol{\mu})^2/\boldsymbol{\sigma}^2\right]dy$ 2) 실제 A_i의 도수인 X_i → $Q_{k-1}=\sum_{i=1}^{k}\frac{(X_i-np_i)^2}{np_i}\xrightarrow{D}\boldsymbol{\chi^2(k-3)}$ **최소화하는 $\hat{\mu},\hat{\sigma}^2$**
	r x c cells	**동질성 검정**	1) 상황: **2개의 k항 다항분포** *각 모수: $(n_1,p_{11},p_{21},\cdots,p_{k1}),(n_2,p_{12},p_{22},\cdots,p_{k2})$ ⇒ $\sum_{j=1}^{2}\sum_{i=1}^{k}\frac{(X_{ij}-n_jp_{ij})^2}{n_jp_{ij}}\xrightarrow{D}[\chi^2(k-1)+\chi^2(k-1)]=\chi^2(2k-2)$ 2) 귀무가설: $H_0: p_{11}=p_{12}$, $p_{21}=p_{22}$, $\cdots$, $p_{k1}=p_{k2}$ (둘은 구간 별 비율이 동일) ⇒ $p_{m1}=p_{m2}$의 MLE: $\frac{X_{m1}+X_{m2}}{n_{m1}+n_{m2}}$ (총 $k-1$개 점추정값 필요) 3) 검정통계량: $\sum_{j=1}^{2}\sum_{i=1}^{k}\frac{\left[\boldsymbol{X_{ij}-n_j\left(\frac{X_{i1}+X_{i2}}{n_{i1}+n_{i2}}\right)}\right]^2}{\boldsymbol{n_j\left(\frac{X_{i1}+X_{i2}}{n_{i1}+n_{i2}}\right)}}\xrightarrow{D}\boldsymbol{\chi^2(k-1)}$ (귀무가설 하)
		독립성 검정	1) 상황: 확률실험 n회 결과 → 가로 (A) a항 / 세로 (B) b항 **두 종류 범주**로 구분 ➔ $p_{ij}=P(A_i\cap B_j)$, X_{ij}는 $A_i\cap B_j$ 도수 ⇒ $Q_{ab-1}=\sum_{j=1}^{b}\sum_{i=1}^{a}\frac{(X_{ij}-np_{ij})^2}{np_{ij}}\xrightarrow{D}\chi^2(ab-1)$ 2) 귀무가설: $H_0: P(A_i\cap B_j)=P(A_i)P(B_j)$ for all (i,j) (속성 A, B는 독립) ⇒ $p_{i*}=P(A_i)$의 MLE: $\hat{p}_{i*}=\frac{X_{i*}}{n}=\frac{\sum_{j=1}^{b}X_{ij}}{n}$ [총 (a − 1) + (b − 1)개 점추정값 필요] 3) 검정통계량: $\sum_{j=1}^{b}\sum_{i=1}^{a}\frac{\left[\boldsymbol{X_{ij}-n\left(\frac{X_{i*}}{n}\right)\left(\frac{X_{*j}}{n}\right)}\right]^2}{\boldsymbol{n\left(\frac{X_{i*}}{n}\right)\left(\frac{X_{*j}}{n}\right)}}\xrightarrow{D}\boldsymbol{\chi^2[(a-1)(b-1)]}$ (귀무가설 하)
비중심 분포	**비중심 χ^2**		$\boldsymbol{Y=\sum_{i=1}^{n}\left(\frac{X_i^2}{\sigma^2}\right)}$, $X_i\sim N(\mu_i,\sigma^2)$ $*\,\mu_i=0$이면 $Y\sim\chi^2(n)$ $M(t)=E[\exp(tX_i^2/\sigma^2)]=\frac{1}{(1-2t)^{n/2}}\exp\left[\frac{t\sum_1^n\mu_i^2}{\sigma^2(1-2t)}\right]=\frac{1}{(1-2t)^{n/2}}\exp\left[\frac{t}{1-2t}\theta\right]$ $\left(t<\frac{1}{2}\right)$ $\boldsymbol{Y=\sum_{i=1}^{n}\left(\frac{X_i^2}{\sigma^2}\right)\sim\chi^2(n,\theta)}$ $\left(\theta=\frac{\sum_1^n\mu_i^2}{\sigma^2}\right)$ $pf)$ MGF 적분 활용 → 치환 하여 정규분포 PDF꼴로 정리
			* R codes 1) dchisq (x,r,a): f(X=x) 2) pchisq (x,r,a): P(X≤x)
	비중심 F		$U\sim\chi^2(n_1,\theta)$ & $V\sim\chi^2(n_2)$ $*\,U,V$는 독립 $Y=\frac{U/n_1}{V/n_2}\sim F(n_1,n_2,\theta)$

4. 통계적 추론 – MLE / 신뢰구간 / 가설검정

Monte Carlo	*** 몬테카를로 생성: 특정 "Known" 표본/분포 ➔ 관측값 생성** (Resampling, Bayesian 등에서 중요) **1. 균등분포 (Uniform distribution)**: $\mathrm{unif}(a, b)$; $\mathrm{pdf} = 1/(b-a)$ ← R codes: runif(횟수)		
	unif(0,1)⇔CDF "관측치 생성"		$\boldsymbol{X = F^{-1}(U)}$는 **cdf** $\boldsymbol{F(X)}$ 따름⇔ 역: $Z = F(X) \sim \mathrm{unif}(0,1)$
		지수	$F(x) = 1 - e^{-x/\beta}$, $(x > 0)$ $\therefore X = F^{-1}(U) = -\beta \ln(1-U)$ 는 지수분포 생성
		푸아송	$m = \lambda w$ → $T_i \sim \exp(1/\lambda)$ 에 대해 $[X = k] \Leftrightarrow \sum_{i=1}^{k} T_i \le w$ & $\sum_{i=1}^{k+1} T_i > w$
			* 구간 w 동안 난수로 T_i 생성 → 횟수 카운트 (초기 $X = 0, T = 0$) 1) $\Delta T = -(1/\lambda)\ln(1-U)$ 2) $T \leftarrow T + \Delta T$ 3) if $\mathrm{T} \le \mathrm{w}$: $X \leftarrow X + 1$ elif $\mathrm{T} > \mathrm{w}$: return X
		정규 분포	<**Box & Muller** (1958)> → 일반화: Marsaglia & Bray (1964) $X_1 = (-2\ln Y_1)^{1/2}\cos(2\pi Y_2)$; $X_2 = (-2\ln Y_1)^{1/2}\sin(2\pi Y_2) \Leftarrow Y_1, Y_2 \sim \mathrm{unif}(0,1)$ $f(X_1, X_2) = \lvert J \rvert g(Y_1, Y_2) = \frac{1}{2\pi}\exp\left[-\frac{1}{2}(x_1^2 + x_2^2)\right]$
	채택-기각 (A-R) 알고리즘 (어려운 CDF)		$X = F^{-1}(U)$를 closed form 계산 불가.⇐ $g(x)$ 이용: ①Easy ②$f(x)$ 유사 ③$\frac{f(x)}{g(x)} \le k$ (유계) ① $Y \sim g(y)$ & U 생성 ② $U \le \frac{f(Y)}{kg(Y)} \le 1$이면 $\mathrm{X = Y}$, 아니면 ①로 돌아가 재 생성 ⇒ 조금 더 넓은 $kg(x)$로 근사 ($f(x) = cf_1(x)$와 $g(x) = dg_1(x)$적당히 상수배 하여 k 무시 가능)
		감마 CDF $\boldsymbol{\Gamma(\alpha, \beta)}$	$Y_i \sim \Gamma(1,1)$ ➔ $X = \sum_{i=1}^{\alpha} Y_i \sim \Gamma(\alpha, 1)$ (α 정수: CDF 생성 쉬움) $X \sim \Gamma(\alpha, 1)$ ➔$\beta X \sim \Gamma(\alpha, \beta)$ (α 실수 → **문제!**)
			① $X \sim \Gamma(\alpha, 1)$ & $Y \sim \Gamma([\alpha], 1/b)$ ② $\frac{f(x)}{g(x)} = b^{-[\alpha]} x^{\alpha - [\alpha]} e^{-(1-b)x} \le b^{-[\alpha]} \left\{\frac{\alpha - [\alpha]}{(1-b)e}\right\}^{\alpha - [\alpha]}$ (by x로 미분) ③ 위 식을 b로 미분 ⇒ $\frac{f(x)}{g(x)} \le ([\alpha]/\alpha)^{-[\alpha]} \left\{\frac{\alpha - [\alpha]}{(1 - [\alpha]/\alpha)e}\right\}^{\alpha - [\alpha]} = M$
		정규 CDF $\boldsymbol{N(0,1)}$	① **Y~Cauchy (역 CDF 알려짐) ➔ X~N(0,1)** ② $\frac{\phi(x)}{g(x)} = \frac{(2\pi)^{-1/2}\exp\{-x^2/2\}}{\{\pi(1+x^2)\}^{-1}} \Rightarrow \frac{\phi_1(x)}{g_1(x)} = (1 + x^2)\exp\{-x^2/2\} \le 2e^{-1/2} = M$
	Monte Carlo t-검정 (오염된 정규)		$W \sim N(0, 1^2)$ & $W \sim N(0, \sigma_c^2)$ $(\varepsilon: 0.25, \sigma_c = 25)$ ← $W = Z$ or $\sigma_c Z$; $E(W) = 0$
		* 가설: $H_0: \mu = 0$, $H_1: \mu > 0$ 1) $\mathrm{n} = 20$, 2) $t_{0.05,19} = 1.729$	* 추정 알고리즘 (N: 시뮬레이션 수) 1) $\mathrm{n} = 20$ 표본 생성 ⇐ X (오염 정규; μ 모름)분포 2) $T = (\bar{X} - \mu)/(S/\sqrt{n})$계산 ➔ 1),2) N번 반복 3) **유의수준 실험적 추정량:** $\boldsymbol{\hat{\alpha} = I/N}$ ($I: T > t_{0.05,10}$ 도수) $\mathbf{SE} = \sqrt{\boldsymbol{\hat{\alpha}(1-\hat{\alpha})/N}}$ 예시) $\hat{\alpha} = 0.0412 \pm 0.0039$
	Monte Carlo 적분		적분가능한 $g(x)$의 closed form 역도함수 (≈부정적분) 존재X ➔ **수치적 적분** $\int_a^b g(x)\,dx = (b-a)\int_a^b g(x)\left(\frac{1}{b-a}\right)dx = \boldsymbol{(b-a)E[g(X)]} \Leftarrow X \sim \mathrm{unif}(a, b)$ $\therefore \boldsymbol{\bar{Y}} = \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{Y_i} = \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{(b-a)g(X_i)}$ 는 정적분의 unbiased estimator ⇐ $X_i \sim \mathrm{unif}(a, b)$

4. 통계적 추론 – MLE / 신뢰구간 / 가설검정

Boot-strap 기본

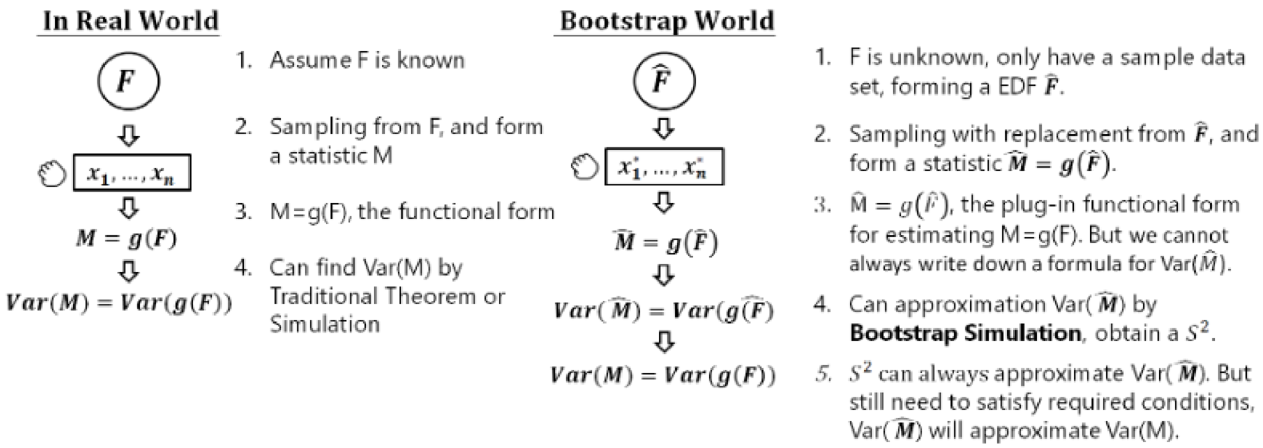
비교

1) 중심극한정리: 표본 통계량 $(\hat{\theta})$ 의 pivotal statistic이 극한 정규분포따름 ➔ 모수 θ 추정
2) 몬테카를로 기법: **X의 known 분포 (CDF)➔** 균등분포 난수추출기로 관측값 $X = F^{-1}(U)$ 생성
3) 부트스트랩: **X의 unknown 분포 ➔ 표본** $(X_1, \cdots X_n)$의 EDF $(\hat{F}_n)$→ 무작위 추출로 X_i^* 생성
$\hat{\theta}^*$의 분포 → $\hat{\theta}$의 신뢰구간 추정 → θ의 근사적 신뢰구간

원리

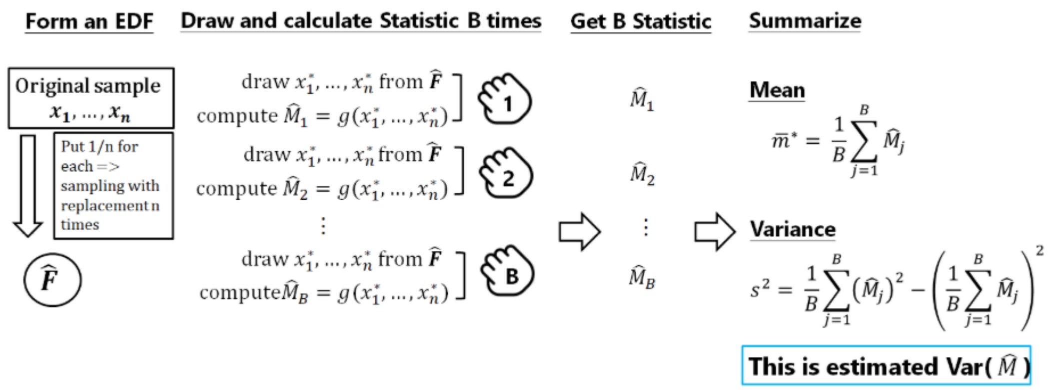
일반적인 통계적 추론에서는 estimator → parameter를 추정함.
Standard error는 estimator의 자체적인 변동성 (표준편차) (e.g. $SE(\bar{X}) = \sqrt{Var(\bar{X})} = \sigma/\sqrt{n}$)
Estimated SE는 SE에 unknown parameter가 들어가 있을 때, 다른 estimator를 이용 $(S/\sqrt{n})$

*문제: ①일반적인 확률변수 Y에 대해 분포 (PDF, CDF)를 알기 어렵고
② 통계량 g(Y)의 S.E.를 $\sigma/\sqrt{n}$처럼 정확한 수식으로 알아낼 수 있는 경우는 많지 않음.



EDF $\hat{F}(x) = \frac{1}{n}\sum_{i=1}^{n} I(x_i \le x)$ ⇔ (x보다 작은 표본내 실현값 수) / n ⇔ PMF는 $\frac{1}{n}$ (for every x_i)

Statistical functional (통계적 범함수): 모수가 [분포함수]의 함수로 표현됨. (평균,분산,중위수,백분위수,etc)
e.g. $E(X) = \int xf dx = \int x dF$, $Var(X) = \int x^2 dF - \left(\int x dF\right)^2$
Plug-in principle: $\hat{\theta} = g(\hat{F}) = \int r(x) d\hat{F}$ ↔ $\theta = g(F) = \int r(x) dF$ (전자는 후자의 plug-in estimate)



2.Variance of $\hat{M}$ with EDF $\hat{F}$

$$\underbrace{s^2 = \frac{1}{B}\Sigma_{j=1}^{B}\left(\hat{M}_j\right)^2 - \left(\frac{1}{B}\Sigma_{j=1}^{B}\hat{M}_j\right)^2}_{\text{1.Bootstrap Variance Estimation}} \underset{\text{1.Simulation Error}}{\approx} \mathbf{Var}(\hat{M}; \hat{F}) \overset{\text{2. Approximation Error}}{\approx} \underbrace{Var(M; F)}_{\text{3.Variance of M with true F}}$$

1번 simulation error는 결국 큰 수의 법칙에 의해 확률 수렴하므로 B↑으로 최소화 가능
2번 approximation error는 $\hat{F}$이 F에 근사 (n↑) 하면 최소화 (n↑면 자연스럽게 $\hat{M} \xrightarrow{P} M$ 성질도...)

4. 통계적 추론 – MLE / 신뢰구간 / 가설검정

<table>
<tr>
<td rowspan="3">Boot-strap 응용</td>
<td>모평균 추정</td>
<td colspan="2">$E(X_i^*) = \sum_{j=1}^{n}\frac{1}{n}X_j = \bar{X}$, $\mathrm{Var}(X_i^*) = \sum_{j=1}^{n}\frac{1}{n}\left(X_j - \bar{X}\right)^2 = \frac{n}{n-1}S^2$
$E\left(\bar{X}_j^*\right) = \bar{X}$, $\mathrm{Var}\left(\bar{X}_j^*\right) = \frac{S^2}{n-1}$
B회 시뮬레이션 평균 $\frac{1}{B}\sum_{i=1}^{B}\bar{X}_j^* \xrightarrow{P} E\left(\bar{X}_j^*\right) = \bar{X} \xrightarrow{P} \boldsymbol{\mu}$, 분산 $\frac{1}{B}\sum_{i=1}^{B}\left(\bar{X}_j^*\right)^2 - \left(\frac{1}{B}\sum_{i=1}^{B}\bar{X}_j^*\right)^2 \xrightarrow{P} \mathrm{Var}\left(\bar{X}_j^*\right) = \frac{S^2}{n-1} \xrightarrow{P} \frac{\boldsymbol{\sigma^2}}{\boldsymbol{n}}$
➔ B 회 부트스트랩 $\bar{X}$ 신뢰구간 (비모수적 counting) ≈ $\left[\bar{X} - z_{\alpha/2}\frac{S^2}{n},\ \bar{X} + z_{\alpha/2}\frac{S^2}{n}\right]$ ≈ [μ의 CLT 신뢰 구간]

- 위의 정규가정을 통한 $z_{\alpha/2}$ 근사는 책 참고 4.9.1를 참조
- 다른 모수 추정도 크게 다르지 않음. ($\bar{X}$처럼 precise한 분산식이 존재하지 않으면 시뮬레이션 효과↑)
- 통계량의 분포가 다른 모수에 종속되지 않게 pivot화하면 부트스트랩 정확성 향상 가능</td>
</tr>
<tr>
<td>Boot-strap 검정</td>
<td><1표본 평균>
$H_0: \mu = \mu_0$
$H_1: \mu > \mu_0$

<2표본 평균>
$H_0: \mu_2 = \mu_1$
$H_1: \mu_2 > \mu_1$</td>
<td>1. 상황: 1) 검정통계량: $\bar{X}$ 2) $\hat{p} = P_{H_0}[\bar{X} \geq \bar{x}]$
2. H_0 가정 ⇒ $\boldsymbol{z_i = x_i - \bar{x} + \mu_0}$ $\left(E(z_i^*) = E\left(\bar{z}_j^*\right) = \mu_0\right)$ ⇒ 복원으로 n개 부트스트랩
3. Empirical P-value 산출: $\boldsymbol{\hat{p} = I/B}$ $\left(I: \left\{\bar{z}_j^* > \bar{x}\right\}\right)$

1. 상황: 1) 검정통계량: $V = \bar{Y} - \bar{X}$ 2) $\hat{p} = P_{H_0}[V \geq \bar{y} - \bar{x}]$
2. H_0 가정 ⇒ 표본 합침 ($\mathrm{n = n_1 + n_2}$) ⇒ 복원으로 $\left(\mathbf{X_i^*}, \mathrm{n}_\mathbf{1}\text{개}\right), \left(\mathbf{Y_i^*}, \mathrm{n}_\mathbf{2}\text{개}\right)$ 추출
3. Empirical P-value 산출: $\boldsymbol{\hat{p} = I/B}$ $\left(I: \{\bar{y}_i^* - \bar{x}_i^* > \bar{y} - \bar{x}\}\right)$
* 부연: H_0 가정 했기 때문에 생성값 ($\bar{y}_i^* - x_i^*$)은 H_0하 통계량임.</td>
</tr>
<tr>
<td>Perm test</td>
<td colspan="2">2표본 perm test: 통합 표본 (n=n$_1$+n$_2$)에서 비복원으로 추출된 x,y 모든 가능한 표본➔ 검정</td>
</tr>
</table>

5. 일치성 / 극한분포 ("통계학적 수렴")

중요한 부등식

1. **Markov**: $\boldsymbol{P[u(X) \ge c] \le E[u(X)]/c}$ (for u(X)≥0, c>0; E[u(X)]존재)
   *증명: $E[u(x)] = \int_{-\infty}^{\infty} u(x)f(x)dx \ge \int_{u(x)\ge c} u(x)f(x)dx \ge c\int_{u(x)\ge c} f(x)dx = c\,P[u(x) \ge c]$
2. **Chevyshev**: $\boldsymbol{P(|X-\mu| \ge k\sigma) \le 1/k^2}$ (for k>0; X가 μ,σ²(유한) 가짐)
   *증명: Markov에서 $u(X) = (X-\mu)^2,\ c = k^2\sigma^2$

확률 수렴

1. **정의**: $\boldsymbol{X_n \xrightarrow{P} X} \iff \forall\epsilon > 0,\ \boldsymbol{\lim_{n\to\infty} P[|X_n - X| \ge \epsilon] = 0}\ \left(\Leftrightarrow \lim_{n\to\infty} P[|X_n - X| < \epsilon] = 1\right)$ "함수열의 수렴"
   ($\boldsymbol{X_n \xrightarrow{P} a}$, if X가 상수 a ➔ "퇴화확률변수, p(a)=1, 나머지 0")

2. **대수의 약법칙**: iid $\{X_n\} \sim \left(\textbf{평균}: \boldsymbol{\mu}, \textbf{분산}: \boldsymbol{\sigma^2 < \infty}\right)$, $\boldsymbol{\bar{X}_n \xrightarrow{P} \mu}$
   *증명: By Chevyshev's ineq, $\boldsymbol{P(|\bar{X}_n - \mu| \ge \epsilon) \le \sigma^2/(n\epsilon^2)} \to 0$ (when $n \to \infty$)

3. **정리**

	정리	증명
선형	$* X_n \xrightarrow{P} X, Y_n \xrightarrow{P} Y$ ① $\boldsymbol{(X_n + Y_n) \xrightarrow{P} (X+Y)}$ ② $\boldsymbol{aX_n \xrightarrow{P} aX}$	①P는 집합오염에 단조 (=공간 커지면 확률 커짐); 삼각부등식 $P[|(X_n+Y_n)-(X-Y)| \ge \epsilon] \le P[|X_n - X| + |Y_n - Y| \ge \epsilon]$ $\le P[|X_n - X| \ge \epsilon/2] + P[|Y_n - Y| \ge \epsilon/2]$
함수	* 받침 상 연속 $g(x)$ ③ $\boldsymbol{X_n \xrightarrow{P} a \Rightarrow g(X_n) \xrightarrow{P} g(a)}$ ④$\boldsymbol{X_n \xrightarrow{P} X \Rightarrow g(X_n) \xrightarrow{P} g(X)}$	① $|g(x) - g(a)| \ge \epsilon \Rightarrow |x-a| \ge \delta\ (\epsilon > 0, \delta > 0)$ $\therefore P[|g(X_n) - g(a)| \ge \epsilon] \le P[|X_n - a| \ge \delta]$
곱	$* X_n \xrightarrow{P} X, Y_n \xrightarrow{P} Y$ ⑤ $\boldsymbol{X_nY_n \xrightarrow{P} XY}$	$* X_nY_n = \frac{1}{2}X_n^2 + \frac{1}{2}Y_n^2 - \frac{1}{2}(X_n - Y_n)^2 \xrightarrow{P} \frac{1}{2}X^2 + \frac{1}{2}Y^2 - \frac{1}{2}(X-Y)^2 = XY$

4. **일치성**: $\boldsymbol{T_n \xrightarrow{P} \theta}$ 면 ⇔ T_n은 θ의 **일치 추정량** $* F(x;\theta)$에서 추출한 $iid\ \{X_1, \cdots, X_n\}$의 통계량 T_n

분산 추정량	① $\boldsymbol{S_n^2 \xrightarrow{P} \sigma^2}$ (일치 & 불편) ② $\boldsymbol{S_{mle}^2} = \frac{n-1}{n}S_n^2 \xrightarrow{P} \boldsymbol{\sigma^2}$ (일치 & MLE)
균등분포 모수 추정량	$X_1, \cdots, X_n \sim \mathrm{unif}\,(0,\theta)$, $Y_n = \max\{X_1, \cdots, X_n\}$ $g_n(Y_n) = \frac{n!}{(n-1)!\,0!\,1!}F(Y_n)^{n-1}\left(1 - F(Y_n)\right)^0 f(y_n) = n\frac{t^{n-1}}{\theta^n}\quad (0 < t \le \theta)$ $E(Y_n) = \int_0^\theta t\left(n\frac{t^{n-1}}{\theta^n}\right)dt = \frac{n}{n+1}\theta$ ∴ 최대값 Y_n은 θ의 일치 추정량
	$\bar{X}_n$은 $\theta/2$의 일치 추정량 ⇒ $2\bar{X}_n$은 θ의 일치 추정량

분포 수렴

1. **정의**: $\boldsymbol{X_n \xrightarrow{D} X} \iff \forall x \in \{F_X$ 연속 점$\}$, $\boldsymbol{\lim_{n\to\infty} F_n(x) = F(x)}$, $\left(F: X\text{의 cdf},\ F_n: X_n\text{의 cdf}\right)$ "극한분포"

2. $\boldsymbol{t}$**분포** ⇒ $\boldsymbol{z}$**분포** $(n \to \infty)$
   ① $f_n(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\,\Gamma\left(\frac{n}{2}\right)}\left(1 + \frac{x^2}{n}\right)^{-\left(\frac{n+1}{2}\right)}$, $\int_{-\infty}^t |f_n(y)|dy \le \int_{-\infty}^t 10 f_1(y)dy = \frac{10}{\pi}\tan^{-1}t < \infty$ (르벡 DCT)
   ② $\lim_{n\to\infty} F_n(t) = \lim_{n\to\infty}\int_{-\infty}^t f_n(x)dx = \int_{-\infty}^t \lim_{n\to\infty} f_n(x)dx = \int_{-\infty}^t \lim_{n\to\infty}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\,\Gamma\left(\frac{n}{2}\right)}\left(1 + \frac{x^2}{n}\right)^{-\left(\frac{n+1}{2}\right)} dx = \int_{-\infty}^t \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)dx = \Phi(t)$

3. **정리**

① $\boldsymbol{X_n \xrightarrow{P} X \Rightarrow X_n \xrightarrow{D} X}$	https://freshrimpsushi.tistory.com/175?category=696570
② $\boldsymbol{X_n \xrightarrow{P} b \Leftrightarrow X_n \xrightarrow{D} b}$	if 분포수렴 $\Rightarrow \lim_{n\to\infty} P[|X_n - b| \le \epsilon] = \lim_{n\to\infty} F_{X_n}(b+\epsilon) - F_{X_n}(b-\epsilon) = 1 - 0 = 0$
③ $\boldsymbol{X_n \xrightarrow{D} X}$ & $\boldsymbol{(A_n \xrightarrow{P} a,\ B_n \xrightarrow{P} b)}$ $\boldsymbol{\Rightarrow A_n + B_nX_n \xrightarrow{D} a + bX}$	<**Slutsky 정리**> e.g. $P_n - Q_n \xrightarrow{P} 0,\ Q_n \xrightarrow{D} X \Rightarrow P_n = (P_n - Q_n) + Q_n \xrightarrow{D} X$
* 받침 상 연속 $g(x)$ ④ $\boldsymbol{X_n \xrightarrow{D} X \Rightarrow g(X_n) \xrightarrow{D} g(X)}$	$\boldsymbol{Z_n \xrightarrow{D} Z \Rightarrow Z_n^2 \xrightarrow{D} \chi^2(1)}$
⑤ $\boldsymbol{X_n \xrightarrow{D} X \Leftrightarrow \lim_{n\to\infty} M_n(t) = M(t)}$	$\boldsymbol{Y_n \sim b(n,p) \Rightarrow \lim_{n\to\infty} M_n(t)} = \lim_{n\to\infty} E(e^{tY_n}) = \lim_{n\to\infty}[(1-p) + (pe^t)]^n = \boldsymbol{e^{\mu(e^t - 1)}}$ ∴ **이항분포** $\boldsymbol{b(n,p)} \xrightarrow{D}$ **푸아송분포** $\boldsymbol{(\mu = np)}$

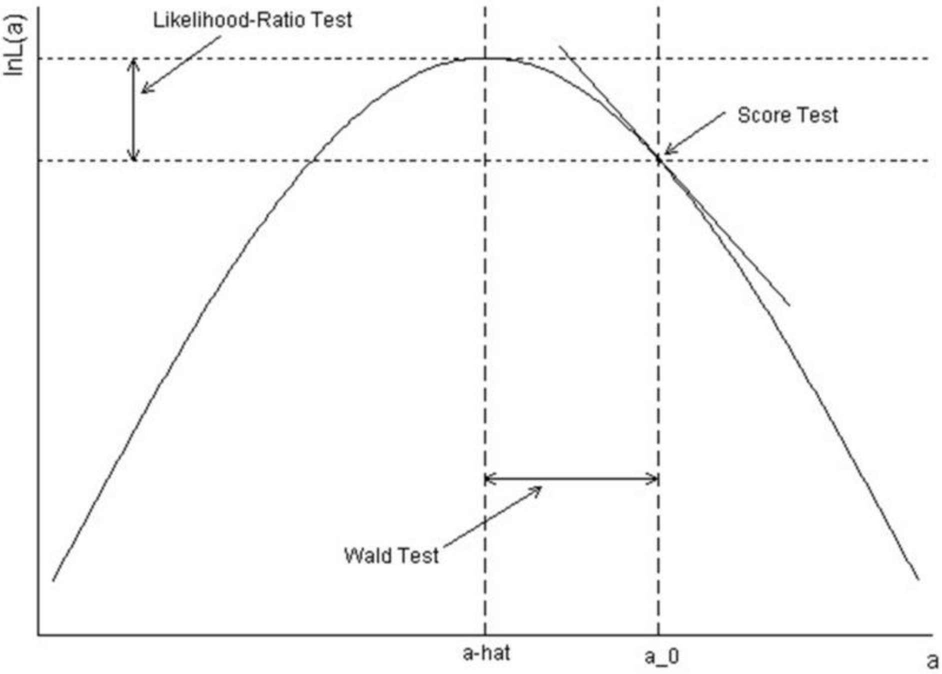
5. 일치성 / 극한분포 (“통계학적 수렴”)

Δ-방법		$\boldsymbol{\sqrt{n}(X_n-\theta)\xrightarrow{D}N(0,\sigma^2)}$ 이고, $g(x)$가 θ에서 미분 가능 & $g'(\theta)\neq 0$ 이면 $\boldsymbol{\sqrt{n}(g(X_n)-g(\theta))\xrightarrow{D}N(0,g'(\theta)^2\sigma^2)}$ **[Δ-method를 잘 이용하면 모수에 종속되지 않는 통계량 분산 만듦]** pf) 테일러 정리에 의해 $g(X_n)=g(\theta)+g'(\theta)(X_n-\theta)+o(\lvert X_n-\theta\rvert)$ 이므로 $\sqrt{n}\big(g(X_n)-g(\theta)\big)=\sqrt{n}g'(\theta)(X_n-\theta)+o(\sqrt{n}\lvert X_n-\theta\rvert)\xrightarrow{P}\sqrt{n}g'(\theta)(X_n-\theta)\xrightarrow{D}N(0,g'(\theta)^2\sigma^2)$ (중간에 little-o를 0으로 확률수렴 시키는 전개는 확률 유계인 Y_n에 대해 $o(Y_n)\xrightarrow{P}0$ 임을 이용)
중심극한정리 (CLT)		1. 중심극한정리: $\boldsymbol{Z_n=\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\xrightarrow{D}N(0,1)}$ ← iid $X_i\sim$(평균: μ, 분산: σ^2) 2. 대표본 추론 통계량: $\boldsymbol{\frac{\bar{X}-\mu}{S/\sqrt{n}}\xrightarrow{D}N(0,1)}$ ∵ $S\xrightarrow{P}\sigma \Leftrightarrow \frac{S}{\sigma}\xrightarrow{P}1$, CLT & Slutsky에 의해 $\left(\frac{\sigma}{S}\right)\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\xrightarrow{D}N(0,1)$
		2. 증명: MGF 이용 (특성함수 $\varphi(t)=E(e^{itx})$ 이용해야 더 정확함) $m(t)\coloneqq E\left[e^{t(X-\mu)}\right]=e^{-\mu t}M(t) \Rightarrow m(0)=1,\ m'(0)=E(X-\mu)=0,\ m''(0)=E[(X-\mu)^2]+m'(0)^2=\sigma^2$ 테일러 정리에 의해 $m(t)=m(0)+m'(0)t+\frac{1}{2}m''(\xi)t^2=1+\frac{1}{2}m''(\xi)t^2=1+\frac{1}{2}\sigma^2t^2+\frac{1}{2}(m''(\xi)-\sigma^2)t^2,\quad \xi\in[-t,t]$ $M(t;n)\coloneqq E(e^{tZ_n})=E\left(\exp\left(t\frac{(1/n)\sum X_i-\mu}{\sigma/\sqrt{n}}\right)\right)=E\left(\exp\left(t\frac{\sum_{i=1}^n(X_i-\mu)}{\sigma\sqrt{n}}\right)\right)=\prod_{i=1}^n E\left(\exp\left(t\frac{X_i-\mu}{\sigma\sqrt{n}}\right)\right)$ $=\left[E\left(\exp\left(\frac{t(X-\mu)}{\sigma\sqrt{n}}\right)\right)\right]^n=\left[m\left(\frac{t}{\sigma\sqrt{n}}\right)\right]^n,\quad -h<\frac{t}{\sigma\sqrt{n}}<h$ $M(t;n)=\left[m\left(\frac{t}{\sigma\sqrt{n}}\right)\right]^n=\left\{1+\frac{t^2}{2n}+\frac{[m''(\xi)-\sigma^2]t^2}{2n\sigma^2}\right\}^n,\quad \xi\in\left[-\frac{t}{\sigma\sqrt{n}},\frac{t}{\sigma\sqrt{n}}\right]$ $\therefore \lim_{n\to\infty}M(t;n)=\lim_{n\to\infty}\left\{1+\frac{t^2}{2n}+\frac{[m''(\xi)-\sigma^2]t^2}{2n\sigma^2}\right\}^n=\lim_{n\to\infty}\left(1+\frac{t^2}{2n}\right)^n=\exp\left(\frac{1}{2}t^2\right)\quad ∵ \lim_{n\to\infty}[m''(\xi)-\sigma^2]=0\ (∵\ \xi\to 0)$ Z_n의 mgf $M(t;n)$의 $n\to\infty$ 극한값은 $N(0,1)$의 mgf $\exp\left(\frac{1}{2}t^2\right) \Rightarrow \therefore \boldsymbol{Z_n\xrightarrow{D}N(0,1)}$
다변량 분포 확장	**수렴성 다변량 확장**	1) 확률수렴: $\{\mathbf{X_n}\}\in\mathbb{R}^p$일 때, 벡터의 각 성분이 수렴하는 경우가 전체 벡터의 수렴과 동치이다. 즉, $\mathbf{X_n}\xrightarrow{P}\mathbf{X} \Leftrightarrow X_{nj}\xrightarrow{P}X_j$ (모든 $j=1,\cdots,p$에서 성립) 2) 분포수렴: $\mathbf{X_n}\xrightarrow{D}\mathbf{X} \Leftrightarrow \forall\mathbf{x}\in\{F(\mathbf{x})$ 연속 점$\}$, $\lim_{n\to\infty}\boldsymbol{F_n}(\mathbf{x})=\boldsymbol{F}(\mathbf{x})$, $\left(F:\mathbf{X}\text{의 cdf},\ F_n:\mathbf{X_n}\text{의 cdf}\right)$ ① $\mathbf{X_n}\xrightarrow{D}\mathbf{X}\Rightarrow \boldsymbol{g}(\mathbf{X_n})\xrightarrow{D}\boldsymbol{g}(\mathbf{X})$ (corollary: $g(\mathbf{x})=x_j$로 두면 분포수렴이 **주변 (marginal)** 수렴 수반 ② $\mathbf{X_n}\xrightarrow{D}\mathbf{X} \Leftrightarrow \lim_{n\to\infty}\boldsymbol{M_n}(\mathbf{t})=\boldsymbol{M}(\mathbf{t})$
	다변량 표본	$\{\mathbf{X_n}\}\in\mathbb{R}^p$ 인 평균 $\boldsymbol{\mu}$, 공분산행렬 $\boldsymbol{\Sigma}$ 인 iid 확률벡터열 ① 표본평균벡터: $\bar{\mathbf{X}}_n=\frac{1}{n}\sum_{i=1}^n\mathbf{X}_i=\left(\bar{X}_1,\cdots,\bar{X}_p\right)^T$ ② 표본공분산행렬: $S_j^2=\frac{1}{n-1}\sum_{i=1}^n\left(X_{ij}-\bar{X}_j\right)^2,\ S_{jk}=\frac{1}{n-1}\sum_{i=1}^n\left(X_{ij}-\bar{X}_j\right)(X_{ik}-\bar{X}_k)$; p × p 행렬 $\therefore\ \bar{\mathbf{X}}_n\xrightarrow{P}\boldsymbol{\mu},\ \mathbf{S}_n\xrightarrow{P}\boldsymbol{\Sigma}$ (4차 적률 유한할 때 대수 약법칙)
	CLT	$\mathbf{Y_n}=\sqrt{n}(\bar{\mathbf{X}}_n-\boldsymbol{\mu})\xrightarrow{D}N_p(\mathbf{0},\boldsymbol{\Sigma}) \Leftrightarrow$ 근사적으로 $\bar{\mathbf{X}}_n\sim N_p\left(\boldsymbol{\mu},\frac{1}{n}\boldsymbol{\Sigma}\right)$
	Δ방법	$\sqrt{n}(\mathbf{X_n}-\boldsymbol{\mu_0})\xrightarrow{D}N_p(\mathbf{0},\boldsymbol{\Sigma})$ ($\mathbf{g}$는 $\mathbb{R}^p\to\mathbb{R}^k$ 로의 변환 $(k\le p)$; 미분행렬 $\mathbf{B}=\left[\frac{\partial g_i}{\partial x_j}\right]$이 연속, $\mathbf{B}\neq\mathbf{0}$ in $\boldsymbol{\mu_0}$ 근방 $\sqrt{n}(\mathbf{g}(\mathbf{X_n})-\mathbf{g}(\boldsymbol{\mu_0}))\xrightarrow{D}N_p(\mathbf{0},\mathbf{B_0}\boldsymbol{\Sigma}\mathbf{B_0}^T)$ $\mathbf{B_0}=\mathbf{B}(\boldsymbol{\mu_0})$

6. 최대가능도방법 (Maximum Likelihood Methods)

MLE **(R0)~(R2)**	**MLE 핵심**	(R0), (R1) 하에서 $\lim_{n\to\infty} P_{\theta_0}[L(\theta_0,\mathbf{X}) > L(\theta,\mathbf{X})] = 1 \quad (\forall\,\theta \neq \theta_0)$ $pf)\ \frac{1}{n}\sum_{i=1}^{n}\ln\left[\frac{f(X_i;\theta)}{f(X_i;\theta_0)}\right] \xrightarrow{P} E_{\theta_0}\left(\ln\left[\frac{f(X_1;\theta)}{f(X_1;\theta_0)}\right]\right) < \ln E_{\theta_0}\left[\frac{f(X_1;\theta)}{f(X_1;\theta_0)}\right]$ by 대수의 법칙, 젠센 부등식 $E_{\theta_0}\left[\frac{f(X_1;\theta)}{f(X_1;\theta_0)}\right] = \int \frac{f(x;\theta)}{f(x;\theta_0)} f(x;\theta_0)dx = 1$ ($R1$ 공통 받침 하에서) $\therefore\ \frac{1}{n}\sum_{i=1}^{n}\ln\left[\frac{f(X_i;\theta)}{f(X_i;\theta_0)}\right] < 0 \Leftrightarrow\ L(\theta_0,\mathbf{X}) > L(\theta,\mathbf{X})$ ∴ 근사적으로 **참값 $\boldsymbol{\theta_0}$에서 우도함수 $\boldsymbol{L(\theta,\mathbf{X})}$가 최대**가 된다. ($\hat\theta = \mathrm{Argmax}[L(\theta)] \xrightarrow{P} \theta_0$)
	불변성	$\boldsymbol{\eta = g(\theta) \Leftrightarrow \hat\eta = g(\hat\theta)}$ $pf)$ ① $g \in$ 1대1 함수: $\max L(\theta) = \max L\left(g^{-1}(\eta)\right)$ 이므로 $\hat\theta = g^{-1}(\hat\eta)$에서 우도 최대화 ② $g \notin$ 1대1 함수: $g^{-1}(\eta) := \{\theta: g(\theta) = \eta\}$ 새로 정의 → $\hat\theta \in g^{-1}(\hat\eta)$ 에서 우도최대화
	추정 방정식	*추정방정식 (estimating equation; EE): $\partial l(\theta)/\partial\theta = 0$ (R0)~(R2) 하에서 $\partial l(\theta)/\partial\theta = 0$ 는 $\hat\theta \xrightarrow{P} \theta_0$ 인 $\hat\theta$를 가짐 (Corollary: EE가 유일해를 가지면 그 해는 $\hat\theta \xrightarrow{P} \theta_0$)
Cramér Rao Bound **(R0)~(R4)**	**스코어 함수 & 피셔정보**	① **Score 함수** $s(\theta) = \frac{\partial \ln f}{\partial\theta}$ ② **Fisher information** $I(\theta) = \mathrm{Var}\left(\frac{\partial \ln f}{\partial\theta}\right) = \mathrm{E}\left[\left(\frac{\partial \ln f}{\partial\theta}\right)^2\right] = -\mathrm{E}\left[\frac{\partial^2 \ln f}{\partial\theta^2}\right]$
		$1 = \int_{-\infty}^{\infty} f dx\ \rightarrow$ 양변 θ로 i) 한번 미분 ii) 두번 미분 하면 i) $0 = \int_{-\infty}^{\infty}(\partial f/\partial\theta)dx = \int_{-\infty}^{\infty}\frac{(\partial f/\partial\theta)}{f} f dx = \int_{-\infty}^{\infty}\left(\frac{\partial \ln f}{\partial\theta}\right) f dx \quad \therefore \boldsymbol{E\left(\frac{\partial \ln f}{\partial\theta}\right) = 0}$ ii) $0 = \int_{-\infty}^{\infty}\frac{\partial^2 \ln f}{\partial\theta^2} f dx + \int_{-\infty}^{\infty}\left(\frac{\partial \ln f}{\partial\theta}\right)\left(\frac{\partial \ln f}{\partial\theta}\right) f dx \quad \therefore \boldsymbol{\mathrm{E}\left[\frac{\partial^2 \ln f}{\partial\theta^2}\right] + \mathrm{E}\left[\left(\frac{\partial \ln f}{\partial\theta}\right)^2\right] = 0}$
		iid $[X_1,\cdots,X_n]$ 에 대해서 ① **Score 함수** $s_n(\theta) = \frac{\partial l}{\partial\theta} = \frac{\partial \ln L}{\partial\theta} = \sum_{i=1}^{n}\frac{\partial \ln f(X_i;\theta)}{\partial\theta}$ ② **Fisher 정보** $I_n(\theta) = \mathrm{Var}\left(\frac{\partial l}{\partial\theta}\right) = \mathrm{Var}\left(\frac{\partial \ln L}{\partial\theta}\right) = n\,I(\theta)$
	Cramér-Rao Bound (CRB)	① $\boldsymbol{\mathrm{Var}(T) \geq \frac{[\partial E(T)/\partial\theta]^2}{nI(\theta)}}$ for 임의의 통계량 $T = g(X_1,\cdots,X_n)$ ② $\boldsymbol{\mathrm{Var}(T) \geq \frac{1}{nI(\theta)}}$ for 불편추정량 T ($\because E(T) = \theta$)
		$pf)\ E(T) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}[T]f(x_1;\theta)\cdots f(x_n;\theta)dx_1\cdots dx_n$ $\Leftrightarrow\ \partial E(T)/\partial\theta = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}[T]\left[\sum_{i=1}^{n}\frac{\partial \ln f(x_i;\theta)}{\partial\theta}\right]f(x_1;\theta)\cdots f(x_n;\theta)\,dx_1\cdots dx_n \quad ,Z := \left[\sum_{i=1}^{n}\frac{\partial \ln f(x_i;\theta)}{\partial\theta}\right]$ $\Leftrightarrow\ \partial E(T)/\partial\theta = E(TZ) = E(T)E(Z) + \rho\,\sigma_T\sigma_Z = \rho\sqrt{\mathrm{Var}(T)}\sqrt{nI(\theta)} \quad \therefore \rho^2 \leq 1 \Leftrightarrow \mathrm{Var}(T) \geq \frac{[\partial E(T)/\partial\theta]^2}{nI(\theta)}$
	효율성	*효율성: 통계량 T의 효율성은 CRB(T)/Var(T) $*$ ARE (근사 상대효율성) $= \mathrm{e(T,W)} = \frac{\mathrm{Var(W)}}{\mathrm{Var(T)}}$ (if $\mathrm{T} \xrightarrow{P} \theta_0, \mathrm{W} \xrightarrow{P} \theta_0$이며 둘다 정규근사 될 때)
	MLE 정규근사 (R0)~(R5)	① **정규 근사**: $\boldsymbol{\sqrt{n}(\hat\theta - \theta_0) \xrightarrow{D} N\left(0, \frac{1}{I(\theta_0)}\right)}$ for 유한 피셔정보 $\mathrm{I}(\theta_0)$ $*\,pf)\ l'(\hat\theta)$를 θ_0 테일러 전개 … ➔ **<u>MLE의 근사 정규 신뢰 구간</u> 구할 수 있음.** ∴ $\mathrm{Var}(\hat{\boldsymbol\theta}) \xrightarrow{P} \frac{\mathbf{1}}{\boldsymbol{nI(\theta_0)}}$ (mle는 근사적으로 효율적 or mle의 분산은 CRB에 근사) ② **Δ방법**: $\boldsymbol{\sqrt{n}\left(g(\hat\theta) - g(\theta_0)\right) \xrightarrow{D} N\left(0, \frac{g'(\theta_0)^2}{I(\theta_0)}\right)}$ ($g(x)$가 θ에서 미분 가능 & $g'(\theta) \neq 0$ 이면) ③ 정규 근사: $\hat\theta - \theta_0 = \frac{1}{nI(\theta_0)}\sum_{i=1}^{n}\frac{\partial \ln f(X_i;\theta_0)}{\partial\theta} + \frac{R_n}{\sqrt{n}} = -\frac{l'(\theta_0)}{l''(\theta_0)} + \frac{R_n}{\sqrt{n}}$ $\left(R_n \xrightarrow{P} 0\right)$
	MLE Newton's	$\hat\theta_1 = \hat\theta_0 - \frac{l'(\hat\theta_0)}{l''(\hat\theta_0)}$ 과정 반복 $*\ \hat\theta_0$이 일치 추정량이면 $\hat\theta_1$은 mle 임$\left(\xrightarrow{D} \boldsymbol{N\left(0, \frac{1}{I(\theta_0)}\right)}\right)$

6. 최대가능도방법 (Maximum Likelihood Methods)

<table>
<tr><td rowspan="5">최대 가능도 검정 (ML tests)</td><td>전개</td><td colspan="2">우도비 (LR): $\boldsymbol{\Lambda} = \frac{\boldsymbol{L(\theta_0)}}{\boldsymbol{L(\hat{\theta})}}$ ($\boldsymbol{\Lambda \le c}$ 에서 기각)
$-\frac{1}{n} l''(\theta_0) \xrightarrow{P} I(\theta_0)$, $\frac{l'(\theta_0)}{\sqrt{n}} = \sqrt{n}(\hat{\theta} - \theta_0) I(\theta_0) + R_n$ 이므로
$l(\hat{\theta}) = l(\theta_0) + (\hat{\theta} - \theta_0) l'(\theta_0) + \frac{1}{2}(\hat{\theta} - \theta_0)\, l''(\theta_n^*)$ 의 미분계수 항들에 대입해주면
$-2\ln\Lambda = 2[l(\hat{\theta}) - l(\theta_0)] = \left[\sqrt{nI(\theta_0)}(\hat{\theta} - \theta_0)\right]^2 + R_n^*$ $\left(R_n^* \xrightarrow{P} 0\right)$
$\therefore\ \boldsymbol{-2\ln\Lambda \xrightarrow{D} \chi^2(1)} \Leftarrow \sqrt{nI(\theta_0)}(\hat{\theta} - \theta_0) \xrightarrow{D} N(0,1)$</td></tr>
<tr><td>우도비 검정</td><td>$\boldsymbol{\chi_L^2 = -2\ln\Lambda}$</td><td rowspan="3">$\chi^2 \ge \chi_\alpha^2(1)$ 에서 단측 검정 기각역
$(H_0: \theta = \theta_0, H_1: \theta \ne \theta_0)$</td></tr>
<tr><td>Wald 검정</td><td>$\chi_W^2 = \left[\sqrt{nI(\hat{\theta})}(\hat{\theta} - \theta_0)\right]^2$</td></tr>
<tr><td>Score 검정</td><td>$\chi_R^2 = \left(\frac{l'(\theta_0)}{\sqrt{nI(\theta_0)}}\right)^2$</td></tr>
<tr><td colspan="3">

have the following relationship Wald ≥ LR ≥ score (Johnston and DiNardo 1997 p. 150). That is, the Wald test statistic will always be greater than the LR test statistic, which will, in turn, always be greater than the test statistic from the score test. When computing power was much more limited, and many models took a long time to run, being able to approximate the LR test using a single model was a fairly major advantage. Today, for most of the models researchers are likely to want to compare, computational time is not an issue, and we generally recommend running the likelihood ratio test in most situations. This is not to say that one should never use the Wald or score tests. For example, the Wald test is commonly used to perform multiple degree of freedom tests on sets of dummy variables used to model categorical predictor variables in regression (for more information see our webbooks on Regression with Stata, SPSS, and SAS, specifically Chapter 3 – Regression with Categorical Predictors.) The advantage of the score test is that it can be used to search for omitted variables when the number of candidate variables is large.</td></tr>
<tr><td>정칙 조건

Regularity conditions</td><td colspan="3">(R0): pdf $f(x;\theta)$는 서로 distinct 하다. i.e. $\theta_1 \ne \theta_2 \Rightarrow f(x_i;\theta_1) \ne f(x_i;\theta_2)$
(R1): pdf $f(x;\theta)$는 모든 θ에 대해 공통된 support를 갖는다. (θ에 의존적이지 않다.)
(R2): θ_0 (참값) $\in \Omega$
(R3): pdf $f(x;\theta)$는 θ로 두 번 미분 가능
(R4): $\int f(x;\theta)dx$는 θ로 두 번 미분 가능
(R5): pdf $f(x;\theta)$는 θ로 세 번 미분 가능, 모든 θ에 대해 $|\partial^3 \ln f / \partial\theta^3| \le M(x)$ $\left(E_{\theta_0}[M(X)] < \infty\right)$ $in\ \theta_0$근방 $\forall x$</td></tr>
</table>

6. 최대가능도방법 (Maximum Likelihood Methods)

다중모수			*정칙조건 ~(R9) 까지 추가됨. (기존 정칙의 다변량 확장) 맨위 "MLE 핵심" 정리는 벡터 $\boldsymbol{\theta} = [\theta_1,\cdots,\theta_p]^T \in \mathbb{R}^p$에 대해서도 똑같이 성립함. ⇔ $\nabla l(\boldsymbol{\theta}) = \mathbf{0}$의 해 구하기
	다중 모수 추정	피셔정보량	$\nabla \ln f(X;\boldsymbol{\theta}) = \left(\frac{\partial \ln f(X;\boldsymbol{\theta})}{\partial \theta_1},\cdots,\frac{\partial \ln f(X;\boldsymbol{\theta})}{\partial \theta_p}]\right)^T$ 피셔 정보량: $\mathbf{I}(\boldsymbol{\theta}) = \mathrm{Cov}(\nabla \ln f(X;\boldsymbol{\theta})) = -E\left[\frac{\partial^2}{\partial\theta_j\,\partial\theta_k}\ln f\right]_{jk} = E\left[\left(\frac{\partial \ln f}{\partial \theta_j}\right)\left(\frac{\partial \ln f}{\partial \theta_k}\right)\right]_{jk}$
		피셔정보량 (표본 n 확장)	$\nabla l = \nabla \ln L = \sum_{i=1}^{n} \nabla \ln f$ 피셔 정보량: $\mathbf{I_n}(\boldsymbol{\theta}) = \mathrm{Cov}(\nabla l) = \mathrm{Cov}(\nabla \ln L) = n\mathbf{I}(\boldsymbol{\theta})$
		CRB	$Var(T_j) \ge \frac{1}{n}[\mathbf{I}^{-1}(\boldsymbol{\theta})]_{jj}$ (T_j가 θ_j의 불편 추정량)
		정규근사	$\sqrt{n}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta_0}) \xrightarrow{D} N_p(\mathbf{0}, \mathbf{I}^{-1}(\boldsymbol{\theta_0}))$ ⇒ $\sqrt{n}(\hat{\theta}_j - \theta_j) \xrightarrow{D} N(0, [\mathbf{I}^{-1}(\boldsymbol{\theta_0})]_{jj})$
		Δ방법	$\sqrt{n}(\mathbf{g}(\hat{\boldsymbol{\theta}}) - \mathbf{g}(\boldsymbol{\theta_0})) \xrightarrow{D} N_p(\mathbf{0}, \mathbf{B}[\mathbf{I}^{-1}(\boldsymbol{\theta_0})]\mathbf{B}^T)$ ($\mathbf{g}$는 $\mathbb{R}^p \to \mathbb{R}^k$ 로의 변환 ($k \le p$); 미분행렬 $\mathbf{B} = \left[\frac{\partial g_i}{\partial \theta_j}\right]$이 연속, $\mathbf{B} \ne \mathbf{0}$ in $\boldsymbol{\theta_0}$ 근방)
	다중 모수 검정	기본	$\boldsymbol{H_0}: \boldsymbol{\theta} \in \boldsymbol{\omega}$, $\boldsymbol{H_1}: \boldsymbol{\theta} \in (\boldsymbol{\omega^c} \cap \boldsymbol{\Omega})$ (Ω: p차원 전체 모수 공간; ω: p-q차원 귀무가설 모수공간 (q: 제약된 모수 개수)) 우도비 (LR): $\boldsymbol{\Lambda} = \frac{\boldsymbol{L}(\hat{\boldsymbol{\omega}})}{\boldsymbol{L}(\hat{\boldsymbol{\Omega}})} = \frac{\max_{\boldsymbol{\theta}\in\boldsymbol{\omega}} \boldsymbol{L}(\boldsymbol{\theta})}{\max_{\boldsymbol{\theta}\in\boldsymbol{\Omega}} \boldsymbol{L}(\boldsymbol{\theta})}$ $\chi_L^2 = -2\ln\Lambda \xrightarrow{D} \chi^2(q)$ (Wald, Score 검정통계량도 가능)
		예시: 정규 μ	$H_0: \mu = \mu_0, H_1: \mu \ne \mu_0$ $\{X_n\} \overset{\text{iid}}{\sim} N(\mu, \sigma^2)$ $L(\hat{\Omega}) = \frac{1}{(2\pi\hat{\sigma}^2)^{n/2}} \exp\left\{-\frac{1}{2}\sum_{i=1}^{n} \frac{(X_i - \bar{X})^2}{\frac{1}{n}(\sum_{i=1}^{n}(X_i-\bar{X})^2)}\right\} = \frac{\mathbf{1}}{(2\pi\hat{\sigma})^{n/2}} \exp\left(-\frac{n}{2}\right)$ $L(\hat{\omega}) = \frac{\mathbf{1}}{(2\pi\hat{\sigma}_0^2)^{n/2}} \exp\left(-\frac{n}{2}\right)$ $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$, $\hat{\sigma}_0{}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu_0)^2$ $\left(\frac{1}{\Lambda}\right)^{\frac{2}{n}} = \left(\frac{L(\hat{\Omega})}{L(\hat{\omega})}\right)^{\frac{2}{n}} = \frac{\sum_{i=1}^{n}(X_i-\mu_0)^2}{\sum_{i=1}^{n}(X_i-\bar{X})^2} = 1 + \frac{n(\bar{X}-\mu_0)^2}{\sum_{i=1}^{n}(X_i-\bar{X})^2} = 1 + \frac{1}{n-1}\left\{\frac{(\bar{X}-\mu_0)}{S/\sqrt{n}}\right\}^2$ $\left(\frac{1}{\Lambda}\right)^{\frac{2}{n}} \ge c'$ ⇔ $\|T\| \ge c^* = \sqrt{(c'-1)(n-1)}$ ∴ 양측 t검정과 동치
		예시: 다항 p	$H_0: p_1 = p_2$, $H_1: p_1 \ne p_2$ (유력후보1 vs 유력후보2 vs 나머지 군소후보) 3항 베르누이 $(X_{i1}, X_{i2}) \sim p_1^{x_1} p_2^{x_2} (1 - p_1 - p_2)^{1-x_1-x_2}$ $(X_{i1}, X_{i2}) \in \{(0,0),(0,1),(1,0)\}$ $\hat{p}_j = \frac{\sum_{i=1}^{n} X_{ij}}{n}$ for $j = 1,2$ (표본: $\{(X_{n1}, X_{n2})\}$) **LR**: $\frac{1}{\Lambda} = \left(\frac{2\hat{p}_1}{\hat{p}_1+\hat{p}_2}\right)^{n\hat{p}_1}\left(\frac{2\hat{p}_2}{\hat{p}_1+\hat{p}_2}\right)^{n\hat{p}_2}$, $-2\ln\Lambda > \chi_\alpha^2(1)$ 에서 기각 **Wald**: $\begin{bmatrix}\hat{p}_1\\ \hat{p}_2\end{bmatrix} \overset{a}{\sim} N_2\left(\begin{bmatrix}p_1\\ p_2\end{bmatrix}, \frac{1}{n}\begin{bmatrix}p_1(1-p_1) & -p_1p_2\\ -p_1p_2 & p_2(1-p_2)\end{bmatrix}\right)$ $W = \hat{p}_1 - \hat{p}_2 = g\left(\begin{bmatrix}\hat{p}_1\\ \hat{p}_2\end{bmatrix}\right)$, Δ방법에서 $\mathbf{B} = \left[\frac{\partial g_i}{\partial p_j}\right] = [1, -1]$ $Var(W) = \frac{1}{n}\mathbf{B}\mathbf{I}^{-1}\mathbf{B}^T = \frac{p_1 + p_2 - (p_1 - p_2)^2}{n}$ ∴ $W \overset{a}{\sim} N\left(p_1 - p_2, \frac{p_1 + p_2 - (p_1 - p_2)^2}{n}\right)$ ⇒ 근사 Z 검정 or 카이제곱
		예시: 2표본 이항 p	$H_0: p_1 = p_2$, $H_1: p_1 \ne p_2$, $\{X_{n1}\} \overset{\text{iid}}{\sim} B(1, p_1)$, $\{Y_{n2}\} \overset{\text{iid}}{\sim} B(1, p_2)$ **Wald**: $\hat{p}_1 \overset{a}{\sim} N\left(p_1, \frac{p_1(1-p_1)}{n_1}\right)$, $\hat{p}_2 \overset{a}{\sim} N\left(p_2, \frac{p_2(1-p_2)}{n_2}\right)$, $\mathrm{Cov}(\hat{p}_1, \hat{p}_2) = 0$ $\hat{p}_1 - \hat{p}_2 \overset{a}{\sim} N\left(p_1 - p_2, \frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}\right)$ & $Slutsky$ ($\hat{p}_1 \xrightarrow{P} p_1, \hat{p}_2 \xrightarrow{P} p_2$) ⇒ 근사 Z 검정 or 카이제곱

7. 충분성 (Sufficiency) – 통계량의 성질

통계량 Review	**통계량**	① 점추정: $\boldsymbol{\theta} \in \Omega$**에 대한 추정량** $\hat{\boldsymbol{\theta}}$ *통계량 (Statistic): $T = T(X_1,\cdots,X_n)$ (표본에 대한 함수) ② 95% CI: $0.95 = P_\theta[\theta \in (\hat\theta_L,\hat\theta_U)]$ *$\theta \in (\hat\theta_L,\hat\theta_U)$인 베르누이 사건 ~ $B(1,0.95)$
	성질	1) 일치추정량: $\boldsymbol{T_n \xrightarrow{P} \theta}$ 면 ⇔ T_n은 θ의 **일치 추정량**
		2) 불편추정량: $E(T)=\theta$ ⇔ T는 θ의 불편 추정량 (bias = 0) ① **MVUE**: 분산 최소인 불편추정량 (UE) ➔ 유일 ② **CRB**: $\mathbf{Var}(\boldsymbol{T}) \ge \mathbf{1}/\{\boldsymbol{nI(\theta)}\}$
		3) MLE: $\hat\theta = \text{Argmax}[L(\theta)] = \text{Argmax}[\prod_{i=1}^n f(x_i,\theta)]$ ① MLE는 근사적으로 효율적 ② $\hat{\boldsymbol\theta} \overset{a}{\sim} \boldsymbol{N}\left(\boldsymbol{\theta_0}, \frac{\mathbf{1}}{\boldsymbol{nI(\theta_0)}}\right)$ ⇒ Z or χ^2화 하면 Wald statistic
		4) $ARE(T_1,T_2) = \dfrac{Var(T_2)}{Var(T_1)}$
	Bias MSE	1) **bias** $(\hat{\boldsymbol\theta}) = \boldsymbol{E}(\hat{\boldsymbol\theta} - \boldsymbol\theta) = E(\hat\theta) - \theta$ $\quad * \text{bias}(g(\hat\theta)) = E\left(g(\hat\theta) - g(\theta)\right)$ 2) Mean square error (MSE): $\boldsymbol{mse}(\hat{\boldsymbol\theta}) = \boldsymbol{E}\left\{(\hat{\boldsymbol\theta} - \boldsymbol\theta)^2\right\} = Var(\hat\theta) + \{\text{bias}(\hat\theta)\}^2$ 3) Mean absolute error (MAE): $\boldsymbol{mse}(\hat{\boldsymbol\theta}) = \boldsymbol{E}\{\lvert\hat{\boldsymbol\theta} - \boldsymbol\theta\rvert\}$
	적률 추정법 (MoM)	r차 표본적률 $\xrightarrow{P}$ r차 모적률 (➔ 연립하여 모수 추정량 구함; 일반적으로 비선호)
		ex) $\{X_i\} \overset{iid}{\sim} \text{Gamma}(k,\theta)$ $m_1 = \frac{\sum_{i=1}^n X_i}{n} = \hat k\hat\theta,\ m_2 = \frac{\sum_{i=1}^n X_i^2}{n} = \hat k(\hat\theta)^2 + (\hat k\hat\theta)^2$ $\hat\theta = \frac{\sum_{i=1}^n (X_i - \bar X)^2}{n\bar X} = \left(\frac{n-1}{n}\right)\frac{S^2}{\bar X} = \frac{S_{mle}^2}{\bar X},\quad \hat k = \frac{n(\bar X)^2}{\sum_{i=1}^n (X_i - \bar X)^2} = \left(\frac{n}{n-1}\right)\frac{\bar X}{S^2} = \frac{\bar X^2}{S_{mle}^2}$
충분성	**정의**	$Y = u(X_1,\cdots,X_n)$에 대해 $\boldsymbol{X}\vert\boldsymbol{y}$**가 θ와 무관**함 ⇔ Y가 θ에 대한 모든 정보 다 포함 (e.g. $Y = \Sigma_{i=1}^n X_i$) $\frac{\prod_{i=1}^n f(x_i,\theta)}{f_Y(y;\theta)} = H(x_1,\cdots,x_n)$ (f_Y: Y의 pdf)
	Neyman-Fisher	[Y가 θ의 SS] ⇔ $\prod_{i=1}^n f(x_i,\theta) = f_Y(y;\theta)\, H(x_1,\cdots,x_n) = g(y;\theta)\, h(x_1,\cdots,x_n)$ (임의의 g,h로 **인수분해**)
	Rao-Blackwell	θ의 충분통계량 Y_1, 불편추정량 Y_2에 대해, 새로운 불편추정량 $\boldsymbol{\varphi(y_1)} = \boldsymbol{E(Y_2\vert y_1)}$ $E(\varphi(Y_1)) = E[E(Y_2\vert Y_1)] = E(Y_2) = \theta$ 2) $\text{Var}(\varphi(Y_1)) = \text{Var}(E(Y_2\vert Y_1)) \le \text{Var}(Y_2)$ ∴ **New** $\boldsymbol{UE}$ $\varphi(y_1) = E(Y_2\vert y_1)$는 **Old** $\boldsymbol{UE}$ Y_2 보다 분산이 작다. *실전: $E(\varphi(Y_1)) = \theta$인 $\varphi(y_1)$찾기
	Lehmann-Scheffe	① 통계량 Y는 **complete (완비)** if 모든 θ에서 $E(h(Y)) = 0$ ⇒ $h(t) = 0$ 만 가능함 ② **레만-셰페: CSS**인 $\boldsymbol{Y_1}$으로 Rao-Blackwellization ➔ $\boldsymbol{\varphi(y_1)} = \boldsymbol{E(Y_2\vert y_1)}$는 유일한 $\boldsymbol{MVUE}$ **of** $\boldsymbol\theta$ pf) CSS인 Y_1에 대해 불편추정량 $\varphi(Y_1),\psi(Y_1)$존재 ⇒ $E(\varphi(Y_1) - \psi(Y_1)) = \theta - \theta = 0$ 완비족 $\{f_{Y_1}(y;\theta): \theta \in \Omega\}$에 대해 위 등식은 $\varphi(Y_1) = \psi(Y_1)$에서만 성립 (더 이상 분산 못 줄임)
	지수족 Exponential Family	$f(x;\theta) = \exp[\eta(\theta)T(x) + H(x) - A(\eta(\theta))]$ $(x \in S)$ ($\eta = \eta(\theta)$는 자연 모수) *정칙: 1) $\boldsymbol{S}$**가** $\boldsymbol\theta$**에 종속 X**, 2) $\boldsymbol{\eta(\theta)}$**연속**, 3) (연속이면) $\boldsymbol{H(x)}$**연속** in $\{K'(x) \ne 0\}$ ① 지수족: 이산 (포아송, 이항, 기하, 음이항, 다항 등) / 연속 (감마, 베타, 정규 등) ② $Y = \sum_{i=1}^n T(x)$ 는 θ의 **CSS** ③ $E(T(X)) = A'(\eta)$, $\text{Var}(T(X)) = A''(\eta)$
	결합 충분통계량 (다중모수)	$\mathbf{Y} = (Y_1,\cdots,Y_m)^T \in \mathbb{R}^m$ & $\boldsymbol\theta = (\theta_1,\cdots,\theta_p) \in \mathbb{R}^p$에 대해 (일반적으로 m = p) $\prod_{i=1}^n f(x_i,\boldsymbol\theta) = f_{\mathbf Y}(\mathbf y;\boldsymbol\theta)H(x_1,\cdots,x_n) = g(\mathbf y;\boldsymbol\theta)\, h(x_1,\cdots,x_n)$ *순서통계량 $\mathbf{Y} = (Y_1,\cdots,Y_n)^T$; $Y_1 < \cdots < Y_n$ → 모든 연속분포의 결합충분통계량
	보조통계량 (Ancillary)	$A = a(X_1,\cdots,X_n)$가 θ와 **무관** ex) 정규분포 iid의 $\boldsymbol{S^2}$: $\boldsymbol\mu$에 대해 ancillary 1) **Basu** 정리:{$\boldsymbol Y$가 θ 의 **CSS**} & {$\boldsymbol Z$가 θ 의 **ancillary**} ⇔ {$\boldsymbol Y$와 $\boldsymbol Z$**는 독립**} ex) $\bar{\boldsymbol X} \perp \boldsymbol{S^2}, \{X_i\} \overset{iid}{\sim} N(\mu,\sigma^2)$ 2) ①**위치불변**: $Z = u(W_1+\theta,\cdots,W_n+\theta) = u(W_1,\cdots,W_n)$ ex) S^2, $\max\{X_i\} - \min\{X_i\}$, $\frac{1}{n}\sum_1^n \lvert X_i - Q_2\rvert$ ②**척도불변**: $Z = u(\theta W_1,\cdots,\theta W_n) = u(W_1,\cdots,W_n)$ ex) $X_1/(X_1+X_2)$, $X_1^2/\sum_1^n X_i^2$, $\min\{X_i\}/\max\{X_i\}$ ③**위치척도불변**: $Z = u(\theta_1 W_1 + \theta_2,\cdots,\theta_1 W_n + \theta_2) = u(W_1,\cdots,W_n)$ ex) $(X_i - \bar X)/S^2$
	MLE	$L(\theta;x_1,\cdots,x_n) = \prod_{i=1}^n f(x_i,\theta) = f_Y(y;\theta)\, H(x_1,\cdots,x_n)$ ➔ L과 f_y 동시에 극대화 by θ ① $MLE\ \hat\theta$이 유일 ⇔ $\hat\theta$는 충분통계량 Y의 함수 ∵ $\hat\theta = \text{argmax}(L(\theta,\mathbf x)) = \text{argmax}(f_Y(y;\theta))$ ② $MLE\ \hat\theta$가 충분통계량 ⇔ $\hat\theta$는 최소 충분통계량 (MSS) *최소충분:reduced from 다른 충분통계량

7. 충분성 (Sufficiency) – 통계량의 성질

지수족 확장

지수족

$f(x;\theta)=\exp[\eta(\theta)T(x)+H(x)-A(\eta(\theta))]$ $(x\in S)$ $(\eta=\eta(\theta)$는 자연 모수)

*정칙: 1) $\boldsymbol{S}$**가** $\boldsymbol{\theta}$**에 종속X**, 2) $\boldsymbol{\eta(\theta)}$**연속**, 3) (연속이면) $\boldsymbol{H(x)}$**연속** in $\{K'(x)\neq 0\}$

① 지수족: 이산 (포아송, 이항, 기하, 음이항, 다항 등) / 연속 (감마, 베타, 정규 등)

② $Y=\sum_{i=1}^{n}T(x)$ 는 θ의 **CSS** ③ $E\big(T(X)\big)=A'(\eta)$, $\mathrm{Var}\big(T(X)\big)=A''(\eta)$

다변량 확장

1변수 1모수	$f(x;\theta)=\exp[\eta(\theta)T(x)+H(x)-A(\eta)]$
1변수 다중모수	$f(x;\boldsymbol{\theta})=\exp[\boldsymbol{\eta}(\boldsymbol{\theta})\cdot\mathbf{T}(x)+H(x)-A(\boldsymbol{\eta})]$
다변량 다중모수	$f(\mathbf{x};\boldsymbol{\theta})=\exp[\boldsymbol{\eta}(\boldsymbol{\theta})\cdot\mathbf{T}(\mathbf{x})+H(\mathbf{x})-A(\boldsymbol{\eta})]$
기대값	$\nabla A(\boldsymbol{\eta})=E[\mathbf{T}(\mathbf{x})]$, $\mathbf{H}[A(\boldsymbol{\eta})]=\mathrm{Cov}(\mathbf{T}(\mathbf{x}))$ $\nabla A(\boldsymbol{\eta}_{mle})=\frac{1}{n}\sum_{i=1}^{n}\mathbf{T}(\mathbf{x}_i)$

예시

분포	모수 $\boldsymbol{\theta}$	자연모수 $\boldsymbol{\eta}$	역모수	$\mathbf{T}(\mathbf{x})$	$A(\boldsymbol{\eta})$
베르누이	p	$\ln\frac{p}{1-p}$	$\frac{1}{1+e^{-\eta}}$ * logistic function	x	$\ln(1+e^{\mu})$
이항					$n\ln(1+e^{\mu})$
푸아송	m	$\ln m$	e^{η}	x	e^{η}
음이항(r)	p	$\ln(1-p)$	$1-e^{\eta}$	x	$-r\ln(1-e^{\mu})$
다항(n)	$\begin{bmatrix}p_1\\ \vdots\\ p_{k-1}\end{bmatrix}$	$\begin{bmatrix}\ln\frac{p_1}{p_k}\\ \vdots\\ \ln\frac{p_{k-1}}{p_k}\end{bmatrix}$	$\begin{bmatrix}\frac{\exp(\eta_1)}{1+\sum_{j=1}^{k-1}\exp(\eta_j)}\\ \vdots\\ \frac{\exp(\eta_{k-1})}{1+\sum_{j=1}^{k-1}\exp(\eta_j)}\end{bmatrix}$ * softmax function	$\begin{bmatrix}x_1\\ \vdots\\ x_{k-1}\end{bmatrix}$	$n\ln(1+\sum_{j=1}^{k-1}\exp(\eta_j))$
	$p_k=1-\sum_{j=1}^{k-1}p_j$, $\eta_k=0$, $\exp(\eta_k)=1$				
감마	$\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$	$\begin{bmatrix}\alpha-1\\ -\frac{1}{\beta}\end{bmatrix}$	$\begin{bmatrix}\eta_1+1\\ -\frac{1}{\eta_2}\end{bmatrix}$	$\begin{bmatrix}\ln x\\ x\end{bmatrix}$	$\ln\Gamma(\eta_1+1)-(\eta_1+1)\ln(-\eta_2)$
지수	β	$-\frac{1}{\beta}$	$-\frac{1}{\eta}$	x	$-\ln(-\eta)$
카이제곱	ν	$\frac{\nu}{2}-1$	$2(\eta+1)$	$\ln x$	$\ln\Gamma(\eta+1)+(\eta+1)\ln 2$
베타	$\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$	$\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$	$\begin{bmatrix}\eta_1\\ \eta_2\end{bmatrix}$	$\begin{bmatrix}\ln x\\ \ln(1-x)\end{bmatrix}$	$\ln\mathrm{B}(\alpha,\beta)=\ln\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$
정규 기지 $\boldsymbol{\sigma^2}$	μ	$\frac{\mu}{\sigma^2}$	$\sigma^2\eta$	x	$\frac{1}{2}\sigma^2\eta^2$
정규 미지 $\boldsymbol{\sigma^2}$	$\begin{bmatrix}\mu\\ \sigma^2\end{bmatrix}$	$\begin{bmatrix}\frac{\mu}{\sigma^2}\\ -\frac{1}{2\sigma^2}\end{bmatrix}$	$\begin{bmatrix}-\frac{\eta_1}{2\eta_2}\\ -\frac{1}{2\eta_2}\end{bmatrix}$	$\begin{bmatrix}x\\ x^2\end{bmatrix}$	$-\frac{\eta_1^2}{4\eta_2}-\frac{1}{2}\ln(-2\eta_2)$
다변량 정규	$\begin{bmatrix}\boldsymbol{\mu}\\ \boldsymbol{\Sigma}\end{bmatrix}$	$\begin{bmatrix}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\\ -\frac{1}{2}\boldsymbol{\Sigma}^{-1}\end{bmatrix}$	$\begin{bmatrix}-\frac{1}{2}\boldsymbol{\eta}_2^{-1}\boldsymbol{\eta}_1\\ -\frac{1}{2}\boldsymbol{\eta}_2^{-1}\end{bmatrix}$	$\begin{bmatrix}\mathbf{x}\\ \mathbf{x}\mathbf{x}^T\end{bmatrix}$	$-\frac{1}{4}\boldsymbol{\eta}_1^T\boldsymbol{\eta}_2^{-1}\boldsymbol{\eta}_1-\frac{1}{2}\ln\lvert-2\boldsymbol{\eta}_2\rvert$

*. 가설의 최적검정

8. 정규모형 추론: ANOVA

1-way ANOVA

확률표본 $X_{ij} \overset{iid}{\sim} N(\mu, \sigma^2)$, $n = ab$ ➔ **정규성 가정 (H₀) (등분산성)**

$$\bar{X}_{\cdot\cdot} = \frac{\sum_{i=1}^{a}\sum_{j=1}^{b} X_{ij}}{ab} \sim N\left(\mu, \frac{\sigma^2}{\boldsymbol{ab}}\right)$$

$$\bar{X}_{i\cdot} = \frac{\sum_{j=1}^{b} X_{ij}}{b} \sim N\left(\mu, \frac{\sigma^2}{\boldsymbol{b}}\right) \quad * Var(\bar{X}_{i\cdot}) = \frac{1}{b^2} Var\left(\sum_{j=1}^{b} X_{ij}\right) = \frac{\sigma^2}{b}$$

$$\bar{X}_{\cdot j} = \frac{\sum_{i=1}^{a} X_{ij}}{a} \sim N\left(\mu, \frac{\sigma^2}{\boldsymbol{a}}\right)$$

	***b*개** (j) 처리 개수						
***a*개** (i) 처리 별 표본 크기	X_{11}	X_{12}	$\cdots$	X_{1j}	$\cdots$	X_{1b}	$\bar{X}_{1\cdot}$
	X_{21}	X_{22}	$\cdots$	X_{2j}	$\cdots$	X_{2b}	$\bar{X}_{2\cdot}$
	$\vdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\vdots$	$\vdots$
	X_{i1}	X_{i2}	$\cdots$	X_{ij}	$\cdots$	X_{ib}	$\boldsymbol{\bar{X}_{i\cdot}}$
	$\vdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\vdots$	$\vdots$
	X_{a1}	X_{a2}	$\cdots$	X_{aj}	$\cdots$	X_{ab}	$\bar{X}_{a\cdot}$
	$\bar{X}_{\cdot 1}$	$\bar{X}_{\cdot 2}$	$\cdots$	$\boldsymbol{\bar{X}_{\cdot j}}$	$\cdots$	$\bar{X}_{\cdot b}$	$\boldsymbol{\bar{X}_{\cdot\cdot}}$

$$(\boldsymbol{ab-1})\boldsymbol{S^2} = \sum_{i=1}^{a}\sum_{j=1}^{b}\left(X_{ij}-\bar{X}_{\cdot\cdot}\right)^2 = \sum_{i=1}^{a}\sum_{j=1}^{b}\left\{\left(X_{ij}-\bar{X}_{i\cdot}\right)+\left(\bar{X}_{i\cdot}-\bar{X}_{\cdot\cdot}\right)\right\}^2 = \sum_{i=1}^{a}\sum_{j=1}^{b}\left(X_{ij}-\bar{X}_{i\cdot}\right)^2 + \sum_{i=1}^{a}\sum_{j=1}^{b}\left(\bar{X}_{i\cdot}-\bar{X}_{\cdot\cdot}\right)^2$$

$$\because\ 2\sum_{i=1}^{a}\sum_{j=1}^{b}\left(X_{ij}-\bar{X}_{i\cdot}\right)\left(\bar{X}_{i\cdot}-\bar{X}_{\cdot\cdot}\right) = 2\sum_{i=1}^{a}\left(\bar{X}_{i\cdot}-\bar{X}_{\cdot\cdot}\right)\left\{\sum_{j=1}^{b}\left(X_{ij}-\bar{X}_{i}\right)\right\} = 0$$

$$\boldsymbol{Q = Q_1 + Q_2} \quad \Leftrightarrow \quad (\boldsymbol{ab-1})\boldsymbol{S^2} = \sum_{i=1}^{a}\sum_{j=1}^{b}\left(X_{ij}-\bar{X}_{i\cdot}\right)^2 + b\sum_{i=1}^{a}\left(\bar{X}_{i\cdot}-\bar{X}_{\cdot\cdot}\right)^2$$

$$\boldsymbol{Q = Q_3 + Q_4} \quad \Leftrightarrow \quad (\boldsymbol{ab-1})\boldsymbol{S^2} = \sum_{j=1}^{b}\sum_{i=1}^{a}\left(X_{ij}-\bar{X}_{\cdot j}\right)^2 + a\sum_{j=1}^{b}\left(\bar{X}_{\cdot j}-\bar{X}_{\cdot\cdot}\right)^2$$

	$\boldsymbol{Q}$	$\boldsymbol{Q_1}$	$\boldsymbol{Q_2}$	$\boldsymbol{Q_3}$	$\boldsymbol{Q_4}$	$\boldsymbol{Q_5}$
정의	$(\boldsymbol{ab-1})\boldsymbol{S^2}$	$\sum_{i=1}^{a}\sum_{j=1}^{b}\left(X_{ij}-\bar{X}_{i\cdot}\right)^2$	$b\sum_{i=1}^{a}\left(\bar{X}_{i\cdot}-\bar{X}_{\cdot\cdot}\right)^2$	$\sum_{j=1}^{b}\sum_{i=1}^{a}\left(X_{ij}-\bar{X}_{\cdot j}\right)^2$	$a\sum_{j=1}^{b}\left(\bar{X}_{\cdot j}-\bar{X}_{\cdot\cdot}\right)^2$	$\sum_{j=1}^{b}\sum_{i=1}^{a}\left(X_{ij}-\bar{X}_{i\cdot}-\bar{X}_{\cdot j}+\bar{X}_{\cdot\cdot}\right)^2$
$\boldsymbol{\chi^2}$ **자유도** $\boldsymbol{Q_k/\sigma^2}$	$\boldsymbol{ab-1}$	$\boldsymbol{a(b-1)}$	$\boldsymbol{a-1}$	$\boldsymbol{b(a-1)}$	$\boldsymbol{b-1}$	$\boldsymbol{(a-1)(b-1)}$
제곱합	**총 제곱합**	**행-내부**	**행 평균간**	**열-내부**	**열 평균간**	**나머지 (Q2, Q4 제외)**

$$\frac{\boldsymbol{Q}}{\boldsymbol{\sigma^2}} = (ab-1)\frac{S^2}{\sigma^2} \sim \chi^2(\boldsymbol{ab-1})$$

$$\frac{\boldsymbol{Q_1}}{\boldsymbol{\sigma^2}} = \sum_{i=1}^{a}(b-1)\left[\frac{\sum_{j=1}^{b}\left(X_{ij}-\bar{X}_{i\cdot}\right)^2/(b-1)}{\sigma^2}\right] \sim \chi^2\{\boldsymbol{a(b-1)}\} \quad \& \quad \frac{\boldsymbol{Q_2}}{\boldsymbol{\sigma^2}} = (a-1)\frac{\sum_{i=1}^{a}(\bar{X}_{i\cdot}-\bar{X}_{\cdot\cdot})^2/(a-1)}{\sigma^2/b} \sim \chi^2(\boldsymbol{a-1})$$

*핵심 질문: 요인 b개의 평균이 모두 동일 한가? ($\boldsymbol{H_0: \mu_1 = \cdots = \mu_b}$)

$\left[X_{1j}, \cdots, X_{ij}, \cdots, X_{aj}\right]^T$ $\quad * \boldsymbol{X_{ij} \sim N(\mu_j, \sigma^2)}$ (샘플 크기 a) ➔ $\boldsymbol{X_{ij} = \mu_j + e_{ij}}$ $\quad * \boldsymbol{e_{ij} \sim N(0, \sigma^2)}$

$$H_0: L(\omega) = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{ab}{2}} \exp\left[-\frac{1}{2\sigma^2}\sum_{j=1}^{b}\sum_{i=1}^{a}\left(x_{ij}-\boldsymbol{\mu}\right)^2\right] \quad \omega = \{(\mu_1, \mu_2, \cdots, \mu_b, \sigma^2): -\infty < \mu_1 = \mu_2 = \cdots = \mu_b = \mu < \infty, 0 < \sigma^2 < \infty\}$$

$$H_1: L(\Omega) = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{ab}{2}} \exp\left[-\frac{1}{2\sigma^2}\sum_{j=1}^{b}\sum_{i=1}^{a}\left(x_{ij}-\boldsymbol{\mu_j}\right)^2\right] \quad \Omega = \{(\mu_1, \mu_2, \cdots, \mu_b, \sigma^2): -\infty < \mu_j < \infty, \quad 0 < \sigma^2 < \infty\}$$

➔ 각각 $(\mu, \sigma^2), (\mu_1, \cdots, \mu_j, \cdots, \mu_b, \sigma^2)$에 대해 편미분 후 MLE 구하면

$$\Lambda = \frac{L(\widehat{\omega})}{L(\widehat{\Omega})} = \left[\frac{1/\sum_{j=1}^{b}\sum_{i=1}^{a}\left(x_{ij}-\boldsymbol{\bar{x}}_{\cdot\cdot}\right)^2}{1/\sum_{j=1}^{b}\sum_{i=1}^{a}\left(x_{ij}-\boldsymbol{\bar{x}}_{\cdot j}\right)^2}\right]^{\frac{ab}{2}} = \left[\frac{\sum_{j=1}^{b}\sum_{i=1}^{a}\left(x_{ij}-\boldsymbol{\bar{x}}_{\cdot j}\right)^2}{\sum_{j=1}^{b}\sum_{i=1}^{a}\left(x_{ij}-\boldsymbol{\bar{x}}_{\cdot\cdot}\right)^2}\right]^{\frac{ab}{2}} = \left[\frac{Q_3}{Q}\right]^{\frac{ab}{2}} \qquad \therefore \Lambda^{2/ab} = \frac{Q_3}{Q_3+Q_4} = \frac{1}{1+Q_4/Q_3}$$

$$\alpha = P_{H_0}\left[\frac{1}{1+Q_4/Q_3} \le \lambda_0^{2/ab}\right] = P_{H_0}\left[\boldsymbol{F} = \frac{Q_4/(b-1)}{Q_3/[b(a-1)]} \ge \boldsymbol{c} = \frac{b(a-1)}{b-1}\left(\lambda_0^{-2/ab}-1\right)\right]$$

<**모델의 재구성**> 처리에 의한 변동 (SSA, Q_4) / 내부 오차에 의한 변동을 χ^2로 표현 (SSE=Q_4) → F-통계량 구현 시 $\boldsymbol{F} = \frac{Q_4/(b-1)}{Q_3/[b(a-1)]} \ge c$ 이면 H_0 기각 (≈ 처리에 의한 변동이 임계치를 넘음)

변동요인	제곱합	수식	$\boldsymbol{\chi^2}$ 자유도 $(\boldsymbol{Q_k/\sigma^2})$
처리	**SSA**	$Q_4 = a\sum_{j=1}^{b}\left(\bar{X}_{\cdot j}-\bar{X}_{\cdot\cdot}\right)^2$	$\boldsymbol{b-1}$
오차	**SSE**	$Q_3 = \sum_{j=1}^{b}\sum_{i=1}^{a}\left(X_{ij}-\bar{X}_{\cdot j}\right)^2$	$\boldsymbol{b(a-1)}$
합	**SST**	$Q = \sum_{i=1}^{a}\sum_{j=1}^{b}\left(X_{ij}-\bar{X}_{\cdot\cdot}\right)^2$	$\boldsymbol{ab-1}$

8. 정규모형 추론: ANOVA / 회귀분석

2-way ANOVA

1-way ANOVA: b개 처리 별 → 동일 크기 (a) 샘플 간 평균 동일성

2-way ANOVA: 요인 A (n=1, ..., a) + 요인 B (n=1, ..., b)

① $X_{ij} \overset{iid}{\sim} N(\mu_{ij}, \sigma^2), n = ab$

② $\mu_{ij} = \mu + (\bar{\mu}_{i.} - \mu) + (\bar{\mu}_{.j} - \mu) = \mu + \boldsymbol{\alpha_i} + \boldsymbol{\beta_j}$ $\quad (\sum_{i=1}^{a} \alpha_i = \sum_{j=1}^{b} \beta_j = 0)$

*Additive model $\quad (\because \sum_{j=1}^{b} \sum_{i=1}^{a} \mu_{ij} = \sum_{i=1}^{a} b\bar{\mu}_{i.} + \sum_{j=1}^{b} a\bar{\mu}_{.j} - ab\mu = ab\mu)$

	b개 (j) **처리** 개수						
a개 (i) **처리** 개수	X_{11}	X_{12}	$\cdots$	X_{1j}	$\cdots$	X_{1b}	$\bar{X}_{1\cdot}$
	X_{21}	X_{22}	$\cdots$	X_{2j}	$\cdots$	X_{2b}	$\bar{X}_{2\cdot}$
	$\vdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\vdots$	$\vdots$
	X_{i1}	X_{i2}	$\cdots$	X_{ij}	$\cdots$	X_{ib}	$\boldsymbol{\bar{X}_{i\cdot}}$
	$\vdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\vdots$	$\vdots$
	X_{a1}	X_{a2}	$\cdots$	X_{aj}	$\cdots$	X_{ab}	$\bar{X}_{a\cdot}$
	$\bar{X}_{\cdot 1}$	$\bar{X}_{\cdot 2}$	$\cdots$	$\boldsymbol{\bar{X}_{\cdot j}}$	$\cdots$	$\bar{X}_{\cdot b}$	$\boldsymbol{\bar{X}_{\cdot\cdot}}$

$$\boldsymbol{Q = Q_2 + Q_4 + Q_5} \Leftrightarrow \boldsymbol{(ab-1)S^2} = b\sum_{i=1}^{a}(\bar{X}_{i\cdot} - \bar{X}_{\cdot\cdot})^2 + a\sum_{j=1}^{b}(\bar{X}_{\cdot j} - \bar{X}_{\cdot\cdot})^2 + \sum_{j=1}^{b}\sum_{i=1}^{a}(X_{ij} - \bar{X}_{i\cdot} - \bar{X}_{\cdot j} + \bar{X}_{\cdot\cdot})^2$$

*핵심 질문: 요인 a,b의 effect ① $H_{0A}: \alpha_1 = \cdots = \alpha_a = 0$

② $H_{0B}: \beta_1 = \cdots = \beta_b = 0$

*열간 차이 모델 (H_{0B}, H_{1B})

$$H_{0B}: \hat{\sigma}_\omega^2 = \frac{Q_4 + Q_5}{ab} = \sum_{i=1}^{a}\sum_{j=1}^{b}\frac{(X_{ij} - \bar{X}_{i.})^2}{ab}$$

$$H_{1B}: \hat{\sigma}_\Omega^2 = \frac{Q_5}{ab}$$

$$\Lambda = (\hat{\sigma}_\Omega^2/\hat{\sigma}_\omega^2)^{\frac{ab}{2}} \Leftrightarrow F = \frac{Q_4/(b-1)}{Q_5/[(a-1)(b-1)]} \geq \boldsymbol{c}$$

* 행간 차이 모델 $(H_{0A}, H_{1A}) \Leftrightarrow F = \dfrac{Q_2/(b-1)}{Q_5/[(a-1)(b-1)]} \geq \boldsymbol{c}$

2-way ANOVA 일반화: 칸당 c>1개 관측값. (위 모델은 칸 당 1개) ➔ 총 n=abc (3차원 구조)

① $X_{ijk} \overset{iid}{\sim} N(\boldsymbol{\mu_{ij}}, \sigma^2), n = abc$

② $\mu_{ij} = \mu + (\bar{\mu}_{i.} - \mu) + (\bar{\mu}_{.j} - \mu) + (\mu_{ij} - \bar{\mu}_{i.} - \bar{\mu}_{.j} + \mu) = \mu + \boldsymbol{\alpha_i} + \boldsymbol{\beta_j} + \boldsymbol{\gamma_{ij}}$ $\quad (\sum_{i=1}^{a} \alpha_i = \sum_{j=1}^{b} \beta_j = 0, \sum_{i=1}^{a} \gamma_{ij} = \sum_{j=1}^{b} \gamma_{ij} = 0)$

$(\gamma_{ij}$: 교호작용 모수, interaction parameter$)$

*핵심 가설: 교호작용 모수=0 $H_{0AB}: \boldsymbol{\gamma_{ij} = 0}$, for all $(i,j) \quad \Leftrightarrow \quad F$통계량 $\geq c$

회귀분석

① 미래의 관측값 $Y = y$를 예측할 수 없지만, $E(Y)$는 예측 가능하다.

② 선형 모형: $E(Y) = \alpha + \beta x + \gamma x^3 + \delta \log x$ (선형 모델: 모수에 대한 선형성)

*추후 정리 with 몽고메리 서적

9. 비모수-로버스트 통계학: 추후 정리

10. 베이지안 통계학 기본

<table>
<tr><td rowspan="9">베이지안 절차</td><td rowspan="2">사전/사후 분포</td><td>①기본: $X_i|\theta \overset{iid}{\sim} f(x|\theta)$ (X 는 θ에 의존적인 확률 분포에서 추출)
②사전분포: $\Theta \sim h(\theta)$ *모수의 prior
③우도: $L(\mathbf{x}|\theta) = f(x_1|\theta)\cdots f(x_n|\theta)$ *표본 $\mathbf{X}^{\mathbf{T}} = (X_1,\cdots,X_n)$
④결합 PDF: $g(\mathbf{x},\theta) = L(\mathbf{x}|\theta)h(\theta)$

∴ 사후분포: $k(\theta|\mathbf{x}) = \frac{g(\mathbf{x},\theta)}{g_1(\mathbf{x})} = \frac{L(\mathbf{x}|\theta)\,h(\theta)}{g_1(\mathbf{x})} = \frac{L(\mathbf{x}|\theta)\,h(\theta)}{\int_{-\infty}^{\infty} L(\mathbf{x}|\theta)\,h(\theta)\,d\theta}$ $\Rightarrow k(\theta|\mathbf{x}) \propto L(\mathbf{x}|\theta)h(\theta)$

*공액분포족: 사전⇔사후 분포가 같은 족에 속함 (e.g. X|θ가 푸아송 → 공액족은 감마분포
X|θ 가 이항분포 → 공액족은 베타분포)</td></tr>
<tr><td><다변량 조건부/주변 분포 이용한 유도>
1) $h(\theta) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} g(x_1,\cdots,x_n,\theta)\,dx_1\cdots dx_n$
2) $L(\mathbf{x}|\theta) = L_{\mathbf{x}|\theta}(x_1,\cdots,x_n\,|\theta) = g(\mathbf{x},\theta)/h(\theta)$
3) $g_1(\mathbf{x}) = g_1(x_1,x_2,\cdots,x_n) = \int_{-\infty}^{\infty} g(\mathbf{x},\theta)\,d\theta$ (초등 적분으로 해결 안되는 경우 ➔ MCMC 등의 근거)</td></tr>
<tr><td rowspan="2">베이지안 추정</td><td>1) 점추정: Maximum A Posteriori (MAP): 사후확률을 최대화하는 값으로 모수 추정 ($\hat{\theta}_{MAP}$)
① MLE 는 모수 미지의 정해진 분포에서 iid 추출한 값들 -> 샘플 기준으로 모수 추정
∴ $\hat{\theta}_{MLE} = \underset{\theta}{\operatorname{argmax}}\, L(\mathbf{x}|\theta)$
② MAP 는 모수~사전분포 + iid 추출 샘플들 -> 두 가지를 혼합한 사후 분포에서 모수 추정
∴ $\hat{\theta}_{MAP} = \underset{\theta}{\operatorname{argmax}}\, L(\mathbf{x}|\theta)h(\theta)$
③ 균등 Prior [=Beta(1,1)] → $h(\theta)$가 상수이므로 $\underset{\theta}{\operatorname{argmax}}\, L(\mathbf{x}|\theta)h(\theta) = \underset{\theta}{\operatorname{argmax}}\, L(\mathbf{x}|\theta)$ ⇔ $\boldsymbol{\hat{\theta}_{MLE} = \hat{\theta}_{MAP}}$
(기타 점추정 방식: 1) MAP, 2) 사후 평균, 3) 사후 중간값 등)</td></tr>
<tr><td>2) 구간 추정: $P[u(\mathbf{x}) < \Theta < v(\mathbf{x}) \mid \mathbf{X} = \mathbf{x}] = \int_{u(\mathbf{x})}^{v(\mathbf{x})} k(\theta|\mathbf{x})\,d\theta = 0.95$
⇔ Credible interval (신용구간) or 사후확률구간 0.95
① HPDI: 사후분포에서 가장 짧은 길이 / ② Equal-tail CI (양측 꼬리 넓이 동일)</td></tr>
<tr><td>베이지안 검정</td><td>$\boldsymbol{P}(\Theta \in \omega_0|\mathbf{x})$ vs. $\boldsymbol{P}(\Theta \in \omega_1|\mathbf{x})$ ➔ 더 큰 쪽으로 가설 채택 (사후 분포 상 가설의 모수 영역 더 넓은 쪽)</td></tr>
<tr><td>계층적 베이즈</td><td>① $X|\theta \sim f(x|\theta)$ ② $\Theta|\gamma \sim h(\theta|\gamma)$ ③ $\Gamma \sim \psi(\gamma)$ (γ: 초모수)
$k(\theta|\mathbf{x}) = \frac{\int_{-\infty}^{\infty} L(\mathbf{x}|\theta)h(\theta|\gamma)\psi(\gamma)d\gamma}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} L(\mathbf{x}|\theta)h(\theta|\gamma)\psi(\gamma)d\gamma d\theta}$</td></tr>
<tr><td>경험적 베이즈</td><td>① $X|\theta \sim f(x|\theta)$ ② $\Theta|\gamma \sim h(\theta|\gamma)$
$m(\mathbf{x}|\gamma) = \int_{-\infty}^{\infty} g(\mathbf{x},\theta|\gamma)\,d\theta = \int_{-\infty}^{\infty} L(\mathbf{x}|\theta)\,h(\theta|\gamma)d\theta$ ➔ 새로운 우도 최대화 하는 $\hat{\gamma}$ 구함
➔ 위 조건부 사전분포에 대입하여 진행</td></tr>
<tr><td rowspan="2">몬테 카를로 기법</td><td>깁스 샘플러</td><td></td></tr>
<tr><td></td><td></td></tr>
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