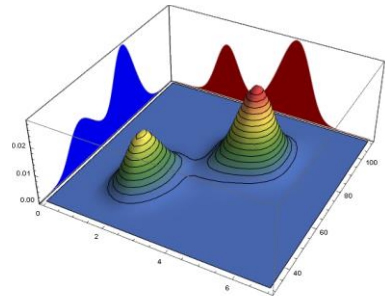
1. 확률, 확률분포

조건부 확률 **독립성**	$C_1 \subset S$ 에서 C_1를 새로운 표본 공간으로 설정 ➔ $C_2 \subset S$에 대해 1. $P(C_2 \mid C_1) = P(C_1 \cap C_2 \mid C_1) = {P(C_1 \cap C_2)}/{P(C_1)}$ ⇔ $P(C_1 \cap C_2) = P(C_2 \mid C_1)\, P(C_1)$ 2. Bayes (C_i는 상호 배반=disjoint, S의 partition) 1) Law of total prob: $P(A) = \sum P(A \cap C_i) = \sum P(A \mid C_i)\, P(C_i)$ 2) Bayes' theorem: $P(C_i \mid A) = {P(A \cap C_i)}/{P(A)} = {P(A \cap C_i)}/{\sum P(A \cap C_i)} = {P(A \mid C_i)\, P(C_i)}/{\sum P(A \mid C_i)\, P(C_i)}$ ① $P(C_i)$: C_i 사전확률 (prior) ② $P(C_i \mid A)$: C_i 사후확률 (posterior) ← 표본 A에서 관찰된 C_i 3) 독립성: $P(A \cap B \cap C) = P(A)P(B)P(C)$ 이면 A,B,C는 statistically independent
확률 변수	1. Prob mass function; PMF (discrete) ➔ CDF of PMF: $F(x) = P\big((-\infty, x]\big) = \sum_{(-\infty,x]} p(x)$ ***변환**: $p_y(y) = p_X(w(y))$; 1-on-1 function $x = w(y)$
	2. Prob density function; PDF (continuous) >0 1) $F(x) = \int_{-\infty}^{x} f(t)dt$ ⇔ 2) $\frac{\mathrm{d}}{\mathrm{dx}}F(x) = f(x)$ (F는 f의 CDF) 3) $P[(a,b)] = \int_{\mathrm{a}}^{\mathrm{b}} f(x)dx = F(b) - F(a)$ ***변환**: X가 pdf f_X on S_X **&** Y가 pdf f_Y on S_Y ; 1-on-1 $\boldsymbol{w(y) = x}$ ➔ $f_Y(y) = f_x\big(w(y)\big)\lvert dx/dy \rvert$ ⇔ $f_Y(y) = f_x\big(w(y)\big)\, \mathbf{abs}(\boldsymbol{J})$ (Jacobian: J=|w'(y)| / J=$\begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix}$) *J는 행렬식
	* Support(받침): PDF $\neq 0$인 space // ***CDF는 유일** for PDF, PMF
기대값	1. 조건: $\boldsymbol{E(\lvert X \rvert)}$ **존재** ⇔ ① 연속 pdf 존재 ② $\int_{-\infty}^{\infty} \lvert \boldsymbol{x} \rvert \boldsymbol{f(x)dx} < \infty$ (이산 pmf 존재 ➔ $\sum \lvert x_i \rvert\, p_i(x) < \infty$) 2. 기대값: 1) $E(X) = \int_{-\infty}^{\infty} x\, f(x)dx$ 2) $E(X) = \sum x_i\, p(x_i)$ 3. $y = g(x)$: 1) $E\big(g(x)\big) = \int_{-\infty}^{\infty} g(x)f(x)dx$ & $E\big(g(x)\big) = \sum g(x_i)p(x_i)$ 2) $E\big(k_1 g_1(x) + k_2 g_2(x)\big) = k_1 E\big(g_1(x)\big) + k_2 E\big(g_2(x)\big)$
	1. 평균: $\mu = E(X)$ 2. 분산: $\sigma^2 = Var(X) = E[(X-\mu)^2] = E(X^2) - \mu^2 = M''(0) - M'(0)^2$ *$\mathrm{Var}(aX+b) = a^2\mathrm{Var}(X)$ 3. 적률생성함수 (MGF) *조건: $\mathrm{t} \in (-\mathrm{h}, \mathrm{h})$ for $\forall h > 0$ ← 0을 포함하는 개구간에서 mgf 존재 1) $\boldsymbol{M(t) = E(e^{tX})}$ ➔ $M_X(0)^{(r)} = E(X^r)$ *분포의 r차 moment ① $M(0)^{(r)} = \frac{d^r}{dt^r}\int_{-\infty}^{\infty} e^{tx}f(x)dx\,\rvert_{t=0} = \int_{-\infty}^{\infty} \frac{d^r}{dt^r} e^{tx}f(x)dx\,\rvert_{t=0} = \int_{-\infty}^{\infty} x^r e^{tx} f(x)dx\,\rvert_{t=0} = \int_{-\infty}^{\infty} x^r\, f(x)dx = E(X^r)$ ② $M(0)^{(r)} = \frac{d^r}{dt^r}\sum e^{tx_i}\, p(x_i)\rvert_{t=0} = \sum \frac{d^r}{dt^r} e^{tx_i}\, p(x_i) = \sum x_i^r e^{tx_i}\, p(x_i)\rvert_{t=0} = \sum x_i^r\, p(x_i) = E(X^r)$ 2) 성질 ① MGF의 **유일성**: $\boldsymbol{M_x(t) = M_y(t)}$ ⇔ $\boldsymbol{X = Y}$ (pdf 동일) ② $M_{X+\alpha}(t) = e^{\alpha t}\, M_X(t)$ $\because M_{X+\alpha}(t) = E\big(e^{t(x+\alpha)}\big) = e^{\alpha t}\, E(e^{tx}) = e^{\alpha t}\, M_X(t)$ ③ $M_{\alpha X}(t) = M_X(\alpha t)$ $\because M_{\alpha X}(t) = E\big(e^{t(\alpha X)}\big) = E\big(e^{(\alpha t)X}\big) = M_X(\alpha t)$ ④ $M_{aX+b}(t) = e^{bt} M_X(at)$ ⑤ $M_Y(t) = \prod M_{X_i}(k_i t)$, $t < \lvert \min(h_i) \rvert$ (for $Y = \sum k_i X_i$, X_i는 모두 독립) ⑥ $M_Y(t) = [M(t)^n]$ (for $Y = \sum X_i$; X_i는 **iid** 확률변수)
중요한 부등식	1. $E(X^m)$이 존재하면 ➔ $E(X^k)$ 존재 for $\mathrm{k} \le \mathrm{m}$ *증명: $E(X^k) = \int_{-\infty}^{\infty} \lvert x \rvert^k f(x)dx = \int_{\lvert x \rvert \le 1} \lvert x \rvert^k f(x)dx + \int_{\lvert x \rvert \ge 1} \lvert x \rvert^k f(x)dx \le \int_{\lvert x \rvert \le 1} f(x)dx + \int_{\lvert x \rvert \ge 1} \lvert x \rvert^m f(x)dx$ $\le \int_{-\infty}^{\infty} f(x)dx + \int_{-\infty}^{\infty} \lvert x \rvert^m f(x)dx \le 1 + E(X^m)$ ∴ 유한함 2. **Markov**: $\boldsymbol{P[u(X) \ge c] \le E[u(X)]/c}$ (for u(X)≥0, c>0; E[u(X)]존재) *증명: $E[u(x)] = \int_{-\infty}^{\infty} u(x)f(x)dx \ge \int_{u(x)\ge c} u(x)f(x)dx \ge c \int_{u(x)\ge c} f(x)dx = c\, P[u(x) \ge c]$ 3. **Chevyshev**: $\boldsymbol{P(\lvert X - \mu \rvert \ge k\sigma) \le 1/k^2}$ (for k>0; X가 μ,σ^2(유한) 가짐) *증명: Markov에서 $u(X) = (X-\mu)^2$, $c = k^2\sigma^2$

2-1. 이변량분포

이변수	1) Joint CDF: $F(x,y)=P[\{X\le x\}\cap\{Y\le y\}]$ * $\mathbf{X}=(X,Y)^T\in D$; **Random vector X** *$P\big((a_1,a_2]\times(b_1,b_2]\big)=F(a_2,b_2)-F(a_1,b_2)-F(a_2,b_1)+F(a_1,b_1) = \int_{b_1}^{b_2}\int_{a_1}^{a_2} f(x,y)\,dxdy$ 2) Joint PMF: $\sum_y\sum_x p(x,y)=1$ 3) Joint PDF: $F(x,y)=\int_{-\infty}^{x}\int_{-\infty}^{y} f(x,y)\,dydx$ $\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dxdy=1\right)$ ⇔ $\partial^2(F)/\partial x\partial y = f(x,y)$ 4) Marginal dist: 한 변수의 효과만 봄; 다른 변수는 (-∞, ∞) 전부 포괄 * $F_X(x)=P(\{X\le x\})=P(\{X\le x\}\cap\{-\infty<Y<\infty\})$ ①PMF of x: $F_X(x)=\sum_{(-\infty,x]}\{\sum_{y\in(-\infty,\infty)}p(x,y)\}$ ➔ $p_X(x)=\sum_{y\in(-\infty,\infty)}p(x,y)$ ②PDF of x: $F_x(x)=\int_{-\infty}^{x}\{\int_{-\infty}^{\infty}\boldsymbol{f(x,y)dy}\}dx$ ➔ $f_X(x)=\int_{-\infty}^{\infty}f(x,y)dy$ 
이변수 기대값	*$E(g(X,Y))$ 존재 조건 ⇔ $E(\lvert g(X,Y)\rvert)<\infty$ 기대값: $E\big(g(X,Y)\big)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g(x,y)\,f(x,y)\,dxdy$ (이산형: $E\big(g(X,Y)\big)=\sum\sum g(x,y)p(x,y)$ 1) $E(k_1g_1+k_2g_2)=k_1E(g_1)+k_2E(g_2)$ 2) $E(\mathbf{X})=[E(X)\ E(Y)]^{\mathrm{T}}=\left[\int_{-\infty}^{\infty}x\,f_x(x)dx\ \ \int_{-\infty}^{\infty}y\,f_y(y)dy\right]^{\mathrm{T}}$ 3) $M(t_1,t_2)=E(\exp(t_1X+t_2Y))$ ➔ $\mathbf{t}=(t_1,t_2)^T$에 대해 $M(\mathbf{t})=E(\exp(\mathbf{t}^T\mathbf{X}))$ $E(X^kY^m)=\frac{\partial^{k+m}}{\partial t_1^k\,\partial t_2^m}M(0,0)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x^ky^m\exp(t_1x+t_2y)\,f(x,y)\,dxdy$
이변수 변환	*변환 조건: 1) $\mathbf{X}=(X_1,X_2)$의 받침 S 2) S→T 사상하는 일대일대응: $y_1=u_1(x_1,x_2)$ & $y_2=u_2(x_1,x_2)$ 3) T→S 사상하는 위 대응 역: $x_1=w_1(y_1,y_2)$ & $x_2=w_2(y_1,y_2)$
	1. 이산형 변환: $p_{\mathbf{Y}}(y_1,y_2)=p_{\mathbf{X}}[w_1(y_1,y_2),w_2(y_1,y_2)]$ for $(y_1,y_2)\in T$ & 나머지 pmf 0 * X_1, X_2 ➔ Y로만 변환 시, dummy 변수를 하나 더 만들어 Y_2로 지정해주고 marginal Y dist를 구함
	2. 연속형 변환: $f_{\mathbf{Y}}(y_1,y_2)=f_{\mathbf{X}}[w_1(y_1,y_2),w_2(y_1,y_2)]\lvert\boldsymbol{J}\rvert$ for $(y_1,y_2)\in T$ & 나머지 pdf 0
	* MGF 이용 변환: $E\big(\exp(tY)\big)=E(\exp(t(X_1+X_2))$ ➔ MGF 유일성으로 Y의 PMF/PDF 구함
조건부	1. 조건부 PMF: $p_{2\|1}(x_2\|x_1)=p(x_1,x_2)/p_1(x_1)$
	2. 조건부 PDF: $f_{2\|1}(x_2\|x_1)=f(x_1,x_2)/f_1(x_1)$ (f_1는 $f_{1,2}$의 marginal 분포) 1) $P(a<Y<b\,\|X=x)=\int_a^b f_{Y\|X}(y\|x)\,dy$ & $P(c<X<d\|Y=y)=\int_c^d f_{X\|Y}(x\|y)\,dx$ 2) $P(-\infty<Y<\infty\,\|X=x)=\int_{-\infty}^{\infty}f_{Y\|X}(y\|x)\,dy=\int_{-\infty}^{\infty}\frac{f(x,y)}{f_X(x)}dy=\frac{1}{f_X(x)}\int_{-\infty}^{\infty}f(x,y)\,dy=1$ 3) 조건부 기대값: $E[\boldsymbol{u(Y)}\|x]=\int_{-\infty}^{\infty}u(y)\,f_{Y\|X}(y\|x)\,dy$ ➔ **x의 함수** ① 조건부 평균: $E(Y\|x)=\int_{-\infty}^{\infty}y\,f_{Y\|X}(y\|x)\,dy$ ② 조건부 분산: $\mathrm{Var}(Y\|x)=E(Y^2\|x)-[E(Y\|x)]^2$
	* 정리: μ_Y 추정 ← $E(Y\|X)$이 Y보다 더 신뢰도 높음 (Rao & Blackwell) 1) $E[E(Y\|X)]=E(Y)$ 2) $\mathrm{Var}\big(E(Y\|X)\big)\le\mathrm{Var}(Y)$ *Y 분산 유한 * 증명: $E(Y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}yf(x,y)\,dydx=\int_{-\infty}^{\infty}[\int_{-\infty}^{\infty}yf_{Y\|X}(y\|x)dy]\,f_X(x)dx=\int_{-\infty}^{\infty}E(Y\|X)\,f_X(x)dx=E(E(Y\|X))$
공분산 / 상관 계수	1.공분산: $\mathbf{Cov}(\boldsymbol{X},\boldsymbol{Y})=\boldsymbol{E}[(\boldsymbol{X}-\boldsymbol{\mu_X})(\boldsymbol{Y}-\boldsymbol{\mu_Y})]=\boldsymbol{E}(\boldsymbol{XY})-\boldsymbol{E}(\boldsymbol{X})\boldsymbol{E}(\boldsymbol{Y})$ *독립이면 $\mathrm{Cov}(X,Y)=0\Leftrightarrow E(XY)=E(X)E(Y)$
	2.상관계수: $\boldsymbol{\rho}=\mathbf{Cov}(\boldsymbol{X},\boldsymbol{Y})/\boldsymbol{\sigma_X\sigma_Y}=\boldsymbol{\sigma_{XY}}/(\boldsymbol{\sigma_X\sigma_Y})$ $(-1\le\rho\le1)$➔$y=a+bx\ (b>0)$에 ρ의 강도로 집중 $(0<\rho\le1)$
	3.선형조건부평균: $E(Y\|X)=a+bx$ ➔ $E(Y\|X)=\mu_y+\rho\frac{\sigma_y}{\sigma_x}(X-\mu_x)$ & $E\big(\mathrm{Var}(Y\|X)\big)=\sigma_y^2(1-\rho^2)$ *회귀분석 모회귀계수 $\beta=\rho\big(\sigma_y/\sigma_x\big)=\mathrm{Cov}(X,Y)/Var(X)$; *X,Y 분산 유한
독립	*정의: $\boldsymbol{f(x,y)=f_x(x)f_y(y)}$ ⇔ **X,Y는 독립** [$x\in(a,b)\,\&\,y\in(c,d)$] (받침이 수평/수직선 box에 존재해야 함) 1. 조건부 증명: $f_y(y)=\int_{-\infty}^{\infty}f(x,y)\,dx=\int_{-\infty}^{\infty}f_{y\|x}(y\|x)\,f_x(x)\,dx=f_{y\|x}(y\|x)\int_{-\infty}^{\infty}f_x(x)dx=f_{y\|x}(y\|x)$ 2. 동치류 1) $\boldsymbol{f(x,y)=f_x(x)\,f_y(y)}$ 2) $\boldsymbol{F(x,y)=F_x(x)\,F_y(y)}$ *증명: $\partial^2F/\partial x\partial y=f_x(x)f_y(y)$ 3) $\boldsymbol{P(a<X<b,\ c<Y<d)=P(a<X<b)\,P(c<Y<d)}$ *증명: $P(a<X<b,c<Y<d)=F(b,d)-F(a,d)-F(b,c)+F(a,c)=[F_x(b)-F_x(a)][F_y(d)-F_y(c)]$ 4) $\boldsymbol{E[u(X)v(Y)]=E[u(X)]\,E[v(Y)]}$ $\Rightarrow E(XY)=E(X)E(Y)\Leftrightarrow\mathrm{Cov}(X,Y)=0$ 5) $\boldsymbol{M(t_1,t_2)=M(t_1,0)\,M(0,t_2)}$ *증명: $M(t_1,t_2)=E(e^{t_1X+t_2Y})=E(e^{t_1X})E(e^{t_2Y})=M(t_1,0)\,M(0,t_2)$ * $\boldsymbol{M(t_1,0)}$는 **x에 대한 marginal 분포의 MGF**

2-2. 다변량분포

다변수	* $\mathbf{x} = (x_1, x_2, \cdots, x_p)^T = \left(X_1(c), X_2(c), \cdots, X_p(c)\right)^T$ for 확률 실험 c ∈ C 1. 결합 확률 함수들 1) Joint CDF: $F(\mathbf{x}) = P[\{X_1 \le x_1\} \cap \{X_2 \le x_2\} \cap \cdots \cap \{X_p \le x_p\}]$ 2) Joint PMF $F(\mathbf{x}) = \sum_{w_1 \le x_1} \cdots \sum_{w_p \le x_p} p(w_1, \cdots, w_p)$ 3) Joint PDF: $F(\mathbf{x}) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \cdots \int_{-\infty}^{x_p} f(x_1, \cdots, x_p)\, dx_p \cdots dx_1$ $\quad (\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \cdots, x_p)\, dx_p \cdots dx_1 = 1)$ $\Leftrightarrow \partial^p\{F(\mathbf{x})\} / \partial x_1 \cdots \partial x_p = f(\mathbf{x})$
	2. Marginal/Conditional 1) $f_1(x_1) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \cdots, x_p)\, dx_2 \cdots dx_p$ ➔ $f_{2,\cdots,p\,\vert 1}(x_2, \cdots, x_p \,\vert x_1) = f(\mathbf{x}) / f_1(x_1)$ 2) $f_{2,4,5}(x_2, x_4, x_5) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\mathbf{x})\, dx_1 dx_3 dx_6$ $\leftarrow \mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6)^T$ ➔ $f_{1,3,6\,\vert 2.4.5}(x_1, x_3, x_6 \,\vert x_2, x_4, x_5) = f(\mathbf{x}) / f_{2,4,5}(x_2, x_4, x_5)$
	3. 기대값 1) $E(u(\mathbf{x})) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} u(x_1, \cdots, x_p)\, dx_1 \cdots dx_p$ (존재성: $\exists E(\vert u(\mathbf{x})\vert)$) *이산: $E(u(\mathbf{x})) = \sum_{x_1} \cdots \sum_{x_p} u(x_1, \cdots, x_p)$ 2) $E(\sum k_i Y_i) = \sum k_i E(Y_i)$ 3) $E[u(X_2, \cdots, X_p) \,\vert x_1] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \boldsymbol{u}(\boldsymbol{x_2}, \cdots, \boldsymbol{x_p})\, f_{2,\cdots,p\,\vert 1}(x_2, \cdots, x_p \,\vert x_1)\, dx_2 \cdots dx_p$
	4. 독립: $E(\prod u_i(X_i)) = \prod E(u_i(X_i))$ 등 동치류 (f, F, P, E, M) ***iid** (independent and identically distributed): 여러 확률 변수가 서로 독립 & 동일한 분포
	5. 변환: **X**의 받침 S에 대해, X⇔Y가 일대일이 되는 $S_1, \cdots, S_k$의 부분 공간 상 각각의 야코비안 J_i 정의 $$g(\mathbf{y}) = \sum_{i=1}^{k} \vert \boldsymbol{J_i} \vert\, f[w_{1i}(\mathbf{x}), \cdots, w_{pi}(\mathbf{x})]$$
Random matrix	* **Random matrix** $\mathbf{W} = [W_{ij}]$, W_{ij} $(1 \le i \le m,\ 1 \le j \le n)$ 1. $E(\mathbf{W}) = [E(W_{ij})]$ (일렬로 배열하여 mn x 1의 벡터로 생각) 1) $E[\mathbf{AW} + \mathbf{BV}] = \mathbf{A}\, E[\mathbf{W}] + \mathbf{B}\, E[\mathbf{V}]$ (A,B: k x m 상수 행렬, W,V: m x n 확률 행렬) 2) $E[\mathbf{AWB}] = \mathbf{A}\, E(\mathbf{W})\, \mathbf{B}$ (A: k x m, W: m x n, B: n x l) 2. 분산-공분산 행렬 (Variance-Covariance matrix) $\quad * \mathbf{X} = (X_1, X_2, \cdots, X_p)^{\mathrm{T}}$; 모든 VCM는 양의 반정부호(psd) 1) 정의: $\mathrm{Cov}(\mathbf{X}) = E[(\mathbf{X} - \boldsymbol{\mu})(\mathbf{X} - \boldsymbol{\mu})^{\mathrm{T}}] = [\sigma_{ij}]$ $\quad (\boldsymbol{\mu} = E(\mathbf{X}))$ ➔ $\sigma_i^2 = \mathrm{Var}(X_i)$ & $\sigma_{ij} = \mathrm{Cov}(X_i, X_j)$ 2) 정리 ① $\mathrm{Cov}(\mathbf{X}) = \mathrm{E}(\mathbf{X}\mathbf{X}^{\mathrm{T}}) - \boldsymbol{\mu}\boldsymbol{\mu}^{\mathrm{T}}$ $\quad (\sigma_i^2 < \infty)$ ② $\mathrm{Cov}(\mathbf{AX}) = \mathbf{A}\mathrm{Cov}(\mathbf{X})\mathbf{A}^{\mathrm{T}}$ $\quad (\sigma_i^2 < \infty,\ \mathrm{A{:}\ m \times p})$ 3. MGF: $M(\mathbf{t}) = E[\exp(\mathbf{t}^T\mathbf{X})] = E[\prod_{i=1}^{p} \exp(t_i X_i)]$ (*X_i 독립➔ $M(\mathbf{t}) = M(t_1, \cdots, 0) \cdots M(0, \cdots, t_p) = \prod_{i=1}^{p} E[\exp(t_i X_i)]$ 1) $M_{\mathbf{Y}}(\mathbf{t}) = \prod M_{\mathbf{X}_i}(\mathbf{t})$ $\quad (\mathbf{Y} = \sum \mathbf{X}_i$, 각 $\mathbf{X}_i \in \mathbb{R}^n$은 독립) 2) $M_{\mathbf{Y}}(\mathbf{t}) = \mathrm{e}^{\mathbf{b}^T t} M_{\mathbf{X}}(\mathbf{A}^T\mathbf{t})$ $\quad (\mathbf{Y} = \mathbf{AX} + \mathbf{b};\ \ \mathbf{A}{:}\, m \times p;\ \ \mathbf{t} \in \mathbb{R}^m;\ \ \mathbf{b} \in \mathbb{R}^m)$
	3. 선형결합: $T = \sum_{i=1}^{n} a_i X_i,\ \ W = \sum_{i=1}^{m} b_i Y_i$ 1) $E(T) = \sum_{i=1}^{n} a_i E(X_i)$ $\quad (*E[\vert X_i \vert] < \infty)$ 2) $\mathrm{Cov}(T, W) = \sum\sum a_i b_j\, \mathrm{Cov}(X_i, Y_j)$ $\quad (*E[X_i^2] < \infty,\ \ E[Y_{ij}^2] < \infty)$ ① $Var(T) = \mathrm{Cov}(T, T) = \sum_{i=1}^{n} a_i^2\, \mathrm{Var}(X_i) + 2\sum_{i<j} a_i b_j\, \mathrm{Cov}(X_i, X_j)$ $\quad (*E[X_i^2] < \infty)$ ② $Var(T) = \mathrm{Cov}(T, T) = \sum_{i=1}^{n} a_i^2\, \mathrm{Var}(X_i)$ (*$X_1, \cdots, X_n$이 유한 분산, 독립) 3) 표본 추정량: $X_1, \cdots, X_n$이 μ, σ^2 가지는 **iid** 확률변수 ① 표본평균: $\boldsymbol{\bar{X}} = \sum_{i=1}^{n} \boldsymbol{X_i} / \boldsymbol{n}$ & 표본분산: $\boldsymbol{S^2} = \sum_{i=1}^{n} (\boldsymbol{X_i} - \boldsymbol{\bar{X}})^2 / \boldsymbol{n-1} = \sum X_i^2 - n\bar{X}^2 / n-1$ ② $\boldsymbol{E(\bar{X})} = \sum_{i=1}^{n} E(X_i)/n = n\mu/n = \boldsymbol{\mu}$ & $\mathbf{Var}(\boldsymbol{\bar{X}}) = \sum_{i=1}^{n} \left(\frac{1}{n}\right)^2 \mathrm{Var}(X_i) = n\left(\frac{1}{n}\right)^2 \sigma^2 = \boldsymbol{\sigma^2/n}$ ③ $\boldsymbol{E(S^2)} = \sum E(X_i^2) - nE(\bar{X}^2) / n-1 = n(\mu^2 + \sigma^2) - n(\mu^2 + \frac{\sigma^2}{n}) / n-1 = \boldsymbol{\sigma^2}$ ($\bar{X}$, S^2는 독립 by Student's정리)

3-1. 주요 분포: 이산

<table>
<tr><td rowspan="6">이항
분포</td><td>* Bernoulli experiment: 성공/실패로 서로 배반인 확률 실험
* Bernoulli trial: 베르누이 실험을 독립적으로 반복 (성공 확률 p 동일)
* Bernoulli distribution의 유도: X(성공)=1, X(실패)=0 ➔ PMF: $p(x)=p^x(1-p)^{1-x}$ *$\mu=p,\ \sigma^2=p(1-p)$</td></tr>
<tr><td>* Binomial distribution (이항분포): n회 반복한 베르누이 시행에서 성공한 총 횟수 분포
1. PMF: $\boldsymbol{p(x)=\binom{n}{x}p^x(1-p)^{n-x}\sim b(n,p)}$ $(x=0,1,\cdots,n)$
2. MGF: $M(t)=\sum e^{tx}p(x)=[(1-p)+(pe^t)]^n$ $(t\in\mathbb{R})$
3. 기대값 1) $\boldsymbol{\mu=np}$ *$\mu=M'(0)=n[(1-p)+pe^t]^{n-1}(pe^t)|_{t=0}=np$
2) $\boldsymbol{\sigma^2=np(1-p)}$ *$\sigma^2=M''(0)-\left(M'(0)\right)^2=n(n-1)p^2+np-(np)^2=np(1-p)$
4. 가법성: $Y=\sum X_i,\ X_i\sim B(n_i,\boldsymbol{p})$ ➔ $Y\sim B(\sum n_i,\boldsymbol{p})$
(증명) $M_Y(t)=\prod M_{X_i}(t)=\prod[(1-p)+(pe^t)]^{n_i}=[(1-p)+(pe^t)]^{\sum n_i}$

$*\,p(x)$가 성공확률(= 평균) p인 Bernoulli분포 ↔ $X\sim B(1,p)$
→ $Y=\sum_{i=1}^{20}X_i$ (iid) 에 대해 $p(y)$는 20회 시행 중 평균 $20p$회 성공하는 Bernoulli ↔ $Y\sim B(20,p)$</td></tr>
<tr><td>* Multinomial distribution (다항분포)
1. PMF: $p(x_1,\cdots,x_{k-1})=\frac{n!}{(x_1)!\cdots(x_k)!}(p_1)^{x_1}\cdots(p_k)^{x_k}$ ➔ $p_k=1-\sum_{i=1}^{k-1}p_i$ & $x_k=n-\sum_{i=1}^{k-1}x_i$
2. MGF: $M(t_1,\cdots,t_{k-1})=(p_1e^{t_1}+\cdots+p_{k-1}e^{t_{k-1}}+p_k)^n$</td></tr>
<tr><td>* R codes 1) dbinom (k,n,p): P(X=k) 2) pbinom (k,n,p): P(X≤k)</td></tr>
<tr><td>* Negative binomial distribution (음이항분포): X번 실패 후 r번 성공 (베르누이 시행) *r번 성공시 나감
1. PMF: $p(x)=\binom{x+r-1}{r-1}p^r(1-p)^x$ 2. MGF: $M(t)=p^r[1-(1-p)e^t]^{-r}$ $(e^t \lt 1/(1-p))$
⬄ [이항:x+(r-1)번 중 (r-1)번 성공] x [p] *$\binom{-n}{k}=(-n)(-n-1)\cdots(-n-k+1)/k!=(-1)^k\binom{n+k-1}{k}$
* Geometric distribution (기하 분포): X번 실패 후 처음 성공 (베르누이 시행) ⬄ r=1인 음이항분포
1. PMF: $p(x)=p(1-p)^x$ 2. MGF: $M(t)=p[1-(1-p)e^t]^{-1}$</td></tr>
<tr><td>* Hypergeometric distribution (초기하분포)
1. PMF: $p(x)=\frac{\binom{D}{x}\binom{N-D}{n-x}}{\binom{N}{n}}$ *N개 중 D개가 성공 & 비복원추출: n번 시행 ➔ x번 성공 확률
2. 기대값: 1) $\mu=n\left(\frac{D}{N}\right)$ 2) $\sigma^2=n\left(\frac{D}{N}\right)\left(1-\frac{D}{N}\right)\left(\frac{N-n}{N-1}\right)$ N>>n이면 이항분포로 근사 가능</td></tr>
<tr><td rowspan="3">Poisson
분포</td><td>* Poisson process: 일정한 구간 (시간, 공간)에서 독립적으로 발생하는 event를 생성하는 과정 (비기억성)
* Poisson postulate: 짧은 구간 h (h->0)에 대해
1) $g(1,h)=\lambda h+o(h)$ $*\,g(x,w)$는 구간 길이 w 내에 x회 발생 확률
2) $\sum_{x=2}^{\infty}g(x,h)=o(h)$ (≒미소 구간 h에 둘 이상은 본질적 불가) * $\lim_{h\to0}o(h)/h=0$ (little-o)
3) 겹치지 않는 구간 ➔ 확률적으로 독립 $\therefore g(x,w)=\frac{e^{-\lambda w}(\lambda w)^x}{x!}$</td></tr>
<tr><td>1. PMF: $\boldsymbol{p(x)=\frac{e^{-\mu}\mu^x}{x!}}$ $(x=0,1,2,\cdots)$ (*$\sum p(x)=e^{-\mu}\sum\left(\frac{\mu^x}{x!}\right)=e^{-\mu}e^{\mu}=1$) ← 주어진 시간에 x회 발생 분포
2. 기댓값: $\boldsymbol{\mu=\sigma^2=\lambda w}$ (λ: 단위 길이당 발생률, w: 주어진 영역 크기)
3. MGF: $M(t)=e^{\mu(e^t-1)}$ $(t\in\mathbb{R})$
4. 가법성: $Y=\sum X_i,\ X_i\sim\mathrm{Poi}(m_i)$ ➔ $Y\sim\mathrm{Poi}(\sum m_i)$
(증명) $M_Y(t)=\prod M_{X_i}(t)=\prod e^{m_i(e^t-1)}=e^{(\sum m_i)(e^t-1)}$
$*\,p(x)$가 주어진 100초당 평균 μ회 발생 Poisson ↔ $X\sim\mathrm{Poi}(\mu)$
→ $Y=\sum_{i=1}^{20}X_i$ (iid) 에 대해 $p(y)$는 주어진 20×100초당 평균 20μ회 발생 Poisson ↔ $Y\sim\mathrm{Poi}(20\mu)$
* 이항분포 $\boldsymbol{b(n,p)}\xrightarrow{D}$ 푸아송분포 $(\boldsymbol{\mu=np})$ (MGF의 극한으로 분포수렴 증명)</td></tr>
<tr><td>* R codes 1) dpois (k,m): P(X=k) 2) ppois (k,m): P(X≤k)</td></tr>
</table>

3-2. 주요 분포: 감마 연관 분포

Γ 분포	* 감마함수: $\boldsymbol{\Gamma(\alpha)=\int_0^\infty y^{\alpha-1}e^{-y}dy}$ $(\alpha>0)$ * $\boldsymbol{\Gamma(\alpha)=(\alpha-1)\Gamma(\alpha-1)}$ ➔ $\boldsymbol{\Gamma(n)=(n-1)!}$ for 자연수 n *$\Gamma(1)=1,\ \Gamma(1/2)=\sqrt{\pi}$ $*$ 스털링 근사: $\Gamma(k+1)\approx\sqrt{2\pi k}\left(\frac{k}{e}\right)^k$
	***Gamma distribution (감마분포): α** $(\in\mathbb{R})$ **번째** Poisson event 발생까지 걸리는 **대기 시간** 1. PDF**:** $\boldsymbol{f(x)=\frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}e^{-x/\beta}}$ $\sim\boldsymbol{\Gamma(\alpha,\beta)}$ $(0\le x<\infty)$ (감마함수 식에 $y=x/\beta$ 대입; α>0 & β>0) 2. 기댓값: 1) $\boldsymbol{\mu=\alpha\beta}$, 2) $\boldsymbol{\sigma^2=\alpha\beta^2}$ 3. MGF: $M(t)=1/(1-\beta t)^\alpha$ (t < 1/β) 4. 가법성: $Y=\sum X_i,\ X_i\sim\Gamma(\alpha_i,\beta)$ ➔ $Y\sim\Gamma(\sum\alpha_i,\beta)$ (증명) $M_Y(t)=\prod M_{X_i}(t)=\prod(1-\beta t)^{-\alpha_i}=(1-\beta t)^{-\sum\alpha_i}$ 5. 스칼라배: $X\sim\Gamma(\alpha,\beta)\Rightarrow kX\sim\Gamma(\alpha,k\beta)$ (*증명: 야코비안 변수변환) 6. 유도: k번 Poisson event 발생까지 시간을 T_i로 분할 ➔ 각 $T_i\sim\Gamma(1,\frac{1}{\lambda})$ ➔ $Y=\sum_{i=1}^k T_i\sim\Gamma(k,\frac{1}{\lambda})$ (*Erlang 분포: 자연수 k인 감마 분포)
	* R codes 1) dgamma (x,shape=a,scale=b): f(X=x) 2) pgamma (x, shape=a, scale=b): P(X≤x)
지수 분포	***Exponential distribution (지수분포): 1번째** Poisson event 발생까지 대기 시간 $=\Gamma(1,\beta)$ 1. PDF**:** $\boldsymbol{f(x)=\frac{1}{\beta}e^{-x/\beta}}$ 2. 기댓값: 1) $\boldsymbol{\mu=\beta}$, 2) $\boldsymbol{\sigma^2=\beta^2}$ 3. 유도: W가 첫 번째 Poisson event 까지 걸린 시간 ➔ w시간 내 푸아송 사건 없을 확률: $P(W>w)=\frac{e^{-\lambda w}(\lambda w)^0}{0!}=e^{-\lambda w}$ ⇔ $P(0<W<w)=1-e^{-\lambda w}$ $\therefore f(w)=\lambda e^{-\lambda w}$ (β=1/λ)
χ^2 분포	***Chi-square distribution (카이제곱 분포)**: 자유도 r에 대해, $\boldsymbol{\chi^2(r)=\Gamma(\frac{r}{2},2)}$ 1. PDF**:** $\boldsymbol{f(x)=\frac{1}{\Gamma(\frac{r}{2})2^{(\frac{r}{2})}}x^{(\frac{r}{2})-1}e^{-\frac{x}{2}}}$ $\sim\boldsymbol{\chi^2(r)}$ $(0\le x<\infty)$ 2. 기댓값: 1) $\boldsymbol{\mu=r}$, 2) $\boldsymbol{\sigma^2=2r}$ 3. MGF: $M(t)=1/(1-2t)^{r/2}$ (t < 1/2) 4. $E(X^k)=2^k\frac{\Gamma(\frac{r}{2}+k)}{\Gamma(\frac{r}{2})}$, $k>-\frac{r}{2}$ 5. 가법성 (corollary): $Y=\sum X_i,\ X_i\sim\chi^2(r_i)$ ➔ $Y\sim\chi^2(\sum r_i)$
	* R codes 1) dchisq (x,r): f(X=x) 2) pchisq (x,r): P(X≤x)
β 분포	*베타함수: $\mathrm{B}(\alpha,\beta)=\int_0^1 y^{\alpha-1}(1-y)^{\beta-1}dy$ $(\alpha>0,\beta>0)$ ①$\mathrm{B}(\alpha,\beta)=\mathrm{B}(\beta,\alpha)$, ②$\mathrm{B}(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$
	*결합 PDF: $h(x_1,x_2)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}x_1^{\alpha-1}x_2^{\beta-1}e^{-(x_1+x_2)};$ $0\le x_1<\infty,0\le x_2<\infty$ (X_1, X_2독립) $*$ $Y_1=X_1/(X_1+X_2)$ & $Y_2=X_1+X_2$ ➔ Y_1에 대한 marginal distribution이 beta(α,β) 1. PDF**:** $\boldsymbol{f(x)=\frac{1}{B(\alpha,\beta)}x^{\alpha-1}(1-x)^{\beta-1}}$ $(0<x<1)$
	* R codes 1) dbeta(x,a,b): f(X=x) 2) pbeta (x,a,b): P(X≤x)
Dirichlet 분포 (β 확장)	*결합 PDF: $h(x_1,\cdots,x_{k+1})=\prod_{i=1}^{k+1}\frac{1}{\Gamma(\alpha_i)}x_i^{\alpha_i-1}e^{-x_i};$ $0\le x_i<\infty$ (X_1, … ,X_k 독립) $*$ $Y_i=\frac{X_i}{\sum_{i=1}^{k+1}X_i}(i=1,2,\cdots,k)$ & $Y_{k+1}=\sum_{i=1}^{k+1}X_i$ ➔$Y_1,\cdots,Y_k$에 대한 marginal dist이 Dirichlet(α_1,…α_{k+1}) 1. PDF: $g(\boldsymbol{y_1},\cdots,\boldsymbol{y_{k+1}})=\frac{\Gamma(\alpha_1+\cdots+\alpha_{k+1})}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_{k+1})}\boldsymbol{y_i^{\alpha_i-1}\cdots y_k^{\alpha_k-1}[1-(y_1+\cdots+y_k)]^{\alpha_{k+1}-1}}$ $(0\le y_k,\ \sum_{i=1}^k y_i<1)$

3-3. 주요 분포: 정규 분포

정규 분포

*표준정규분포: $I=\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{2}\right)dz$ → $0<\exp\left(-\frac{z^2}{2}\right)<\exp(-|z|+1)$ 유계 ($\int_{-\infty}^{\infty}e^{-|z|+1}dz=2e$)

→ $I^2=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(-\frac{x^2+y^2}{2}\right)dxdy=\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty}e^{-\frac{r^2}{2}}\,r\,drd\theta=1$ ∴ $f(z)=\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{2}\right)$

*정규분포: $X=\sigma Z+\mu$ 로 변수 변환 $f(x)=\frac{1}{\sigma\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}$

*Bell shape 분포: location 모수 (μ), scale 모수 (σ^2) vs. 감마분포 등: shape 모수 (α), scale 모수 (β)

***표준 정규 분포 $N(0, 1^2)$**

1. PDF: $\phi(z)=\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}z^2\right)$ $(-\infty<z<\infty)$
2. MGF: $M(t)=\exp\left(\frac{1}{2}t^2\right)$, $t\in\mathbb{R}$ 3. 기대값: $E(Z)=0$, $\mathrm{Var}(Z)=1$
4. $E(Z^k)=k!\Big/2^{\frac{k}{2}}\left(\frac{k}{2}\right)!$ (k가 짝수), $E(Z^k)=0$ (k가 홀수) $*\,M(t)=\exp\left(\frac{1}{2}t^2\right)=\sum_{m=0}^{\infty}\left(\frac{t^2}{2}\right)^m/m!$

***정규 분포 $N(\mu, \sigma^2)$**

1. PDF: $f(x)=\frac{1}{\sigma\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}$ $(-\infty<x<\infty\ ,\ \sigma>0)$
2. MGF: $M(t)=\exp\left(\mu t+\frac{1}{2}\sigma^2t^2\right)$, $t\in\mathbb{R}$ 3. 기대값: $E(Z)=\mu$, $\mathrm{Var}(Z)=\sigma^2$
4. $E(X^k)=E\left[(\sigma Z+\mu)^k\right]=\sum_{j=0}^{k}\binom{k}{j}\sigma^jE(Z^j)\mu^{k-j}$
5. 가법성: $Y=\sum a_iX_i$, $X_i\sim N(\mu_i,\sigma_i^2)$ → $Y\sim N[\sum(a_i\mu_i),\sum(a_i\sigma_i)^2]$
 (증명) $M_Y(t)=\prod M_{a_iX_i}(t)=\prod M_{X_i}(a_it)=\prod\exp\left(\mu_i(a_it)+\frac{1}{2}\sigma_i^2(a_it)^2\right)=\exp\left((\sum a_i\mu_i)t+\frac{1}{2}(\sum a_i^2\sigma_i^2)t^2\right)$
6. Corollary: $\bar{X}=\sum_{i=1}^{n}X_i\big/n$, $X_i\sim N(\mu,\sigma^2)$ (iid) → $\bar{X}\sim N(\mu,\ \sigma^2/n)$

*** 정리: $Z^2\sim\chi^2(1)$**

pf) $W=Z^2$일 때, $F(x)=P(W\le x)=P(Z^2\le x)=P\left(-\sqrt{x}\le Z\le\sqrt{x}\right)$, $x\ge 0$

→ $y=\sqrt{w}$ 변환 시, $F(x)=2\int_0^{\sqrt{x}}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{y^2}{2}\right)dy=\int_0^{x}\frac{1}{\sqrt{2\pi}\sqrt{w}}\exp\left(-\frac{w}{2}\right)dw$

→ $f(x)=\frac{1}{\sqrt{2\pi}}x^{-\frac{1}{2}}e^{-\frac{x}{2}}=\frac{1}{\Gamma\left(\frac{1}{2}\right)2^{1/2}}x^{\frac{1}{2}-1}e^{-\frac{x}{2}}$ $\sim\chi^2(1)$ $(0\le x<\infty)$

* **따름 정리**: $Y=\sum_{i=1}^{n}Z_i^2\sim\chi^2(n)$ (가법성 of χ^2 using MGF; for iid Z ~ $N(0,1^2)$)

*** Contaminated normal distribution**: 대부분 $Z\sim N(0,1^2)$, 일부 outlier $\sim N(0,\sigma_c^2)$ (오염 비율: ε)

1) $W=KZ+(1-K)\,\sigma_cZ$ for $K=\begin{cases}1 & \text{확률 } 1-\varepsilon\\ 0 & \text{확률 } \varepsilon\end{cases}$ (Z, K는 독립)

$F_W(w)=P(W\le w)=P(W\le w, I=1)+P(W\le w, I=0)=P(Z\le w)(1-\varepsilon)+P\left(Z\le\frac{w}{\varepsilon_c}\right)\varepsilon=(1-\varepsilon)\Phi(w)+\varepsilon\,\Phi\left(\frac{w}{\sigma_c}\right)$

① PDF: $f_W(w)=(1-\varepsilon)\phi(w)+\frac{\varepsilon}{\sigma_c}\phi\left(\frac{w}{\sigma_c}\right)$ ② $E(W)=0$, $\mathrm{Var}(W)=1+\varepsilon(\sigma_c^2-1)$

2) $X=a+bW$ $(b>0)$

① PDF: $f_X(x)=(1-\varepsilon)\phi\left(\frac{X-a}{b}\right)+\frac{\varepsilon}{\sigma_c}\phi\left(\frac{X-a}{b\sigma_c}\right)$ ② $E(W)=a$, $\mathrm{Var}(W)=b^2[1+\varepsilon(\sigma_c^2-1)]$

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* R codes 1) dnorm(x,a,b): f(X=x) 2) pnorm (x,a,b): P(X≤x) (평균 a, 표준편차 b)

3-3. 주요 분포: 정규 분포

다변량 정규 분포

* $\mathbf{Z} \sim N_p(\mathbf{0}, \mathbf{I_p})$ / $\mathbf{z} = (Z_1, \cdots, Z_p)^T \in \mathbb{R}^p \sim \text{iid } N(0,1)$

1) PDF: $f_{\mathbf{Z}}(\mathbf{z}) = \left(\frac{1}{2\pi}\right)^{p/2} \exp\left(-\frac{1}{2}\mathbf{z}^T\mathbf{z}\right)$ pf) $f_{\mathbf{Z}}(\mathbf{z}) = \prod \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}z_i^2\right) = \left(\frac{1}{2\pi}\right)^{n/2} \exp\left(-\frac{1}{2}\sum z_i^2\right)$

2)MGF: $M_{\mathbf{Z}}(\mathbf{t}) = \exp\left(\frac{1}{2}\mathbf{t}^T\mathbf{t}\right)$ $(\mathbf{t} \in \mathbb{R}^{\boldsymbol{p}})$ pf) $M_{\mathbf{Z}}(\mathbf{t}) = E\{\exp(\mathbf{t}^T\mathbf{Z})\} = E\{\prod \exp(t_i Z_i)\} = \prod E\{\exp(t_i Z_i)\} = \exp\left(\frac{1}{2}\sum t_i^2\right)$

3) 기대값: $E[\mathbf{Z}] = \mathbf{0}$, $\text{Cov}[\mathbf{Z}] = \mathbf{I}_{\boldsymbol{p}}$

* $\mathbf{X} \sim N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ / $\text{Cov}[\mathbf{X}] = \boldsymbol{\Sigma}$가 psd (양반정치)

⇔ p개의 **의존관계**인 정규분포 확률변수의 **결합 분포**

0) 변환: $\mathbf{X} = \boldsymbol{\Sigma}^{1/2}\mathbf{Z} + \boldsymbol{\mu}$ & $\mathbf{Z} = \boldsymbol{\Sigma}^{-1/2}(\mathbf{X} - \boldsymbol{\mu})$

1) PDF: $f_{\mathbf{X}}(\mathbf{x}) = \dfrac{1}{(2\pi)^{\frac{p}{2}}|\boldsymbol{\Sigma}|^{\frac{1}{2}}} \exp\left\{-\dfrac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T(\boldsymbol{\Sigma}^{-1})(\mathbf{x} - \boldsymbol{\mu})\right\}$

2) MGF: $M_{\mathbf{X}}(\mathbf{t}) = \exp\left\{\mathbf{t}^T\boldsymbol{\mu} + \dfrac{1}{2}\mathbf{t}^T(\boldsymbol{\Sigma})\mathbf{t}\right\}$, $(\mathbf{t} \in \mathbb{R}^{\boldsymbol{p}})$

3) 기대값: $E[\mathbf{X}] = \boldsymbol{\mu}$, $\text{Cov}[\mathbf{X}] = \boldsymbol{\Sigma}$

<유도> $\boldsymbol{\Sigma}$가 psd & 대칭 ➔ EVD 가능

$\boldsymbol{\Sigma} = \boldsymbol{\Gamma}^T\boldsymbol{\Lambda}\boldsymbol{\Gamma}$ $(\boldsymbol{\Lambda} = \text{diag}(\lambda_1, \cdots, \lambda_p);\ \lambda_1 \geq \cdots \geq \lambda_p)$

$\boldsymbol{\Sigma}^{1/2} = \boldsymbol{\Gamma}^T\boldsymbol{\Lambda}^{1/2}\boldsymbol{\Gamma}$, $\boldsymbol{\Sigma}^{-1/2} = \boldsymbol{\Gamma}^T\boldsymbol{\Lambda}^{-1/2}\boldsymbol{\Gamma}$ (if $\boldsymbol{\Sigma}$ is pd)

$E[\mathbf{X}] = E[\boldsymbol{\Sigma}^{1/2}\mathbf{Z}] + \boldsymbol{\mu} = \boldsymbol{\Sigma}^{1/2}E[\mathbf{Z}] + \boldsymbol{\mu} = \boldsymbol{\mu}$

$\text{Cov}[\mathbf{X}] = E[(\mathbf{X} - \boldsymbol{\mu})(\mathbf{X} - \boldsymbol{\mu})^T] = E[(\boldsymbol{\Sigma}^{1/2}\boldsymbol{Z})(\boldsymbol{\Sigma}^{1/2}\boldsymbol{Z})^{\boldsymbol{T}}]$

$= \left(\boldsymbol{\Sigma}^{\frac{1}{2}}\right)E(\mathbf{Z}\mathbf{Z}^T)\left(\boldsymbol{\Sigma}^{\frac{1}{2}}\right) = \boldsymbol{\Sigma}$ $*\ E[\mathbf{Z}\mathbf{Z}^T] = \text{Cov}(\mathbf{Z}) + \mathbf{0} = \mathbf{I}_p$

1-1. Theorem $*\ \mathbf{X} \sim N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$; $\mathbf{A}: \text{m} \times \text{p}, \mathbf{b} \in \mathbb{R}^{\text{m}}$

$\mathbf{Y} = \mathbf{AX} + \mathbf{b} \ \rightarrow \mathbf{Y} \sim N_{\text{m}}(\mathbf{A}\boldsymbol{\mu} + \mathbf{b}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^T)$ (MGF로 증명)

1-2. Corollary (m개 변수에 대한 **주변 분포**)

$*\mathbf{X} \rightarrow \mathbf{X_1} \in \mathbb{R}^m$, $\mathbf{X_2} \in \mathbb{R}^q$ $(\text{p} = \text{m} + \text{q})$ 분할

- $\mathbf{X} = \begin{bmatrix}\mathbf{X_1}\\ \mathbf{X_2}\end{bmatrix}$, $\boldsymbol{\mu} = \begin{bmatrix}\boldsymbol{\mu_1}\\ \boldsymbol{\mu_2}\end{bmatrix}$, $\boldsymbol{\Sigma} = \begin{bmatrix}\boldsymbol{\Sigma_{11}} & \boldsymbol{\Sigma_{12}}\\ \boldsymbol{\Sigma_{21}} & \boldsymbol{\Sigma_{22}}\end{bmatrix}$
- $\mathbf{A} = [\mathbf{I}_m \quad \mathbf{O}_{mq}] \rightarrow \mathbf{X_1} = \mathbf{AX}$

➔ $\mathbf{X} \sim N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \rightarrow \mathbf{X_1} \sim N_m(\boldsymbol{\mu_1}, \boldsymbol{\Sigma_{11}})$

$(\because A\mu = \mu_1,\ \text{A}\Sigma\text{A}^T = [\text{I}_m \quad \text{O}_{mq}]\begin{bmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{bmatrix}\begin{bmatrix}\text{I}_m\\ \text{O}_{mq}\end{bmatrix} = \Sigma_{11})$

<MGF 유도>

$M_{\mathbf{X}}(t) = \exp(\mathbf{t}^T\boldsymbol{\mu})\,M_{\mathbf{Z}}\{(\boldsymbol{\Sigma}^{\frac{1}{2}})^T\mathbf{t}\}$

$= \exp(\mathbf{t}^T\boldsymbol{\mu})\exp\{(1/2)[(\boldsymbol{\Sigma}^{1/2})^T\mathbf{t}]^{\boldsymbol{T}}[(\boldsymbol{\Sigma}^{1/2})^T\mathbf{t}]\}$

$= \exp(\mathbf{t}^T\boldsymbol{\mu})\exp[(1/2)\mathbf{t}^T(\boldsymbol{\Sigma}^{1/2})^T(\boldsymbol{\Sigma}^{1/2})\,\mathbf{t}] = e^{\mathbf{t}^T\boldsymbol{\mu} + \frac{1}{2}\mathbf{t}^T(\boldsymbol{\Sigma})\mathbf{t}}$

2. 주변분포 독립성: $\mathbf{X_1}, \mathbf{X_2}$ 독립 ⇔ $\boldsymbol{\Sigma_{12}} = \boldsymbol{\Sigma_{21}} = \mathbf{0}$

pf) $M_{\mathbf{X_1},\mathbf{X_2}}(\mathbf{t_1}, \mathbf{t_2}) = \exp\left\{[\mathbf{t_1} \quad \mathbf{t_2}]\begin{bmatrix}\boldsymbol{\mu_1}\\ \boldsymbol{\mu_2}\end{bmatrix} + \frac{1}{2}[\mathbf{t_1} \quad \mathbf{t_2}]\begin{bmatrix}\boldsymbol{\Sigma_{11}} & \boldsymbol{\Sigma_{12}}\\ \boldsymbol{\Sigma_{21}} & \boldsymbol{\Sigma_{22}}\end{bmatrix}\begin{bmatrix}\mathbf{t_1}\\ \mathbf{t_2}\end{bmatrix}\right\}$

$M_{\mathbf{X_1}}(\mathbf{t_1})M_{\mathbf{X_2}}(\mathbf{t_2}) = \exp\{\mathbf{t_1}\boldsymbol{\mu_1} + \mathbf{t_2}\boldsymbol{\mu_2} + \dfrac{1}{2}(\mathbf{t_1^T}\boldsymbol{\Sigma_{11}}\mathbf{t_1} + \mathbf{t_2^T}\boldsymbol{\Sigma_{22}}\mathbf{t_2})\}$ $\therefore M_{\mathbf{X_1},\mathbf{X_2}}(\mathbf{t_1}, \mathbf{t_2}) = M_{\mathbf{X_1}}(\mathbf{t_1})M_{\mathbf{X_2}}(\mathbf{t_2}) \leftrightarrow \boldsymbol{\Sigma_{12}} = \boldsymbol{\Sigma_{21}} = \mathbf{0}$

3. 조건부 분포: $\text{X}_1|\text{X}_2 \sim N_m(\boldsymbol{\mu_1} + \boldsymbol{\Sigma_{12}}\boldsymbol{\Sigma_{22}^{-1}}(\mathbf{X_2} - \boldsymbol{\mu_2}),\ \boldsymbol{\Sigma_{11}} - \boldsymbol{\Sigma_{12}}\boldsymbol{\Sigma_{22}^{-1}}\boldsymbol{\Sigma_{21}})$ ($\boldsymbol{\Sigma}$는 양정치)

pf) $\mathbf{W} = \mathbf{X_1} - \boldsymbol{\Sigma_{12}}\boldsymbol{\Sigma_{22}^{-1}}\mathbf{X_2} \ \rightarrow \ \begin{bmatrix}\mathbf{W}\\ \mathbf{X_2}\end{bmatrix} = \begin{bmatrix}\mathbf{I_m} & -\boldsymbol{\Sigma_{12}}\boldsymbol{\Sigma_{22}^{-1}}\\ \mathbf{O_{qm}} & \mathbf{I}_{\boldsymbol{q}}\end{bmatrix}\begin{bmatrix}\mathbf{X_1}\\ \mathbf{X_2}\end{bmatrix}$ $\left(\mathbf{A} = \begin{bmatrix}\mathbf{I_m} & -\boldsymbol{\Sigma_{12}}\boldsymbol{\Sigma_{22}^{-1}}\\ \mathbf{O_{qm}} & \mathbf{I}_{\boldsymbol{q}}\end{bmatrix}\right)$

$\begin{bmatrix}\mathbf{W}\\ \mathbf{X_2}\end{bmatrix} \sim N_p(\mathbf{A}\boldsymbol{\mu}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A^T})$; $\mathbf{A}\boldsymbol{\Sigma}\mathbf{A^T} = \begin{bmatrix}\boldsymbol{\Sigma_{11}} - \boldsymbol{\Sigma_{12}}\boldsymbol{\Sigma_{22}^{-1}}\boldsymbol{\Sigma_{21}} & \mathbf{O_{mq}}\\ \mathbf{O_{qm}} & \boldsymbol{\Sigma_{22}}\end{bmatrix}$ ➔ **W, $\mathbf{X_2}$ 독립**

$\mathbf{W}|\mathbf{X_2} = \mathbf{W} \sim N_m(\boldsymbol{\mu_1} - \boldsymbol{\Sigma_{12}}\boldsymbol{\Sigma_{22}^{-1}}\boldsymbol{\mu_1}, \boldsymbol{\Sigma_{11}} - \boldsymbol{\Sigma_{12}}\boldsymbol{\Sigma_{22}^{-1}}\boldsymbol{\Sigma_{21}}) \rightarrow \therefore \boldsymbol{X_1}|\boldsymbol{X_2} \sim N_m(\boldsymbol{\mu_1} + \boldsymbol{\Sigma_{12}}\boldsymbol{\Sigma_{22}^{-1}}(\mathbf{X_2} - \boldsymbol{\mu_2}),\ \boldsymbol{\Sigma_{11}} - \boldsymbol{\Sigma_{12}}\boldsymbol{\Sigma_{22}^{-1}}\boldsymbol{\Sigma_{21}})$

4. 카이 제곱: $W = (\mathbf{X} - \boldsymbol{\mu})^T(\boldsymbol{\Sigma}^{-1})(\mathbf{X} - \boldsymbol{\mu}) = \mathbf{Z}^T\mathbf{Z} \sim \chi^2(p)$ ($\boldsymbol{\Sigma}$는 양정치)

pf) $W = \mathbf{Z}^T\mathbf{Z} = \sum_{i=1}^p Z_i^2 \sim \chi^2(p)$ $*$ 가법성 of $\chi 2$ using MGF; for iid $\text{Z} \sim \text{N}(0, 1^2)$ ➔ $\sum_{i=1}^p[(X_i - \mu_i)/\sigma_i]^2 \sim \chi^2(p)$

*** Bivariate normal distribution (이변량 정규 분포)**

1) **기댓값**: $\boldsymbol{\mu} = \begin{bmatrix}\boldsymbol{\mu_1}\\ \boldsymbol{\mu_2}\end{bmatrix}$, $\boldsymbol{\Sigma} = \begin{bmatrix}\boldsymbol{\sigma_1^2} & \boldsymbol{\sigma_{12}}\\ \boldsymbol{\sigma_{12}} & \boldsymbol{\sigma_2^2}\end{bmatrix}$, $\boldsymbol{\sigma_{12}} = \boldsymbol{\rho}\,\boldsymbol{\sigma_1}\boldsymbol{\sigma_2}$

2) PDF: $f(x, y) = \dfrac{1}{2\pi\sigma_1\sigma_2\sqrt{1 - \rho^2}}\exp\left\{-\dfrac{1}{2(1 - \rho^2)}\left[\left(\dfrac{x - \mu_1}{\sigma_1}\right)^2 + \left(\dfrac{y - \mu_2}{\sigma_2}\right)^2 - 2\rho\left(\dfrac{x - \mu_1}{\sigma_1}\right)\left(\dfrac{y - \mu_2}{\sigma_2}\right)\right]\right\}$

3) 조건부 분포: $Y|X \sim N[\mu_2 + \rho\dfrac{\sigma_2}{\sigma_1}(x - \mu_1), \sigma_2^2(1 - \rho^2)]$

PCA 기본

$\mathbf{Y} = \boldsymbol{\Gamma}\mathbf{X} = (\mathbf{PC_1}, \mathbf{PC_2}, \cdots, \mathbf{PC}_{\boldsymbol{p}})^T$ ➔ $\mathbf{PC_1} = \mathbf{v_1^T}\mathbf{X}$ ($\mathbf{v_1}$: $\text{Cov}(\mathbf{X}) = \boldsymbol{\Sigma}$의 λ_1 대응 고유벡터)

pf)$\mathbf{Y} \sim N_p(\boldsymbol{\Gamma\mu}, \boldsymbol{\Gamma\Sigma\Gamma}^T) = N_p(\boldsymbol{\Gamma\mu}, \boldsymbol{\Lambda})$ → $\text{TV}(\mathbf{X}) = \sum\sigma_i^2 = tr(\boldsymbol{\Sigma}) = tr(\boldsymbol{\Lambda}) = \sum\lambda_i = \text{TV}(\mathbf{Y})$

어떤 $\|\mathbf{a}\|^2 = 1$, $\mathbf{a} = \sum_{j=1}^p a_j\mathbf{v}_j$ *에 대해* $\mathbf{a}^T\mathbf{v}_1 = a_i$

→ $\text{Var}(\mathbf{a}^T\mathbf{X}) = \text{Cov}(\mathbf{a}^T\mathbf{X}) = \mathbf{a}^T\Sigma\mathbf{a} = (\boldsymbol{\Gamma}\mathbf{a})^T\boldsymbol{\Lambda}(\boldsymbol{\Gamma}\mathbf{a}) = \sum_i^p\lambda_i(\mathbf{a}^T\mathbf{v}_i)^2 = \sum_i^p\lambda_i a_i^2 \leq \lambda_1 = \text{Var}(Y_1)$, 등호조건: $\mathbf{a} = \mathbf{v}_1$

$\therefore Y_1 = \mathbf{v_1^T}\mathbf{X}$ (고유벡터 v_1으로 총 데이터 X 사영): 총분산 $\sum\lambda_i$ 중 최대 분산 λ_1 을 설명하는 $\mathbf{PC_1}$

→ $\mathbf{X} = \boldsymbol{\Gamma}^{\mathbf{T}}\mathbf{Y}$ 에서 $X_k = (v_{1k})\mathbf{PC_1} + (v_{2k})\mathbf{PC_2} + \cdots$ (각 v_{ik}는 X_k의 PC_i에 대한 PC score)

3-4. 주요 분포: t-분포, F-분포

t-분포

Joint PDF: $h(w,v)=\left\{\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}w^2\right)\right\}\left\{\frac{1}{\Gamma\left(\frac{r}{2}\right)2^{\left(\frac{r}{2}\right)}}v^{\left(\frac{r}{2}\right)-1}e^{-\frac{v}{2}}\right\}$ $w\in(-\infty,\infty), v\in(0,\infty)$; 서로 독립

$T=\frac{W}{\sqrt{V/r}}$ 일때 $t=\frac{w}{\sqrt{v/r}}, u=v$로 변환 시 $|J|=\frac{\sqrt{u}}{\sqrt{r}}$ $\rightarrow g(t,u)$ 에서 T의 marginal PDF $g_1(w)=\int_0^\infty g(t,u)du$

$T=\frac{\boldsymbol{W}}{\sqrt{\boldsymbol{V}/\boldsymbol{r}}}$ $\leftarrow W\sim N(0,1^2), V\sim\chi^2(r)$; 서로 독립

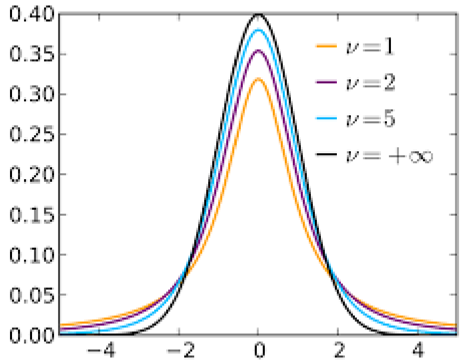
1. PDF: $f(x)=\frac{\Gamma\left(\frac{r+1}{2}\right)}{\sqrt{r\pi}\,\Gamma\left(\frac{r}{2}\right)}\left(1+\frac{x^2}{r}\right)^{-\left(\frac{r+1}{2}\right)}$, $x\in\mathbb{R}$

2. 기대값: $E(T)=0\ (r>1)$, $\mathrm{Var}(T)=\frac{r}{r-2}\ (r>2)$

$*\ E(T^k)=E\left[W^k\left(\frac{V}{r}\right)^{-k/2}\right]=E(W^k)E\left[\left(\frac{V}{r}\right)^{-k/2}\right]=\frac{1}{r^{-\frac{k}{2}}}E(W^k)E(V^{-k/2})$

$=E(W^k)\left(\frac{r}{2}\right)^{\frac{k}{2}}\frac{\Gamma\left(\frac{r}{2}-\frac{k}{2}\right)}{\Gamma\left(\frac{r}{2}\right)}$, $r>k$

(Cauchy 분포: df = 1인 t 분포, $f(x)=\frac{1}{\pi(1+x^2)}$)



3. Student's theorem *iid $X_i\sim N(\mu,\sigma^2)$ $\boldsymbol{n}$개 **[직관: $\boldsymbol{\sigma^2}$를 모르나, $\boldsymbol{X_i\sim N(\mu,\sigma^2)}$ ➔ $\boldsymbol{\bar{X}}$, $\boldsymbol{S}$로 $\boldsymbol{\mu}$ 정확 추정]**

1) $\frac{(n-1)S^2}{\sigma^2}\sim\chi^2(n-1)$, 2) $T=\frac{\bar{X}-\mu}{S/\sqrt{n}}\sim t\,(n-1)$ $\leftarrow T=\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right)\Big/\sqrt{\frac{(n-1)S^2}{\sigma^2}/(n-1)}$

pf) $V=\sum_{i=1}^n\left(\frac{X_i-\mu}{\sigma}\right)^2=\sum_{i=1}^n\left(\frac{(X_i-\bar{X})+(\bar{X}-\mu)}{\sigma}\right)^2=\sum_{i=1}^n\left(\frac{X_i-\bar{X}}{\sigma}\right)^2+\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right)^2=\frac{(n-1)S^2}{\sigma^2}+\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right)^2$

$\sum_{i=1}^n\left(\frac{X_i-\mu}{\sigma}\right)^2\sim\chi^2(n)$, $\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right)^2\sim\chi^2(1)$ $\rightarrow\bar{X}, S^2$ 독립; 양변 mgf 취하면 $(1-2t)^{-\frac{n-1}{2}}=E\left(\exp\left\{\frac{(n-1)S^2}{\sigma^2}\right\}t\right)$

* R codes 1) dt(x,r): f(X=x) 2) pt(x,r): P(X≤x) (자유도 r) // 3) qt(0.975,r): 97.5%인 t값

F-분포

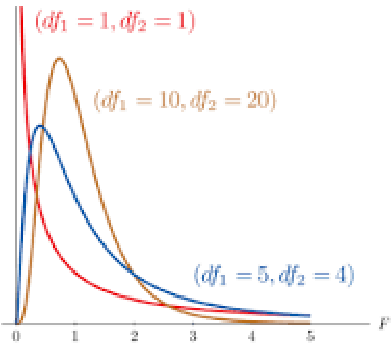
$F=\frac{U/r_1}{V/r_2}$ $\leftarrow U\sim\chi^2(r_1), V\sim\chi^2(r_2)$; 서로 독립

1. PDF: $f(x)=\frac{1}{B\left(\frac{r_1}{2},\frac{r_2}{2}\right)}\left(\frac{r_1}{r_2}\right)^{\frac{r_1}{2}}\left(1+\frac{r_1}{r_2}x\right)^{-\frac{r_1+r_2}{2}}x^{\frac{r_1}{2}-1}$ $0<x<\infty$

2. 기댓값: $E(F)=\frac{r_2}{r_2-2}$ $(r_2>2)$

$*\ E(F^k)=E\left[\left(\frac{U/r_1}{V/r_2}\right)^k\right]=\left(\frac{r_2}{r_1}\right)^kE(U^k)E(V^{-k})$

$=\left(\frac{r_2}{r_1}\right)^k\frac{\Gamma\left(\frac{r_1}{2}+k\right)}{\Gamma\left(\frac{r_1}{2}\right)}\frac{\Gamma\left(\frac{r_2}{2}-k\right)}{\Gamma\left(\frac{r_2}{2}\right)}$ $(r_2>2k)$



* R codes 1) df(x,a,b): f(X=x) 2) pf(x,a,b): P(X≤x) (자유도 a,b) // 3) qf(0.975,a,b): 97.5%인 F값

3-5. 혼합 분포

혼합 분포

1. 혼합분포: $f(x)=\sum_{i=1}^k p_i f_i(x)$ (각각 받침 S_i, 평균 μ_i, 분산 σ_i^2 , $\sum_{j=1}^k p_j=1$)

1) $E(X)=\sum_{i=1}^k p_i\int_{-\infty}^{\infty}xf_i(x)dx=\sum_{i=1}^k p_i\mu_i=\bar{\mu}$

2) $Var(X)=\sum_{i=1}^k p_i\sigma_i^2+\sum_{i=1}^k p_i(\mu_i-\bar{\mu})^2$

$*\ Var(X)=\sum_{i=1}^k p_i\int_{-\infty}^{\infty}(x-\bar{\mu})^2f_i(x)dx=\sum_{i=1}^k p_i\int_{-\infty}^{\infty}(x-\mu_i)^2f_i(x)dx+\sum_{i=1}^k p_i(\mu_i-\bar{\mu})^2\int_{-\infty}^{\infty}f_i(x)dx$

2. 응용: 유한 개로 제한할 필요 X

1) **감마(θ,α,β)+푸아송(x,θ) ➔ 음이항 분포**

$p(x)=\int_0^\infty\left[\frac{1}{\Gamma(\alpha)\beta^\alpha}\theta^{\alpha-1}e^{-\theta/\beta}\right]\left[\frac{e^{-\theta}\theta^x}{x!}\right]d\theta=\frac{1}{\Gamma(\alpha)\beta^\alpha x!}\int_0^\infty\theta^{\alpha+x-1}e^{-\theta(\alpha+\beta)/\beta}d\theta=\frac{\Gamma(\alpha+x)}{\Gamma(\alpha)x!}\frac{\beta^x}{(1+\beta)^{\alpha+x}}$

$\rightarrow p(x)=\frac{(r+x-1)!}{(r-1)!}\frac{p^r(1-p)^x}{x!}$, $\alpha=r\in\mathbb{Z}^+$, $\beta=\frac{1-p}{p}$

2) **베이지안 추론**: $h(x)=\int_\theta g(\theta)f(x|\theta)d\theta$; $g(\theta)$: Conjugate prior, $h(x)$: 무조건부

① $X|\theta\sim N(0,\ 1/\theta)$, $\theta\sim\Gamma(r/2, 2/r)\rightarrow X\sim t(r)$ ② 이항분포 (p모름)→베타분포 β(p)로 추출 $\int_0^1 p(x|p)g(p)dp$

4. 통계적 추론 – MLE / 신뢰구간 / 가설검정

표본 / 통계량

* 표본 → 1) 분포 f(x), p(x)의 추론 // 2) θ 추론 ← f(x), p(x)는 알고 있음 (X_i: 확률변수, x_i: 실현값)

1. 확률 표본 (Random sample): $\text{iid}\,[X_1,\cdots,X_n]$
2. 통계량 (Statistic): $T = T(X_1,\cdots,X_n)$ (표본에 대한 함수)

   ➔ **$\boldsymbol{\theta} \in \Omega$에 대한 추정량**이면 T: 점추정량 (point estimator), 실현값 t: 점추정값 (point estimate)
3. **불편추정량 (Unbiased estimator)**: $E(T) = \theta$ $\quad [E(\bar{X}) = \mu,\ \ E(S^2) = \sigma^2]$
4. **Maximum likelihood estimator (mle)**
   1) **가능도 함수:** $L(\theta) = \prod_{i=1}^{n} f(x_i;\theta)$ ← **MLE:** $\hat{\theta} = \text{Argmax}[L(\theta)]$
   2) **로그우도 함수:** $l(\theta) = \sum_{i=1}^{n} \ln f(x_i,\theta)$ ← **MLE:** $\hat{\theta} = \text{Argmax}[l(\theta)] \leftarrow \partial l/\partial\theta = 0$

지수	$l(\beta) = \sum_{i=1}^{n} \ln \frac{1}{\beta} e^{-\frac{x_i}{\beta}} = -\frac{1}{\beta}\sum x_i - n\ln\beta = -\text{n}\left(\frac{1}{\beta}\bar{X} + \ln\beta\right) \rightarrow \frac{\partial l}{\partial \beta} = \text{n}\left(\frac{\bar{\text{X}}}{\beta^2} - \frac{1}{\beta}\right) \rightarrow \hat{\beta} = \bar{X}$ (also 불편)
이항	$l(p) = \sum_{i=1}^{n} \ln p^{x_i}(1-p)^{1-x_i} = n\bar{X}\ln p + (n - n\bar{X})\ln(1-p) \rightarrow \frac{\partial l}{\partial p} = n\left(\frac{\bar{\text{X}}}{p} - \frac{1-\bar{X}}{1-p}\right) \rightarrow \hat{p} = \bar{X}$ (also 불편)
정규	$l(\mu,\sigma) = -\frac{n}{2}\ln 2\pi - n\ln\sigma - \frac{1}{2}\sum\left(\frac{x_i-\mu}{\sigma}\right)^2 \rightarrow \nabla l(\mu,\sigma) = \left[\frac{1}{\sigma}\sum\left(\frac{x_i-\mu}{\sigma}\right), -\frac{n}{\sigma} + \frac{1}{\sigma^3}\sum(x_i-\mu)^2\right]^T$ $\rightarrow \hat{\mu} = \bar{X},\ \ \hat{\sigma}^2 = \frac{1}{n}\sum_{\text{i}=1}^{\text{n}}(X_i - \bar{X})^2 = \frac{n-1}{n}S^2$

CLT

1) Pivot 확률변수: (추정량-모수)/표준오차
2) 중심극한정리: $Z_n = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \xrightarrow{D} N(0,1) \ \leftrightarrow\ $ 근사적으로 $N(0,1)$에 수렴

신뢰 구간

*신뢰구간: **모수 $\boldsymbol{\theta}$가 추정량 $\hat{\boldsymbol{\theta}}$에서** 얼마나 벗어났는가?

1. 신뢰구간: $1-\alpha = P_\theta[\theta \in (\hat{\theta}_L, \hat{\theta}_U)]$ → 100(1-α)% 신뢰구간 (같은 신뢰계수 → 구간 길이 최소화)

*해석: **모수 $\boldsymbol{\theta}$가 추정량 $(\hat{\boldsymbol{\theta}}_L, \hat{\boldsymbol{\theta}}_U)$ 구간에 있는 사건 $\sim\boldsymbol{B(1, 1-\alpha)}$ (95% CI: $\boldsymbol{\theta}$가 $(\hat{\boldsymbol{\theta}}_L, \hat{\boldsymbol{\theta}}_U)$에 평균 19회/20회)**

2. 평균 신뢰 구간 ($\boldsymbol{z}_{\alpha/2}$: 상위 $\alpha/2$에서 z값) ⇔ $(z_{\alpha/2} = \xi_{1-\alpha/2} \leftrightarrow F(z_{\alpha/2}) = F(\xi_{1-\alpha/2}) = 1-\alpha/2)$

상황	가정	Pivot statistic	$\boldsymbol{\mu}$의 100(1-α)% CI
대표본 (근사;CLT)	평균 μ 분산 σ^2	$Z = \frac{\bar{X}-\mu}{S/\sqrt{n}} \sim N(0,1)$	$1-\alpha \approx P\left(-\boldsymbol{z}_{\alpha/2} < \frac{\bar{X}-\mu}{S/\sqrt{n}} < \boldsymbol{z}_{\alpha/2}\right)$
t-구간 정규성 (정확)	$X_i \sim N(\mu,\sigma^2)$	$T = \frac{\bar{X}-\mu}{S/\sqrt{n}}\ \ (df = n-1)$	$1-\alpha = P\left(-\boldsymbol{t}_{\alpha/2,n-1} < \frac{\bar{X}-\mu}{S/\sqrt{n}} < \boldsymbol{t}_{\alpha/2,n-1}\right)$

3. 평균 차이 $(\bar{\boldsymbol{X}} - \bar{\boldsymbol{Y}})$ 신뢰 구간 $\ast\, E(\bar{X}-\bar{Y}) = \mu_1 - \mu_2,\ \text{Var}(\bar{X}-\bar{Y}) = (\sigma_1^2/n_1) + (\sigma_2^2/n_2)$

상황	가정	Pivot statistic	유도/비고
대표본 (근사;CLT)	평균 $\mu_1 - \mu_2$ 분산 $(\sigma_1^2/n_1) + (\sigma_2^2/n_2)$	$\boldsymbol{Z} = \frac{(\bar{\boldsymbol{X}}-\bar{\boldsymbol{Y}}) - (\boldsymbol{\mu_1}-\boldsymbol{\mu_2})}{\sqrt{(\boldsymbol{S_1^2}/\boldsymbol{n_1}) + (\boldsymbol{S_2^2}/\boldsymbol{n_2})}} \sim N(0,1)$	
t-통계량 정규성 (등분산)	$X_i \sim N(\mu_1, \boldsymbol{\sigma^2})$ $Y_i \sim N(\mu_2, \boldsymbol{\sigma^2})$	$\boldsymbol{T} = \frac{(\bar{\boldsymbol{X}}-\bar{\boldsymbol{Y}}) - (\boldsymbol{\mu_1}-\boldsymbol{\mu_2})}{\boldsymbol{S_p}\sqrt{(\boldsymbol{1}/\boldsymbol{n_1}) + (\boldsymbol{1}/\boldsymbol{n_2})}}$ $(df = n_1 + n_2 - 2)$ $S_p^2 = \frac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{(n_1-1)+(n_2-1)}$	1) $\text{E}(S_p^2) = \sigma^2$ 2) $(\text{n}_1 - 1)S_1^2/\sigma^2 \sim \chi^2(n_1-1)$ $\rightarrow (n_1+n_2-2)S_p^2 \sim \chi^2(n_1+n_2-2)$ 3) $S_p^2 \leftrightarrow (\bar{X}-\bar{Y})$ 독립
t-통계량 정규성 (이분산)	$X_i \sim N(\mu_1, \boldsymbol{\sigma_1^2})$ $Y_i \sim N(\mu_2, \boldsymbol{\sigma_2^2})$	$\boldsymbol{T} = \frac{(\bar{\boldsymbol{X}}-\bar{\boldsymbol{Y}}) - (\boldsymbol{\mu_1}-\boldsymbol{\mu_2})}{\sqrt{(\boldsymbol{S_1^2}/\boldsymbol{n_1}) + (\boldsymbol{S_2^2}/\boldsymbol{n_2})}}$ **(Welch's, df<-round(df))**	$df = \frac{\left(\frac{s_x^2}{n_x} + \frac{s_y^2}{n_y}\right)^2}{\frac{\left(\frac{s_x^2}{n_x}\right)^2}{n_x-1} + \frac{\left(\frac{s_y^2}{n_y}\right)^2}{n_y-1}}$

4. 통계적 추론 – MLE / 신뢰구간 / 가설검정

신뢰 구간

4. 비율 차이 (극한 표준정규분포; CLT)

1) 가정: $X \sim b(1, p_1),\ Y \sim b(1, p_2) \rightarrow\ \hat{p}_1 = \bar{X},\ \hat{p}_2 = \bar{Y}$ $\quad\quad \therefore E(\hat{p}_1) = p_1,\ \mathrm{Var}(\hat{p}_1) = p_1(1-p_1)/n_1$

상황	가정	Pivot statistic
대표본 (근사;CLT)	평균 $p_1 - p_2$ 분산 $\dfrac{p_1(1-p_1)}{n_1} + \dfrac{p_2(1-p_2)}{n_2}$	$\boldsymbol{Z} = \dfrac{(\widehat{\boldsymbol{p}}_1 - \widehat{\boldsymbol{p}}_2) - (\boldsymbol{p}_1 - \boldsymbol{p}_2)}{\sqrt{\dfrac{\widehat{\boldsymbol{p}}_1(1-\widehat{\boldsymbol{p}}_1)}{\boldsymbol{n}_1} + \dfrac{\widehat{\boldsymbol{p}}_2(1-\widehat{\boldsymbol{p}}_2)}{\boldsymbol{n}_2}}} \sim N(0,1)$

5. 이산형 모수

1) $F_T(T;\theta)$: 통계량 T의 cdf; θ에 대해 단조 감소 → 신뢰 구간: $F_T(T_{n-1};\ \underline{\theta}) = 1-\alpha_2,\ F_T(T_n; \bar{\theta}) = \alpha_1$

2) **Bisection algorithm**: 순감소 $g(x) = d \in g([a,\ b])\ \rightarrow\ $ 1) if $g\{(a+b)/2\} > d\ \rightarrow$ 구간 $[(a+b)/2,\ b]$ 재설정

$\rightarrow$ 2) if $g\{(a+b)/2\} < d\ \rightarrow$ 구간 $[a,\ (a+b)/2]$ 재설정

상황	조건	유도
Binomial	$X \sim b(1,p)$ $n = 30, \bar{x} = 0.60$ $T = n\bar{X} \sim b(30, p)$ $(T_{n-1} = 17,\quad T_n = 18)$	① **하한:** pbinom(17, 30, 0.4)=0.9787, pbinom(17,30,0.45)=0.9286 ➔ pbinom(17, 30, **0.434**) ≈ 0.95 ② **상한:** pbinom(18, 30, 0.7)=0.1593, pbinom(18,30,0.8)=0.0094 ➔ pbinom(18, 30, **0.747**) ≈ 0.05 $\therefore \boldsymbol{p}$의 90% CI: $[\mathbf{0.434}, \mathbf{0.747}]$
Poisson	$X \sim \mathrm{Poi}(\mu)$ $\mathrm{n} = 25, \bar{x} = 5$ $T = n\bar{X} \sim \mathrm{Poi}(25\mu)$ $(T_{n-1} = 124,\quad T_n = 125)$	① **하한:** ppois(124, 25 x 4)=0.9912, ppois(124, 25 x 4.4)=0.9145 ➔ ppois(124, 25 x **4.287**) ≈ 0.95 ② **상한:** ppois(125, 25 x 5.5)=0.1330, ppois(125, 25 x 6)=0.0204 ➔ ppois(125, 25 x **5.8**) ≈ 0.05 $\therefore \boldsymbol{\mu}$의 90% CI: $[\mathbf{4.287}, \mathbf{5.8}]$

순서 통계량

*정의: $(Y_1 < \cdots < Y_n) \leftarrow [X_1, \cdots, X_n]$ 재배열 $\quad\quad$ *강점: 분포에 종속되지 않음.

1. **PDF**: $g(y_1, \cdots, y_n) = \boldsymbol{n}!\, f(y_1) \cdots f(y_n)$ $\quad$ (on $a < y_1 < \cdots < y_n < b$) $\quad\quad$ pf) $g(y_1, \cdots, y_n) = \sum_{i=1}^{n!} |J_i| f(y_1) \cdots f(y_n)$

2. **Marginal PDF** $\quad$ 1) $\boldsymbol{g_k(y_k)} = \dfrac{\boldsymbol{n}!}{(\boldsymbol{k}-\mathbf{1})!\,(\boldsymbol{n}-\boldsymbol{k})!\,(\mathbf{1})!}[\boldsymbol{F(y_k)}]^{\boldsymbol{k}-\mathbf{1}}[\mathbf{1}-\boldsymbol{F(y_k)}]^{\boldsymbol{n}-\boldsymbol{k}}\boldsymbol{f(y_k)}$

$$\text{pf)}\ \ g_k(y_k) = \int_a^{y_k} \cdots \int_a^{y_2} \int_{y_k}^{b} \cdots \int_{y_{n-1}}^{b} \boldsymbol{n}!\, f(y_1) \cdots f(y_n)\, dy_n \cdots dy_{k+1} dy_1 \cdots dy_{k-1} \quad (y_n \rightarrow y_{k+1}\,; y_1 \rightarrow y_{k-1})$$

2) $\boldsymbol{g_{ij}(y_i, y_j)} = \dfrac{\boldsymbol{n}!}{(\boldsymbol{i}-\mathbf{1})!\,(\boldsymbol{j}-\boldsymbol{i}-\mathbf{1})!\,(\boldsymbol{n}-\boldsymbol{j})!\,(\mathbf{1})!\,(\mathbf{1})!}[\boldsymbol{F(y_i)}]^{\boldsymbol{i}-\mathbf{1}}\left[\boldsymbol{F(y_j)} - \boldsymbol{F(y_i)}\right]^{\boldsymbol{j}-\boldsymbol{i}-\mathbf{1}}\left[\mathbf{1}-\boldsymbol{F(y_j)}\right]^{\boldsymbol{n}-\boldsymbol{j}}\boldsymbol{f(y_i)f(y_j)}$

3. **Quantile (분위수)**: cdf $F(\xi_p) = p \leftrightarrow \boldsymbol{\xi_p} = \boldsymbol{F^{-1}(p)}, \quad \boldsymbol{k} = \mathbf{floor}[\boldsymbol{p}(\boldsymbol{n}+\mathbf{1})]$

1) $\boldsymbol{F(Y_k)}$는 $\dfrac{\boldsymbol{k}}{\boldsymbol{n}+\mathbf{1}}$의 불편추정량: $E(F(Y_k)) = \displaystyle\int_a^b F(y_k) g_k(y_k) dy_k = \int_0^1 \frac{n!}{(k-1)!\,(n-k)!} z^k (1-z)^{n-k} dz = \frac{k}{n+1}$

2) Quartile: 1분위수 $(\boldsymbol{Q_1} = Y_{[0.25(n+1)]})$ ⇔ 중위수 $(\boldsymbol{Q_2} = Y_{[0.5(n+1)]})$ ⇔ 3분위수 $(\boldsymbol{Q_3} = Y_{[0.75(n+1)]})$

*중위수: 홀수→중간값 $Y_{(n+1)/2}$ / 짝수→ $(Y_{(n/2)} + Y_{(n/2)+1})/2$

➔ **Box plot**: $h = 1.5(Q_3 - Q_1),\ LF = Q_1 - h,\ UF = Q_3 + h$ $\quad$ (LF, UF 바깥: 이상값; 정규분포상 P≤0.007)

3) **Q-Q plot**: 표본의 순서통계량 $(Y_1, Y_2, \cdots, Y_{50})$ ⇔ 이론적 분위수 $(Z_{0.02}, Z_{0.04}, \cdots, Z_{1.00})$ ← any 분포

4) 신뢰구간: $1-\alpha = P(Y_i < \xi_p < Y_j) = \sum_{w=i}^{j-1} \binom{n}{w} p^w (1-p)^{n-w}$ $\quad \leftarrow \boldsymbol{p} = \boldsymbol{F(\xi_p)}$ $\quad$ (중위수: $p = 1/2$)

4. 통계적 추론 – MLE / 신뢰구간 / 가설검정

가설 검정

1) 가설 정의: $H_0: \theta \in \boldsymbol{\omega_0}$ (Null) vs. $H_1: \theta \in \boldsymbol{\omega_1}$ (alternative) ← $X \sim f(x;\theta)$에 대해 $\theta \in \Omega = (\omega_0 \cup \omega_1)$, 분할
2) 가설 검정: 표본 $(X_1, \cdots, X_n) \in C$ → H_1채택 (기각역 $C \subset D = \mathrm{span}\{(X_1, \cdots, X_n)\}$)
표본 $(X_1, \cdots, X_n) \notin C$ → H_0유지
3) 유의 수준: $\alpha = \max_{\theta \in \omega_0} P_\theta[(X_1, \cdots, X_n) \in C]$ (복합귀무가설에 대해 모든 null 모수 → 기각역에 속할 확률 최대)
* 1종 오류: H_0 참, but 기각 → H_1 채택 (=FP) ∴ 유의수준(α): 1종 오류 범할 최대 확률
4) 검정력: $\gamma(\theta) = P_\theta[(X_1, \cdots, X_n) \in C]$, $\theta \in \omega_1$ → 유의 수준과 다르게 대립가설 모수에 따라 달라짐
① 2종 오류: H_0 거짓, but 유지 → H_0 유지 (=FN) ∴ β: 2종 오류 범할 확률 (under given $\theta \in \boldsymbol{\omega_1}$)
② 검정력: H_0 거짓 → 알맞게 H_1 채택 (TP)
5) **P-값**: 1) **Upper tail**: **P-값**$= \boldsymbol{P_{H_0}(X \geq x_{obs}) = 1 - F_{H_0}(x_{obs})}$
2) **Lower tail**: **P-값**$= \boldsymbol{P_{H_0}(X \leq x_{obs}) = F_{H_0}(x_{obs})}$
3) **2-sided**: **P-값**$= \boldsymbol{2 \times P_{H_0}(X \geq |x_{obs}|) = 2[1 - F_{H_0}(|x_{obs}|)]}$ (X=0 좌우 대칭)
➔ $X = F^{-1}(U)$ (단조 증가) 정리의 역에 의해 P-값 ~ unif(0,1) under 귀무가설 H_0

예시	분포	가설	유도
단일 이항 단측	$X_i \sim B(1,p)$	$H_0: p = p_0$ $H_1: p < p_0$	*표본통계량: $\boldsymbol{S = \sum_{i=1}^n X_i \sim B(n,p)}$ 1) 기각역 설정: 귀무가설 하에서 $S \sim B(n,p_0) \to \alpha = P_{p_0}[S \leq k]$ ➔ $0.11 = P_{p_0}[S \leq 11]$ $(n = 20,\ p_0 = 0.7)$ 2) 검정력 함수: $\gamma(p) = P_p[S \leq 11]$ (**단조 감소 of p**) ∴ **<u>$\boldsymbol{H_0: p \geq p_0}$로 확장</u>** ← $\max_{p \geq p_0} P_p[S \leq k] = P_{p_0}[S \leq k]$ (**단조성**)
	대표본에서 $\boldsymbol{\dfrac{\hat{p} - p_0}{\sqrt{\hat{p}(1-\hat{p})/n}} \approx \dfrac{\hat{p} - p_0}{\sqrt{p(1-p)/n}} \xrightarrow{D} N(0,1)}$		
대표본			* 표본통계량: $\boldsymbol{\dfrac{\bar{X} - \mu}{S/\sqrt{n}} \approx \dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}} \xrightarrow{D} N(0,1)}$
대표본 단측 (Upper)	X_i ~ **미지** 분포 1) 평균: μ 2) 분산: σ^2	$H_0: \mu = \mu_0$ $H_1: \mu > \mu_0$	1) 기각역 설정: $\alpha = P_{\mu_0}\left[\dfrac{\bar{X} - \mu_0}{S/\sqrt{n}} \geq z_\alpha\right] \approx 1 - \Phi(z_\alpha)$ 2) 검정력 함수: $\gamma(\mu) = P_\mu\left[\dfrac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \geq z_\alpha\right] = P_\mu\left[\dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}} \geq \dfrac{\mu_0 - \mu}{\sigma/\sqrt{n}} + z_\alpha\right]$ $\approx 1 - \Phi\left(\dfrac{\sqrt{n}(\mu_0 - \mu)}{\sigma} + z_\alpha\right) = \boldsymbol{\Phi\left(\dfrac{\sqrt{n}(\mu - \mu_0)}{\sigma} - z_\alpha\right)}$ (**단조 증가 of μ**) *** Power 증가: n↑, 효과크기 (μ-μ₀)↑, α↑ & σ↓**
대표본 단측 (Lower)		$H_0: \mu = \mu_0$ $H_1: \mu < \mu_0$	1) 기각역 설정: $\alpha = P_{\mu_0}\left[\dfrac{\bar{X} - \mu_0}{S/\sqrt{n}} \leq -z_\alpha\right] \approx \Phi(z_\alpha)$ 2) 검정력 함수: $\gamma(\mu) = P_\mu\left[\dfrac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \leq -z_\alpha\right] = P_\mu\left[\dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}} \leq \dfrac{\mu_0 - \mu}{\sigma/\sqrt{n}} - z_\alpha\right]$ $\approx \boldsymbol{\Phi\left(\dfrac{\sqrt{n}(\mu_0 - \mu)}{\sigma} - z_\alpha\right) = \Phi\left(-\dfrac{\sqrt{n}(\mu - \mu_0)}{\sigma} - z_\alpha\right)}$ (**단조 감소 of μ**) *** Power 증가: n↑, 효과크기 (μ-μ₀)↑, α↑ & S↓**
대표본 양측		$H_0: \mu = \mu_0$ $H_1: \mu \neq \mu_0$	1) 기각역 설정: $\boldsymbol{\alpha} = P_{\mu_0}\left[\left\|\dfrac{\bar{X} - \mu_0}{S/\sqrt{n}}\right\| \geq z_{\alpha/2}\right]$ ⇐ (**양측 동일 배분**) 2) 검정력 함수: $\gamma(\mu) = P_\mu\left[\left\|\dfrac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}\right\| \geq z_{\alpha/2}\right]$ $\approx \boldsymbol{\Phi\left(\dfrac{\sqrt{n}(\mu - \mu_0)}{\sigma} - z_\alpha\right) + \Phi\left(-\dfrac{\sqrt{n}(\mu - \mu_0)}{\sigma} - z_\alpha\right)}$ (**U자 함수 of μ**) ➔ (μ₀에서 최소값)
t-검정 정규성	$X_i \sim N(\mu, \sigma^2)$	$H_0: \mu = \mu_0$ $H_1: \mu \neq \mu_0$	* 표본통계량: $\boldsymbol{T = \dfrac{\bar{X} - \mu_0}{S/\sqrt{n}} \sim t(n-1)}$ *** t-분포**는 N(0,1) 보다 누워 있음 ➔ "보수적" // 정규성 하 "정확"
2-표본 t-검정	$X_i \sim N(\mu_1, \boldsymbol{\sigma^2})$ $Y_i \sim N(\mu_2, \boldsymbol{\sigma^2})$ (정규,등분산)	$H_0: \mu_1 = \mu_2$ $H_1: \mu_1 \neq \mu_2$	* 표본통계량: $\boldsymbol{T = \dfrac{(\bar{X} - \bar{Y}) - 0}{S_p\sqrt{(1/n_1) + (1/n_2)}} \sim t(n_1 + n_2 - 2)}$ * $\boldsymbol{\|T\| \geq t_{0.025, n_1+n_2-2}}$ 이면 $\boldsymbol{H_0}$ **기각**

4. 통계적 추론 – MLE / 신뢰구간 / 가설검정

<table>
<tr><td rowspan="8">Pearson χ^2 검정</td><td colspan="3">자유도: n(확률표본)-n(미지수 or 제약)</td></tr>
<tr><td>2 cells</td><td colspan="2">1. 상황: $\boldsymbol{X_1 \sim b(n, p_1)}$, $X_2 = n - X_1$, $p_2 = 1 - p_1$ ⇒ $Y = \frac{X_1 - np_1}{\sqrt{np_1(1-p_1)}} \xrightarrow{D} N(0,1)$; $Q_1 = Y^2 \xrightarrow{D} \chi^2(1)$
2. 검정통계량: $\boldsymbol{Q_1} = \frac{(X_1-np_1)^2}{np_1(1-p_1)} = \frac{(X_1-np_1)^2}{np_1} + \frac{(X_1-np_1)^2}{n(1-p_1)} = \boldsymbol{\frac{(X_1-np_1)^2}{np_1} + \frac{(X_2-np_2)^2}{np_2} \xrightarrow{D} \chi^2(1)}$</td></tr>
<tr><td rowspan="3">k cells</td><td colspan="2">1. 상황: k항;n회 다항분포 ($p_k = 1 - \sum_{i=1}^{k-1} p_i$ & $x_k = n - \sum_{i=1}^{k-1} x_i$)
2. 검정통계량: $\boldsymbol{Q_{k-1} = \sum_{i=1}^{k} \frac{(X_i - np_i)^2}{np_i} \xrightarrow{D} \chi^2(k-1)}$ ⇐ $(k-1)$개 알면 나머지 1개 앎</td></tr>
<tr><td>적합도 검정</td><td>1) 귀무가설: $H_0: p_1 = p_{1,0}$, $p_2 = p_{2,0}$, ⋯ , $p_k = p_{k,0}$
2) 검정통계량: $Q_{k-1} = \sum_{i=1}^k \frac{(O_i - E_i)^2}{E_i} = \sum_{i=1}^k \frac{(X_i - np_{i,0})^2}{np_{i,0}} \xrightarrow{D} \boldsymbol{\chi^2(k-1)}$ (귀무가설 하)</td></tr>
<tr><td>최소 χ^2 추정량</td><td><예시> 정규분포 모수 추정 $N(\mu,\sigma^2)$
1) 상황: 실수구간 → k등분 $(A_1, \cdots, A_k)$; $\boldsymbol{p_i} = \int_{\boldsymbol{A_i}} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}(y-\boldsymbol{\mu})^2/\boldsymbol{\sigma}^2\right] dy$
2) 실제 A_i의 도수인 X_i → $Q_{k-1} = \sum_{i=1}^k \frac{(X_i - np_i)^2}{np_i} \xrightarrow{D} \boldsymbol{\chi^2(k-3)}$ 최소화하는 $\hat{\mu}, \hat{\sigma}^2$</td></tr>
<tr><td rowspan="2">r x c cells</td><td>동질성 검정</td><td>1) 상황: 2개의 k항 다항분포 *각 모수: $(n_1, p_{11}, p_{21}, \cdots, p_{k1}), (n_2, p_{12}, p_{22}, \cdots, p_{k2})$
⇒ $\sum_{j=1}^2 \sum_{i=1}^k \frac{(X_{ij} - n_j p_{ij})^2}{n_j p_{ij}} \xrightarrow{D} [\chi^2(k-1) + \chi^2(k-1)] = \chi^2(2k-2)$
2) 귀무가설: $H_0: p_{11} = p_{12}$, $p_{21}=p_{22}$, ⋯ , $p_{k1} = p_{k2}$ (둘은 구간 별 비율이 동일)
⇒ $p_{m1} = p_{m2}$의 MLE: $\frac{X_{m1}+X_{m2}}{n_{m1}+n_{m2}}$ (총 k − 1개 점추정값 필요)
3) 검정통계량: $\boldsymbol{\sum_{j=1}^2\sum_{i=1}^k \frac{\left[X_{ij} - n_j\left(\frac{X_{i1}+X_{i2}}{n_{i1}+n_{i2}}\right)\right]^2}{n_j\left(\frac{X_{i1}+X_{i2}}{n_{i1}+n_{i2}}\right)} \xrightarrow{D} \chi^2(k-1)}$ (귀무가설 하)</td></tr>
<tr><td>독립성 검정</td><td>1) 상황: 확률실험 n회 결과 → 가로 (A) a항 / 세로 (B) b항 두 종류 범주로 구분
➔ $p_{ij} = P(A_i \cap B_j)$, X_{ij}는 $A_i \cap B_j$ 도수
⇒ $Q_{ab-1} = \sum_{j=1}^b \sum_{i=1}^a \frac{(X_{ij}-np_{ij})^2}{np_{ij}} \xrightarrow{D} \chi^2(ab-1)$
2) 귀무가설: $H_0: P(A_i \cap B_j) = P(A_i)P(B_j)$ for all (i,j) (속성 A, B는 독립)
⇒ $p_{i*} = P(A_i)$의 MLE: $\hat{p}_{i*} = \frac{X_{i*}}{n} = \frac{\sum_{j=1}^b X_{ij}}{n}$ [총 (a − 1) + (b − 1)개 점추정값 필요]
3) 검정통계량: $\boldsymbol{\sum_{j=1}^b\sum_{i=1}^a \frac{\left[X_{ij} - n\left(\frac{X_{i*}}{n}\right)\left(\frac{X_{*j}}{n}\right)\right]^2}{n\left(\frac{X_{i*}}{n}\right)\left(\frac{X_{*j}}{n}\right)} \xrightarrow{D} \chi^2[(a-1)(b-1)]}$ (귀무가설 하)</td></tr>
<tr><td colspan="3">*비중심 카이제곱 분포: H_0 외의 카이제곱 분포 (검정력 계산)</td></tr>
</table>

4. 통계적 추론 – MLE / 신뢰구간 / 가설검정

Monte Carlo	* **몬테카를로 생성: 특정 "Known" 표본/분포 ➔ 관측값 생성** (Resampling, Bayesian 등에서 중요) **1. 균등분포 (Uniform distribution)**: $\mathrm{unif}(a,b)$; $\mathrm{pdf} = 1/(\mathrm{b}-\mathrm{a})$ ← R codes: runif(횟수)		
	unif(0,1)⇔CDF "관측치 생성"	$\boldsymbol{X = F^{-1}(U)}$는 **cdf** $\boldsymbol{F(X)}$ 따름⇔ 역: $Z = F(X) \sim \mathrm{unif}(0,1)$	
		지수	$F(x) = 1 - e^{-x/\beta}$, $(x > 0)$ $\therefore X = F^{-1}(U) = -\beta \ln(1-U)$ 는 지수분포 생성
		푸아송	$m = \lambda w$ → $T_i \sim \exp(1/\lambda)$ 에 대해 $[X = k] \Leftrightarrow \sum_{i=1}^{k} T_i \le w$ & $\sum_{i=1}^{k+1} T_i > w$ * 구간 w 동안 난수로 T_i 생성 → 횟수 카운트 (초기 $X = 0, T = 0$) 1) $\Delta T = -(1/\lambda)\ln(1-U)$ 2) $T \leftarrow T + \Delta T$ 3) if $\mathrm{T} \le \mathrm{w}$: $X \leftarrow X + 1$ elif $\mathrm{T} > \mathrm{w}$: return X
		정규 분포	<**Box & Muller** (1958)> → 일반화: Marsaglia & Bray (1964) $X_1 = (-2\ln Y_1)^{1/2} \cos(2\pi Y_2)$; $X_2 = (-2\ln Y_1)^{1/2} \sin(2\pi Y_2) \Leftarrow Y_1, Y_2 \sim \mathrm{unif}(0,1)$ $f(X_1, X_2) = \lvert J \rvert g(Y_1, Y_2) = \frac{1}{2\pi} \exp\left[-\frac{1}{2}(x_1^2 + x_2^2)\right]$
	채택-기각 (A-R) 알고리즘 (어려운 CDF)	$X = F^{-1}(U)$를 closed form 계산 불가. ⇐ $g(x)$ 이용: ①Easy ②$f(x)$ 유사 ③$\frac{f(x)}{g(x)} \le k$ (유계) ① $Y \sim g(y)$ & U 생성 ② $U \le \frac{f(Y)}{kg(Y)} \le 1$이면 $\mathrm{X} = \mathrm{Y}$, 아니면 ①로 돌아가 재 생성 ⇒ 조금 더 넓은 $kg(x)$로 근사 ($f(x) = cf_1(x)$와 $g(x) = dg_1(x)$적당히 상수배 하여 k 무시 가능)	
		감마 CDF $\boldsymbol{\Gamma(\alpha,\beta)}$	$Y_i \sim \Gamma(1,1)$ ➔ $X = \sum_{i=1}^{\alpha} Y_i \sim \Gamma(\alpha, 1)$ (α 정수: CDF 생성 쉬움) $X \sim \Gamma(\alpha, 1)$ ➔$\beta X \sim \Gamma(\alpha, \beta)$ (α 실수 → **문제!**)
			① $X \sim \Gamma(\alpha, 1)$ & $Y \sim \Gamma([\alpha], 1/b)$ ② $\frac{f(x)}{g(x)} = b^{-[\alpha]} x^{\alpha-[\alpha]} e^{-(1-b)x} \le b^{-[\alpha]} \left\{\frac{\alpha - [\alpha]}{(1-b)e}\right\}^{\alpha-[\alpha]}$ (by x로 미분) ③ 위 식을 b로 미분 ⇒ $\frac{f(x)}{g(x)} \le ([\alpha]/\alpha)^{-[\alpha]} \left\{\frac{\alpha - [\alpha]}{(1-[\alpha]/\alpha)e}\right\}^{\alpha-[\alpha]} = M$
		정규 CDF $\boldsymbol{N(0,1)}$	① **Y~Cauchy (역 CDF 알려짐) ➔ X~N(0,1)** ② $\frac{\phi(x)}{g(x)} = \frac{(2\pi)^{-1/2}\exp\{-x^2/2\}}{\{\pi(1+x^2)\}^{-1}} \Rightarrow \frac{\phi_1(x)}{g_1(x)} = (1+x^2)\exp\{-x^2/2\} \le 2e^{-1/2} = M$
	Monte Carlo t-검정 (오염된 정규)	$W \sim N(0,1^2)$ & $W \sim N(0, \sigma_c^2)$ $(\varepsilon: 0.25, \sigma_c = 25)$ ← $W = Z$ or $\sigma_c Z$; $E(W) = 0$	
		* 가설: $H_0: \mu = 0$, $H_1: \mu > 0$ 1) $\mathrm{n} = 20$, 2) $t_{0.05,19} = 1.729$	* 추정 알고리즘 (N: 시뮬레이션 수) 1) $\mathrm{n} = 20$ 표본 생성 ⇐ X (오염 정규; μ 모름) 분포 2) $T = (\bar{X} - \mu)/(S/\sqrt{n})$계산 ➔ 1),2) N번 반복 3) **유의수준 실험적 추정량:** $\boldsymbol{\hat{\alpha} = I/N}$ ($I: T > t_{0.05,10}$ 도수) $\mathbf{SE} = \sqrt{\boldsymbol{\hat{\alpha}(1-\hat{\alpha})/N}}$ 예시) $\hat{\alpha} = 0.0412 \pm 0.0039$
	Monte Carlo 적분	적분가능한 $g(x)$의 closed form 역도함수 (≈부정적분) 존재X ➔ **수치적 적분** $\int_a^b g(x)\,dx = (b-a)\int_a^b g(x)\left(\frac{1}{b-a}\right)dx = \boldsymbol{(b-a)E[g(X)]} \Leftarrow X \sim \mathrm{unif}(a,b)$ $\therefore \boldsymbol{\bar{Y}} = \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{Y_i} = \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{(b-a)g(X_i)}$ 는 정적분의 unbiased estimator ⇐ $X_i \sim \mathrm{unif}(a,b)$	

4. 통계적 추론 – MLE / 신뢰구간 / 가설검정

Boot-strap 기본

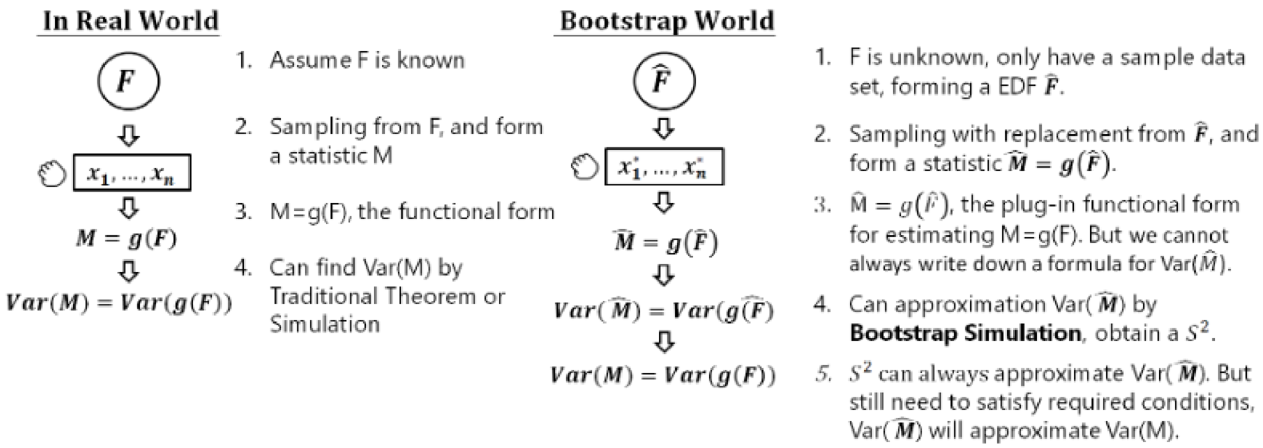
비교

1) 중심극한정리: 표본 통계량 ($\hat{\theta}$) 의 pivotal statistic이 극한 정규분포따름 ➔ 모수 θ 추정
2) 몬테카를로 기법: **X의 known 분포 (CDF)➔** 균등분포 난수추출기로 관측값 $X = F^{-1}(U)$ 생성
3) 부트스트랩: **X의 unknown 분포 ➔ 표본** $(X_1, \cdots X_n)$의 EDF ($\hat{F}_n$)→ 무작위 추출로 X_i^* 생성
$\hat{\theta}^*$의 분포 → $\hat{\theta}$의 신뢰구간 추정 → θ의 근사적 신뢰구간

원리

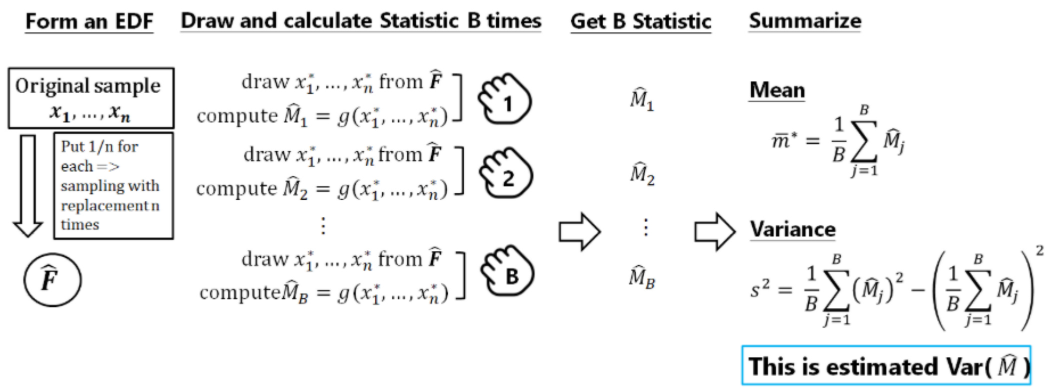
일반적인 통계적 추론에서는 estimator → parameter를 추정함.
Standard error는 estimator의 자체적인 변동성 (표준편차) (e.g. $SE(\bar{X}) = \sqrt{Var(\bar{X})} = \sigma/\sqrt{n}$)
Estimated SE는 SE에 unknown parameter가 들어가 있을 때, 다른 estimator를 이용 ($S/\sqrt{n}$)

*문제: ①일반적인 확률변수 Y에 대해 분포 (PDF, CDF)를 알기 어렵고
② 통계량 g(Y)의 S.E.를 $\sigma/\sqrt{n}$처럼 정확한 수식으로 알아낼 수 있는 경우는 많지 않음.



EDF $\hat{F}(x) = \frac{1}{n}\sum_{i=1}^{n} I(x_i \le x)$ ⇔ $\big($x보다 작은 표본내 실현값 수$\big)$ / n ⇔ PMF는 $\frac{1}{n}$ (for every x_i)

Statistical functional (통계적 범함수): 모수가 [분포함수]의 함수로 표현됨. (평균,분산,중위수,백분위수,etc)
e.g. $E(X) = \int xfdx = \int xdF$, $Var(X) = \int x^2 dF - \left(\int xdF\right)^2$
Plug-in principle: $\hat{\theta} = g(\hat{F}) = \int r(x)d\hat{F} \leftrightarrow \theta = g(F) = \int r(x)dF$ (전자는 후자의 plug-in estimate)



2.Variance of $\hat{M}$ with EDF $\hat{F}$

2. Approximation Error

$$\underbrace{s^2 = \frac{1}{B}\sum_{j=1}^{B}\left(\hat{M}_j\right)^2 - \left(\frac{1}{B}\sum_{j=1}^{B}\hat{M}_j\right)^2}_{\text{1.Bootstrap Variance Estimation}} \approx \mathbf{Var}(\hat{M};\hat{F}) \approx \underbrace{Var(M;F)}_{\text{3.Variance of M with true F}}$$

1.Simulation Error

1번 simulation error는 결국 큰 수의 법칙에 의해 확률 수렴하므로 B↑으로 최소화 가능
2번 approximation error는 $\hat{F}$이 F에 근사 (n↑) 하면 최소화 (n↑면 자연스럽게 $\hat{M} \xrightarrow{P} M$ 성질도...)

4. 통계적 추론 – MLE / 신뢰구간 / 가설검정

Boot-strap 응용	**모평균 추정**		$E(X_i^*) = \sum_{j=1}^{n} \frac{1}{n} X_j = \bar{X}$, $\mathrm{Var}(X_i^*) = \sum_{j=1}^{n} \frac{1}{n}\left(X_j - \bar{X}\right)^2 = \frac{n}{n-1} S^2$ $E(\bar{X}_j^*) = \bar{X}$, $\mathrm{Var}(\bar{X}_j^*) = \frac{S^2}{n-1}$ B회 시뮬레이션 평균 $\frac{1}{B}\sum_{i=1}^{B} \bar{X}_j^* \xrightarrow{P} E(\bar{X}_j^*) = \bar{X} \xrightarrow{P} \boldsymbol{\mu}$, 분산 $\frac{1}{B}\sum_{i=1}^{B} (\bar{X}_j^*)^2 - \left(\frac{1}{B}\sum_{i=1}^{B} \bar{X}_j^*\right)^2 \xrightarrow{P} \mathrm{Var}(\bar{X}_j^*) = \frac{S^2}{n-1} \xrightarrow{P} \boldsymbol{\frac{\sigma^2}{n}}$ ➔ B 회 부트스트랩 $\bar{X}$ 신뢰구간 (비모수적 counting) ≈ $\left[\bar{X} - z_{\alpha/2}\frac{S^2}{n},\ \bar{X} + z_{\alpha/2}\frac{S^2}{n}\right]$ ≈ [μ의 CLT 신뢰 구간] - 위의 정규가정을 통한 $z_{\alpha/2}$ 근사는 책 참고 4.9.1를 참조 - 다른 모수 추정도 크게 다르지 않음. ($\bar{X}$처럼 precise한 분산식이 존재하지 않으면 시뮬레이션 효과↑) - 통계량의 분포가 다른 모수에 종속되지 않게 pivot화하면 부트스트랩 정확성 향상 가능
	Boot-strap 검정	<1표본 평균> $H_0: \mu = \mu_0$ $H_1: \mu > \mu_0$	1. 상황: 1) 검정통계량: $\bar{X}$ 2) $\hat{p} = P_{H_0}[\bar{X} \geq \bar{x}]$ 2. H_0 가정 ⇒ $\boldsymbol{z_i = x_i - \bar{x} + \mu_0}$ $\left(E(z_i^*) = E(\bar{z}_j^*) = \mu_0\right)$ ⇒ 복원으로 n개 부트스트랩 3. **Empirical P-value 산출:** $\boldsymbol{\hat{p} = I/B}$ $(I: \{\bar{z}_j^* > \bar{x}\})$
		<2표본 평균> $H_0: \mu_2 = \mu_1$ $H_1: \mu_2 > \mu_1$	1. 상황: 1) 검정통계량: $V = \bar{Y} - \bar{X}$ 2) $\hat{p} = P_{H_0}[V \geq \bar{y} - \bar{x}]$ 2. H_0 가정 ⇒ 표본 합침 $(\mathrm{n} = \mathrm{n}_1 + \mathrm{n}_2)$ ⇒ 복원으로 $\left(\mathbf{X}_\mathbf{i}^*, \mathrm{n}_\mathbf{1}\text{개}\right), \left(\mathbf{Y}_\mathbf{i}^*, \mathrm{n}_\mathbf{2}\text{개}\right)$ 추출 3. **Empirical P-value 산출:** $\boldsymbol{\hat{p} = I/B}$ $(I: \{\bar{y}_i^* - \bar{x}_i^* > \bar{y} - \bar{x}\})$ * 부연: H_0 가정 했기 때문에 생성값 $(\bar{y}_i^* - x_i^*)$은 H_0하 통계량임.
	Perm test		2표본 perm test: 통합 표본 ($n=n_1+n_2$)에서 비복원으로 추출된 x,y 모든 가능한 표본→ 검정

5. 일치성 / 극한분포 (“통계학적 수렴”)

중요한 부등식

1. Markov: $\boldsymbol{P[u(X) \geq c] \leq E[u(X)]/c}$ (for u(X)≥0, c>0; E[u(X)]존재)

   *증명: $E[u(x)] = \int_{-\infty}^{\infty} u(x)f(x)dx \geq \int_{u(x)\geq c} u(x)f(x)dx \geq c\int_{u(x)\geq c} f(x)dx = c\,P[u(x) \geq c]$

2. Chevyshev: $\boldsymbol{P(|X-\mu| \geq k\sigma) \leq 1/k^2}$ (for k>0; X가 μ,σ²(유한) 가짐)

   *증명: Markov에서 $u(X) = (X-\mu)^2,\ c = k^2\sigma^2$

확률 수렴

1. 정의: $\boldsymbol{X_n \xrightarrow{P} X}$ $\Leftrightarrow \forall\epsilon > 0,\ \boldsymbol{\lim_{n\to\infty} P[|X_n - X| \geq \epsilon] = 0}$ $\left(\Leftrightarrow \lim_{n\to\infty} P[|X_n - X| < \epsilon] = 1\right)$ “함수열의 수렴”

   ($\boldsymbol{X_n \xrightarrow{P} a}$, if X가 상수 a ➔ “퇴화확률변수, p(a)=1, 나머지 0”)

2. 대수의 약법칙: iid $\{X_n\} \sim \left(\textbf{평균}: \boldsymbol{\mu}, \textbf{분산}: \boldsymbol{\sigma^2 < \infty}\right)$, $\boldsymbol{\bar{X}_n \xrightarrow{P} \mu}$

   *증명: By Chevyshev's ineq, $\boldsymbol{P(|\bar{X}_n - \mu| \geq \epsilon) \leq \sigma^2/(n\epsilon^2)} \to 0$ (when $n \to \infty$)

3. 정리

	정리	증명
선형	$* X_n \xrightarrow{P} X, Y_n \xrightarrow{P} Y$ ① $\boldsymbol{(X_n + Y_n) \xrightarrow{P} (X+Y)}$ ② $\boldsymbol{aX_n \xrightarrow{P} aX}$	①P는 집합오염에 단조 (=공간 커지면 확률 커짐); 삼각부등식 $P[|(X_n+Y_n)-(X-Y)| \geq \epsilon] \leq P[|X_n - X| + |Y_n - Y| \geq \epsilon]$ $\leq P[|X_n - X| \geq \epsilon/2] + P[|Y_n - Y| \geq \epsilon/2]$
함수	* 받침 상 연속 $g(x)$ ③ $\boldsymbol{X_n \xrightarrow{P} a \Rightarrow g(X_n) \xrightarrow{P} g(a)}$ ④$\boldsymbol{X_n \xrightarrow{P} X \Rightarrow g(X_n) \xrightarrow{P} g(X)}$	① $|g(x) - g(a)| \geq \epsilon \Rightarrow\ |x - a| \geq \delta\ (\epsilon > 0, \delta > 0)$ $\therefore P[|g(X_n) - g(a)| \geq \epsilon]\ \leq\ P[|X_n - a| \geq \delta]$
곱	$* X_n \xrightarrow{P} X, Y_n \xrightarrow{P} Y$ ⑤ $\boldsymbol{X_nY_n \xrightarrow{P} XY}$	$* X_nY_n = \frac{1}{2}X_n^2 + \frac{1}{2}Y_n^2 - \frac{1}{2}(X_n - Y_n)^2 \xrightarrow{P} \frac{1}{2}X^2 + \frac{1}{2}Y^2 - \frac{1}{2}(X-Y)^2 = XY$

4. 일치성: $\boldsymbol{T_n \xrightarrow{P} \theta}$ 면 ⇔ T_n은 θ의 **일치 추정량** $* F(x;\theta)$에서 추출한 $iid\ \{X_1, \cdots, X_n\}$의 통계량 T_n

분산 추정량	① $\boldsymbol{S_n^2 \xrightarrow{P} \sigma^2}$ (일치 & 불편) ② $\boldsymbol{S_{mle}^2} = \frac{n-1}{n}S_n^2 \xrightarrow{P} \boldsymbol{\sigma^2}$ (일치 & MLE)
균등분포 모수 추정량	$X_1, \cdots, X_n \sim \mathrm{unif}\,(0,\theta)$, $Y_n = \max\{X_1, \cdots, X_n\}$ $g_n(Y_n) = \frac{n!}{(n-1)!\,0!\,1!}F(Y_n)^{n-1}\left(1 - F(Y_n)\right)^0 f(y_n) = n\frac{t^{n-1}}{\theta^n}$ $(0 < t \leq \theta)$ $E(Y_n) = \int_0^\theta t\left(n\frac{t^{n-1}}{\theta^n}\right)dt = \frac{n}{n+1}\theta$ ∴ 최대값 Y_n은 θ의 일치 추정량
	$\bar{X}_n$은 $\theta/2$의 일치 추정량 ⇒ $2\bar{X}_n$은 θ의 일치 추정량

분포 수렴

1. 정의: $\boldsymbol{X_n \xrightarrow{D} X}$ $\Leftrightarrow \forall x \in \{F_X$ 연속 점$\}$, $\boldsymbol{\lim_{n\to\infty} F_n(x) = F(x)}$, $\left(F: X\text{의 cdf},\ F_n: X_n\text{의 cdf}\right)$ “극한분포”

2. $\boldsymbol{t}$**분포** ⇒ $\boldsymbol{z}$**분포** $(n \to \infty)$

   ① $f_n(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\,\Gamma\left(\frac{n}{2}\right)}\left(1 + \frac{x^2}{n}\right)^{-\left(\frac{n+1}{2}\right)}$, $\int_{-\infty}^{t} |f_n(y)|dy \leq \int_{-\infty}^{t} 10f_1(y)dy = \frac{10}{\pi}\tan^{-1}t < \infty$ (르벡 DCT)

   ② $\lim_{n\to\infty} F_n(t) = \lim_{n\to\infty}\int_{-\infty}^{t} f_n(x)dx = \int_{-\infty}^{t}\lim_{n\to\infty} f_n(x)dx = \int_{-\infty}^{t}\lim_{n\to\infty}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\,\Gamma\left(\frac{n}{2}\right)}\left(1 + \frac{x^2}{n}\right)^{-\left(\frac{n+1}{2}\right)}dx = \int_{-\infty}^{t}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)dx = \Phi(t)$

3. 정리

① $\boldsymbol{X_n \xrightarrow{P} X \Rightarrow X_n \xrightarrow{D} X}$	https://freshrimpsushi.tistory.com/175?category=696570
② $\boldsymbol{X_n \xrightarrow{P} b \Leftrightarrow X_n \xrightarrow{D} b}$	if 분포수렴 ⇒ $\lim_{n\to\infty} P[|X_n - b| \leq \epsilon] = \lim_{n\to\infty} F_{X_n}(b+\epsilon) - F_{X_n}(b-\epsilon) = 1 - 0 = 0$
③ $\boldsymbol{X_n \xrightarrow{D} X}$ & $\boldsymbol{(A_n \xrightarrow{P} a,\ B_n \xrightarrow{P} b)}$ $\boldsymbol{\Rightarrow A_n + B_nX_n \xrightarrow{D} a + bX}$	<**Slutsky 정리**> e.g. $P_n - Q_n \xrightarrow{P} 0,\ Q_n \xrightarrow{D} X\ \Rightarrow\ P_n = (P_n - Q_n) + Q_n \xrightarrow{D} X$
* 받침 상 연속 $g(x)$ ④ $\boldsymbol{X_n \xrightarrow{D} X \Rightarrow g(X_n) \xrightarrow{D} g(X)}$	$\boldsymbol{Z_n \xrightarrow{D} Z\ \Rightarrow\ Z_n^2 \xrightarrow{D} \chi^2(1)}$
⑤ $\boldsymbol{X_n \xrightarrow{D} X\ \Leftrightarrow\ \lim_{n\to\infty} M_n(t) = M(t)}$	$\boldsymbol{Y_n \sim b(n,p)} \Rightarrow \lim_{n\to\infty}\boldsymbol{M_n(t)} = \lim_{n\to\infty} E(e^{tY_n}) = \lim_{n\to\infty}[(1-p) + (pe^t)]^n = \boldsymbol{e^{\mu(e^t - 1)}}$ ∴ **이항분포** $\boldsymbol{b(n,p)} \xrightarrow{D}$ **푸아송분포** $(\boldsymbol{\mu = np})$

5. 일치성 / 극한분포 (“통계학적 수렴”)

Δ-방법		$\boldsymbol{\sqrt{n}(X_n-\theta) \xrightarrow{D} N(0,\sigma^2)}$ 이고, $g(x)$가 θ에서 미분 가능 & $g'(\theta)\neq 0$ 이면 $\boldsymbol{\sqrt{n}(g(X_n)-g(\theta)) \xrightarrow{D} N(0,g'(\theta)^2\sigma^2)}$ **[Δ-method를 잘 이용하면 모수에 종속되지 않는 통계량 분산 만듦]** pf) 테일러 정리에 의해 $g(X_n)=g(\theta)+g'(\theta)(X_n-\theta)+o(\lvert X_n-\theta\rvert)$ 이므로 $\sqrt{n}\big(g(X_n)-g(\theta)\big)=\sqrt{n}g'(\theta)(X_n-\theta)+o(\sqrt{n}\lvert X_n-\theta\rvert) \xrightarrow{P} \sqrt{n}g'(\theta)(X_n-\theta) \xrightarrow{D} N(0,g'(\theta)^2\sigma^2)$ (중간에 little-o를 0으로 확률수렴 시키는 전개는 확률 유계인 Y_n에 대해 $o(Y_n)\xrightarrow{P}0$ 임을 이용)
중심극한정리 (CLT)		1. 중심극한정리: $\boldsymbol{Z_n=\dfrac{\bar{X}-\mu}{\sigma/\sqrt{n}} \xrightarrow{D} N(0,1)}$ $\leftarrow$ iid $X_i\sim$(평균: μ, 분산: σ^2) 2. 대표본 추론 통계량: $\boldsymbol{\dfrac{\bar{X}-\mu}{S/\sqrt{n}} \xrightarrow{D} N(0,1)}$ $\because$ $S\xrightarrow{P}\sigma \Leftrightarrow \dfrac{S}{\sigma}\xrightarrow{P}1$, CLT & Slutsky에 의해 $\left(\dfrac{\sigma}{S}\right)\dfrac{\bar{X}-\mu}{\sigma/\sqrt{n}} \xrightarrow{D} N(0,1)$
		2. 증명: MGF 이용 (특성함수 $\varphi(t)=E(e^{itx})$ 이용해야 더 정확함) $m(t)\coloneqq E\left[e^{t(X-\mu)}\right]=e^{-\mu t}M(t) \Rightarrow m(0)=1,\ m'(0)=E(X-\mu)=0,\ m''(0)=E[(X-\mu)^2]+m'(0)^2=\sigma^2$ 테일러 정리에 의해 $m(t)=m(0)+m'(0)t+\dfrac{1}{2}m''(\xi)t^2=1+\dfrac{1}{2}m''(\xi)t^2=1+\dfrac{1}{2}\sigma^2t^2+\dfrac{1}{2}(m''(\xi)-\sigma^2)t^2,\quad \xi\in[-t,t]$ $M(t;n)\coloneqq E(e^{tZ_n})=E\left(\exp\left(t\dfrac{(1/n)\sum X_i-\mu}{\sigma/\sqrt{n}}\right)\right)=E\left(\exp\left(t\dfrac{\sum_{i=1}^n(X_i-\mu)}{\sigma\sqrt{n}}\right)\right)=\prod_{i=1}^n E\left(\exp\left(t\dfrac{X_i-\mu}{\sigma\sqrt{n}}\right)\right)$ $=\left[E\left(\exp\left(\dfrac{t(X-\mu)}{\sigma\sqrt{n}}\right)\right)\right]^n=\left[m\left(\dfrac{t}{\sigma\sqrt{n}}\right)\right]^n,\quad -h<\dfrac{t}{\sigma\sqrt{n}}<h$ $M(t;n)=\left[m\left(\dfrac{t}{\sigma\sqrt{n}}\right)\right]^n=\left\{1+\dfrac{t^2}{2n}+\dfrac{[m''(\xi)-\sigma^2]t^2}{2n\sigma^2}\right\}^n,\quad \xi\in\left[-\dfrac{t}{\sigma\sqrt{n}},\dfrac{t}{\sigma\sqrt{n}}\right]$ $\therefore \lim_{n\to\infty}M(t;n)=\lim_{n\to\infty}\left\{1+\dfrac{t^2}{2n}+\dfrac{[m''(\xi)-\sigma^2]t^2}{2n\sigma^2}\right\}^n=\lim_{n\to\infty}\left(1+\dfrac{t^2}{2n}\right)^n=\exp\left(\dfrac{1}{2}t^2\right)\quad \because \lim_{n\to\infty}[m''(\xi)-\sigma^2]=0\ (\because \xi\to 0)$ Z_n의 mgf $M(t;n)$의 $n\to\infty$ 극한값은 $N(0,1)$의 mgf $\exp\left(\dfrac{1}{2}t^2\right) \Rightarrow \therefore \boldsymbol{Z_n \xrightarrow{D} N(0,1)}$
다변량 분포 확장	**수렴성 다변량 확장**	1) 확률수렴: $\{\mathbf{X_n}\}\in\mathbb{R}^p$일 때, 벡터의 각 성분이 수렴하는 경우가 전체 벡터의 수렴과 동치이다. 즉, $\mathbf{X_n}\xrightarrow{P}\mathbf{X} \Leftrightarrow X_{nj}\xrightarrow{P}X_j$ (모든 $j=1,\cdots,p$에서 성립) 2) 분포수렴: $\mathbf{X_n}\xrightarrow{D}\mathbf{X} \Leftrightarrow \forall\mathbf{x}\in\{F(\mathbf{x})$ 연속 점$\}$, $\boldsymbol{\lim_{n\to\infty}F_n(\mathbf{x})=F(\mathbf{x})}$, $\big(F$: $\mathbf{X}$의 cdf, F_n: $\mathbf{X_n}$의 cdf$\big)$ ① $\boldsymbol{\mathbf{X_n}\xrightarrow{D}\mathbf{X}\Rightarrow g(\mathbf{X_n})\xrightarrow{D}g(\mathbf{X})}$ (corollary: $g(\mathbf{x})=x_j$로 두면 분포수렴이 **주변 (marginal)** 수렴 수반 ② $\boldsymbol{\mathbf{X_n}\xrightarrow{D}\mathbf{X} \Leftrightarrow \lim_{n\to\infty}M_n(\mathbf{t})=M(\mathbf{t})}$
	다변량 표본	$\{\mathbf{X_n}\}\in\mathbb{R}^p$ 인 평균 $\boldsymbol{\mu}$, 공분산행렬 $\boldsymbol{\Sigma}$ 인 iid 확률벡터열 ① 표본평균벡터: $\bar{\mathbf{X}}_n=\dfrac{1}{n}\sum_{i=1}^n\mathbf{X}_i=\left(\bar{X}_1,\cdots,\bar{X}_p\right)^T$ ② 표본공분산행렬: $S_j^2=\dfrac{1}{n-1}\sum_{i=1}^n\left(X_{ij}-\bar{X}_j\right)^2$, $S_{jk}=\dfrac{1}{n-1}\sum_{i=1}^n\left(X_{ij}-\bar{X}_j\right)(X_{ik}-\bar{X}_k)$; p × p 행렬 $\therefore\ \bar{\mathbf{X}}_n\xrightarrow{P}\boldsymbol{\mu},\ \mathbf{S}_n\xrightarrow{P}\boldsymbol{\Sigma}$ (4차 적률 유한할 때 대수 약법칙)
	CLT	$\mathbf{Y_n}=\sqrt{n}(\bar{\mathbf{X}}_n-\boldsymbol{\mu})\xrightarrow{D}N_p(\mathbf{0},\boldsymbol{\Sigma}) \Leftrightarrow$ 근사적으로 $\bar{\mathbf{X}}_n\sim N_p\left(\boldsymbol{\mu},\frac{1}{n}\boldsymbol{\Sigma}\right)$
	Δ방법	$\sqrt{n}(\mathbf{X_n}-\boldsymbol{\mu_0})\xrightarrow{D}N_p(\mathbf{0},\boldsymbol{\Sigma})$ ($\mathbf{g}$는 $\mathbb{R}^p\to\mathbb{R}^k$ 로의 변환 $(k\le p)$; 미분행렬 $\mathbf{B}=\left[\dfrac{\partial g_i}{\partial x_j}\right]$이 연속, $\mathbf{B}\neq\mathbf{0}$ in $\boldsymbol{\mu_0}$ 근방 $\sqrt{n}(\mathbf{g}(\mathbf{X_n})-\mathbf{g}(\boldsymbol{\mu_0}))\xrightarrow{D}N_p(\mathbf{0},\mathbf{B_0}\boldsymbol{\Sigma}\mathbf{B_0}^T)$ $\quad \mathbf{B_0}=\mathbf{B}(\boldsymbol{\mu_0})$

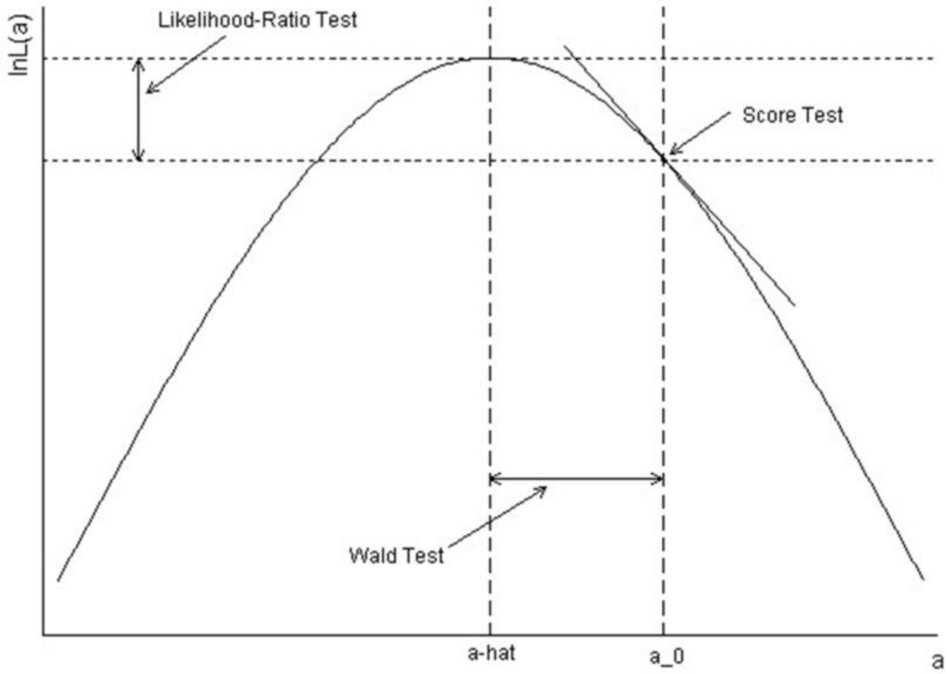
6. 최대가능도방법 (Maximum Likelihood Methods)

<table>
<tr><td rowspan="3">MLE

(R0)~(R2)</td><td>MLE 핵심</td><td>(R0), (R1) 하에서 $\lim_{n\to\infty} P_{\theta_0}[L(\theta_0,\mathbf{X}) > L(\theta,\mathbf{X})] = 1 \quad (\forall\, \theta \neq \theta_0)$
$pf)\ \frac{1}{n}\sum_{i=1}^{n}\ln\left[\frac{f(X_i;\theta)}{f(X_i;\theta_0)}\right] \xrightarrow{P} E_{\theta_0}\left(\ln\left[\frac{f(X_1;\theta)}{f(X_1;\theta_0)}\right]\right) < \ln E_{\theta_0}\left[\frac{f(X_1;\theta)}{f(X_1;\theta_0)}\right]$ by 대수의 법칙, 젠센 부등식
$E_{\theta_0}\left[\frac{f(X_1;\theta)}{f(X_1;\theta_0)}\right] = \int \frac{f(x;\theta)}{f(x;\theta_0)} f(x;\theta_0)dx = 1$ $(R1$ 공통 받침 하에서)
$\therefore\ \frac{1}{n}\sum_{i=1}^{n}\ln\left[\frac{f(X_i;\theta)}{f(X_i;\theta_0)}\right] < 0 \Leftrightarrow\ L(\theta_0,\mathbf{X}) > L(\theta,\mathbf{X})$
∴ 근사적으로 <u>참값 $\boldsymbol{\theta_0}$에서 우도함수 $\boldsymbol{L(\theta,\mathbf{X})}$가 최대</u>가 된다. ($\hat\theta = \text{Argmax}[L(\theta)] \xrightarrow{P} \theta_0$)</td></tr>
<tr><td>불변성</td><td><u>$\boldsymbol{\eta = g(\theta) \Leftrightarrow \hat\eta = g(\hat\theta)}$</u>
$pf)$ ① $g \in$ 1대1 함수: $\max L(\theta) = \max L\left(g^{-1}(\eta)\right)$ 이므로 $\hat\theta = g^{-1}(\hat\eta)$에서 우도 최대화
② $g \notin$ 1대1 함수: $g^{-1}(\eta) :=$ $\{\theta: g(\theta) = \eta\}$ 새로 정의 → $\hat\theta \in g^{-1}(\hat\eta)$ 에서 우도최대화</td></tr>
<tr><td>추정 방정식</td><td>*추정방정식 (estimating equation; EE): $\partial l(\theta)/\partial\theta = 0$
(R0)~(R2) 하에서 $\partial l(\theta)/\partial\theta = 0$ 는 $\hat\theta \xrightarrow{P} \theta_0$ 인 $\hat\theta$를 가짐
(Corollary: EE가 유일해를 가지면 그 해는 $\hat\theta \xrightarrow{P} \theta_0$)</td></tr>
<tr><td rowspan="8">Cramér Rao Bound

(R0)~(R4)</td><td rowspan="3">스코어 함수 & 피셔정보</td><td>① Score 함수 $s(\theta) = \frac{\partial \ln f}{\partial\theta}$
② Fisher information $I(\theta) = \text{Var}\left(\frac{\partial \ln f}{\partial\theta}\right) = \text{E}\left[\left(\frac{\partial \ln f}{\partial\theta}\right)^2\right] = -\text{E}\left[\frac{\partial^2 \ln f}{\partial\theta^2}\right]$</td></tr>
<tr><td>$1 = \int_{-\infty}^{\infty} f dx$ → 양변 θ로 i) 한번 미분 ii) 두번 미분 하면
i) $0 = \int_{-\infty}^{\infty}(\partial f/\partial\theta)dx = \int_{-\infty}^{\infty}\frac{(\partial f/\partial\theta)}{f} f dx = \int_{-\infty}^{\infty}\left(\frac{\partial \ln f}{\partial\theta}\right) f dx \qquad \therefore \boldsymbol{E\left(\frac{\partial \ln f}{\partial\theta}\right) = 0}$
ii) $0 = \int_{-\infty}^{\infty}\frac{\partial^2 \ln f}{\partial\theta^2} f dx + \int_{-\infty}^{\infty}\left(\frac{\partial \ln f}{\partial\theta}\right)\left(\frac{\partial \ln f}{\partial\theta}\right) f dx \quad \therefore \boldsymbol{\text{E}\left[\frac{\partial^2 \ln f}{\partial\theta^2}\right] + \text{E}\left[\left(\frac{\partial \ln f}{\partial\theta}\right)^2\right] = 0}$</td></tr>
<tr><td>iid $[X_1,\cdots,X_n]$ 에 대해서
① Score 함수 $s_n(\theta) = \frac{\partial l}{\partial\theta} = \frac{\partial \ln L}{\partial\theta} = \sum_{i=1}^{n}\frac{\partial \ln f(X_i;\theta)}{\partial\theta}$
② Fisher 정보 $I_n(\theta) = \text{Var}\left(\frac{\partial l}{\partial\theta}\right) = \text{Var}\left(\frac{\partial \ln L}{\partial\theta}\right) = n\, I(\theta)$</td></tr>
<tr><td rowspan="2">Cramér-Rao Bound (CRB)</td><td>① $\boldsymbol{\text{Var}(T) \geq \frac{[\partial E(T)/\partial\theta]^2}{nI(\theta)}}$ for 임의의 통계량 $T = g(X_1,\cdots,X_n)$
② $\boldsymbol{\text{Var}(T) \geq \frac{1}{nI(\theta)}}$ for 불편추정량 T ($\because E(T) = \theta$)</td></tr>
<tr><td>$pf)\ E(T) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}[T]f(x_1;\theta)\cdots f(x_n;\theta)dx_1\cdots dx_n$
$\Leftrightarrow\ \partial E(T)/\partial\theta = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}[T]\left[\sum_{i=1}^{n}\frac{\partial \ln f(x_i;\theta)}{\partial\theta}\right] f(x_1;\theta)\cdots f(x_n;\theta)\, dx_1\cdots dx_n\ \ ,Z := \left[\sum_{i=1}^{n}\frac{\partial \ln f(x_i;\theta)}{\partial\theta}\right]$
$\Leftrightarrow\ \partial E(T)/\partial\theta = E(TZ) = E(T)E(Z) + \rho\,\sigma_T\sigma_Z = \rho\sqrt{\text{Var}(T)}\sqrt{nI(\theta)} \qquad \therefore \rho^2 \leq 1 \Leftrightarrow \text{Var}(T) \geq \frac{[\partial E(T)/\partial\theta]^2}{nI(\theta)}$</td></tr>
<tr><td>효율성</td><td>*효율성: 통계량 T의 효율성은 CRB(T)/Var(T)
$*$ ARE (근사 상대효율성) $= \text{e}(\text{T},\text{W}) = \frac{\text{Var(W)}}{\text{Var(T)}}$ (if $\text{T} \xrightarrow{P} \theta_0, \text{W} \xrightarrow{P} \theta_0$이며 둘다 정규근사 될 때)</td></tr>
<tr><td>MLE 정규근사
(R0)~(R5)</td><td>① 정규 근사: $\boldsymbol{\sqrt{n}(\hat\theta - \theta_0) \xrightarrow{D} N\left(0,\frac{1}{I(\theta_0)}\right)}$ for 유한 피셔정보 $\text{I}(\theta_0)$ $\quad * pf)\ l'(\hat\theta)$를 θ_0 테일러 전개 ...
➔ <u>MLE의 근사 정규 신뢰 구간</u> 구할 수 있음.
$\therefore\ \text{Var}(\hat{\boldsymbol\theta}) \xrightarrow{P} \frac{\mathbf{1}}{\boldsymbol{nI(\theta_0)}}$ (mle는 근사적으로 효율적 or mle의 분산은 CRB에 근사)
② Δ방법: $\boldsymbol{\sqrt{n}\left(g(\hat\theta) - g(\theta_0)\right) \xrightarrow{D} N\left(0,\frac{g'(\theta_0)^2}{I(\theta_0)}\right)}$ $\quad$ ($g(x)$가 θ에서 미분 가능 & $g'(\theta) \neq 0$ 이면)
③ 정규 근사: $\hat\theta - \theta_0 = \frac{1}{nI(\theta_0)}\sum_{i=1}^{n}\frac{\partial \ln f(X_i;\theta_0)}{\partial\theta} + \frac{R_n}{\sqrt{n}} = -\frac{l'(\theta_0)}{l''(\theta_0)} + \frac{R_n}{\sqrt{n}}$ $\left(R_n \xrightarrow{P} 0\right)$</td></tr>
<tr><td>MLE Newton's</td><td>$\hat\theta_1 = \hat\theta_0 - \frac{l'(\hat\theta_0)}{l''(\hat\theta_0)}$ 과정 반복 $\qquad * \ \hat\theta_0$이 일치 추정량이면 $\hat\theta_1$은 mle 임 $\left(\xrightarrow{D} \boldsymbol{N\left(0,\frac{1}{I(\theta_0)}\right)}\right)$</td></tr>
</table>

6. 최대가능도방법 (Maximum Likelihood Methods)

<table>
<tr><td rowspan="5">최대
가능도
검정
(ML tests)</td><td>전개</td><td colspan="2">우도비 (LR): $\boldsymbol{\Lambda = \frac{L(\theta_0)}{L(\hat{\theta})}}$ ($\Lambda \le c$ 에서 기각)
$-\frac{1}{n} l''(\theta_0) \xrightarrow{P} I(\theta_0)$, $\frac{l'(\theta_0)}{\sqrt{n}} = \sqrt{n}(\hat{\theta}-\theta_0)I(\theta_0) + R_n$ 이므로
$l(\hat{\theta}) = l(\theta_0) + (\hat{\theta}-\theta_0)l'(\theta_0) + \frac{1}{2}(\hat{\theta}-\theta_0)\, l''(\theta_n^*)$ 의 미분계수 항들에 대입해주면
$-2\ln\Lambda = 2\left[l(\hat{\theta}) - l(\theta_0)\right] = \left[\sqrt{nI(\theta_0)}(\hat{\theta}-\theta_0)\right]^2 + R_n^*$ $\left(R_n^* \xrightarrow{P} 0\right)$
∴ $\boldsymbol{-2\ln\Lambda \xrightarrow{D} \chi^2(1)}$ ⇐ $\sqrt{nI(\theta_0)}(\hat{\theta}-\theta_0) \xrightarrow{D} N(0,1)$</td></tr>
<tr><td>우도비 검정</td><td>$\boldsymbol{\chi_L^2 = -2\ln\Lambda}$</td><td rowspan="3">$\boldsymbol{\chi^2 \ge \chi_\alpha^2(1)}$ 에서 단측 검정 기각역
$(H_0: \theta = \theta_0, H_1: \theta \ne \theta_0)$</td></tr>
<tr><td>Wald 검정</td><td>$\chi_W^2 = \left[\sqrt{nI(\hat{\theta})}(\hat{\theta}-\theta_0)\right]^2$</td></tr>
<tr><td>Score 검정</td><td>$\chi_R^2 = \left(\frac{l'(\theta_0)}{\sqrt{nI(\theta_0)}}\right)^2$</td></tr>
<tr><td colspan="3">

have the following relationship Wald ≥ LR ≥ score (Johnston and DiNardo 1997 p. 150). That is, the Wald test statistic will always be greater than the LR test statistic, which will, in turn, always be greater than the test statistic from the score test. When computing power was much more limited, and many models took a long time to run, being able to approximate the LR test using a single model was a fairly major advantage. Today, for most of the models researchers are likely to want to compare, computational time is not an issue, and we generally recommend running the likelihood ratio test in most situations. This is not to say that one should never use the Wald or score tests. For example, the Wald test is commonly used to perform multiple degree of freedom tests on sets of dummy variables used to model categorical predictor variables in regression (for more information see our webbooks on Regression with Stata, SPSS, and SAS, specifically Chapter 3 – Regression with Categorical Predictors.) The advantage of the score test is that it can be used to search for omitted variables when the number of candidate variables is large.</td></tr>
<tr><td>정칙
조건

Regularity conditions</td><td colspan="3">(R0): pdf $f(x;\theta)$는 서로 distinct 하다. i.e. $\theta_1 \ne \theta_2 \Rightarrow f(x_i;\theta_1) \ne f(x_i;\theta_2)$
(R1): pdf $f(x;\theta)$는 모든 θ에 대해 공통된 support를 갖는다. (θ에 의존적이지 않다.)
(R2): θ_0 (참값) $\in \Omega$
(R3): pdf $f(x;\theta)$는 θ로 두 번 미분 가능
(R4): $\int f(x;\theta)dx$는 θ로 두 번 미분 가능
(R5): pdf $f(x;\theta)$는 θ로 세 번 미분 가능, 모든 θ에 대해 $|\partial^3 \ln f/\partial\theta^3| \le M(x)$ $\left(E_{\theta_0}[M(X)] < \infty\right)$ $in\ \theta_0$근방 $\forall x$</td></tr>
</table>

6. 최대가능도방법 (Maximum Likelihood Methods)

다중모수					*정칙조건 ~(R9) 까지 추가됨. (기존 정칙의 다변량 확장) 맨위 "MLE 핵심" 정리는 벡터 $\boldsymbol{\theta} = [\theta_1,\cdots,\theta_p]^T \in \mathbb{R}^p$에 대해서도 똑같이 성립함. ⇔ $\nabla l(\boldsymbol{\theta}) = \mathbf{0}$의 해 구하기
	다중 모수 추정		피셔정보량		$\nabla \ln f(X;\boldsymbol{\theta}) = \left(\frac{\partial \ln f(X;\boldsymbol{\theta})}{\partial\theta_1},\cdots,\frac{\partial \ln f(X;\boldsymbol{\theta})}{\partial\theta_p}]\right)^T$ 피셔 정보량: $\mathbf{I}(\boldsymbol{\theta}) = \mathrm{Cov}(\nabla \ln f(X;\boldsymbol{\theta})) = -E\left[\frac{\partial^2}{\partial\theta_j\,\partial\theta_k}\ln f\right]_{jk} = E\left[\left(\frac{\partial \ln f}{\partial\theta_j}\right)\left(\frac{\partial \ln f}{\partial\theta_k}\right)\right]_{jk}$
			피셔정보량 (표본 n 확장)		$\nabla l = \nabla \ln L = \sum_{i=1}^{n}\nabla \ln f$ 피셔 정보량: $\mathbf{I_n}(\boldsymbol{\theta}) = \mathrm{Cov}(\nabla l) = \mathrm{Cov}(\nabla \ln L) = n\mathbf{I}(\boldsymbol{\theta})$
			CRB		$Var(T_j) \ge \frac{1}{n}[\mathbf{I}^{-1}(\boldsymbol{\theta})]_{jj}$ (T_j가 θ_j의 불편 추정량)
			정규근사		$\sqrt{n}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta_0}) \xrightarrow{D} N_p(\mathbf{0}, \mathbf{I}^{-1}(\boldsymbol{\theta_0}))$ ⇒ $\sqrt{n}(\hat{\theta}_j - \theta_j) \xrightarrow{D} N(0, [\mathbf{I}^{-1}(\boldsymbol{\theta_0})]_{jj})$
			Δ방법		$\sqrt{n}(\mathbf{g}(\hat{\boldsymbol{\theta}}) - \mathbf{g}(\boldsymbol{\theta_0})) \xrightarrow{D} N_p(\mathbf{0}, \mathbf{B}[\mathbf{I}^{-1}(\boldsymbol{\theta_0})]\mathbf{B}^T)$ ($\mathbf{g}$는 $\mathbb{R}^p \to \mathbb{R}^k$ 로의 변환 ($k \le p$); 미분행렬 $\mathbf{B} = \left[\frac{\partial g_i}{\partial\theta_j}\right]$이 연속, $\mathbf{B} \ne \mathbf{0}$ in $\boldsymbol{\theta_0}$ 근방)
	다중 모수 검정	기본			$\boldsymbol{H_0}: \boldsymbol{\theta} \in \boldsymbol{\omega}$, $\boldsymbol{H_1}: \boldsymbol{\theta} \in (\boldsymbol{\omega^c} \cap \boldsymbol{\Omega})$ (Ω: p차원 전체 모수 공간; ω: p-q차원 귀무가설 모수공간 (q: 제약된 모수 개수))
					우도비 (LR): $\boldsymbol{\Lambda} = \frac{\boldsymbol{L}(\hat{\boldsymbol{\omega}})}{\boldsymbol{L}(\hat{\boldsymbol{\Omega}})} = \frac{\max_{\boldsymbol{\theta}\in\boldsymbol{\omega}} \boldsymbol{L}(\boldsymbol{\theta})}{\max_{\boldsymbol{\theta}\in\boldsymbol{\Omega}} \boldsymbol{L}(\boldsymbol{\theta})}$ $\boldsymbol{\chi_L^2} = -\mathbf{2}\ln\boldsymbol{\Lambda} \xrightarrow{\boldsymbol{D}} \boldsymbol{\chi^2(q)}$ (Wald, Score 검정통계량도 가능)
		예시	정규 μ		$\boldsymbol{H_0}: \boldsymbol{\mu} = \boldsymbol{\mu_0}, \boldsymbol{H_1}: \boldsymbol{\mu} \ne \boldsymbol{\mu_0}$ $\{\boldsymbol{X_n}\} \overset{\text{iid}}{\sim} \boldsymbol{N}(\boldsymbol{\mu}, \boldsymbol{\sigma^2})$ $L(\hat{\Omega}) = \frac{1}{(2\pi\hat{\sigma}^2)^{n/2}}\exp\left\{-\frac{1}{2}\sum_{i=1}^{n}\frac{(X_i - \bar{X})^2}{\frac{1}{n}(\sum_{i=1}^{n}(X_i - \bar{X})^2)}\right\} = \frac{\mathbf{1}}{(\mathbf{2}\boldsymbol{\pi}\hat{\boldsymbol{\sigma}})^{\boldsymbol{n/2}}}\mathbf{exp}\left(-\frac{\boldsymbol{n}}{\mathbf{2}}\right)$ $L(\hat{\omega}) = \frac{\mathbf{1}}{(\mathbf{2}\boldsymbol{\pi}\hat{\boldsymbol{\sigma}}_0^2)^{\boldsymbol{n/2}}}\mathbf{exp}\left(-\frac{\boldsymbol{n}}{\mathbf{2}}\right)$ $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$, $\hat{\sigma}_0{}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu_0)^2$ $\left(\frac{1}{\Lambda}\right)^{\frac{2}{n}} = \left(\frac{L(\hat{\Omega})}{L(\hat{\omega})}\right)^{\frac{2}{n}} = \frac{\sum_{i=1}^{n}(X_i - \mu_0)^2}{\sum_{i=1}^{n}(X_i - \bar{X})^2} = 1 + \frac{n(\bar{X} - \mu_0)^2}{\sum_{i=1}^{n}(X_i - \bar{X})^2} = 1 + \frac{1}{n-1}\left\{\frac{(\bar{X} - \mu_0)}{S/\sqrt{n}}\right\}^2$ $\left(\frac{1}{\Lambda}\right)^{\frac{2}{n}} \ge c' \Leftrightarrow \lvert T\rvert \ge c^* = \sqrt{(c'-1)(n-1)}$ ∴ 양측 t검정과 동치
			다항 p		$\boldsymbol{H_0}: \boldsymbol{p_1} = \boldsymbol{p_2}$, $\boldsymbol{H_1}: \boldsymbol{p_1} \ne \boldsymbol{p_2}$ (유력후보1 vs 유력후보2 vs 나머지 군소후보) 3항 베르누이 $(X_{i1}, X_{i2}) \sim \boldsymbol{p_1^{x_1}p_2^{x_2}(1 - p_1 - p_2)^{1 - x_1 - x_2}}$ $(X_{i1}, X_{i2}) \in \{(0,0),(0,1),(1,0)\}$ $\hat{p}_j = \frac{\sum_{i=1}^{n}X_{ij}}{n}$ for $j = 1,2$ (표본: $\{(X_{n1}, X_{n2})\}$)
				LR	$\frac{1}{\Lambda} = \left(\frac{2\hat{p}_1}{\hat{p}_1 + \hat{p}_2}\right)^{n\hat{p}_1}\left(\frac{2\hat{p}_2}{\hat{p}_1 + \hat{p}_2}\right)^{n\hat{p}_2}$, $-2\ln\Lambda > \chi^2_\alpha(1)$에서 기각
				Wald	$\begin{bmatrix}\hat{p}_1\\ \hat{p}_2\end{bmatrix} \overset{a}{\sim} N_2\left(\begin{bmatrix}p_1\\ p_2\end{bmatrix}, \frac{1}{n}\begin{bmatrix}p_1(1-p_1) & -p_1p_2\\ -p_1p_2 & p_2(1-p_2)\end{bmatrix}\right)$ $W = \hat{p}_1 - \hat{p}_2 = g\left(\begin{bmatrix}\hat{p}_1\\ \hat{p}_2\end{bmatrix}\right)$, Δ방법에서 $\mathbf{B} = \left[\frac{\partial g_i}{\partial p_j}\right] = [1, -1]$ $Var(W) = \frac{1}{n}\mathbf{B}\mathbf{I^{-1}}\mathbf{B}^T = \frac{p_1 + p_2 - (p_1 - p_2)^2}{\text{n}}$ ∴ $W \overset{a}{\sim} N\left(p_1 - p_2, \frac{p_1 + p_2 - (p_1 - p_2)^2}{\text{n}}\right)$ ⇒ 근사 Z 검정 or 카이제곱
			2표본 이항 p		$\boldsymbol{H_0}: \boldsymbol{p_1} = \boldsymbol{p_2}$, $\boldsymbol{H_1}: \boldsymbol{p_1} \ne \boldsymbol{p_2}$, $\{\boldsymbol{X_{n1}}\} \overset{\text{iid}}{\sim} \boldsymbol{B}(\mathbf{1}, \boldsymbol{p_1})$, $\{\boldsymbol{Y_{n2}}\} \overset{\text{iid}}{\sim} \boldsymbol{B}(\mathbf{1}, \boldsymbol{p_2})$
				Wald	$\hat{p}_1 \overset{a}{\sim} N\left(p_1, \frac{p_1(1-p_1)}{n_1}\right)$, $\hat{p}_2 \overset{a}{\sim} N\left(p_2, \frac{p_2(1-p_2)}{n_2}\right)$, $\mathrm{Cov}(\hat{p}_1, \hat{p}_2) = 0$ $\hat{p}_1 - \hat{p}_2 \overset{a}{\sim} N\left(p_1 - p_2, \frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}\right)$ & $Slutsky$ ($\hat{p}_1 \xrightarrow{P} p_1, \hat{p}_2 \xrightarrow{P} p_2$) ⇒ 근사 Z 검정 or 카이제곱

7. 충분성 (Sufficiency) – 통계량의 성질

통계량 Review	**통계량**	① 점추정: $\boldsymbol{\theta \in \Omega}$**에 대한 추정량** $\boldsymbol{\hat\theta}$ *통계량 (Statistic): $T = T(X_1, \cdots, X_n)$ (표본에 대한 함수) ② 95% CI: $0.95 = P_\theta[\theta \in (\hat\theta_L, \hat\theta_U)]$ *$\theta \in (\hat\theta_L, \hat\theta_U)$인 베르누이 사건 $\sim B(1, 0.95)$
	성질	1) 일치추정량: $\boldsymbol{T_n \xrightarrow{P} \theta}$ 면 $\Leftrightarrow T_n$은 θ의 **일치 추정량**
		2) 불편추정량: $E(T) = \theta \;\; \Leftrightarrow T$는 θ의 불편 추정량 (bias = 0) ① **MVUE**: 분산 최소인 불편추정량 (UE) ➔ 유일 ② **CRB**: $\mathbf{Var}(\boldsymbol{T}) \geq \mathbf{1}/\{\boldsymbol{nI(\theta)}\}$
		3) MLE: $\hat\theta = \mathrm{Argmax}[L(\theta)] = \mathrm{Argmax}[\prod_{i=1}^n f(x_i, \theta)]$ ① MLE는 근사적으로 효율적 ② $\boldsymbol{\hat\theta \overset{a}{\sim} N\left(\theta_0, \frac{1}{nI(\theta_0)}\right)}$ $\Rightarrow$ Z or χ^2화 하면 Wald statistic
		4) $ARE(T_1, T_2) = \dfrac{Var(T_2)}{Var(T_1)}$
	Bias MSE	1) **bias** $(\boldsymbol{\hat\theta}) = \boldsymbol{E(\hat\theta - \theta)} = E(\hat\theta) - \theta$ $*\,\mathrm{bias}(g(\hat\theta)) = E\left(g(\hat\theta) - g(\theta)\right)$ 2) Mean square error (MSE): $\boldsymbol{mse(\hat\theta) = E\left\{(\hat\theta - \theta)^2\right\}} = Var(\hat\theta) + \{\mathrm{bias}(\hat\theta)\}^2$ 3) Mean absolute error (MAE): $\boldsymbol{mse(\hat\theta) = E\{\lvert\hat\theta - \theta\rvert\}}$
	적률 추정법 (MoM)	r차 표본적률 $\xrightarrow{P}$ r차 모적률 (➔ 연립하여 모수 추정량 구함; 일반적으로 비선호)
		ex) $\{X_i\} \overset{\mathrm{iid}}{\sim} \mathrm{Gamma}(k, \theta)$ $m_1 = \dfrac{\sum_{i=1}^n X_i}{n} = \hat k \hat\theta, \;\; m_2 = \dfrac{\sum_{i=1}^n X_i^2}{n} = \hat k (\hat\theta)^2 + (\hat k \hat\theta)^2$ $\hat\theta = \dfrac{\sum_{i=1}^n (X_i - \bar X)^2}{n \bar X} = \left(\dfrac{n-1}{n}\right)\dfrac{S^2}{\bar X} = \dfrac{S_{mle}^2}{\bar X}, \qquad \hat k = \dfrac{n(\bar X)^2}{\sum_{i=1}^n (X_i - \bar X)^2} = \left(\dfrac{n}{n-1}\right)\dfrac{\bar X}{S^2} = \dfrac{\bar X^2}{S_{mle}^2}$
충분성	**정의**	$Y = u(X_1, \cdots, X_n)$에 대해 $\boldsymbol{X \mid y}$**가** $\boldsymbol{\theta}$**와 무관**함 $\Leftrightarrow$ Y가 θ에 대한 모든 정보 다 포함 (e.g. $Y = \Sigma_{i=1}^n X_i$) $\dfrac{\prod_{i=1}^n f(x_i, \theta)}{f_Y(y; \theta)} = H(x_1, \cdots, x_n) \qquad (f_Y$: Y의 pdf)
	Neyman-Fisher	$[Y$가 θ의 $SS] \Leftrightarrow \prod_{i=1}^n f(x_i, \theta) = f_Y(y; \theta)\, H(x_1, \cdots, x_n) = g(y; \theta)\, h(x_1, \cdots, x_n)$ (임의의 g, h로 **인수분해**)
	Rao-Blackwell	θ의 충분통계량 Y_1, 불편추정량 Y_2에 대해, 새로운 불편추정량 $\boldsymbol{\varphi(y_1) = E(Y_2 \mid y_1)}$ $E(\varphi(Y_1)) = E[E(Y_2 \mid Y_1)] = E(Y_2) = \theta$ 2) $\mathrm{Var}(\varphi(Y_1)) = \mathrm{Var}(E(Y_2 \mid Y_1)) \leq \mathrm{Var}(Y_2)$ ∴ **New** $\boldsymbol{UE}$ $\varphi(y_1) = E(Y_2 \mid y_1)$는 **Old** $\boldsymbol{UE}$ Y_2 보다 분산이 작다. *실전: $E(\varphi(Y_1)) = \theta$인 $\varphi(y_1)$찾기
	Lehmann-Scheffe	① 통계량 Y는 **complete (완비)** if 모든 θ에서 $E(h(Y)) = 0 \Rightarrow h(t) = 0$ 만 가능함 ② **레만-셰페: CSS**인 $\boldsymbol{Y_1}$으로 Rao-Blackwellization ➔ $\boldsymbol{\varphi(y_1) = E(Y_2 \mid y_1)}$는 유일한 $\boldsymbol{MVUE}$ **of** $\boldsymbol{\theta}$ $pf)$ CSS인 Y_1에 대해 불편추정량 $\varphi(Y_1), \psi(Y_1)$존재 $\Rightarrow E(\varphi(Y_1) - \psi(Y_1)) = \theta - \theta = 0$ 완비족 $\{f_{Y_1}(y; \theta) : \theta \in \Omega\}$에 대해 위 등식은 $\varphi(Y_1) = \psi(Y_1)$에서만 성립 (더 이상 분산 못 줄임)
	지수족 Exponential Family	$f(x; \theta) = \exp[\eta(\theta) T(x) + H(x) - A(\eta(\theta))]$ $(x \in S)$ $(\eta = \eta(\theta)$는 자연 모수) *정칙: 1) $\boldsymbol{S}$**가** $\boldsymbol{\theta}$**에 종속 X**, 2) $\boldsymbol{\eta(\theta)}$**연속**, 3) (연속이면) $\boldsymbol{H(x)}$**연속** in $\{K'(x) \neq 0\}$ ① 지수족: 이산 (포아송, 이항, 기하, 음이항, 다항 등) / 연속 (감마, 베타, 정규 등) ② $Y = \sum_{i=1}^n T(x)$ 는 θ의 **CSS** ③ $E(T(X)) = A'(\eta)$, $\mathrm{Var}(T(X)) = A''(\eta)$
	결합 충분통계량 (다중모수)	$\mathbf{Y} = (Y_1, \cdots, Y_m)^T \in \mathbb{R}^m$ & $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_p) \in \mathbb{R}^p$에 대해 (일반적으로 $\mathrm{m} = \mathrm{p}$) $\prod_{i=1}^n f(x_i, \boldsymbol{\theta}) = f_{\mathbf{Y}}(\mathbf{y}; \boldsymbol{\theta}) H(x_1, \cdots, x_n) = g(\mathbf{y}; \boldsymbol{\theta})\, h(x_1, \cdots, x_n)$ *순서통계량 $\mathbf{Y} = (Y_1, \cdots, Y_n)^T$; $Y_1 < \cdots < Y_n$ $\rightarrow$ 모든 연속분포의 결합충분통계량
	보조통계량 (Ancillary)	$A = a(X_1, \cdots, X_n)$가 θ와 **무관** ex) 정규분포 iid의 $\boldsymbol{S^2}$: $\boldsymbol{\mu}$에 대해 ancillary 1) **Basu** 정리:$\{\boldsymbol{Y}$가 θ 의 **CSS**$\}$ & $\{\boldsymbol{Z}$가 θ 의 **ancillary**$\} \Leftrightarrow \{\boldsymbol{Y}$와 $\boldsymbol{Z}$는 **독립**$\}$ ex) $\boldsymbol{\bar X \perp S^2}, \{X_i\} \overset{\mathrm{iid}}{\sim} N(\mu, \sigma^2)$ 2) ①**위치불변**:$Z = u(W_1 + \theta, \cdots, W_n + \theta) = u(W_1, \cdots, W_n)$ ex) S^2, $\max\{X_i\} - \min\{X_i\}$, $\frac{1}{n}\sum_1^n \lvert X_i - Q_2 \rvert$ ②**척도불변**: $Z = u(\theta W_1, \cdots, \theta W_n) = u(W_1, \cdots, W_n)$ ex) $X_1/(X_1 + X_2)$, $X_1^2 / \sum_1^n X_i^2$, $\min\{X_i\} / \max\{X_i\}$ ③**위치척도불변**: $Z = u(\theta_1 W_1 + \theta_2, \cdots, \theta_1 W_n + \theta_2) = u(W_1, \cdots, W_n)$ ex) $(X_i - \bar X)/S^2$
	MLE	$L(\theta; x_1, \cdots, x_n) = \prod_{i=1}^n f(x_i, \theta) = f_Y(y; \theta)\, H(x_1, \cdots, x_n)$ ➔ L과 f_y 동시에 극대화 by θ ① $MLE\ \hat\theta$이 유일 $\Leftrightarrow$ $\hat\theta$는 충분통계량 Y의 함수 $\because \hat\theta = \mathrm{argmax}(L(\theta, \mathbf{x})) = \mathrm{argmax}(f_Y(y; \theta))$ ② $MLE\ \hat\theta$가 충분통계량 $\Leftrightarrow$ $\hat\theta$는 최소 충분통계량 (MSS) *최소충분:reduced from 다른 충분통계량

7. 충분성 (Sufficiency) – 통계량의 성질

지수족 확장

지수족

$f(x;\theta) = \exp[\eta(\theta)T(x) + H(x) - A(\eta(\theta))]$ $(x \in S)$ ($\eta = \eta(\theta)$는 자연 모수)

*정칙: 1) $\boldsymbol{S}$**가** $\boldsymbol{\theta}$**에 종속 X**, 2) $\boldsymbol{\eta}(\boldsymbol{\theta})$**연속**, 3) (연속이면) $\boldsymbol{H}(\boldsymbol{x})$**연속** in $\{K'(x) \neq 0\}$

① 지수족: 이산 (포아송, 이항, 기하, 음이항, 다항 등) / 연속 (감마, 베타, 정규 등)

② $Y = \sum_{i=1}^{n} T(x)$ 는 θ의 **CSS** ③ $E\big(T(X)\big) = A'(\eta)$, $\mathrm{Var}\big(T(X)\big) = A''(\eta)$

다변량 확장

1변수 1모수	$f(x;\theta) = \exp[\eta(\theta)T(x) + H(x) - A(\eta)]$
1변수 다중모수	$f(x;\boldsymbol{\theta}) = \exp[\boldsymbol{\eta}(\boldsymbol{\theta}) \cdot \mathbf{T}(x) + H(x) - A(\boldsymbol{\eta})]$
다변량 다중모수	$f(\mathbf{x};\boldsymbol{\theta}) = \exp[\boldsymbol{\eta}(\boldsymbol{\theta}) \cdot \mathbf{T}(\mathbf{x}) + H(\mathbf{x}) - A(\boldsymbol{\eta})]$
기대값	$\nabla A(\boldsymbol{\eta}) = E[\mathbf{T}(\mathbf{x})]$, $\mathbf{H}[A(\boldsymbol{\eta})] = \mathrm{Cov}(\mathbf{T}(\mathbf{x}))$ $\nabla A(\boldsymbol{\eta}_{mle}) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{T}(\mathbf{x}_i)$

예시

분포	모수 $\boldsymbol{\theta}$	자연모수 $\boldsymbol{\eta}$	역모수	$\mathbf{T}(\mathbf{x})$	$A(\boldsymbol{\eta})$
베르누이	p	$\ln\frac{p}{1-p}$	$\frac{1}{1+e^{-\eta}}$ * logistic function	x	$\ln(1+e^{\mu})$
이항					$n\ln(1+e^{\mu})$
푸아송	m	$\ln m$	e^{η}	x	e^{η}
음이항(r)	p	$\ln(1-p)$	$1-e^{\eta}$	x	$-r\ln(1-e^{\mu})$
다항(n)	$\begin{bmatrix} p_1 \\ \vdots \\ p_{k-1} \end{bmatrix}$	$\begin{bmatrix} \ln\frac{p_1}{p_k} \\ \vdots \\ \ln\frac{p_{k-1}}{p_k} \end{bmatrix}$	$\begin{bmatrix} \frac{\exp(\eta_1)}{1+\sum_{j=1}^{k-1}\exp(\eta_j)} \\ \vdots \\ \frac{\exp(\eta_{k-1})}{1+\sum_{j=1}^{k-1}\exp(\eta_j)} \end{bmatrix}$ * softmax function	$\begin{bmatrix} x_1 \\ \vdots \\ x_{k-1} \end{bmatrix}$	$n\ln(1+\sum_{j=1}^{k-1}\exp(\eta_j))$
	$p_k = 1-\sum_{j=1}^{k-1}p_j$, $\eta_k = 0$, $\exp(\eta_k) = 1$				
감마	$\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$	$\begin{bmatrix} \alpha-1 \\ -\frac{1}{\beta} \end{bmatrix}$	$\begin{bmatrix} \eta_1+1 \\ -\frac{1}{\eta_2} \end{bmatrix}$	$\begin{bmatrix} \ln x \\ x \end{bmatrix}$	$\ln\Gamma(\eta_1+1) - (\eta_1+1)\ln(-\eta_2)$
지수	β	$-\frac{1}{\beta}$	$-\frac{1}{\eta}$	x	$-\ln(-\eta)$
카이제곱	ν	$\frac{\nu}{2}-1$	$2(\eta+1)$	$\ln x$	$\ln\Gamma(\eta+1) + (\eta+1)\ln 2$
베타	$\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$	$\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$	$\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$	$\begin{bmatrix} \ln x \\ \ln(1-x) \end{bmatrix}$	$\ln \mathrm{B}(\alpha,\beta) = \ln\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$
정규 기지 $\boldsymbol{\sigma^2}$	μ	$\frac{\mu}{\sigma^2}$	$\sigma^2\eta$	x	$\frac{1}{2}\sigma^2\eta^2$
정규 미지 $\boldsymbol{\sigma^2}$	$\begin{bmatrix} \mu \\ \sigma^2 \end{bmatrix}$	$\begin{bmatrix} \frac{\mu}{\sigma^2} \\ -\frac{1}{2\sigma^2} \end{bmatrix}$	$\begin{bmatrix} -\frac{\eta_1}{2\eta_2} \\ -\frac{1}{2\eta_2} \end{bmatrix}$	$\begin{bmatrix} x \\ x^2 \end{bmatrix}$	$-\frac{\eta_1^2}{4\eta_2} - \frac{1}{2}\ln(-2\eta_2)$
다변량 정규	$\begin{bmatrix} \boldsymbol{\mu} \\ \boldsymbol{\Sigma} \end{bmatrix}$	$\begin{bmatrix} \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} \\ -\frac{1}{2}\boldsymbol{\Sigma}^{-1} \end{bmatrix}$	$\begin{bmatrix} -\frac{1}{2}\boldsymbol{\eta}_2^{-1}\boldsymbol{\eta}_1 \\ -\frac{1}{2}\boldsymbol{\eta}_2^{-1} \end{bmatrix}$	$\begin{bmatrix} \mathbf{x} \\ \mathbf{x}\mathbf{x}^T \end{bmatrix}$	$-\frac{1}{4}\boldsymbol{\eta}_1^T\boldsymbol{\eta}_2^{-1}\boldsymbol{\eta}_1 - \frac{1}{2}\ln\lvert-2\boldsymbol{\eta}_2\rvert$