CSCI 470: Graphs, Depth-First-Search, Topological Sort

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Overview

1. DFS
2. DFS properties
3. Topological sort
4. Strongly connected components
5. Breadth-First Search

DFS

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- Once all of v's edges have been explored, the search "backtracks" to explore edges leaving the vertex from which v was discovered.
- This process continues until we have discovered all the vertices that are reachable from the original source vertex.
- If any undiscovered vertices remain, then depth-first search selects one of them as a new source, and it repeats the search from that source. The algorithm repeats this entire process until it has discovered every vertex.

DFS: pseudocode

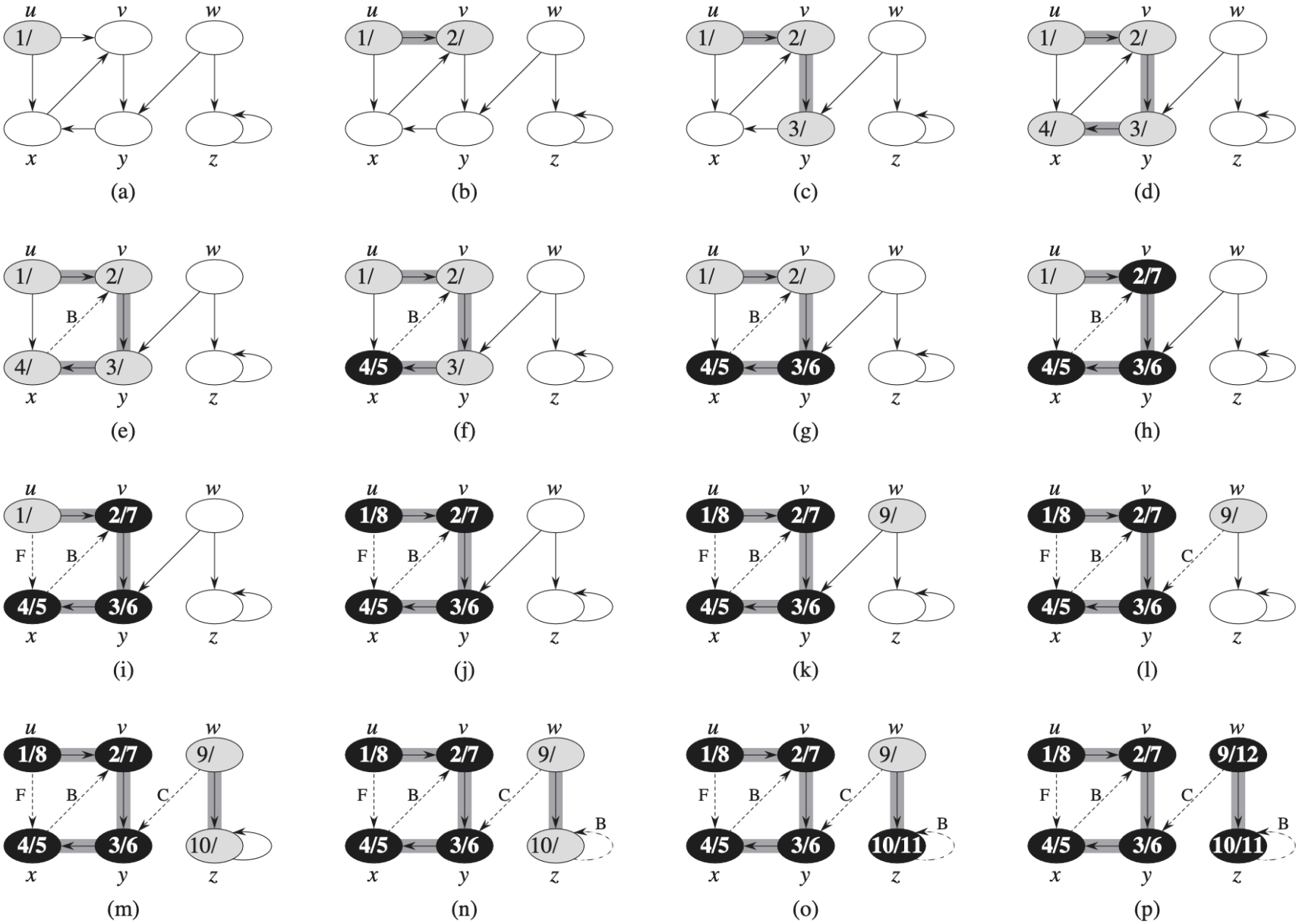
```
DFS(G)
1  for each vertex u ∈ G.V
2        u.color = WHITE
3        u.π = NIL
4  time = 0
5  for each vertex u ∈ G.V
6        if u.color == WHITE
7              DFS-VISIT(G, u)
```

DFS-Visit: pseudocode

DFS-VISIT(G, u)

```
 1  time = time + 1 // white vertex u has just been discovered
 2  u.d = time
 3  u.color = GRAY
 4  for each v ∈ G.Adj[u] // explore edge (u, v)
 5        if v.color == WHITE
 6              v.π = u
 7              DFS-VISIT(G, v)
 8  u.color = BLACK // blacken u; it is finished
 9  time = time + 1
10  u.f = time
```

Illustration

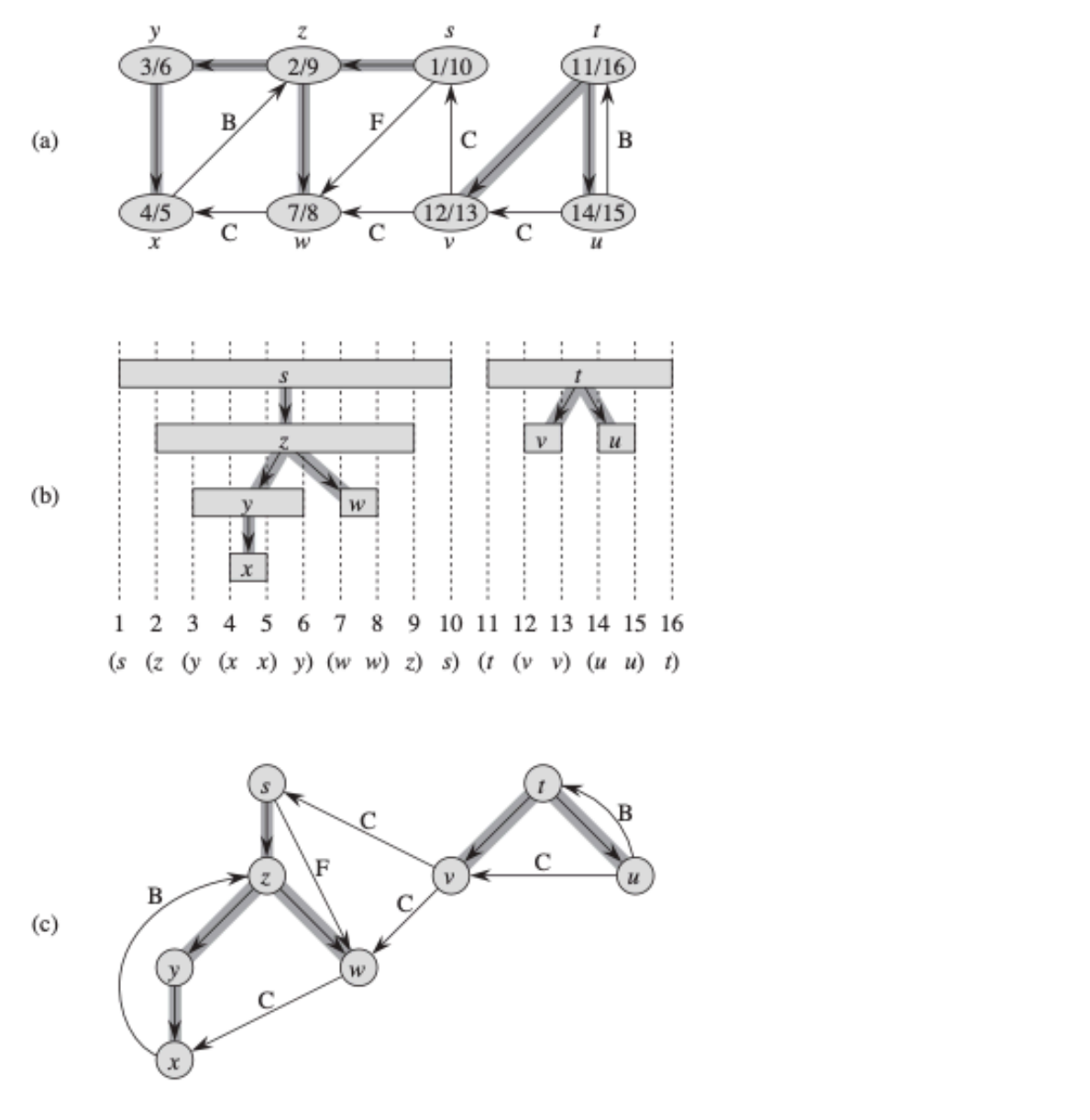


DFS properties

Properties of depth-first search

1. Vertex v is a descendent of vertex u in the depth-first forest if and only if v is discovered during the time that u is gray.
2. **Parenthesis structure**: If v is a descendent of u in the depth-first forest, then $[v.d, v.f]$ is contained entirely within $[u.d, u.f]$. If neither vertex is a descendent of the other, then the intervals are disjoint.
3. **White path theorem**: In a depth-first forest, v is a descendent of u if and only if at the time $u.d$ that the search discovers u, there is a path from u to v consisting entirely of white vertices.

Properties of DFS



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In any depth-first search of a (directed or undirected) graph $G = (V, E)$, for any two vertices u and v, exactly one of the following three conditions holds:

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- the interval $[v.d, v.f]$ is contained entirely within the interval $[u.d, u.f]$, and v is a descendant of u in a depth-first tree.

Parenthesis structure: Proof

We begin with the case in which $u.d < v.d$. We consider two subcases, according to whether $v.d < u.f$ or not.

1. The first case occurs when $v.d < u.f$, so v was discovered while u was still gray, which implies that v is a descendant of u. Moreover, since v was discovered more recently than u, all of its outgoing edges are explored, and v is finished, before the search returns to and finishes u. In this case, therefore, the interval $[v.d, v.f]$ is entirely contained within the interval $[u.d, u.f]$.

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2. In the other subcase, $u.f < v.d$, and $u.d < u.f$, $u.d < u.f < v.d < v.f$; thus the intervals $[u.d, u.f]$ and $[v.d, v.f]$ are disjoint. Because the intervals are disjoint, neither vertex was discovered while the other was gray, and so neither vertex is a descendant of the other.

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    - The case in which $v.d < u.d$ is similar, with the roles u and v reversed in the above argument. □

Corollary 22.8 (Nesting of descendant's intervals)

Corollary: Vertex v is a proper descendant of vertex u in the depth-first search forest for a (directed or undirected) graph G if and only if $u.d < v.d < v.f < u.f$.
This can immediately be shown using the first case in the previous proof.

White path theorem

In a depth-first forest, v is a descendent of u if and only if at the time $u.d$ that the search discovers u, there is a path from u to v consisting entirely of white vertices.

White path theorem: proof ($\Rightarrow$)

$\Rightarrow$:

- If $v = u$, then the path from u to v contains just vertex u, which is still white when we set the value of $u.d$.

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- If $v = u$, then the path from u to v contains just vertex u, which is still white when we set the value of $u.d$.
- Now, suppose that v is a proper descendent of u in the depth-first forest. By Corollary 22.8, $u.d < v.d$, and so v is white at the time $u.d$. Since v can be any descendant of u, all vertices on the unique simple path from u to v in the depth-first forest are white at time $u.d$.

White path theorem: proof ($\Leftarrow$)

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- Suppose that there is a path of white vertices from u to v at time $u.d$, but v does not become a descendant of u in the depth-first tree.

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- Without loss of generality, assume that every vertex other than v along the path becomes a descendant of u. (Otherwise, let v be the closest vertex to u along the path that doesn't become a descendant of u.)

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- Let w be the predecessor of v in the path, so that w is a descendant of u (w and u may in fact be the same vertex).
- By Corollary 22.8, $w.f \leq u.f$. Because v must be discovered after u is discovered, but before w is finished, we have $u.d < v.d < w.f \leq u.f$. Theorem 22.7 then implies that the interval $[v.d, v.f]$ is contained entirely within the interval $[u.d, u.f]$. By Corollary 22.8, v must after all be a descendant of u. $\square$

Classification of edges

- ***Tree edges*** are edges in the depth-first forest G_π. Edge (u, v) is a tree edge if v was first discovered by exploring edge (u, v).
- ***Back edges*** are edges (u, v) connecting a vertex u to an ancestor v in a depth-first tree. We consider self-loops, which may occur in directed graphs, to be back edges.
- ***Forward edges*** are those nontree edges (u, v) connecting a vertex u to be a descendant v in a depth-first tree. Non-tree edge is an edge leading to a marked vertex.
- ***Cross edges*** are all other edges. They can go between vertices in the same depth-first tree, as long as one vertex is not an ancestor of the other, or they can go between vertices in different depth-first trees. Alternatively, Cross edges point from a node to a previously visited node that is neither an ancestor nor a descendant.

Topological sort

Topological sort

- A ***toplogical sort*** of a dag $G = (V, E)$ is a linear ordering of all its vertices such that if G contains an edge (u, v), then u appears before v in the ordering.
- If a graph contains a cycle, then no linear order is possible.
- We can view a topological sort of a graph as an ordering of its vertices along a horizontal line so that all directed edges go from left to right.
- Topological sorting is thus different from the usual kind of "sorting" as compared to what we did in previous lectures.

Topological sort: pseudocode

TOPOLOGICAL-SORT(G)

1 call DFS(G) to compute finishing times $v.f$ for each vertex v
2 as each vertex is finished, insert it onto the front of a linked list
3 **return** the linked list of vertices

- We can perform a topological sort in time $\Theta(V + E)$, since depth-first search takes $O(1)$ time to insert each of the $|V|$ vertices onto the front of the linked list.

Topological sort: Figure

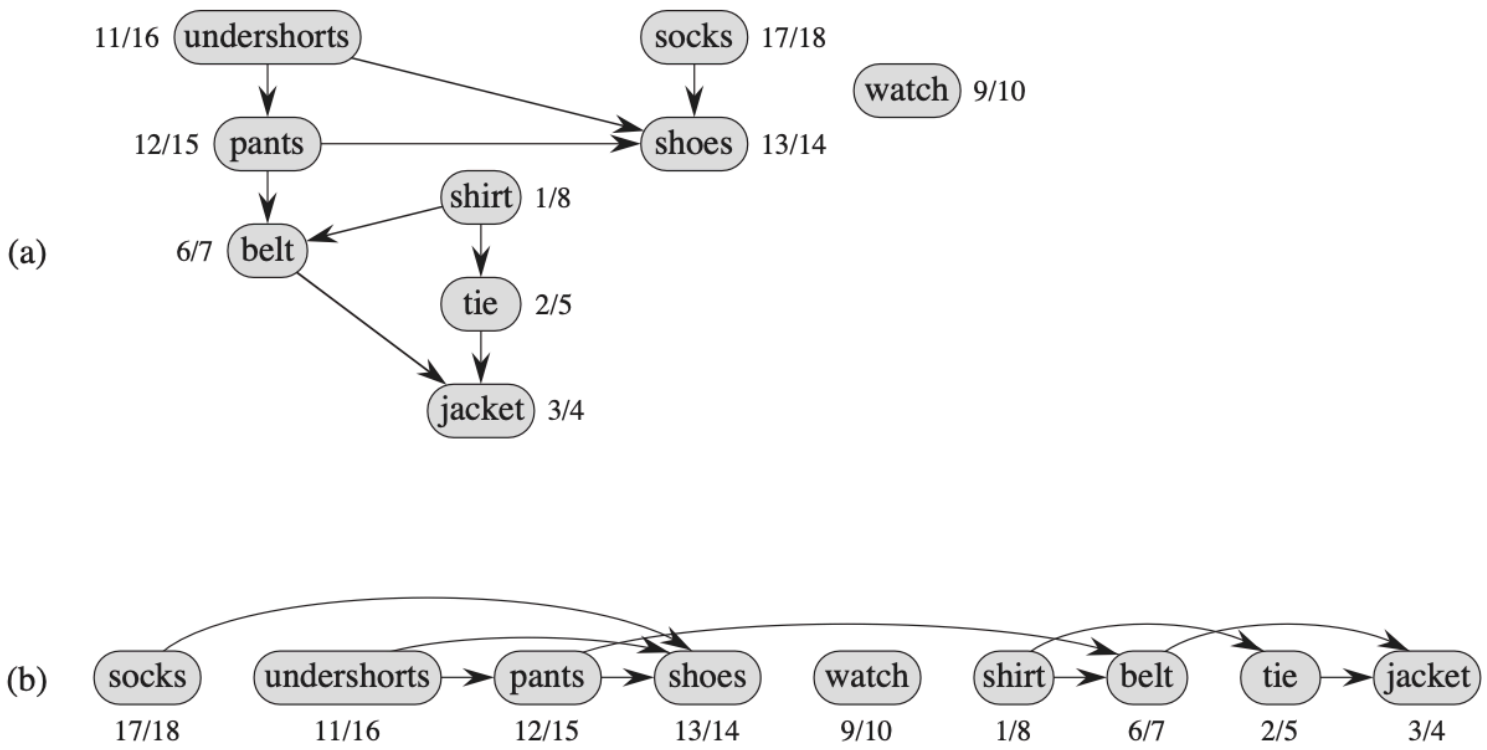

Figure 22.7 **(a)** Professor Bumstead topologically sorts his clothing when getting dressed. Each directed edge (u, v) means that garment u must be put on before garment v. The discovery and finishing times from a depth-first search are shown next to each vertex. **(b)** The same graph shown topologically sorted, with its vertices arranged from left to right in order of decreasing finishing time. All directed edges go from left to right.

Topological sort: Lemma 22.11

Lemma 22.11 A directed graph G is a acyclic if and only if a depth-first search of G yields no back edges.

Lemma 22.11: Proof ($\Rightarrow$)

$\Rightarrow$

- Suppose that a depth-first search produces a back edge (u, v). Then vertex v is an ancestor of vertex u in the depth-first forest. Thus, G contains a path from v to u, and the back edge (u, v) completes a cycle.

Lemma 22.11: Proof ($\Leftarrow$)

$\Leftarrow$

- Suppose that G contains a cycle c. We show that a depth-first search of G yields a back edge.
- Let v be the first vertex to be discovered in c, and let (u, v) be the preceding edge in c. At time $v.d$, the vertices of c form a path white vertices from v to u. By the white-path theorem, vertex u becomes a descendant of v in the depth-first forest. Therefore, (u, v) is a back edge.

Topological sort: Theorem 22.12

TOPOLOGICAL-SORT produces a topological sort of the directed acyclic graph provides as its input.

- Suppose that DFS is run on a given dag $G = (V, E)$ to determine finishing times for its vertices. It suffices to show that for any pair of distinct vertices $u, v \in V$, if G contains an edge from u to v, then $v.f < u.f$.

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- Consider any edge (u, v) explored in DFS(G). When this edge is explored, v cannot be gray, since then v would be an ancestor of u and (u, v) would be a back edge, contradicting Lemma 22.11. Therefore, v must be either white or black.

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- If v is white, it becomes a descendant of u, and so $v.f < u.f$. If v is black, it has already been finished, so that $v.f$ has already been set.
- Because we are still exploring from u, we have yet to assign a timestamp to $u.f$, and so once we do, we will have $v.f < u.f$ as well. Thus, for any edge (u, v) in the dag, we have $v.f < u.f$, proving the theorem.

Strongly connected components

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- An undirected graph is connected if every vertex is reachable from all other vertices. That is, for every pair of vertices $u, v \in V$, there is a path from u to v (and therefore, a path from v to u).
- A directed graph G is strongly connected if every two vertices are reachable from each other. That is, for every pair of vertices $u, v \in V$, there is both a path from u to v and a path from v to u.

Strongly connected components

- A strongly connected component of a directed graph $G = (V, E)$ is a maximal part of the graph that is strongly connected. Formally, it is a maximal set of vertices $C \subseteq V$ such that for every pair of vertices $u, v \in C$, there is both a path from u to v and a path from v to u.
- A connected component of an undirected graph is a maximal set of vertices where every vertex in the set is reachable from every vertex in the set. That is, for every pair of vertices u, v in the connected component C, there is a path from u to v.

SCC: Figure

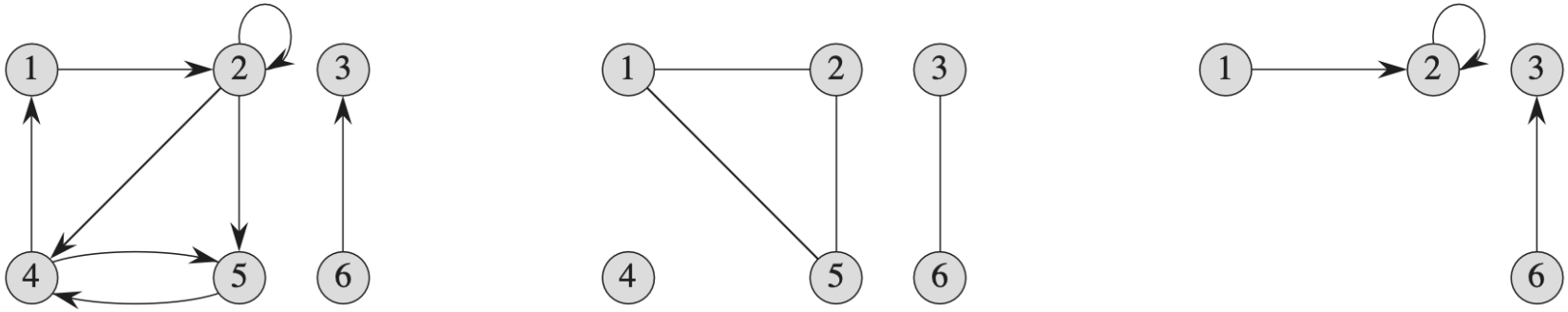


Figure B.2 Directed and undirected graphs. **(a)** A directed graph $G = (V, E)$, where $V = \{1, 2, 3, 4, 5, 6\}$ and $E = \{(1, 2), (2, 2), (2, 4), (2, 5), (4, 1), (4, 5), (5, 4), (6, 3)\}$. The edge $(2, 2)$ is a self-loop. **(b)** An undirected graph $G = (V, E)$, where $V = \{1, 2, 3, 4, 5, 6\}$ and $E = \{(1, 2), (1, 5), (2, 5), (3, 6)\}$. The vertex 4 is isolated. **(c)** The subgraph of the graph in part (a) induced by the vertex set $\{1, 2, 3, 6\}$.

SCC

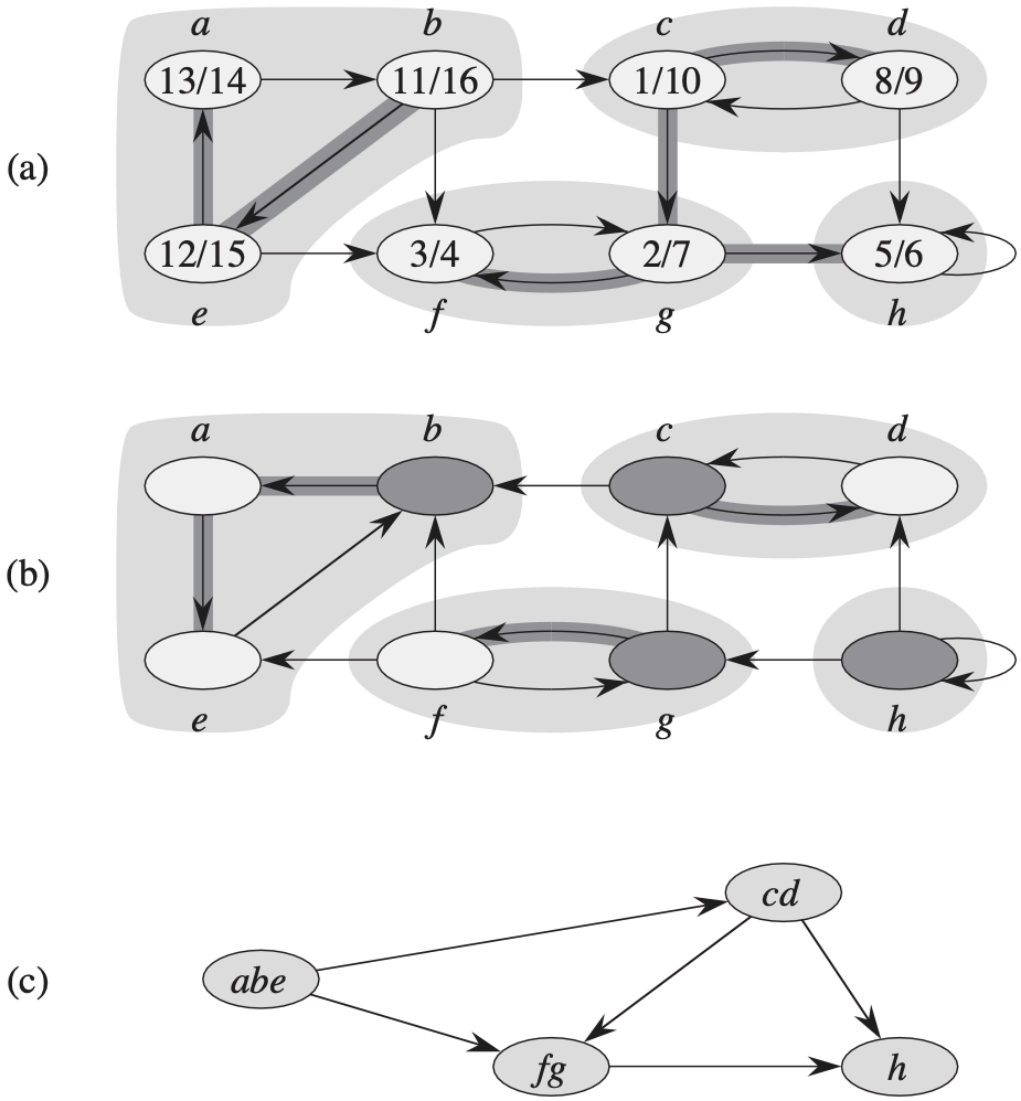
- Our algorithm for finding strongly connected components of a graph $G = (V, E)$ uses the transpose of G.
- We define $G^T = (V, E^T)$, where $E^T = \{(u, v) : (v, u) \in E\}$. That is, E^T consists of the edges of G with their directions reversed.
- It is interesting to observe that G and G^T have exactly the same strongly connected components: u and v are reachable from each other in G if and only if they are reachable from each other in G^T.

SCC: pesudocode

STRONGLY-CONNECTED-COMPONENTS(G)

1 call DFS to compute finishing times $u.f$ for each vertex u
2 compute G^T
3 call DFS(G^T), but in the main loop DFS, consider the vertices in order of decreasing $u.f$ (as computed in line 1)
4 output the vertices of each tree in the depth-first forest formed in line 3 as separate strongly connected component

SCC: Figure


(a)
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Breadth-First Search

BFS