1 Radial Density Function

1.1 Calculation of Distances with Periodicity

Suppose a large chemical structure has uncountably many atoms but the follow a periodic pattern of n atoms every p Angstroms. The atom locations within a period are given by $a_1, a_2, \ldots, a_n$ where $a_i \in \mathbb{R}^3$. The radial density function is the distribution of pairwise distances between these atoms.

The distances d between atoms a_i and a_j where $i \neq j$, atom a_i has been displaced by x, and atom a_j has been displaced by y per the periodicity is

$$\begin{aligned} d^2 &= \langle a_i + x - (a_j + y), a_i + x - (a_j + y) \rangle \\ &= \langle a_i - a_j, a_i - a_j \rangle + \langle x - y, x - y \rangle + 2\langle a_i - a_j, x - y \rangle \end{aligned}$$

where $x = (k_1 p, k_2 p, k_3 p)$ for $k_i \in \mathbb{Z}$ and $y = (l_1 p, l_2 p, l_3 p)$ for $l_i \in \mathbb{Z}$. Here $\langle x, y \rangle$ denotes the inner product between x and y.

Suppose D is a random variable that samples at random the distances, d, in the chemical structure. The radial density function is the probability density function of this random variable. This function can be estimated empirically via a histogram.

1.2 Adding Noise For Atom Vibration

Due to the vibrations of the molecules, the radial density function will not be just the equilibrium positions. We can approximate this fluctuation in distances via a Gaussian filter or Weierstrass transform.

$$F(x) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} f(y) e^{-\frac{(x-y)^2}{4t}} dy$$

Given that the density function is only defined for a finite number of distances, we use a discrete version of the transform making sure to keep the sum of the weights equal to one.

$$F(d_k) = \frac{\sum_{d_i=d_0}^{d_n} f(d_i) \exp\left(-\frac{(d_k-d_i)^2}{4t}\right)}{\sum_{d_i=d_0}^{d_n} \exp\left(-\frac{(d_k-d_i)^2}{4t}\right)}$$

where d_0 is the minimum distance and d_n is the maximum distance.

1.3 Cubane Example

As an example of the above, below are the calculations for cubane (C_8H_8).

Here are the coordinates of the elements in cubane in Angstroms.

```
Element, x, y, z
C, 1.2455, 0.5367,-0.0729
C, 0.9239,-0.9952, 0.0237
C,-0.1226,-0.7041, 1.1548
C, 0.1989, 0.8277, 1.0582
C, 0.1226, 0.7042,-1.1548
C,-0.9239, 0.9952,-0.0237
C,-1.2454,-0.5367, 0.0729
C,-0.1989,-0.8277,-1.0582
H, 2.2431, 0.9666,-0.1313
H, 1.6638,-1.7924, 0.0426
H,-0.2209,-1.2683, 2.0797
H, 0.3583, 1.4907, 1.9059
H, 0.2208, 1.2681,-2.0799
H,-1.6640, 1.7922,-0.0427
H,-2.2430,-0.9665, 0.1313
H,-0.3583,-1.4906,-1.9058
```

RDF before smoothing

RDF after smoothing

Sources

http://en.wikipedia.org/wiki/Atom_vibrations

http://en.wikipedia.org/wiki/Radial_distribution_function