$$\begin{array}{rll}
d_2 - d_1 &= \sqrt{(x-(-c))^2 + (y-0)^2} - \sqrt{(x-c)^2 + (y-0)^2} = 2a & \text{Distance Formula} \\
\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} &= 2a & \text{Simplify expressions.} \\
\sqrt{(x+c)^2 + y^2} &= 2a + \sqrt{(x-c)^2 + y^2} & \text{Move radical to opposite side.} \\
(x+c)^2 + y^2 &= \left(2a + \sqrt{(x-c)^2 + y^2}\right)^2 & \text{Square both sides.} \\
x^2 + 2cx + c^2 + y^2 &= 4a^2 + 4a\sqrt{(x-c)^2 + y^2} + (x-c)^2 + y^2 & \text{Expand the squares.} \\
x^2 + 2cx + c^2 + y^2 &= 4a^2 + 4a\sqrt{(x-c)^2 + y^2} + x^2 - 2cx + c^2 + y^2 & \text{Expand remaining square.} \\
2cx &= 4a^2 + 4a\sqrt{(x-c)^2 + y^2} - 2cx & \text{Combine like terms.} \\
4cx - 4a^2 &= 4a\sqrt{(x-c)^2 + y^2} & \text{Isolate the radical.} \\
cx - a^2 &= a\sqrt{(x-c)^2 + y^2} & \text{Divide by 4.} \\
\left(cx - a^2\right)^2 &= a^2\left(\sqrt{(x-c)^2 + y^2}\right)^2 & \text{Square both sides.} \\
c^2x^2 - 2a^2cx + a^4 &= a^2\left(x^2 - 2cx + c^2 + y^2\right) & \text{Expand the squares.} \\
c^2x^2 - 2a^2cx + a^4 &= a^2x^2 - 2a^2cx + a^2c^2 + a^2y^2 & \text{Distribute } a^2. \\
a^4 + c^2x^2 &= a^2x^2 + a^2c^2 + a^2y^2 & \text{Combine like terms.} \\
c^2x^2 - a^2x^2 - a^2y^2 &= a^2c^2 - a^4 & \text{Rearrange terms.} \\
x^2\left(c^2 - a^2\right) - a^2y^2 &= a^2\left(c^2 - a^2\right) & \text{Factor common terms.} \\
x^2b^2 - a^2y^2 &= a^2b^2 & \text{Set } b^2 = c^2 - a^2. \\
\frac{x^2b^2}{a^2b^2} - \frac{a^2y^2}{a^2b^2} &= \frac{a^2b^2}{a^2b^2} & \text{Divide both sides by } a^2b^2 \\
\frac{x^2}{a^2} - \frac{y^2}{b^2} &= 1 &
\end{array}$$

This equation defines a hyperbola centered at the origin with vertices $(\pm a, 0)$ and co-vertices $(0 \pm b)$.

Standard Forms of the Equation of a Hyperbola with Center (0,0)

The standard form of the equation of a hyperbola with center $(0, 0)$ and transverse axis on the x-axis is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where

- the length of the transverse axis is $2a$
- the coordinates of the vertices are $(\pm a, 0)$
- the length of the conjugate axis is $2b$
- the coordinates of the co-vertices are $(0, \pm b)$
- the distance between the foci is $2c$, where $c^2 = a^2 + b^2$
- the coordinates of the foci are $(\pm c, 0)$
- the equations of the asymptotes are $y = \pm \frac{b}{a}x$

See Figure 5**a**.

The standard form of the equation of a hyperbola with center $(0, 0)$ and transverse axis on the y-axis is

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

where

- the length of the transverse axis is $2a$
- the coordinates of the vertices are $(0, \pm a)$
- the length of the conjugate axis is $2b$
- the coordinates of the co-vertices are $(\pm b, 0)$

- the distance between the foci is $2c$, where $c^2 = a^2 + b^2$
- the coordinates of the foci are $(0, \pm c)$
- the equations of the asymptotes are $y = \pm \frac{a}{b} x$

See Figure 5**b**.

Note that the vertices, co-vertices, and foci are related by the equation $c^2 = a^2 + b^2$. When we are given the equation of a hyperbola, we can use this relationship to identify its vertices and foci.

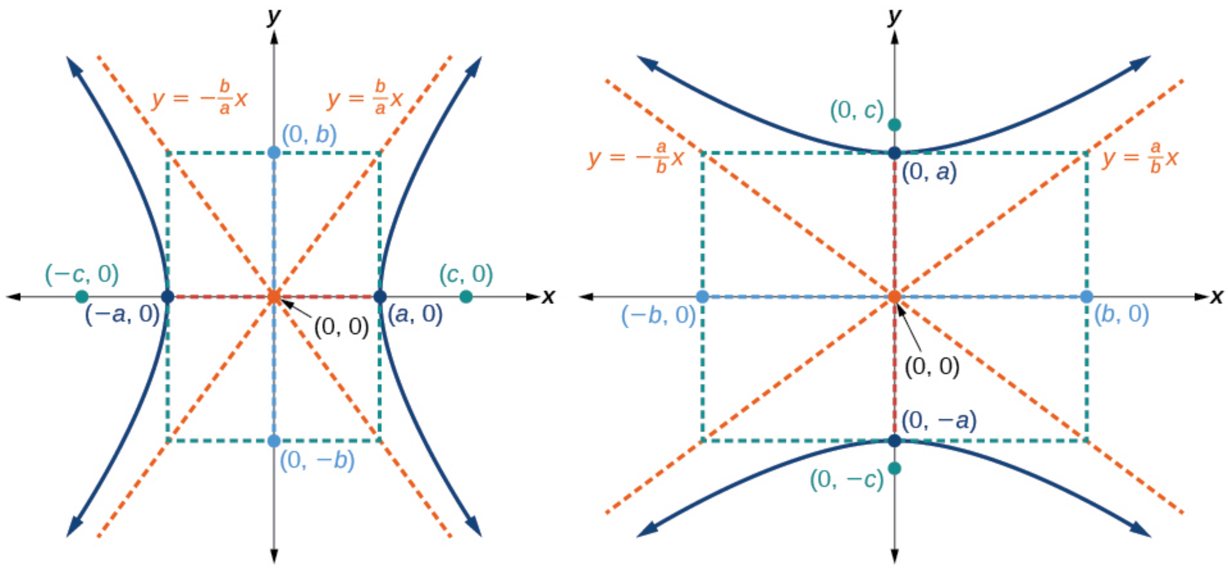

Figure 5 (a) Horizontal hyperbola with center $(0, 0)$ (b) Vertical hyperbola with center $(0, 0)$

HOW TO

Given the equation of a hyperbola in standard form, locate its vertices and foci.

1. Determine whether the transverse axis lies on the *x*- or *y*-axis. Notice that a^2 is always under the variable with the positive coefficient. So, if you set the other variable equal to zero, you can easily find the intercepts. In the case where the hyperbola is centered at the origin, the intercepts coincide with the vertices.
   a. If the equation has the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then the transverse axis lies on the *x*-axis. The vertices are located at $(\pm a, 0)$, and the foci are located at $(\pm c, 0)$.
   b. If the equation has the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, then the transverse axis lies on the *y*-axis. The vertices are located at $(0, \pm a)$, and the foci are located at $(0, \pm c)$.
2. Solve for a using the equation $a = \sqrt{a^2}$.
3. Solve for c using the equation $c = \sqrt{a^2 + b^2}$.

EXAMPLE 1

Locating a Hyperbola's Vertices and Foci

Identify the vertices and foci of the hyperbola with equation $\frac{y^2}{49} - \frac{x^2}{32} = 1$.

Solution

The equation has the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, so the transverse axis lies on the *y*-axis. The hyperbola is centered at the