Two very special right triangles

Mr. Worley

Introduction

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- Their simplicity makes them special. Hmm... this seems demeaning.
- Elegance! Let's call them elegant.
- We can determine their angles AND lengths, without trigonometry!

Triangles

- Triangles are polygons with 3 sides.
- The sum of the interior angles equals 180°, or π radians.
- But you knew all that already, right?

Right Triangles

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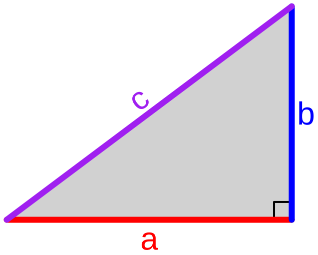
- Common... this is a very important theorem...
- Yup! The Pythagorean Theorem.

Pythagorean Theorem



$$a^2 + b^2 = c^2$$

- We call the two shorter side the “legs”, and the longest side the “hypotenuse”.
- If a triangle is right, then the sum of the squares of the legs equals the square of the hypotenuse.

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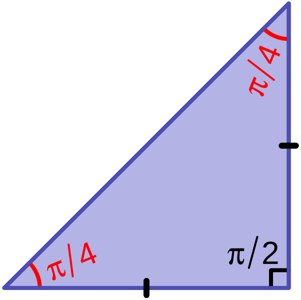
- Can you predict the angle measures of a right isosceles triangle?
- In degrees?
- In radians?

Picture of IRT

- Here's a picture of IRT. Elegant IRT.



- The angle measures are 45°, 45°, and 90°.
- In radians, the angle measures are $\frac{\pi}{4}$, $\frac{\pi}{4}$, and $\frac{\pi}{2}$.
- The two legs are congruent.

The lengths of IRT

- Earlier I said we'd also get the lengths, without trigonometry!

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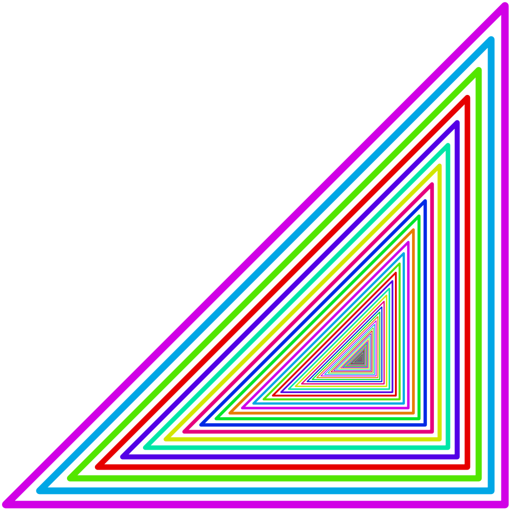
- Earlier I said we'd also get the lengths, without trigonometry!
- But the lengths can be anything. We can double the size and get another IRT.
- There are infinite IRTs of varying sizes!

So many IRTs!



- Ahhh! Let's pick a specific IRT.

Set the hypotenuse length to 1.

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- The unit circle!

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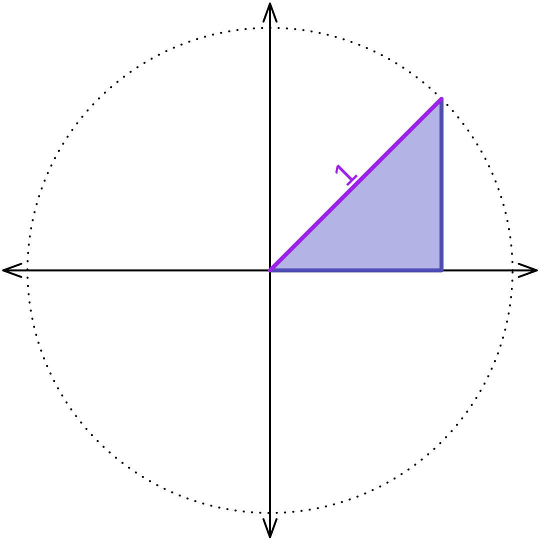
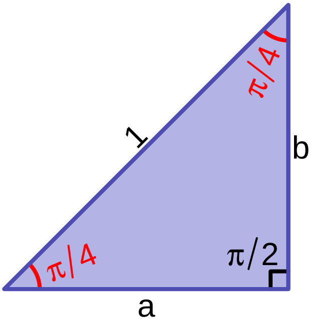
- The unit circle!
- If the hypotenuse is 1, it can connect the origin to the edge of the unit circle. (The unit circle has a radius of 1, and we usually assume it is centered at the origin.)

IRT with hypotenuse as a radius of unit circle



- How do I find the lengths of the legs?

Maybe Pythagoras can help?



- Hmm... we know IRT is isosceles... so...

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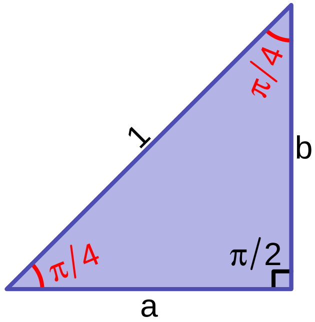

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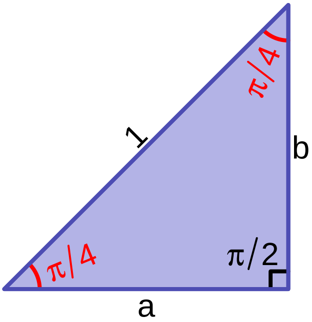
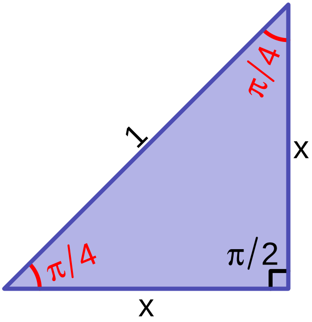


- Hmm... we know IRT is isosceles... so...
- $a = b$
- Let's just use x because x is our favorite variable.

The legs of IRT are congruent!



- It's time for Pythagoras!

$$x^2 + x^2 = 1^2$$

Al-Jabr (Algrebra) for IRT

- Solve $x^2 + x^2 = 1^2$.

Al-Jabr (Algebra) for IRT

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- Combine like terms.

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- Also, $1^2 = 1$.

$$2x^2 = 1$$

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$$x^2 = \frac{1}{2}$$

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$$x = \sqrt{\frac{1}{2}}$$

Al-Jabr (Algrebra) for IRT cont.

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Al-Jabr (Algebra) for IRT cont.

- $$x = \sqrt{\frac{1}{2}}$$
- For reasons, when we simplify, we do not leave fractions inside a radical. Distribute the radical to both factors of the quotient. This is allowed because $\left(\frac{g}{h}\right)^p \equiv \frac{g^p}{h^p}$ and $\sqrt{z} \equiv z^{1/2}$.

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Al-Jabr (Algrebra) for IRT cont..

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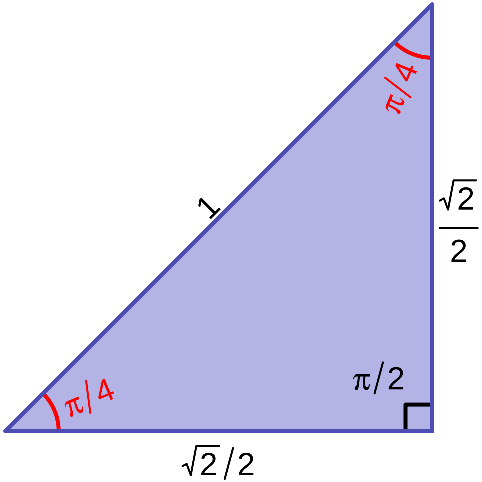
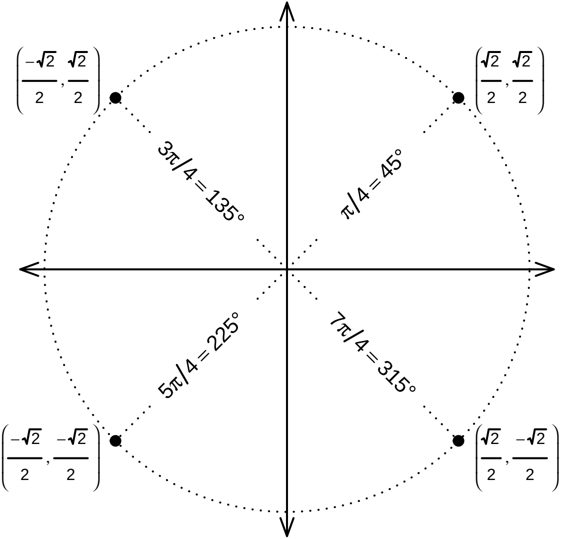
- $$x = \frac{1 \cdot \sqrt{2}}{\sqrt{2} \cdot \sqrt{2}}$$
- Simplify.
- $$x = \frac{\sqrt{2}}{2}$$

Boom! We know everything about IRT, without using trigonometry!



Apply IRT knowledge to the unit circle (use symmetry)



Trigonometric implications of IRT

$$\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \qquad \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \qquad \tan\left(\frac{\pi}{4}\right) = 1$$

$$\cos\left(\frac{3\pi}{4}\right) = \frac{-\sqrt{2}}{2} \qquad \sin\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2} \qquad \tan\left(\frac{3\pi}{4}\right) = -1$$

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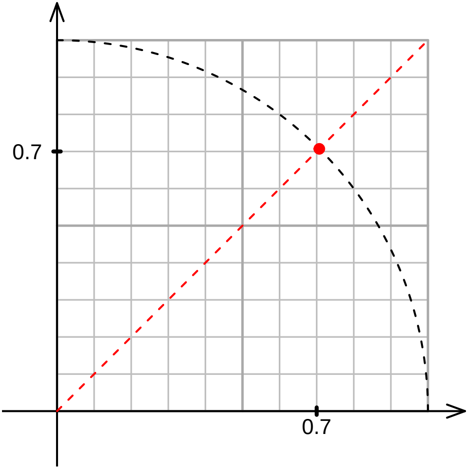
$$\cos\left(\frac{7\pi}{4}\right) = \frac{\sqrt{2}}{2} \qquad \sin\left(\frac{7\pi}{4}\right) = \frac{-\sqrt{2}}{2} \qquad \tan\left(\frac{7\pi}{4}\right) = -1$$

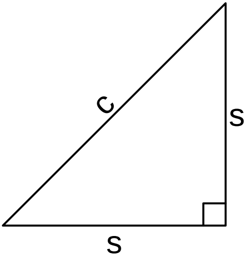
Can I get a decimal please?

$$\frac{\sqrt{2}}{2} \approx 0.7071068$$



Does IRT show up on the SAT?

- IRT is on the SAT formula sheet. But, NOT with hypotenuse=1. Instead...



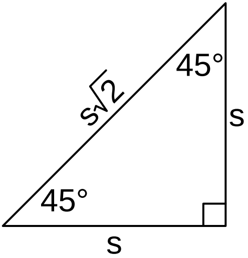
$$s^2 + s^2 = c^2$$

$$2s^2 = c^2$$

$$\sqrt{2} \cdot s = c$$

IRT on the SAT

- Most people will call this the 45-45-90 triangle.



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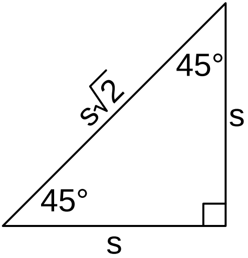

- For example, if a square has a width of 5 meters, how long is the diagonal?

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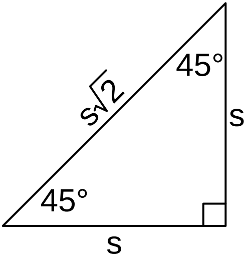

- For example, if a square has a width of 5 meters, how long is the diagonal?
- $5\sqrt{2} \;\approx\; 5 \cdot 1.414 \;\approx\; 7.07$ meters

There's ANOTHER special right triangle????

- Yes.

There's ANOTHER special right triangle????

- Yes.
- We'll cover this one much faster.

Cut an equilateral triangle in half.

- What is an equilateral triangle?

Cut an equilateral triangle in half.

- What is an equilateral triangle?
- An equilateral triangle has three congruent sides and three congruent angles. How big is each interior angle of an equilateral triangle?

Cut an equilateral triangle in half.

- What is an equilateral triangle?
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- Each angle of an equilateral is 60°, which is $\frac{\pi}{3}$ radians.

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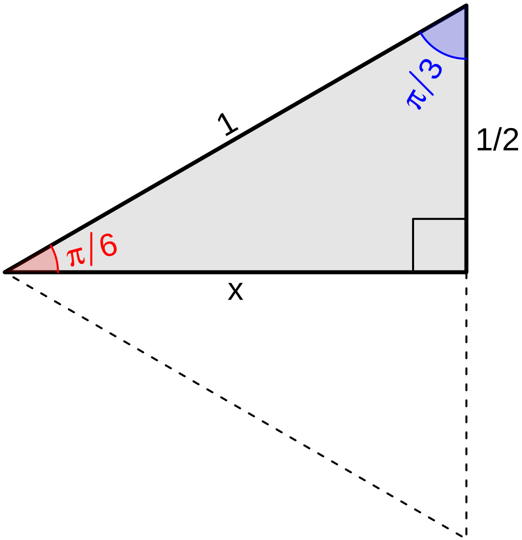
- Each angle of an equilateral is 60°, which is $\frac{\pi}{3}$ radians.
- If we bisect an angle, the resulting angle measure is. . .

Cut an equilateral triangle in half.

- What is an equilateral triangle?
- An equilateral triangle has three congruent sides and three congruent angles. How big is each interior angle of an equilateral triangle?
- Each angle of an equilateral is $60°$, which is $\frac{\pi}{3}$ radians.
- If we bisect an angle, the resulting angle measure is...
- $30°$ or $\frac{\pi}{6}$ radians.

Cut an equilateral triangle in half..



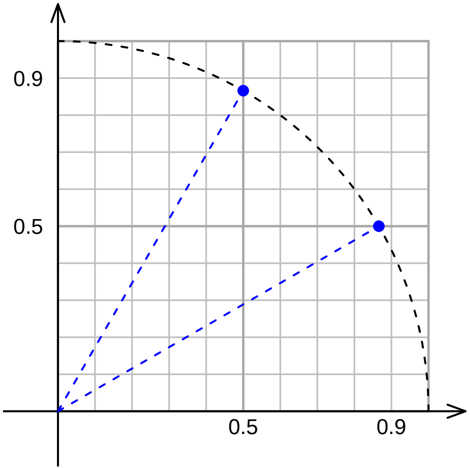
- We can call this one Half of an Equilateral Triangle (HET)

Algebra for HET

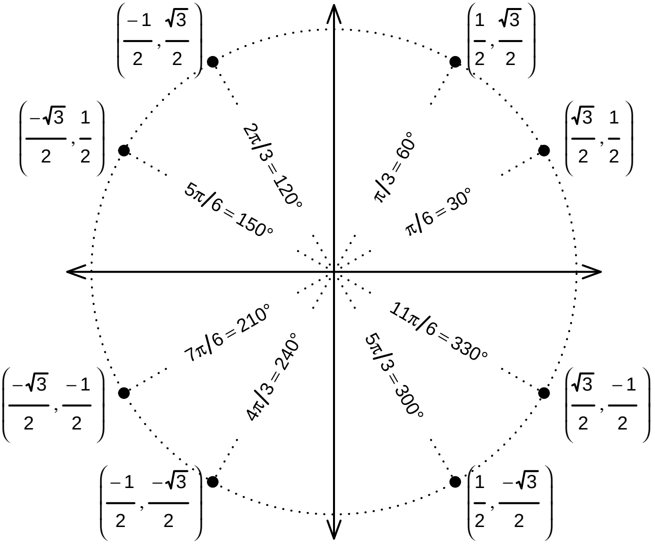
$$
\begin{aligned}
x^2 + \left(\frac{1}{2}\right)^2 &= 1^2 \\
x^2 + \frac{1}{4} &= 1 \\
x^2 &= 1 - \frac{1}{4} \\
x^2 &= \frac{3}{4} \\
x &= \sqrt{\frac{3}{4}} \\
x &= \frac{\sqrt{3}}{\sqrt{4}} \\
x &= \frac{\sqrt{3}}{2}
\end{aligned}
$$

Decimal approximation of $\sqrt{3}/2$

$$\frac{\sqrt{3}}{2} \approx 0.8660254$$



Unit circle coordinates from HET



Some trig implications from HET

$$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \qquad \sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \qquad \tan\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{3}$$

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$$\cos\left(\frac{2\pi}{3}\right) = \frac{-1}{2} \qquad \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2} \qquad \tan\left(\frac{2\pi}{3}\right) = -\sqrt{3}$$

$$\cos\left(\frac{5\pi}{6}\right) = \frac{-\sqrt{3}}{2} \qquad \sin\left(\frac{5\pi}{6}\right) = \frac{1}{2} \qquad \tan\left(\frac{5\pi}{6}\right) = \frac{-\sqrt{3}}{3}$$

How get $\tan(\pi/6)$?

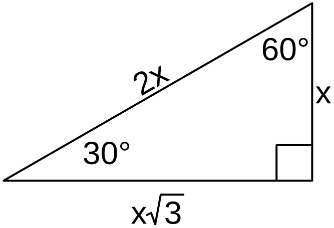
$$\begin{aligned}
\tan(\pi/6) &= \frac{\sin(\pi/6)}{\cos(\pi/6)} \\
&= \frac{1/2}{\sqrt{3}/2} \\
&= \frac{1}{\sqrt{3}} \\
&= \frac{1\cdot\sqrt{3}}{\sqrt{3}\cdot\sqrt{3}} \\
&= \frac{\sqrt{3}}{3}
\end{aligned}$$

How get $\tan(\pi/3)$?

$$\begin{aligned}
\tan(\pi/3) &= \frac{\sin(\pi/3)}{\cos(\pi/3)} \\
&= \frac{\sqrt{3}/2}{1/2} \\
&= \sqrt{3}
\end{aligned}$$

HET on the SAT

- The 30-60-90 triangle is on the SAT formula sheet. Notice, we can multiply our sides by $2x$ to get their version of HET.



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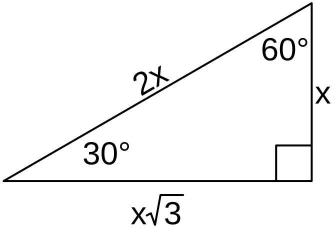

- Example: An equilateral triangle has side lengths of 10 meters; what is the height?

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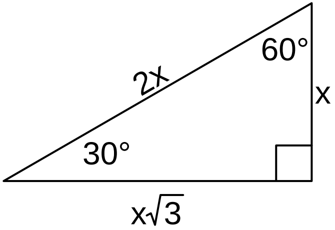
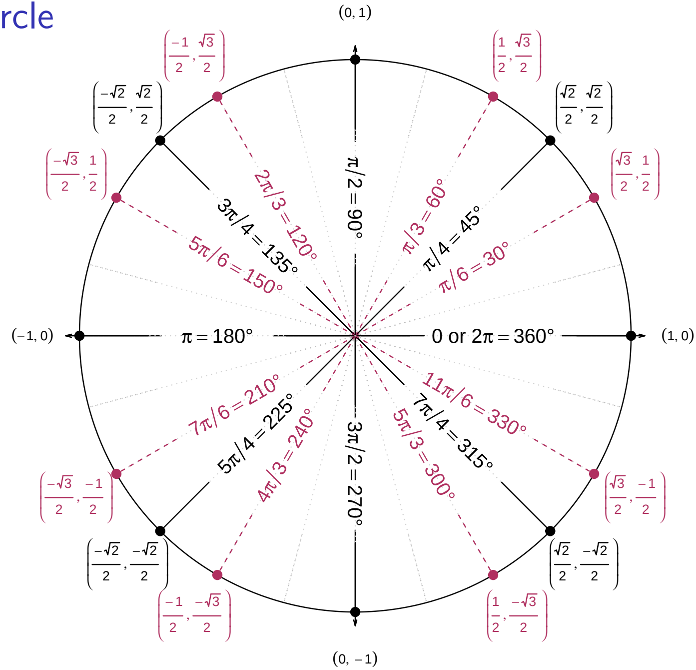


- Example: An equilateral triangle has side lengths of 10 meters; what is the height?
- If $10 = 2x$, then the height equals $5\sqrt{3} \approx 8.660254$

Unit circle



$(0,1)$
$(-1,0)$
$(1,0)$
$(0,-1)$
$\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$
$\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$
$\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$
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$\left(\frac{\sqrt{3}}{2},\frac{-1}{2}\right)$
0 or $2\pi = 360°$
$\pi/6 = 30°$
$\pi/4 = 45°$
$\pi/3 = 60°$
$\pi/2 = 90°$
$2\pi/3 = 120°$
$3\pi/4 = 135°$
$5\pi/6 = 150°$
$\pi = 180°$
$7\pi/6 = 210°$
$5\pi/4 = 225°$
$4\pi/3 = 240°$
$3\pi/2 = 270°$
$5\pi/3 = 300°$
$7\pi/4 = 315°$
$11\pi/6 = 330°$