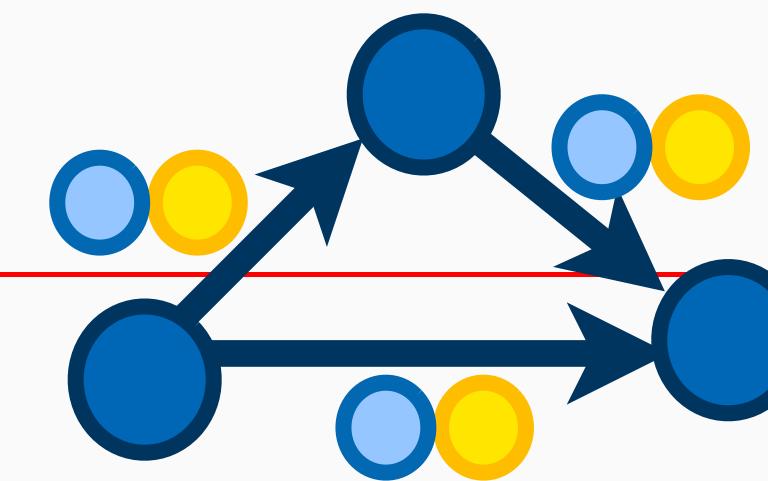
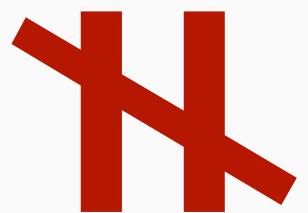
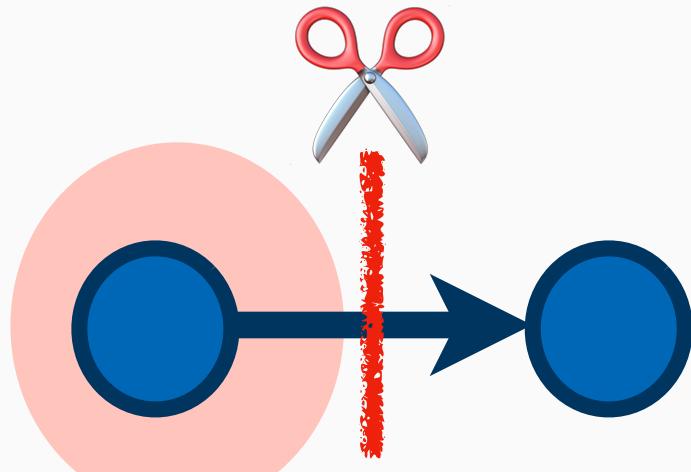


Minimum Cut, Bipartite Matching and Minimum Cost Maximum Flow with BGL

Min Cost Max Flow = MCMF



Daniel Graf (some slides by Andreas Bärtschi)

November 14, 2018

ETH Zürich

Recap: Basic Network Flows – What did we see last time?

One problem to rule them all...

Network Flow Algorithms: Ford-Fulkerson and Edmonds-Karp

- Solution: Keep track of the flow and allow paths that *reroute* units of flow. These are called *augmenting paths* in the *residual network*.
- Ford-Fulkerson (1955): Repeatedly take any augmenting path: running time $\mathcal{O}(m|f|)$.
- Edmonds-Karp (1972) / Dinitz (1970): Repeatedly take any shortest (as in: fewest edges) augmenting path: running time: best of $\mathcal{O}(m|f|)$ and $\mathcal{O}(m(nm))$ [BGL-Doc]. Simple bound for total flow: $|f| \leq n \max c$

6

Common tricks

- Multiple sources/sinks with e.g. $\infty \approx \sum_{e \in E} c(e)$
- Undirected graphs antiparallel edges with flow reducible to one direction
- Vertex capacities split into in-vertex and out-vertex
- Minimum flow per edge how to enforce $c_{\min}(e) \leq f(e) \leq c_{\max}(e)$?

[Exercise] ← 21

see solution for
Kingdom Defence

Flow Application: Edge Disjoint Paths

How many ways are there to get from HB to CAB without using the same street twice?

Map: search.ch, TomTom, swisstopo, OSM

22

Flow Application: Circulation Problem

- Multiple sources with a certain amount of flow to give (**supply**).
- Multiple sinks that want a certain amount of flow (**demand**).
- Model these as negative or positive demand per vertex d_v .
- Question: Is there a feasible flow? Surely not if $\sum_{v \in V} d_v \neq 0$. Otherwise? Add super-source and super-sink to get a maximum flow problem.

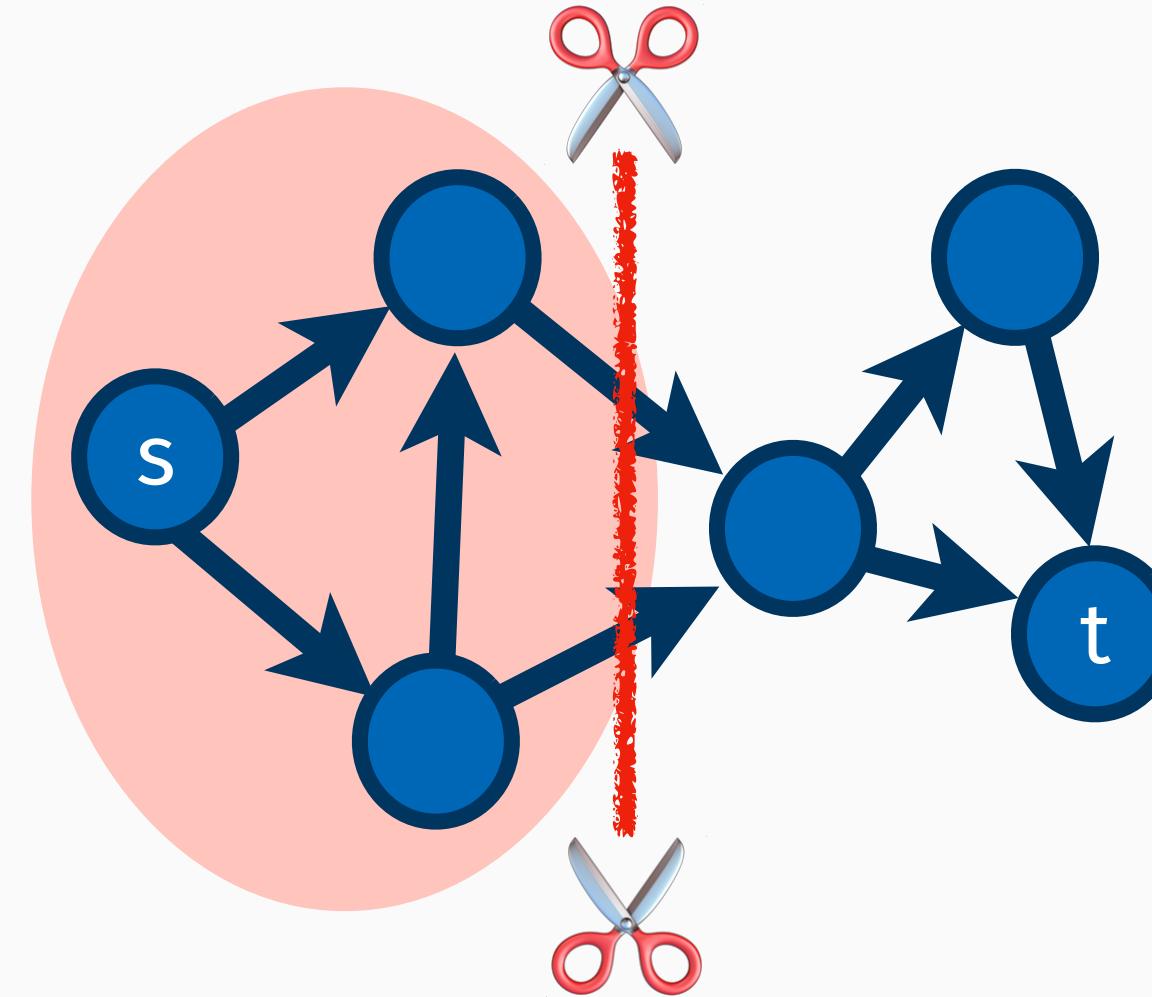
feasible flow exists no feasible flow exists

28

Today: Advanced Network Flow – What else are flows useful for?

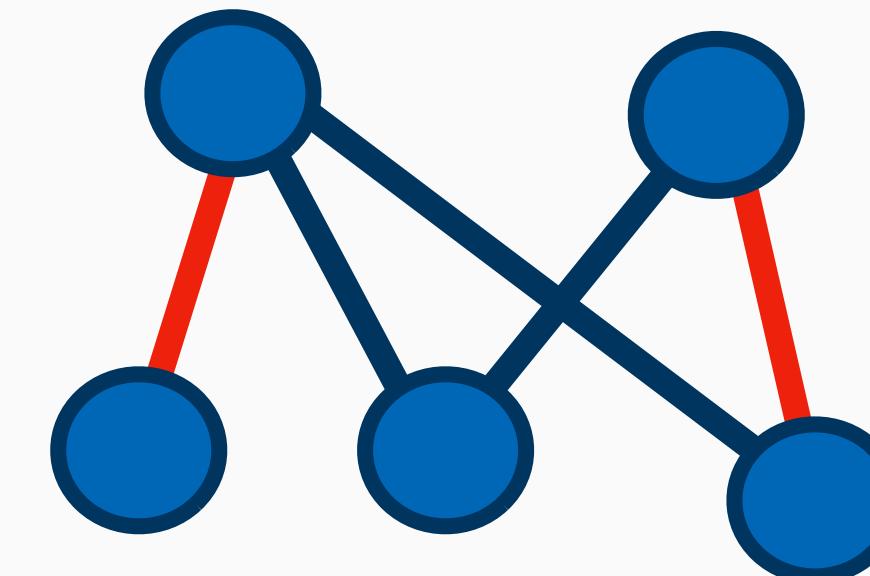
Minimum Cuts

- ▶ How to disconnect t from s cheaply?



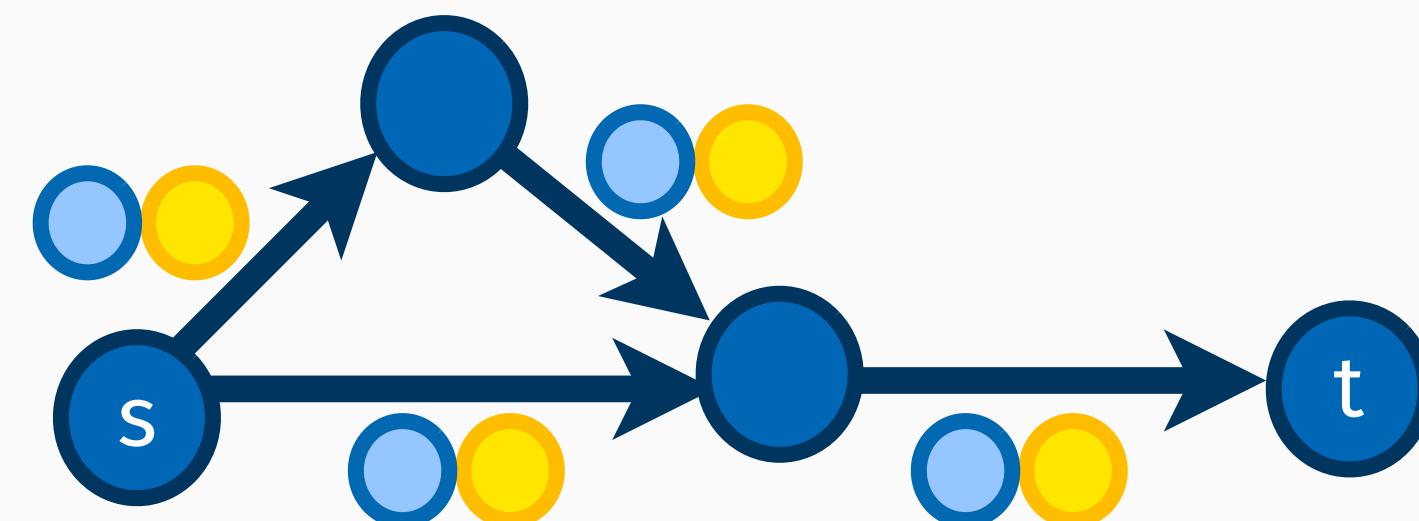
Bipartite Matching

- ▶ How to assign A's to B's effectively?



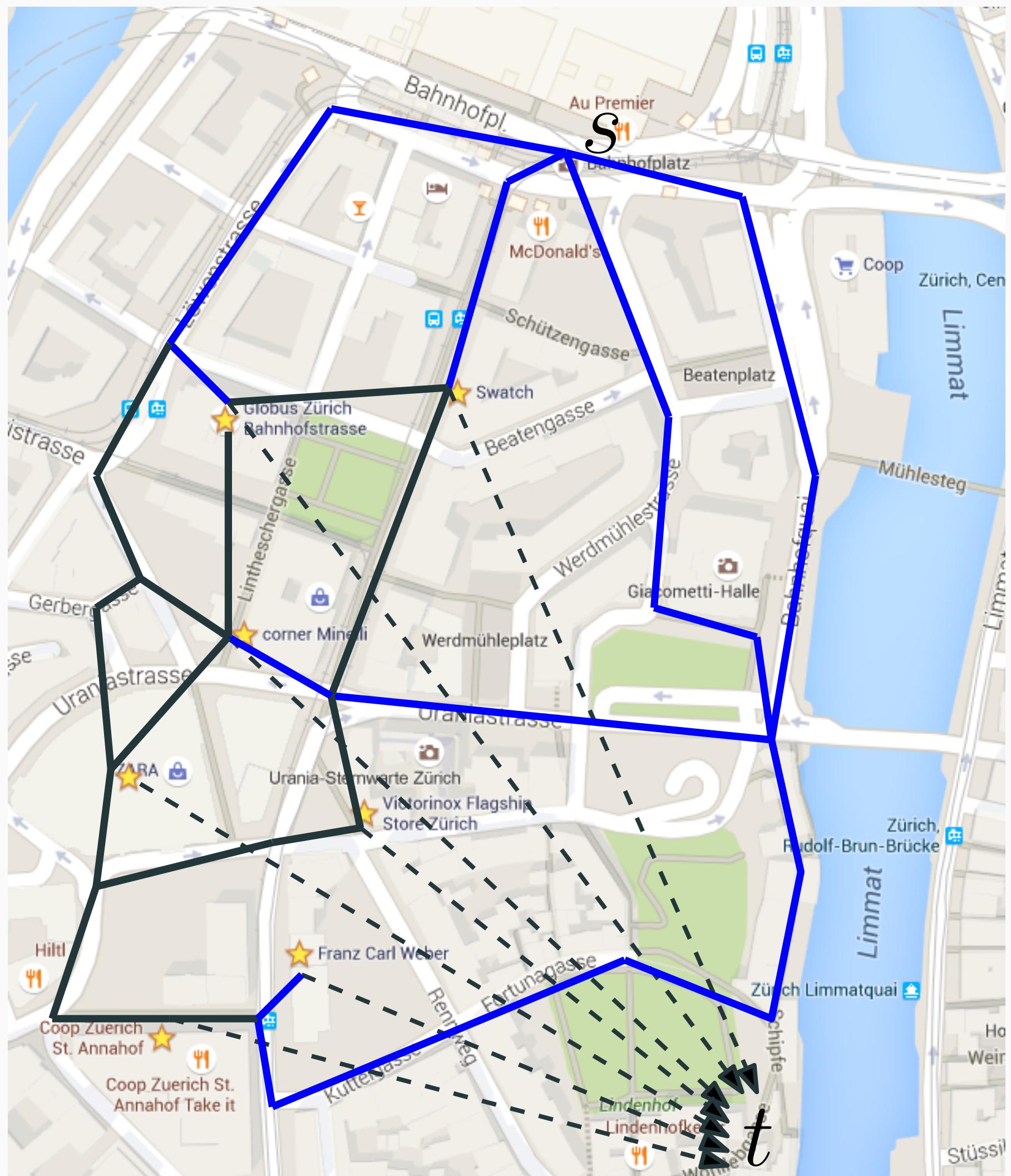
Flows with Costs

- ▶ What if sending flow comes with a price?



Minimum Cut

Minimum Cut: Shopping Trip



Start from HB:

- ▶ Visit as many shops as possible.
- ▶ Return to HB after each shop.

Condition: Use each road on at most one trip.

Compute the bottleneck, i.e. the number of edge-disjoint paths. \Rightarrow Four shops.

Unrealistic condition!

(There are interesting streets in Zürich.)

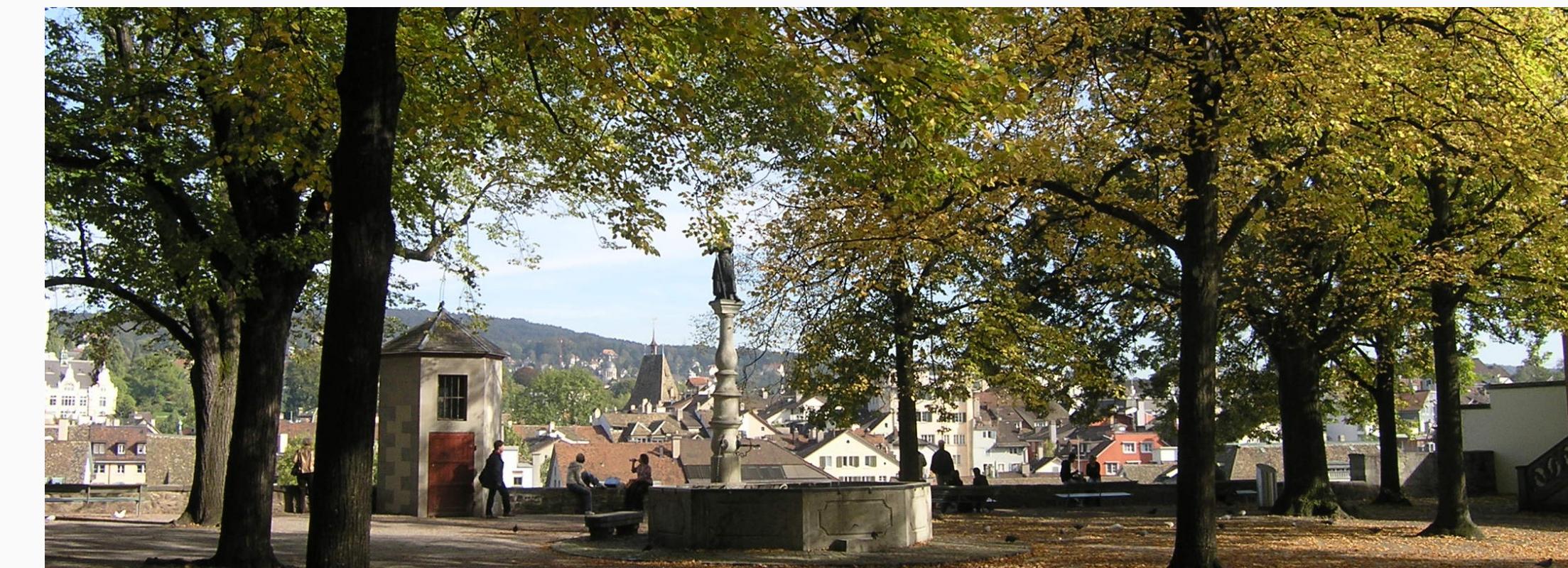
Minimum Cut: Shopping Trip



Start from HB:

- ▶ Visit as many shops as possible.
- ▶ Return to HB after each shop.

Condition: Use beautiful roads more often.



Use route via Lindenhof at most twice.

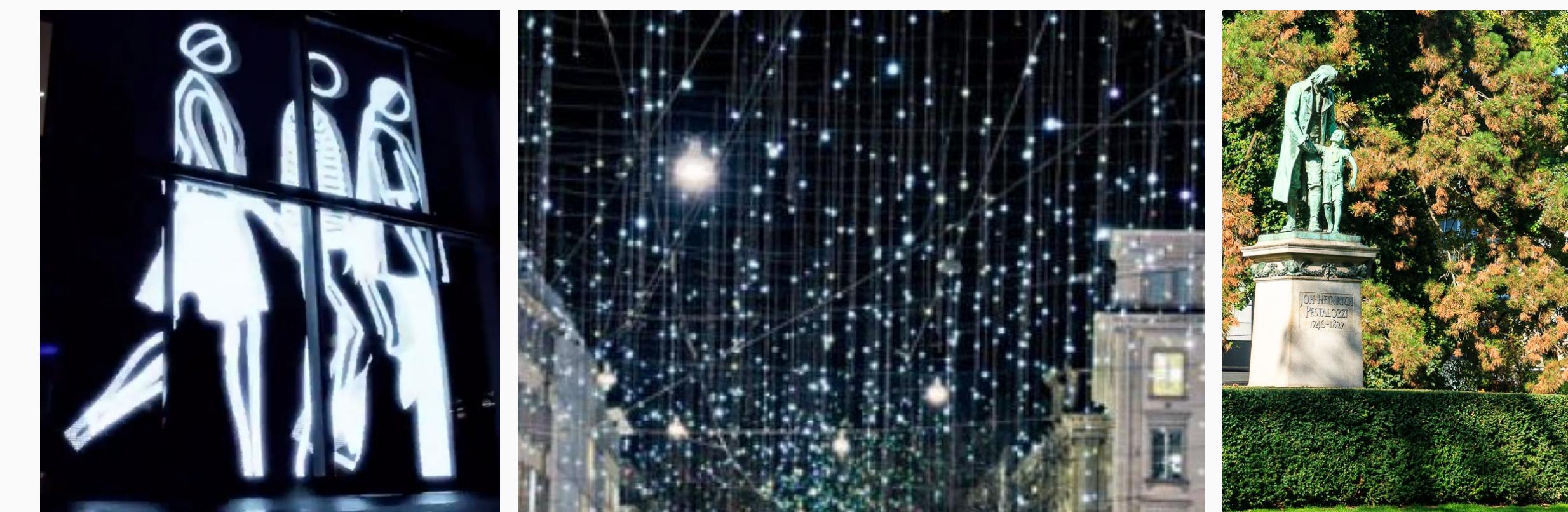
Minimum Cut: Shopping Trip



Start from HB:

- ▶ Visit as many shops as possible.
- ▶ Return to HB after each shop.

Condition: Use beautiful roads more often.



Use Bahnhofstrasse up to three times.

Compute the weighted bottleneck, i.e. the minimum cut between s and t . $\Rightarrow 6$ shops.

Minimum Cut: Cuts and Flows

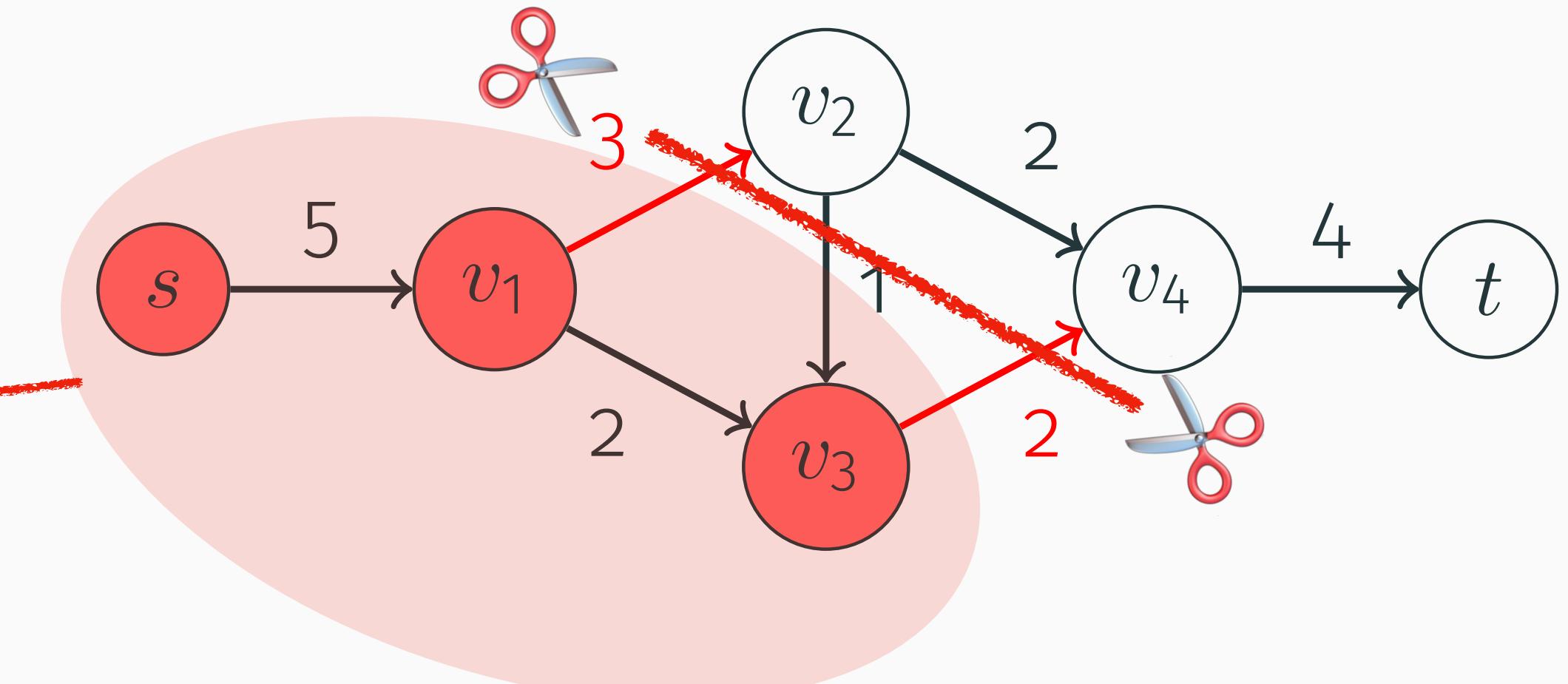
$G = (V, E, s, t)$ a flow network. $S \subset V$ s.t. $s \in S, t \in V \setminus S$, e.g. $S = \{s, v_1, v_3\}$.

The value of the $(S, V \setminus S)$ -cut is

$\text{cap}(S, V \setminus S) :=$ outgoing capacity

$$= \sum_{\substack{e=(u,v) \\ u \in S, v \in V \setminus S}} \text{cap}(e)$$

$$= 3 + 2 = 5.$$

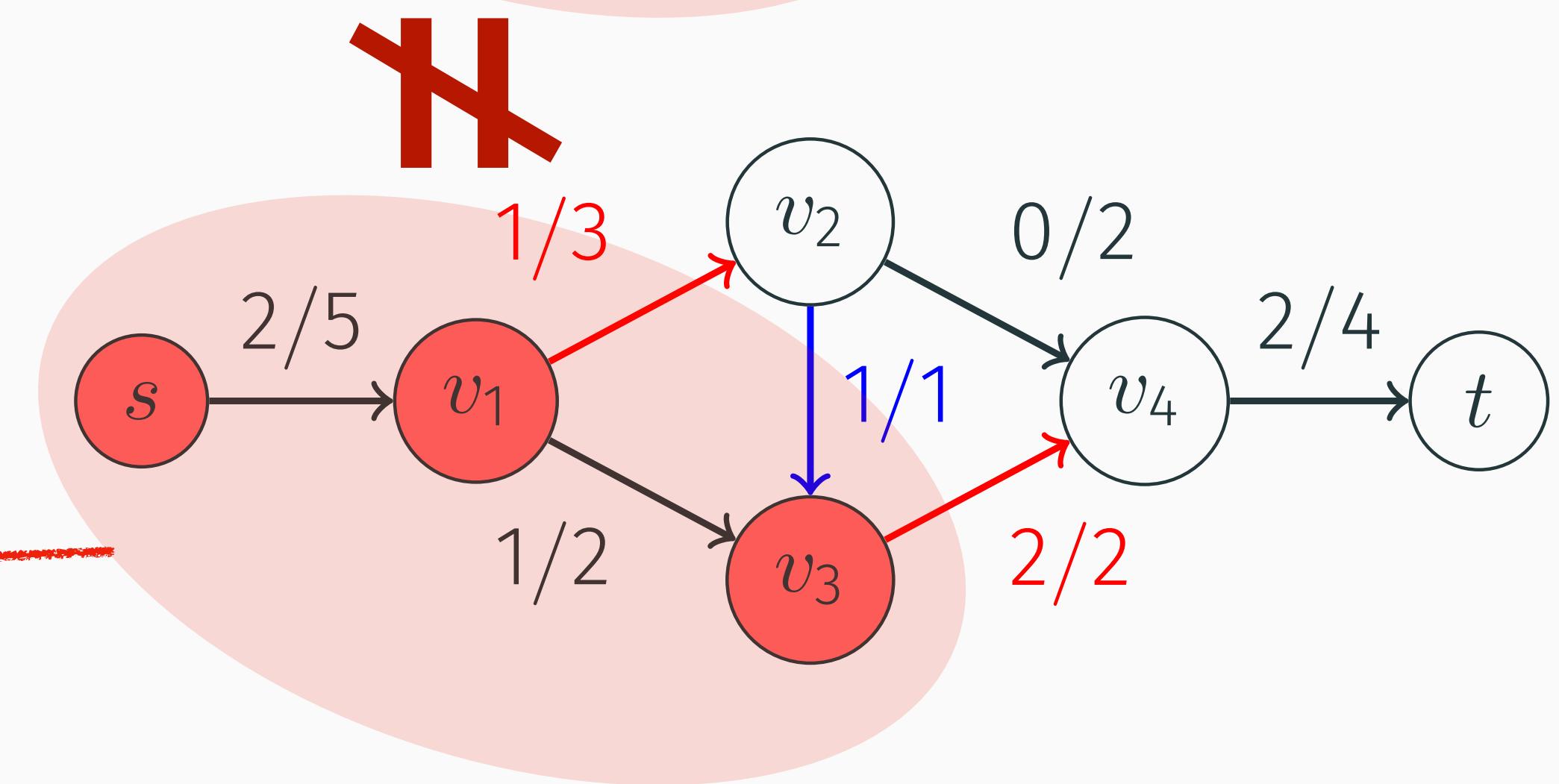


The value of the flow f from S to $V \setminus S$ is

$f(S, V \setminus S) :=$ outgoing flow – incoming flow

$$= \sum_{\substack{e=(u,v) \\ u \in S, v \in V \setminus S}} \text{flow}(e) - \sum_{\substack{e=(v,u) \\ u \in S, v \in V \setminus S}} \text{flow}(e)$$

$$= 1 + 2 - 1 = 2.$$



Minimum Cut: Maxflow-Mincut-Theorem

Theorem (Maxflow-Mincut-Theorem)

Let f be an s - t -flow in a graph G . Then f is a maximum flow if and only if

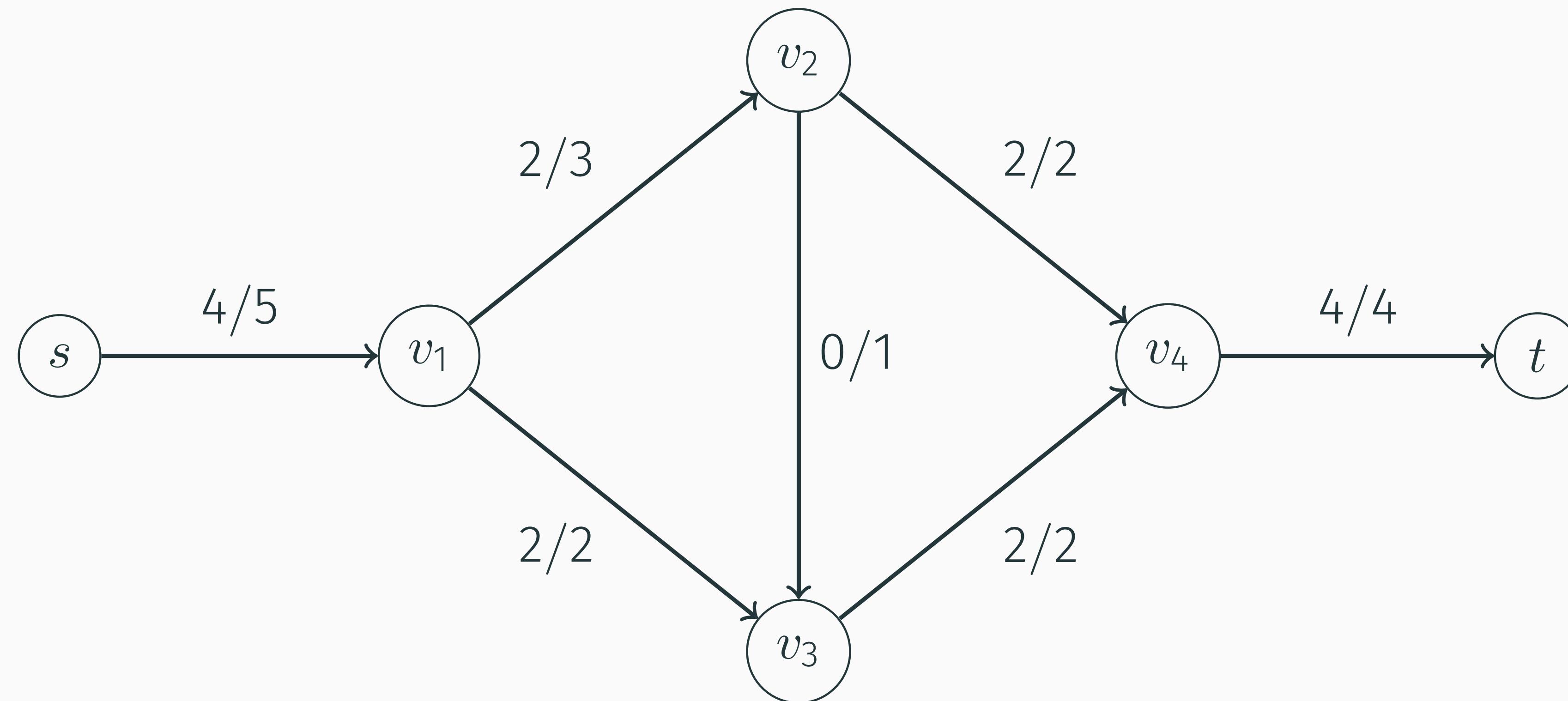
$$|f| = \min_{S: s \in S, t \notin S} \text{cap}(S, V \setminus S).$$

This allows us to easily find a minimum s - t -cut:

- ▶ Construct the residual graph $G_f := (V, E_f)$. For each edge $(u, v) \in G$ we have:
 - An edge $(u, v) \in G_f$ with capacity $\text{cap}(e) - f(e)$, if $\text{cap}(e) - f(e) > 0$.
 - An edge $(v, u) \in G_f$ with capacity $f(e)$, if $f(e) > 0$.
- ▶ Since f is a maximum flow, there is no s - t path in the residual graph G_f .
- ▶ Take S to be all vertices in G_f reachable from s .
 $\Rightarrow (S, V \setminus S)$ is a minimum s - t -cut.

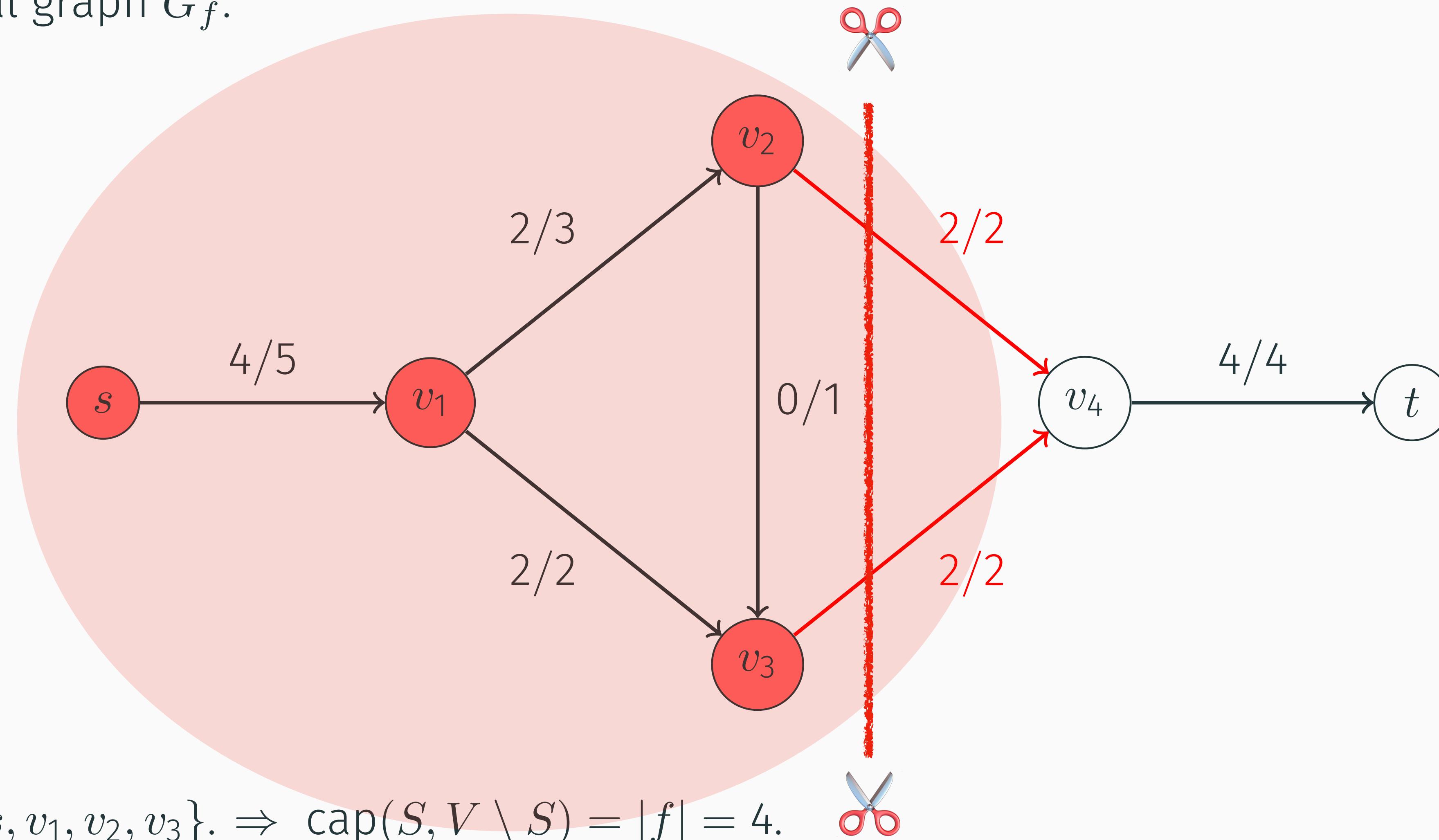
Minimum Cut: Example

Graph G and a maximum flow f .



Minimum Cut: Example

Residual graph G_f .



Minimum Cut: Code

Example code: BFS on the residual graph G_f . → [tut9_bgl_residual_bfs.cpp](#)

```
90 // BFS to find vertex set S
91 std::vector<int> vis(N, false); // visited flags
92 std::queue<int> Q; // BFS queue (from std:: not boost::)
93 vis[src] = true; // Mark the source as visited
94 Q.push(src);
95 while (!Q.empty()) {
96     const int u = Q.front();
97     Q.pop();
98     OutEdgeIt ebeg, eend;
99     for (boost::tie(ebeg, eend) = boost::out_edges(u, G); ebeg != eend; ++ebeg) {
100         const int v = boost::target(*ebeg, G);
101         // Only follow edges with spare capacity
102         if (rescapacitymap[*ebeg] == 0 || vis[v]) continue;
103         vis[v] = true;
104         Q.push(v);
105     }
106 }
```

Minimum Cut: Proof of the Maxflow-Mincut-Theorem

Theorem (Maxflow-Mincut-Theorem)

Let f be an s - t -flow in a graph G . Then the following are equivalent:

1. $|f| = \min_{s \in S, t \notin S} \text{cap}(S, V \setminus S)$.
2. f is a maxflow.
3. There is no s - t path in the residual graph G_f .

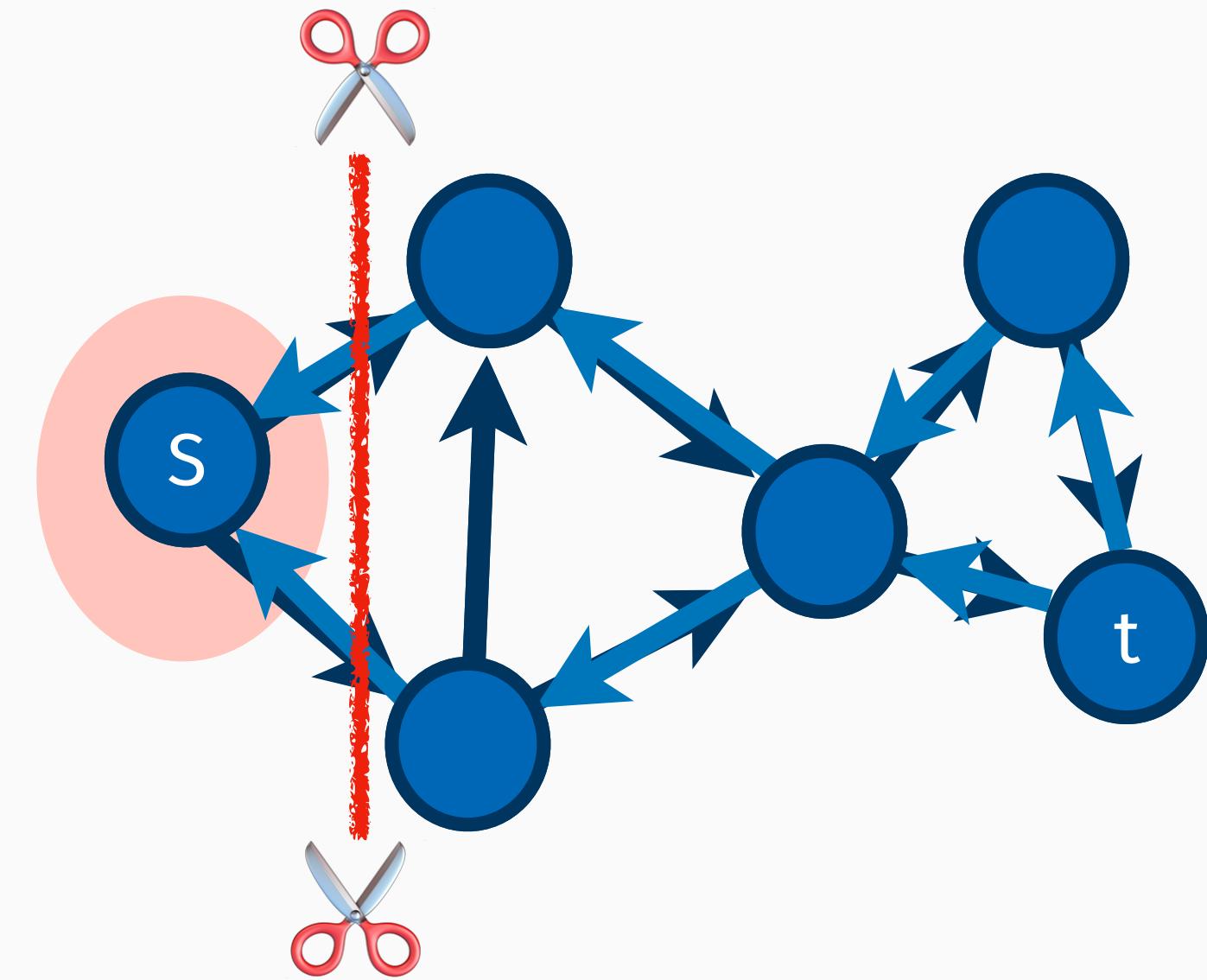
Proof.

- 1) \implies 2) f cannot be bigger than $\min_S \text{cap}(S, V \setminus S)$.
- 2) \implies 3) Indirectly: If there was s - t path in G_f , f could be extended.
- 3) \implies 1) Take S to be all vertices in G_f reachable from s .
Then all edges from S to $V \setminus S$ must be fully saturated by the flow.
But then $|f| = f(S, V \setminus S) = \text{cap}(S, V \setminus S)$. □

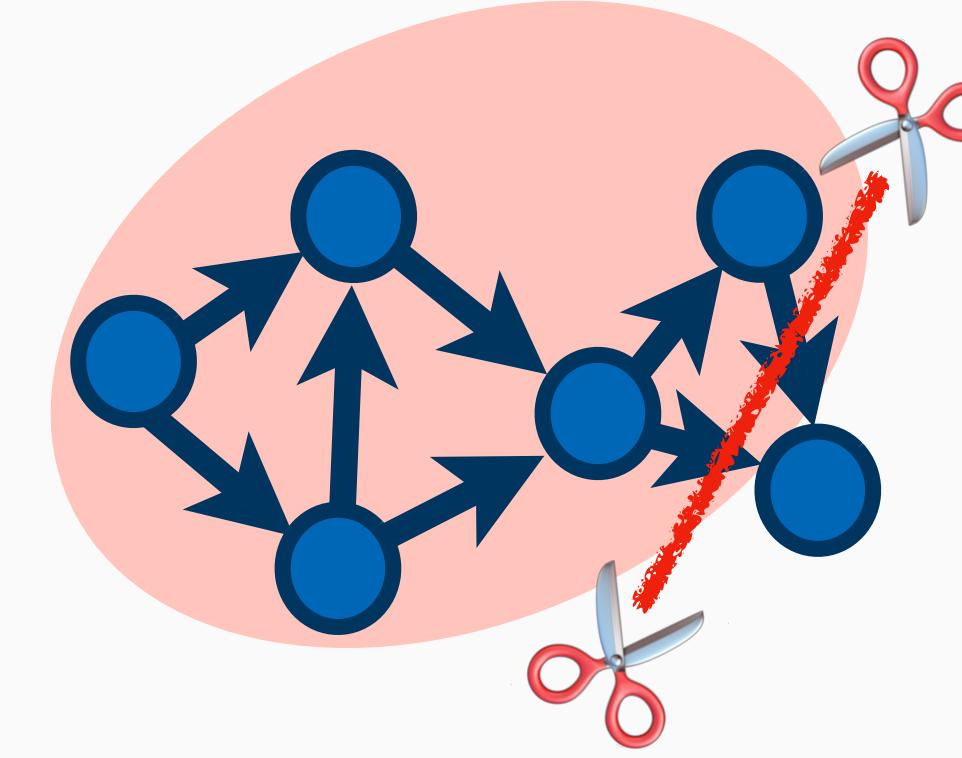
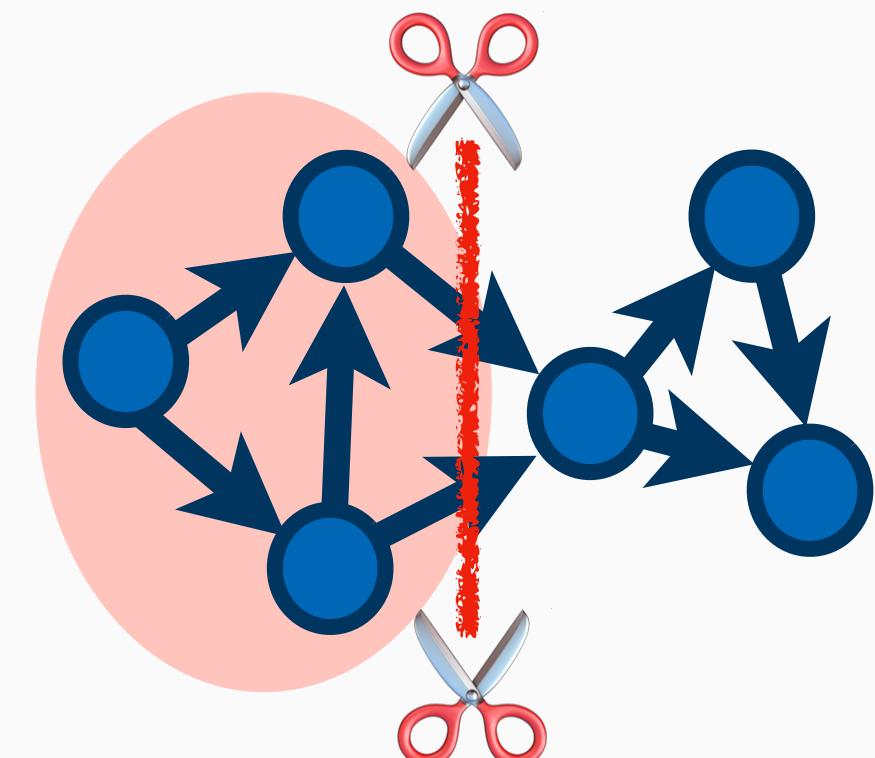
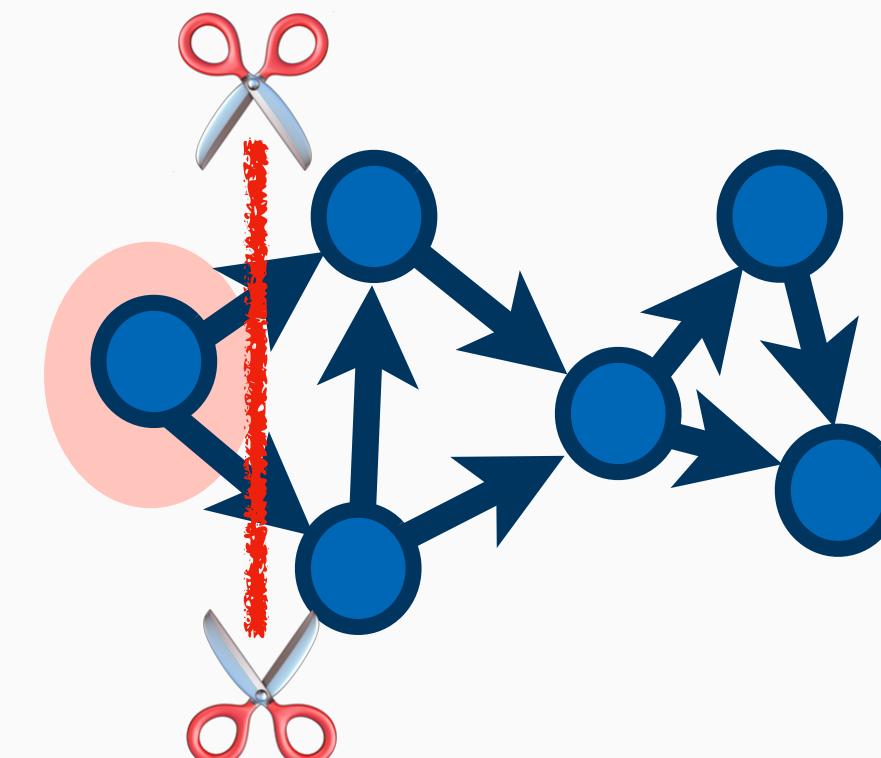
Minimum cut: Algorithm

Summary of what you need to do to find a minimum cut:

1. Compute maximum flow f and the residual graph G_f .
2. Compute the set of vertices S :
 - S is reachable from the source s in G_f .
 - BFS on edges with residual capacity > 0 .
3. Output (depending on the task):
 - All vertices in S .
 - All edges going from S to $V \setminus S$.



Note: Minimum cuts are not necessarily unique. But earliest and latest min-cuts are.

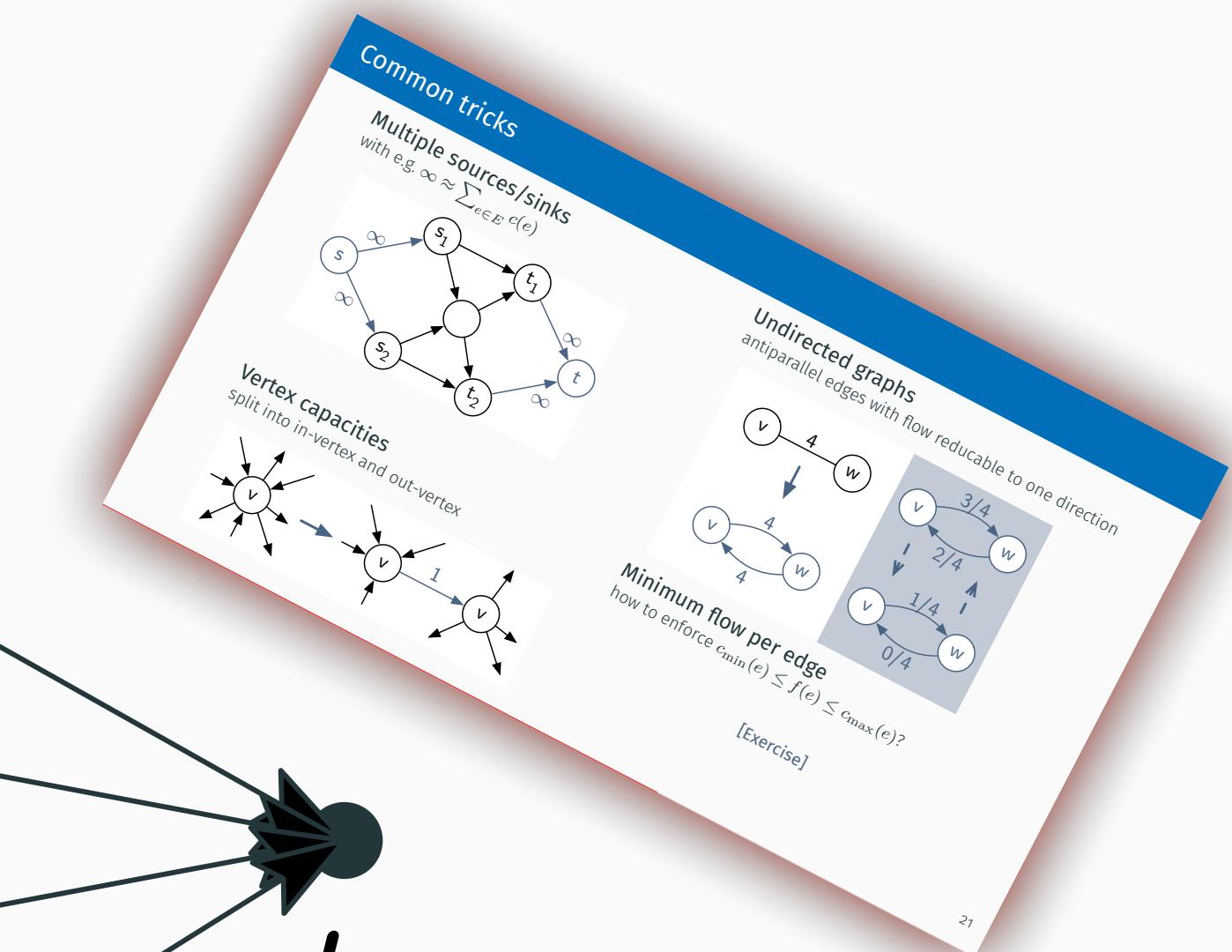
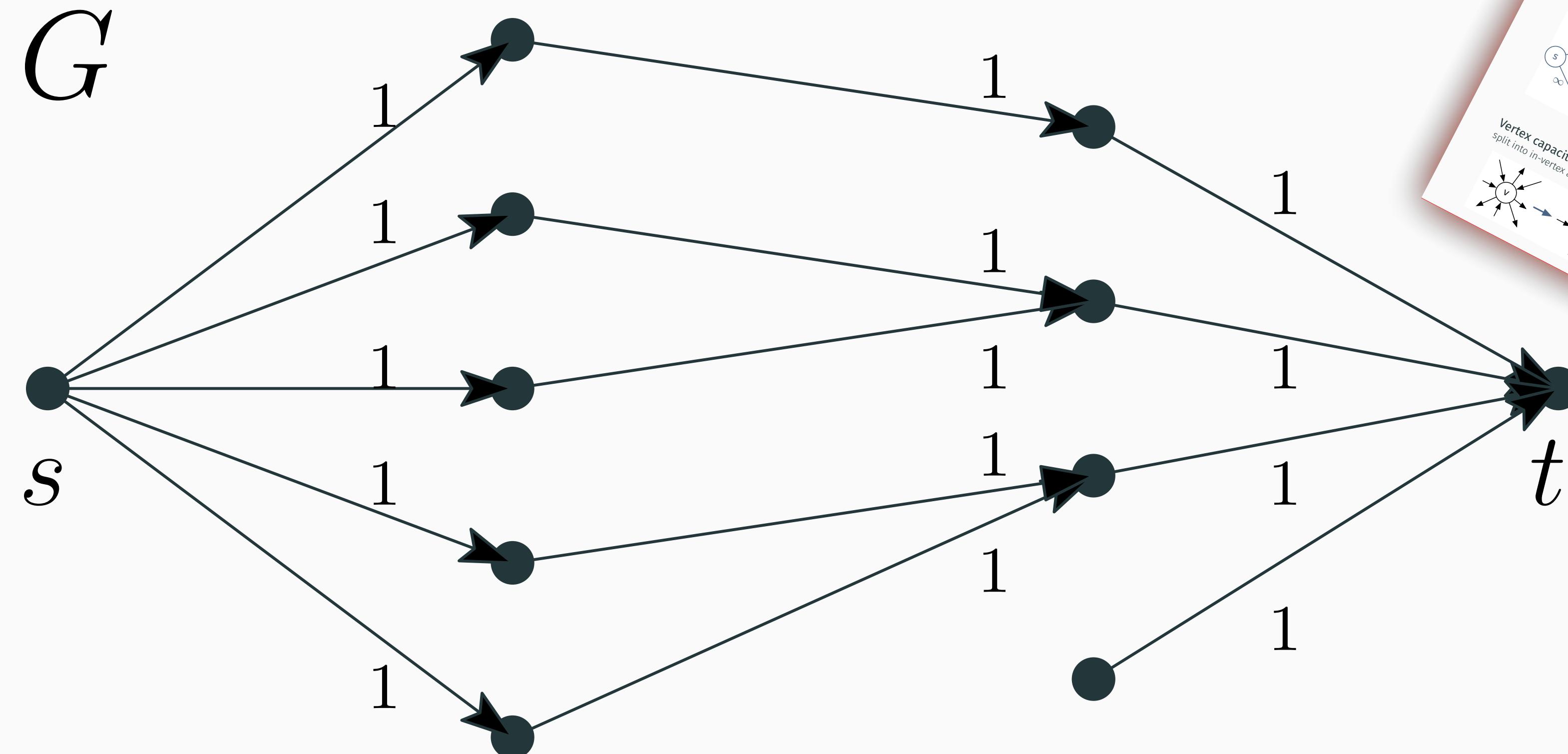


Bipartite Matchings

Maximum Matchings: Bipartite Graphs

Maximum Matching = pick as many non-adjacent edges as possible

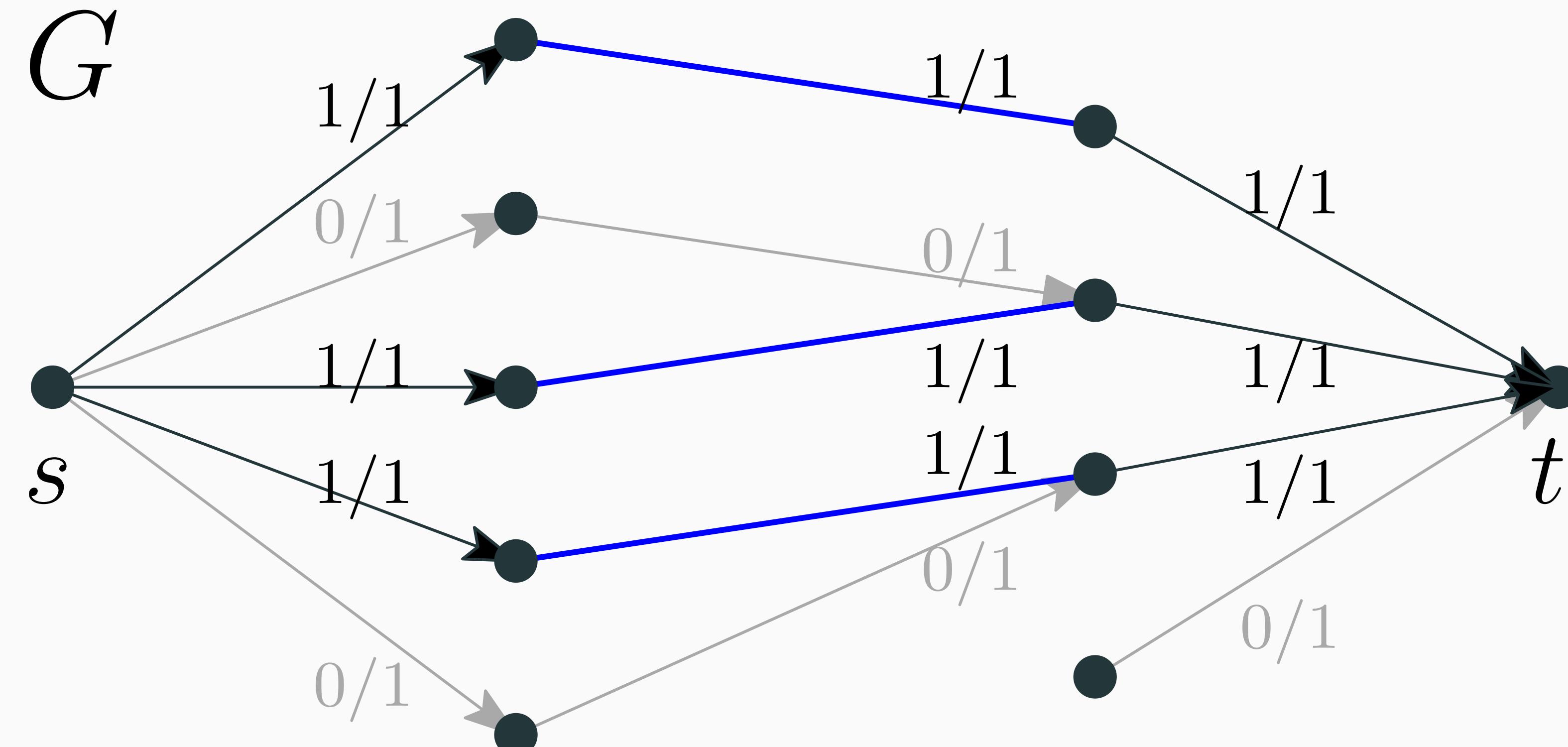
Flow formulation through circulation / vertex capacities / edges-disjoint paths:



Maximum Matchings: Bipartite Graphs

Maximum Matching = pick as many non-adjacent edges as possible

Flow formulation through circulation / vertex capacities / edges-disjoint paths:



Vertex Cover and Independent Set: General Graphs

- ▶ Maximum independent set (MaxIS)

Largest $T \subseteq V$, such that

$$\nexists u, v \in T : (u, v) \in E.$$

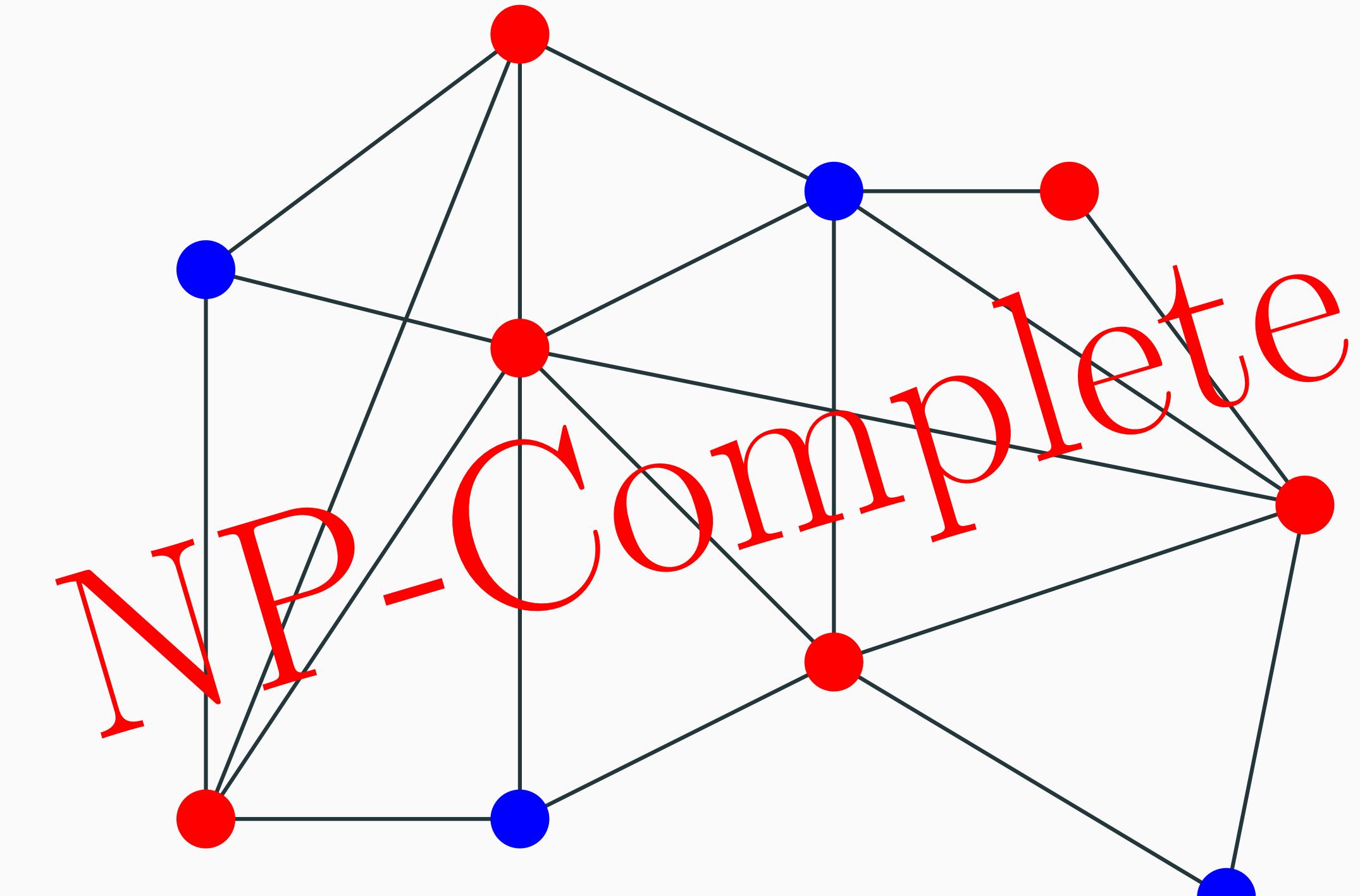
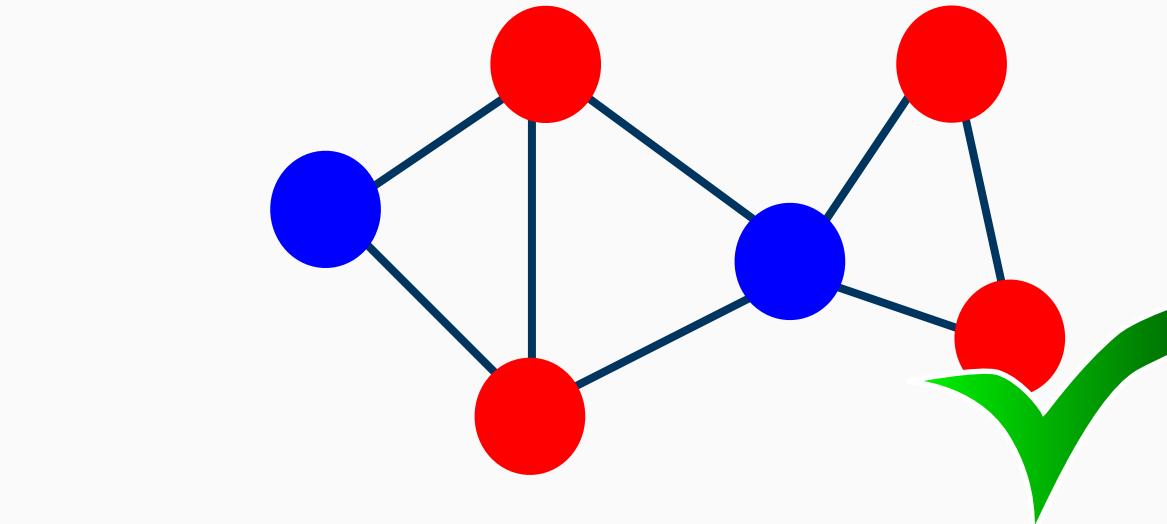
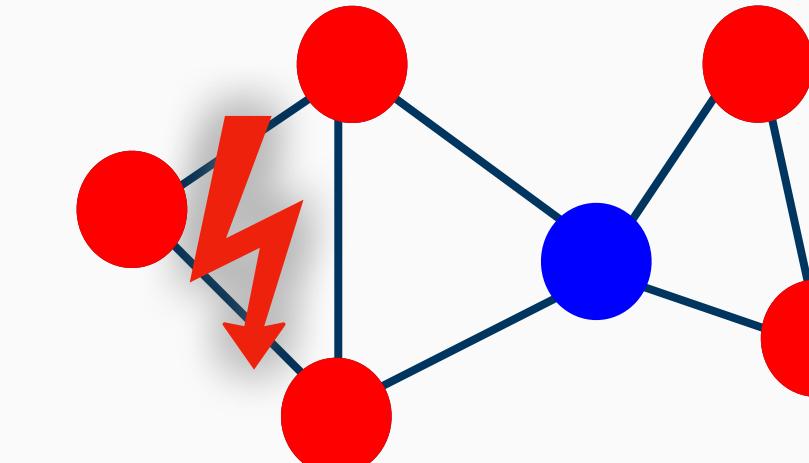
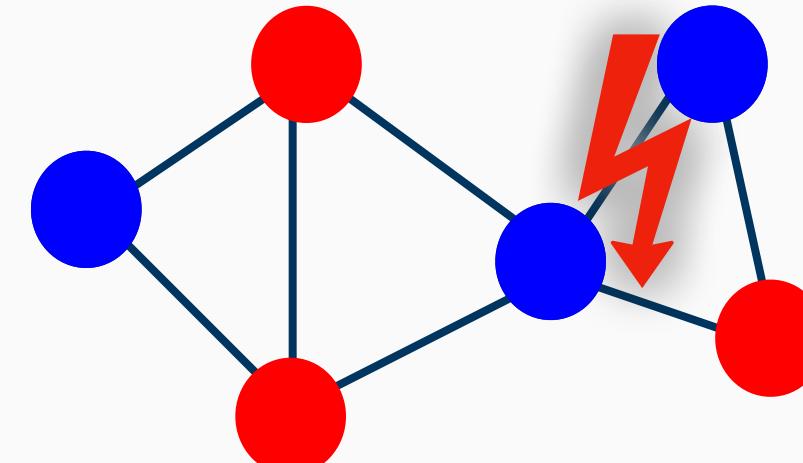
- ▶ Minimum vertex cover (MinVC)

Smallest $S \subseteq V$, such that

$$\forall (u, v) \in E : u \in S \vee v \in S.$$

- ▶ These problems are complementary!

$$\text{MaxIS} = V \setminus \text{MinVC}$$



Brief Excursion: Options for Runtime Analysis

By now, we know many ways of deciding whether an algorithm is fast or slow:

- ▶ look at the input size (the classical way)
- ▶ look at the output size (e.g. fast as the answer is guaranteed to be small)
If you know that MaxIS is very small, then it might be tractable while computing a MinVC directly is too slow
- ▶ look at some special input restrictions (e.g. Attack of the clones)
- ▶ look at detailed structure of the input (e.g. all graphs are trees)

Vertex Cover and Independent Set: Bipartite Graphs

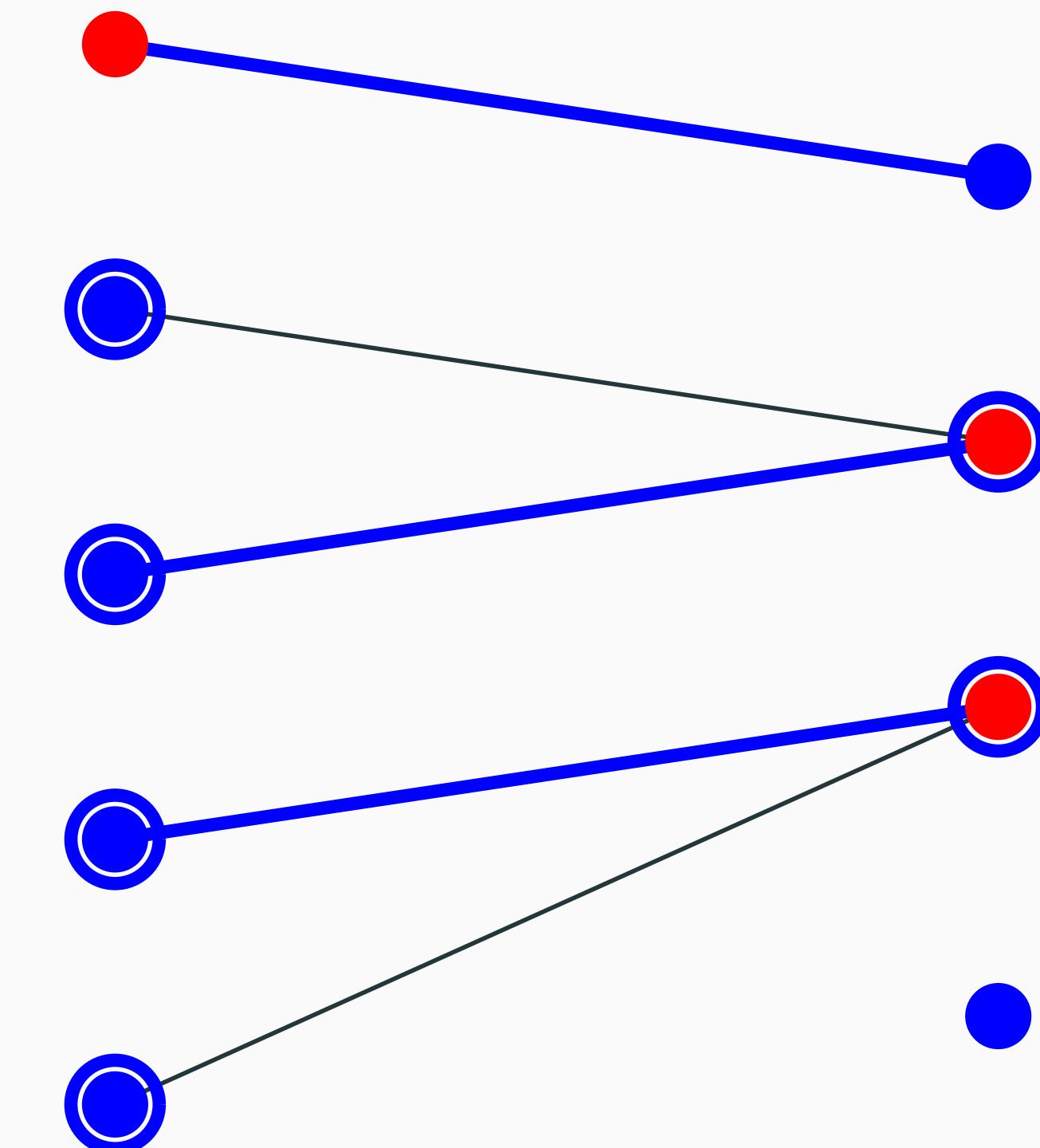
Theorem (König: MinVC and MaxIS is simpler on bipartite graphs!)

In a bipartite graph, the number of edges in a maximum matching is equal to the number of vertices in a minimum vertex cover.

Proof: See [Wikipedia](#) for a nice and short proof.

Algorithm:

1. Maximum matching M , $V = L \cup R$. Find all unmatched vertices in L , label them as visited.
2. Starting at visited vertices search (BFS) left to right along edges from $E \setminus M$ and right to left along edges from M . Label each found vertex as visited.
3. MinVC – all unvisited in L and all visited in R .
MaxIS – all visited in L and all unvisited in R .

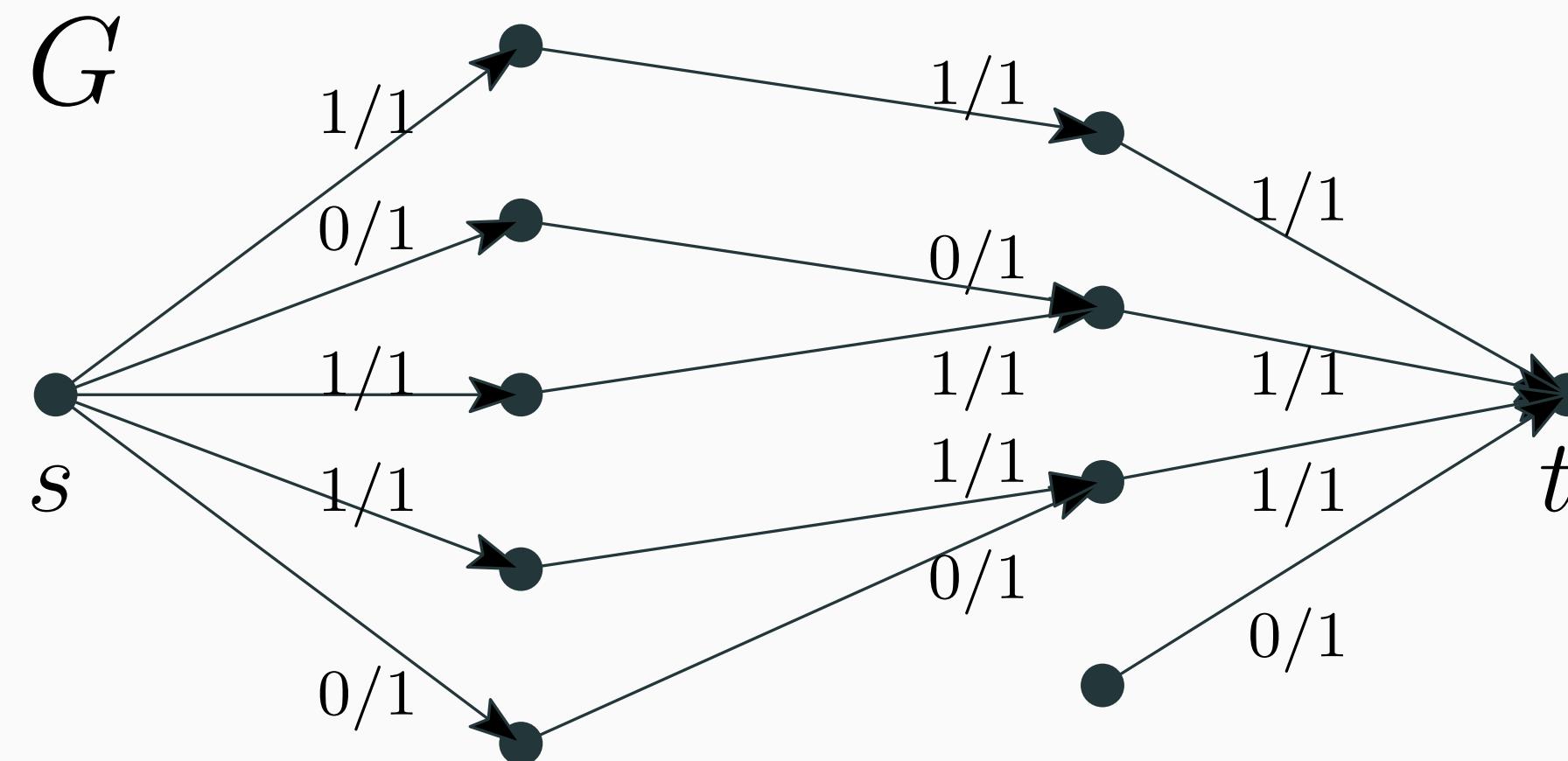


Careful! Step 2 can take several rounds.

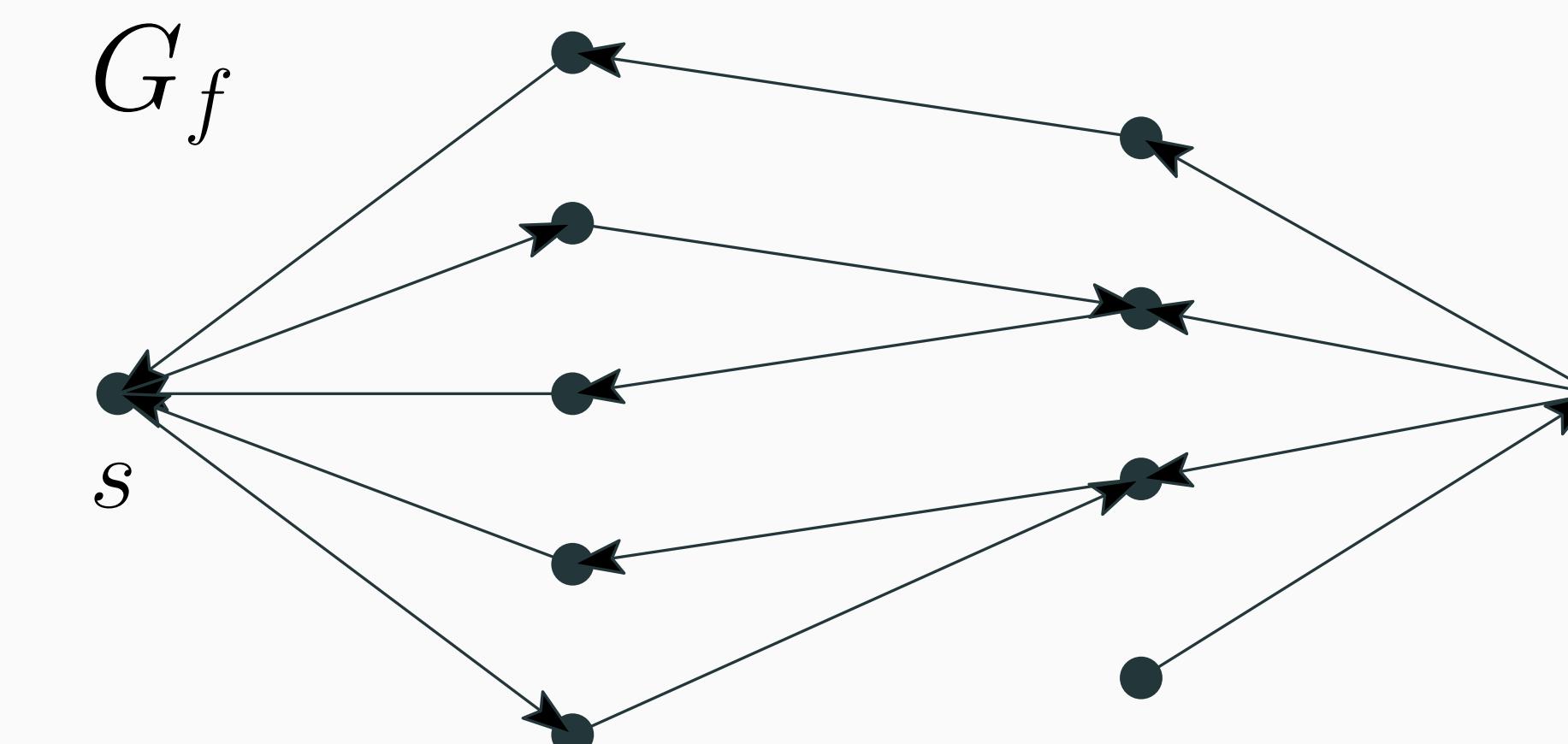
Easy Implementation?

Finding a MinVC or MaxIS in bipartite graphs: step by step

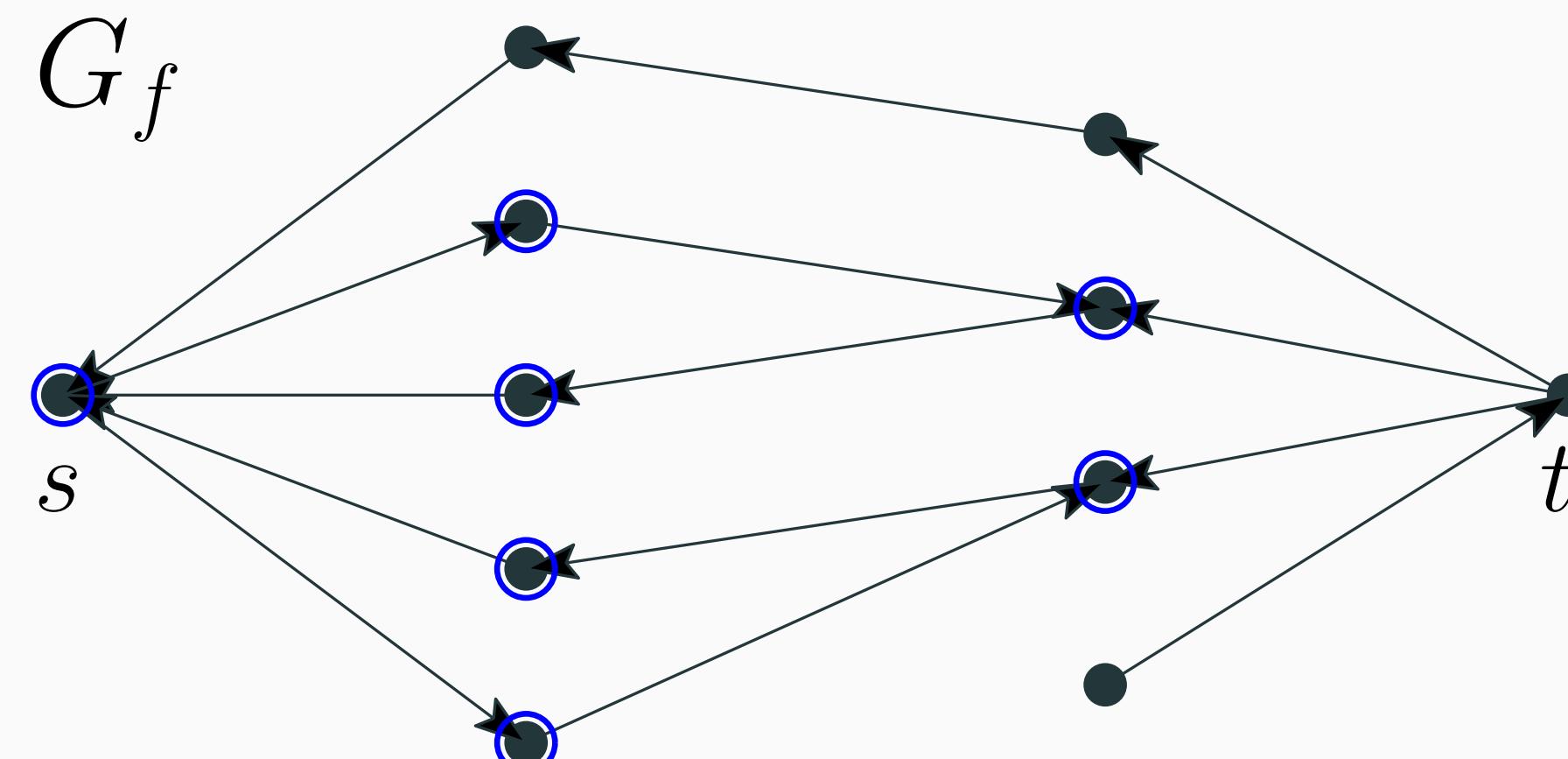
1) Formulate and compute the flow:



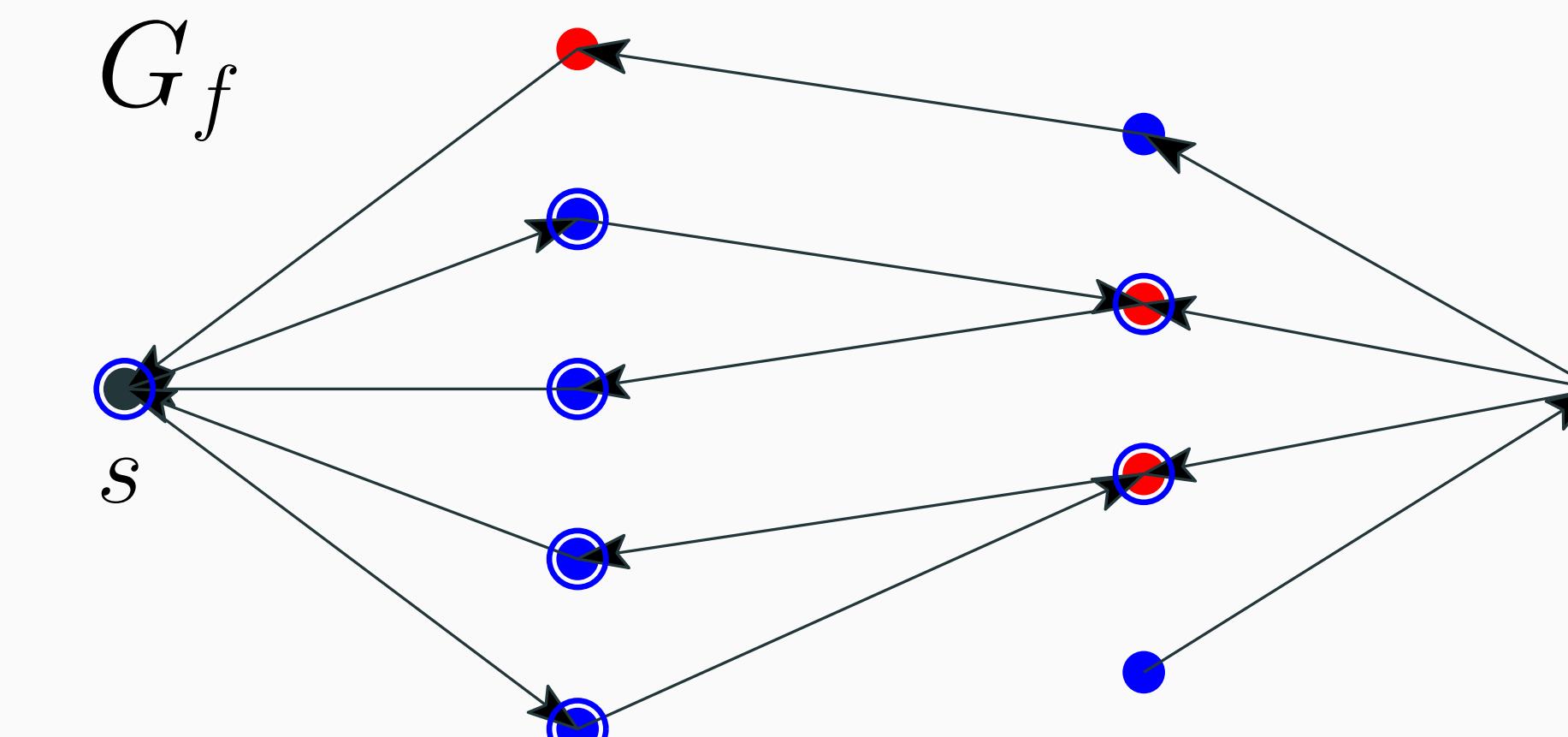
2) Compute the residual graph G_f :



3) Mark reachable vertices from s with BFS:



4) Read the MinVC or MaxIS from the marks:



Summary: MaxFlowMinCut and Bipartite Matching

What you should remember:

Edge Cut

- ▶ Theorem: maximum amount of any s - t -flow = minimum capacity of any s - t -cut
- ▶ Finding the cut: BFS/DFS on residual graph starting from s .

Vertex Cover

- ▶ Minimum vertex cover and maximum independent set are hard problems.
- ▶ Bipartite graphs allow fast MinVC and MaxIS (both on top of maximum matching).
- ▶ Finding the minimum vertex cover: BFS/DFS on residual graph from s .

Min Cost Max Flow

Minimum Cost for a Bipartite Matching

How to pick the *best* maximum matching? How to break ties among equally large ones?

E.g.: Which set of marriages is the most stable?

Or: Which consumer/producer pairing is the most effective?

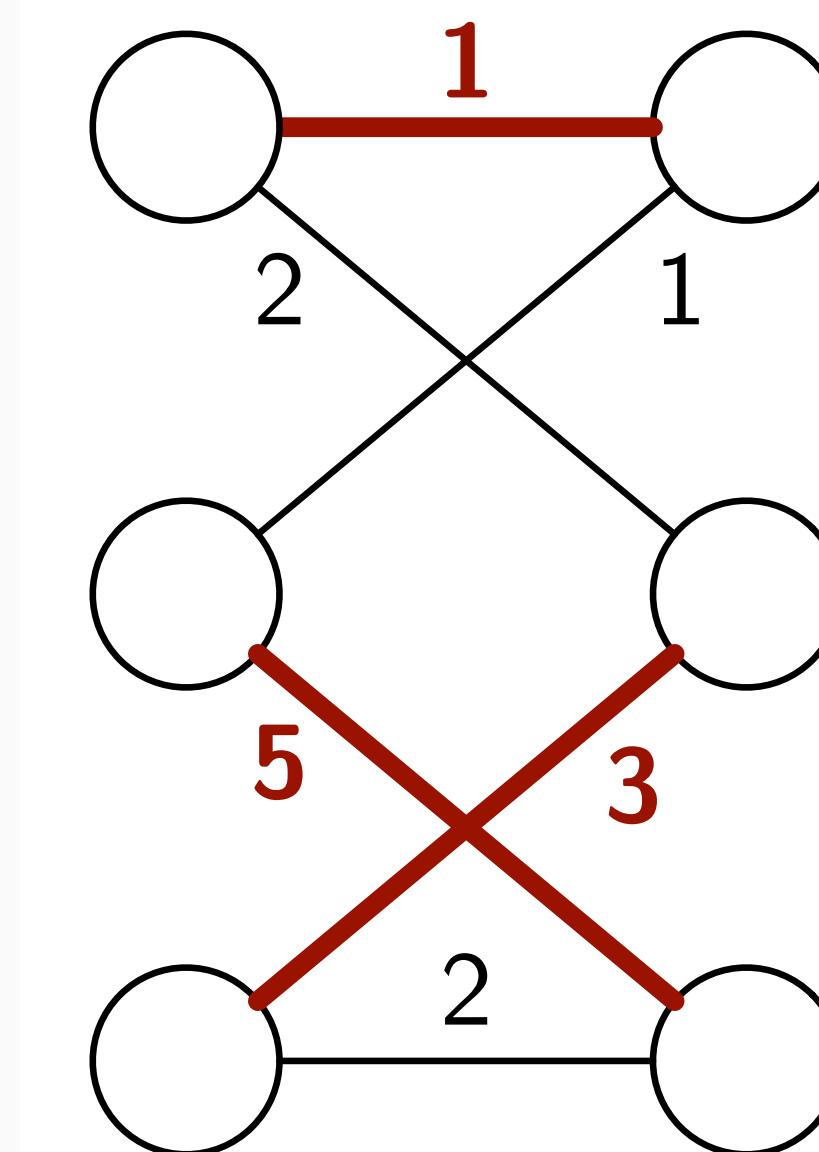
What if the edges in a matching also have costs associated?

- ▶ cardinality of the matching is no longer the only objective
- ▶ second priority: minimize the total cost

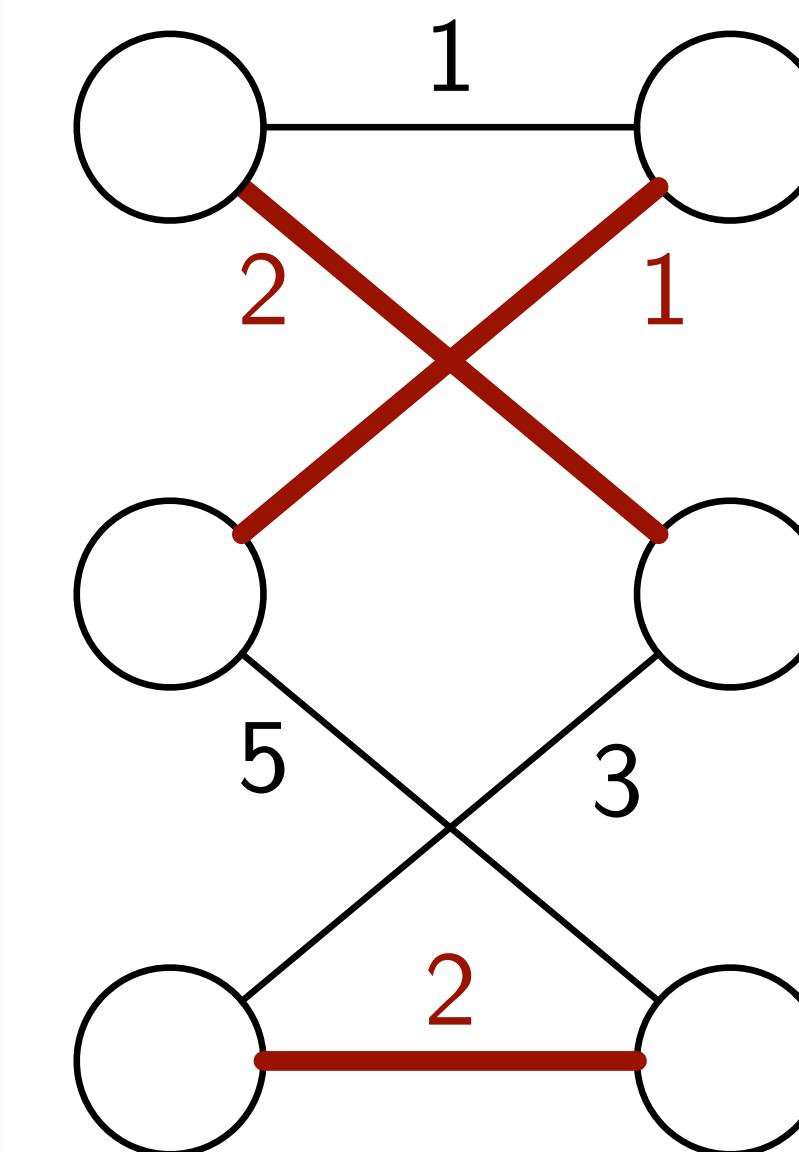
We search for the cheapest among all maximum matchings.

This does not fit into our model of flows.

two maximum matchings of different cost:



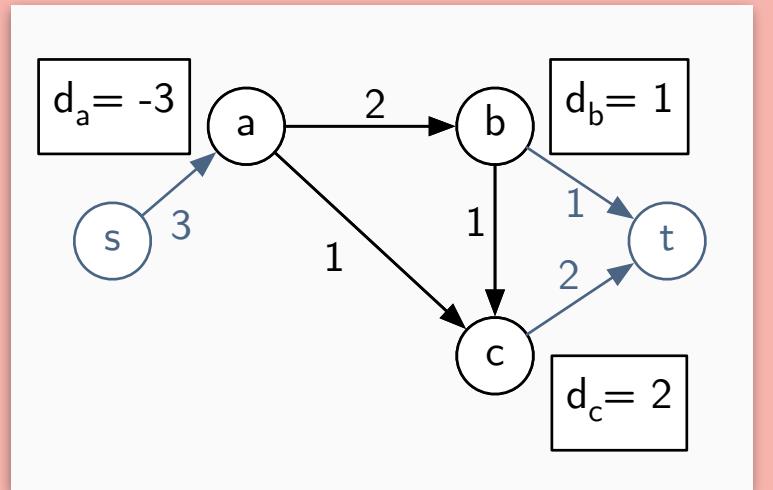
$$1 + 5 + 3 = 9$$



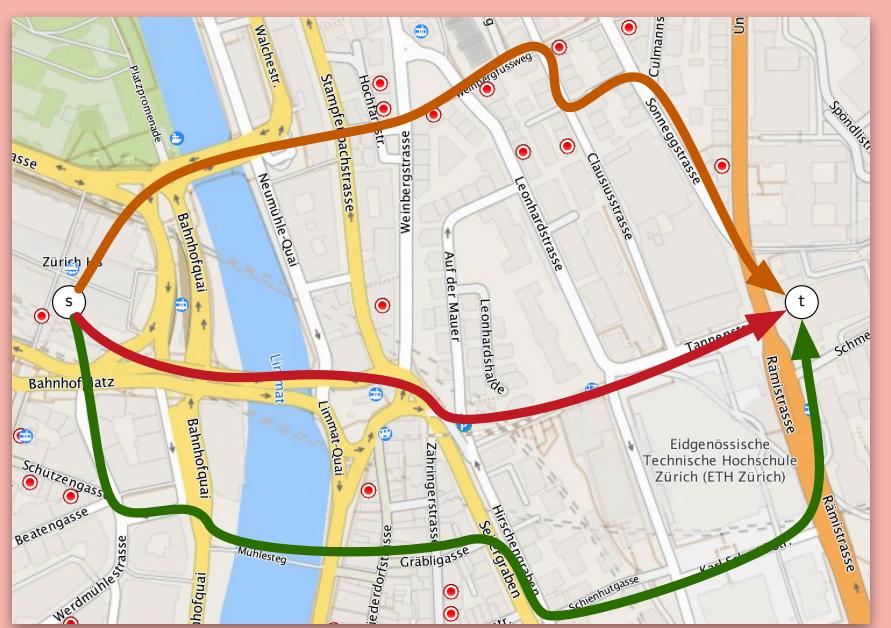
$$2 + 1 + 2 = 5$$

Complexity Landscape

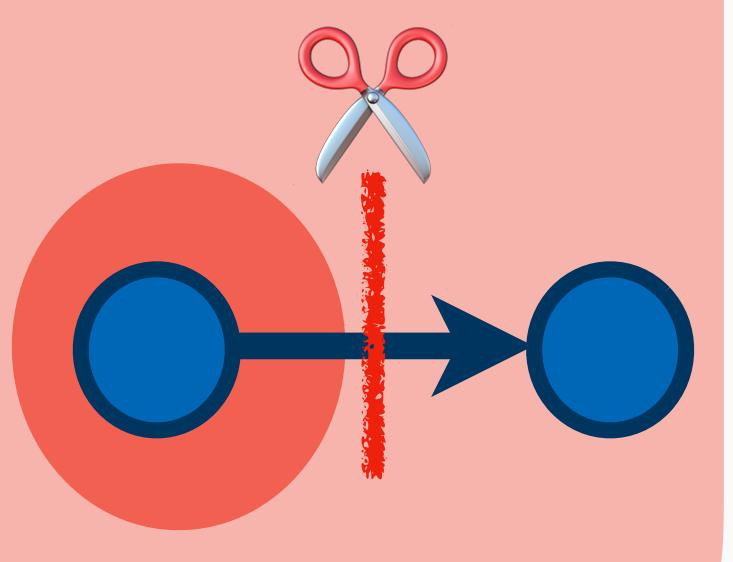
Circulation



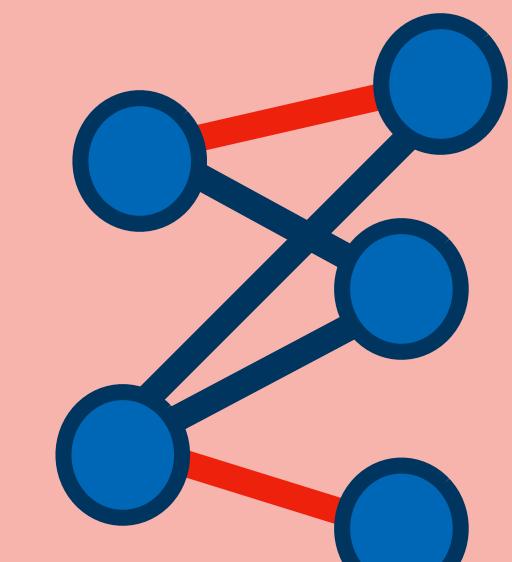
Edge-disjoint Paths



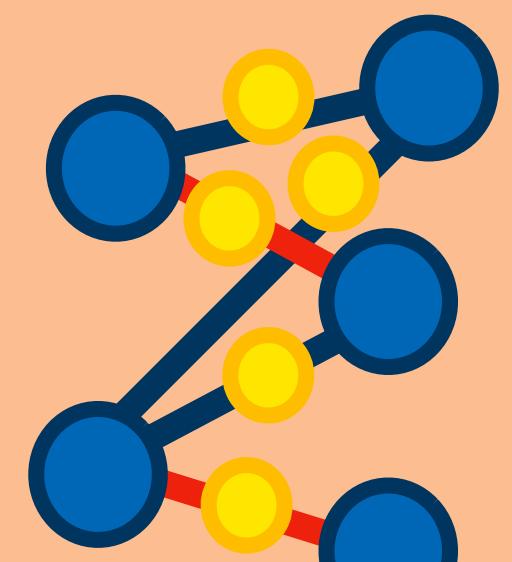
Minimum Cut



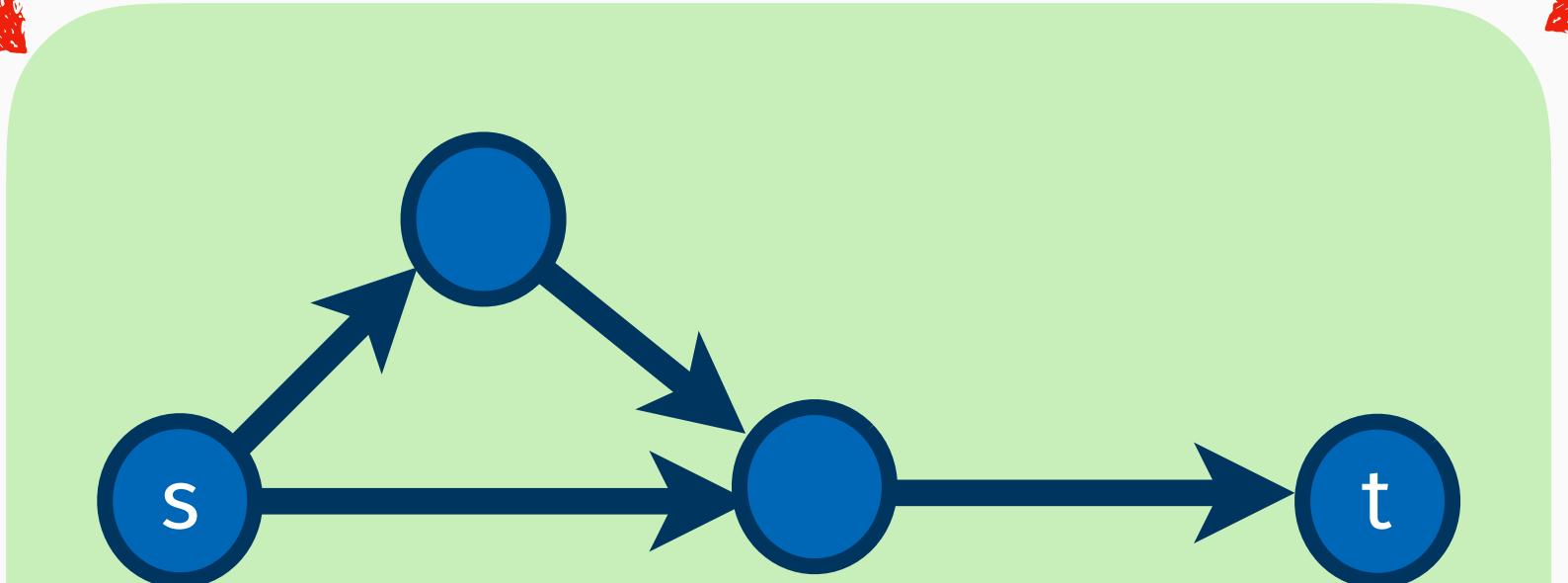
Bipartite Matching



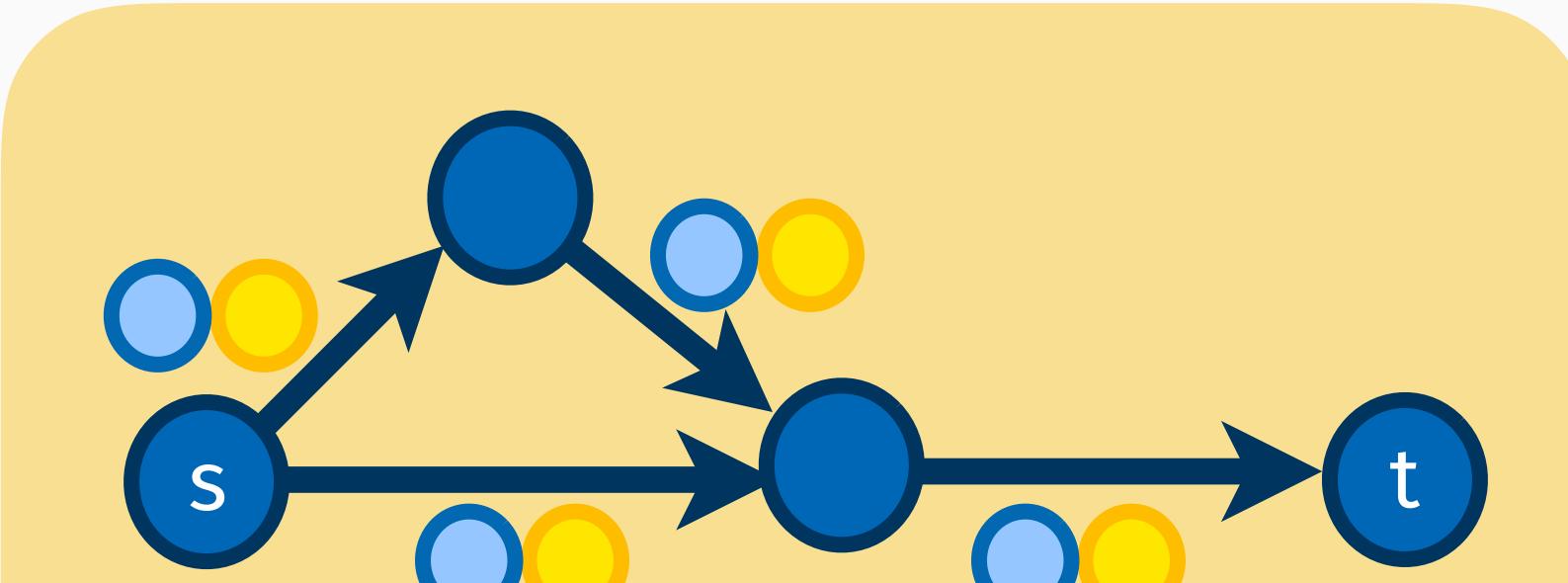
Bipartite Cost Matching



Maximum Flow



Min Cost Max Flow



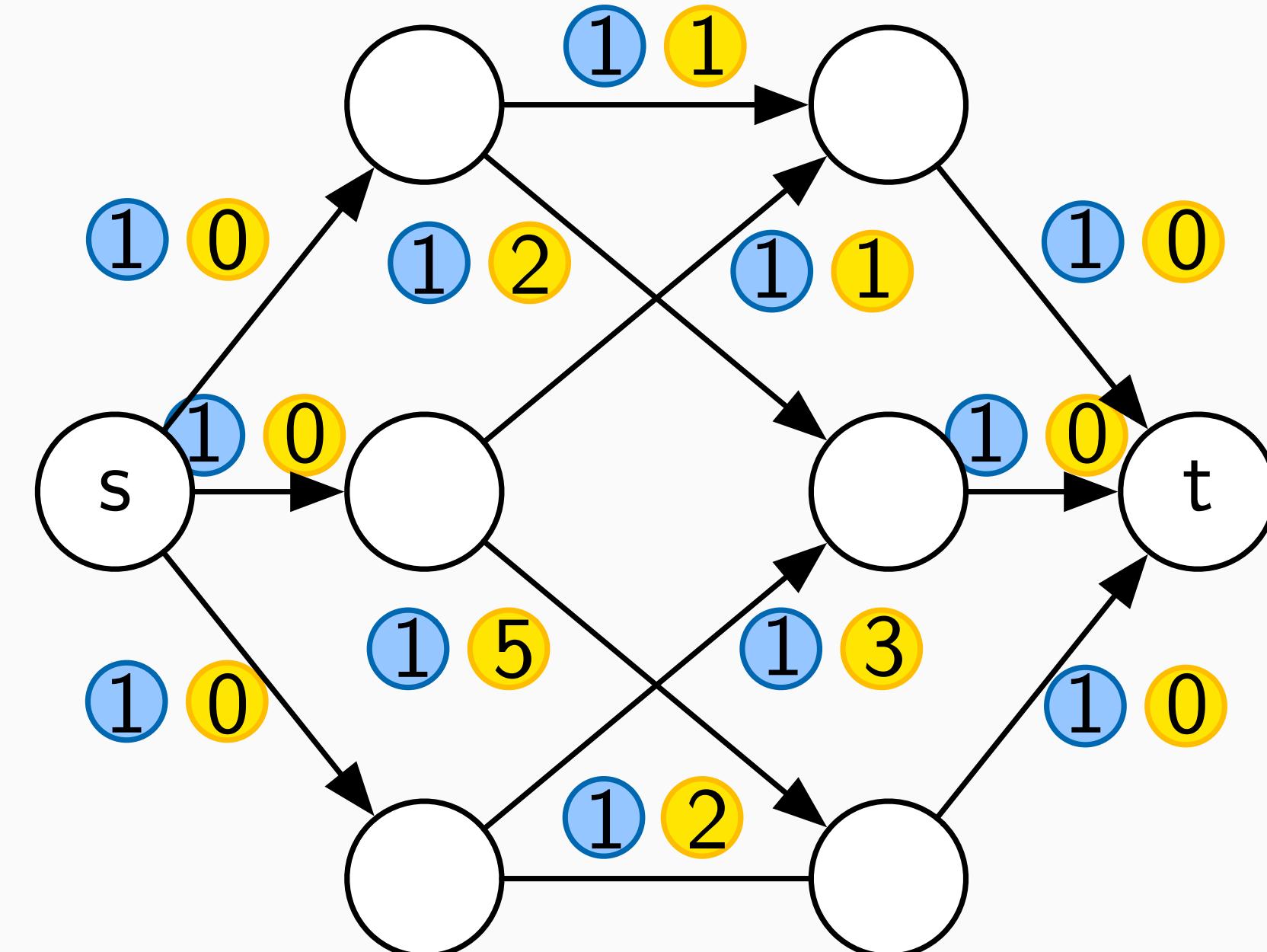
More General Model: Minimum Cost Maximum Flow

We extend the network flow problem by allowing edge costs that occur per unit of flow.

Input: A flow network consisting of

- ▶ directed graph $G = (V, E)$
- ▶ source and sink $s, t \in V$
- ▶ edge capacity $cap : E \rightarrow \mathbb{N}$
- ▶ edge cost $cost : E \rightarrow \mathbb{Z}$.

Output: A flow function f with minimal $cost(f) = \sum_{e \in E} f(e) \cdot cost(e)$ among all flows with maximal $|f|$.



This can model much more than just minimum cost bipartite matching.

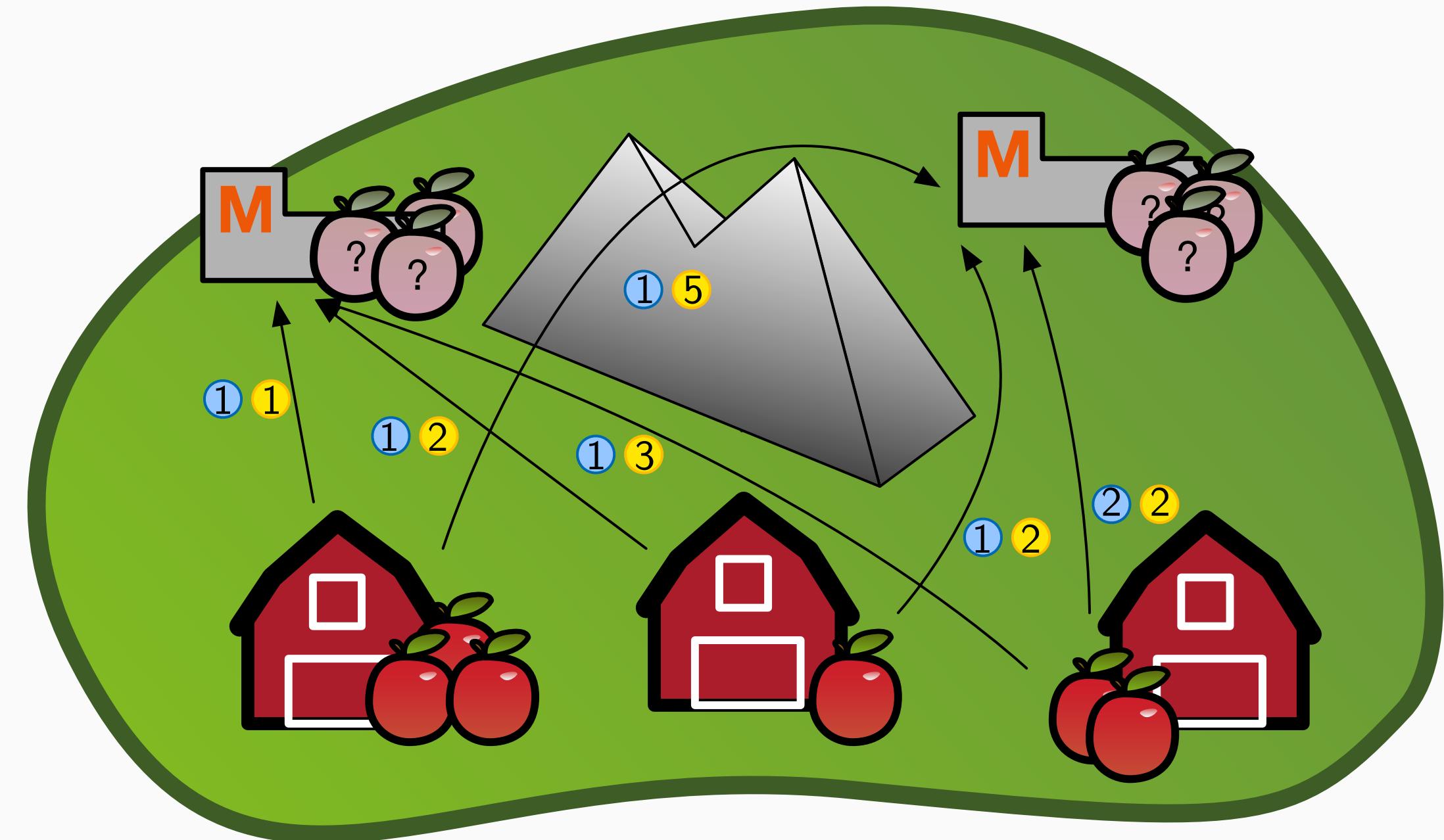
Legend: capacity ● vs ○ cost

Example: Fruit Delivery

Migros wants to schedule fruit deliveries from their farmers to their shops.

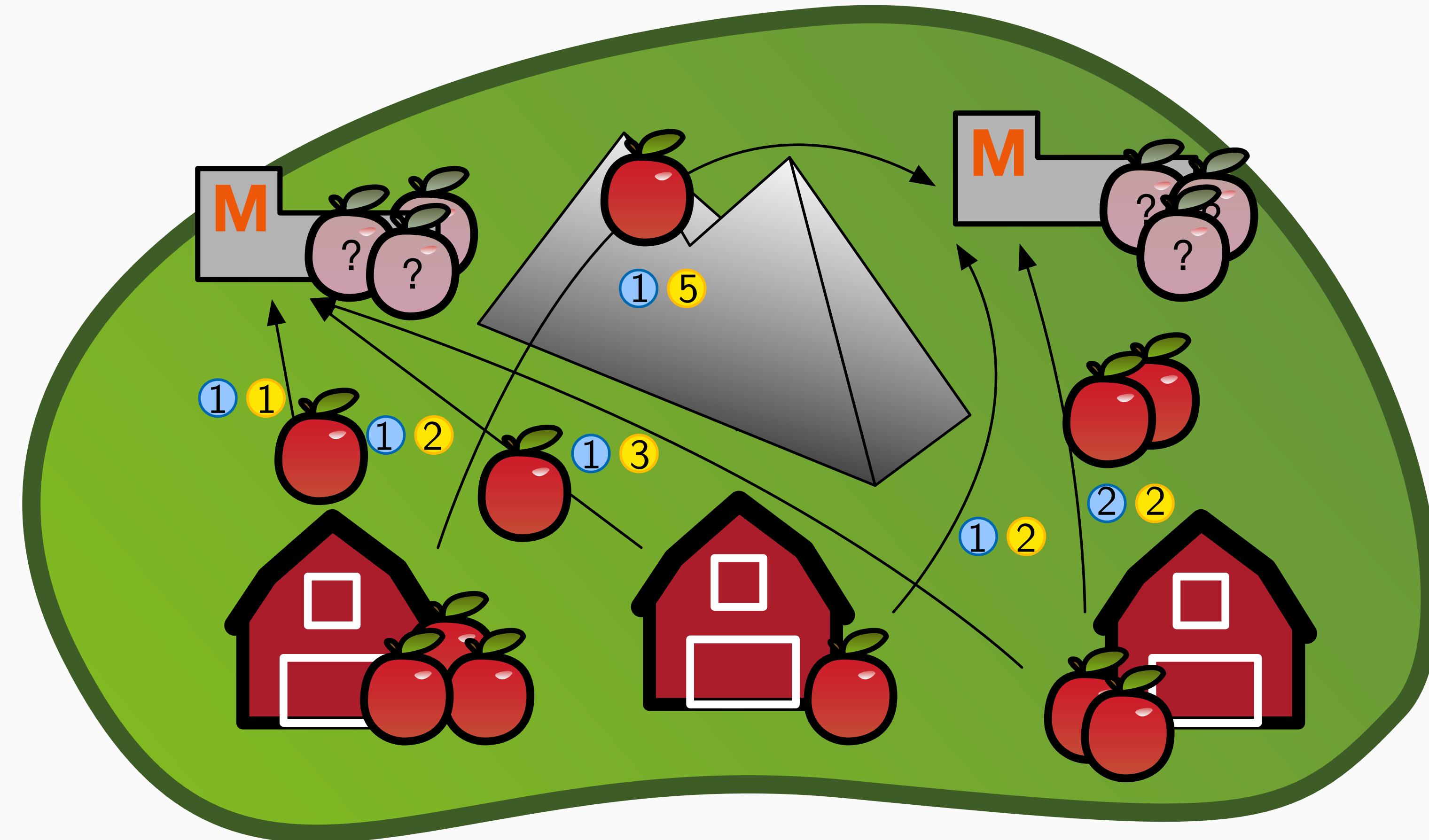
They know all important parameters;

- ▶ production per farm [in kg]
- ▶ demand per shop [in kg]
- ▶ transportation capacity [in kg] and
transportation cost [in Fr. pro kg]
for every farm-shop pair



Note: This is not just a bipartite matching, even though the graph is bipartite.
One farm might deliver to multiple shops (and vice versa).

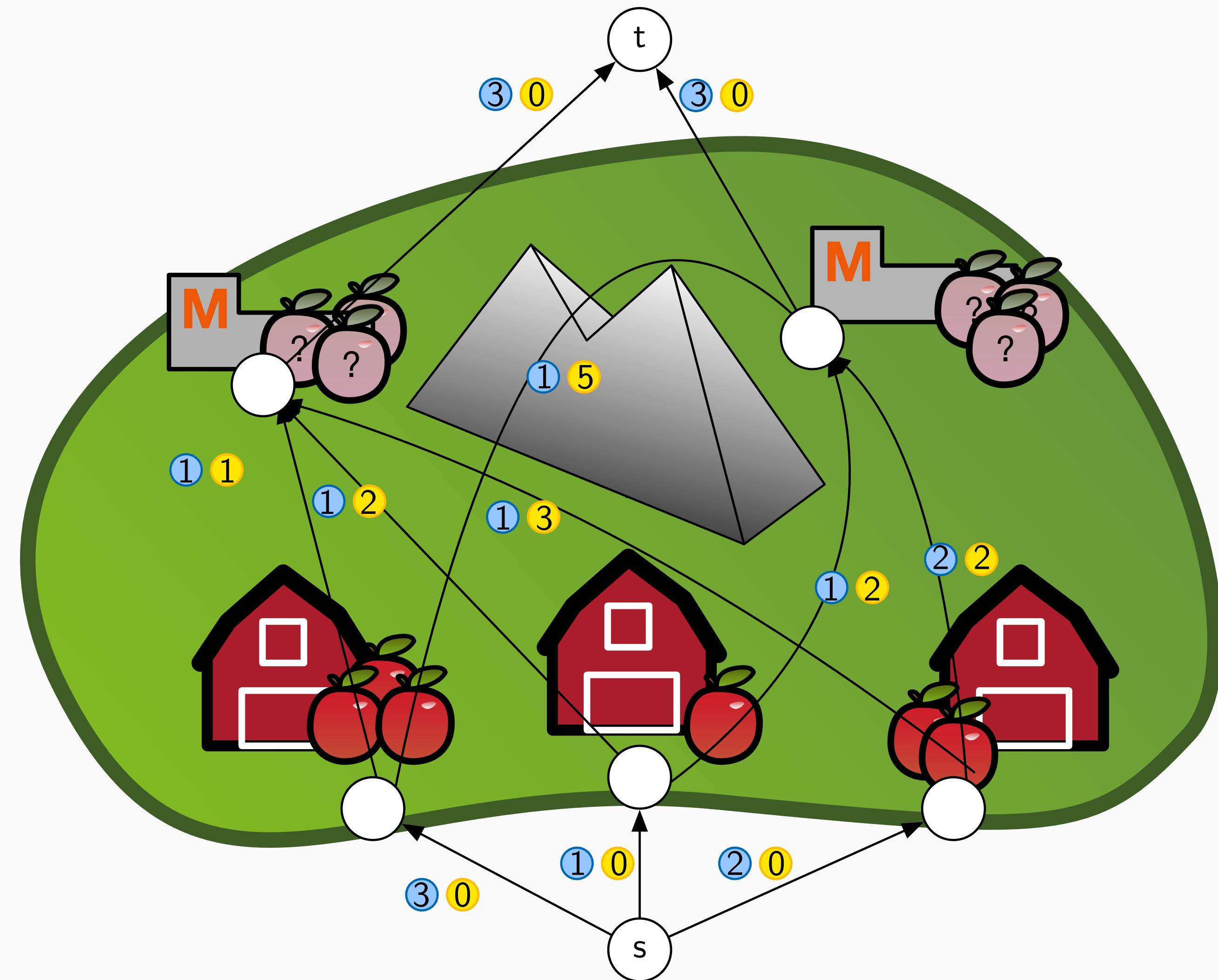
Example: Fruit Delivery



Flow: $2 + 1 + 2 = 5$

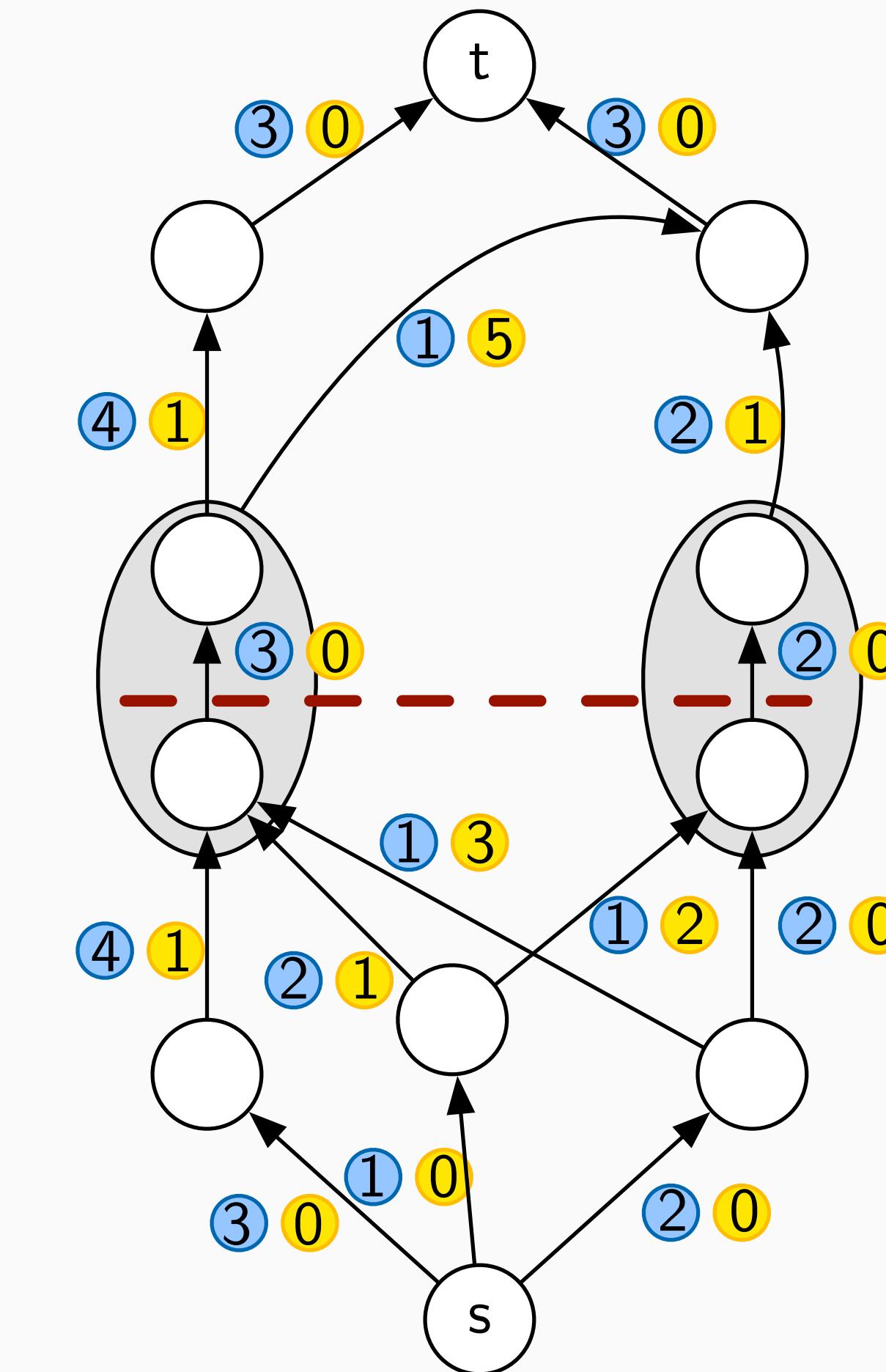
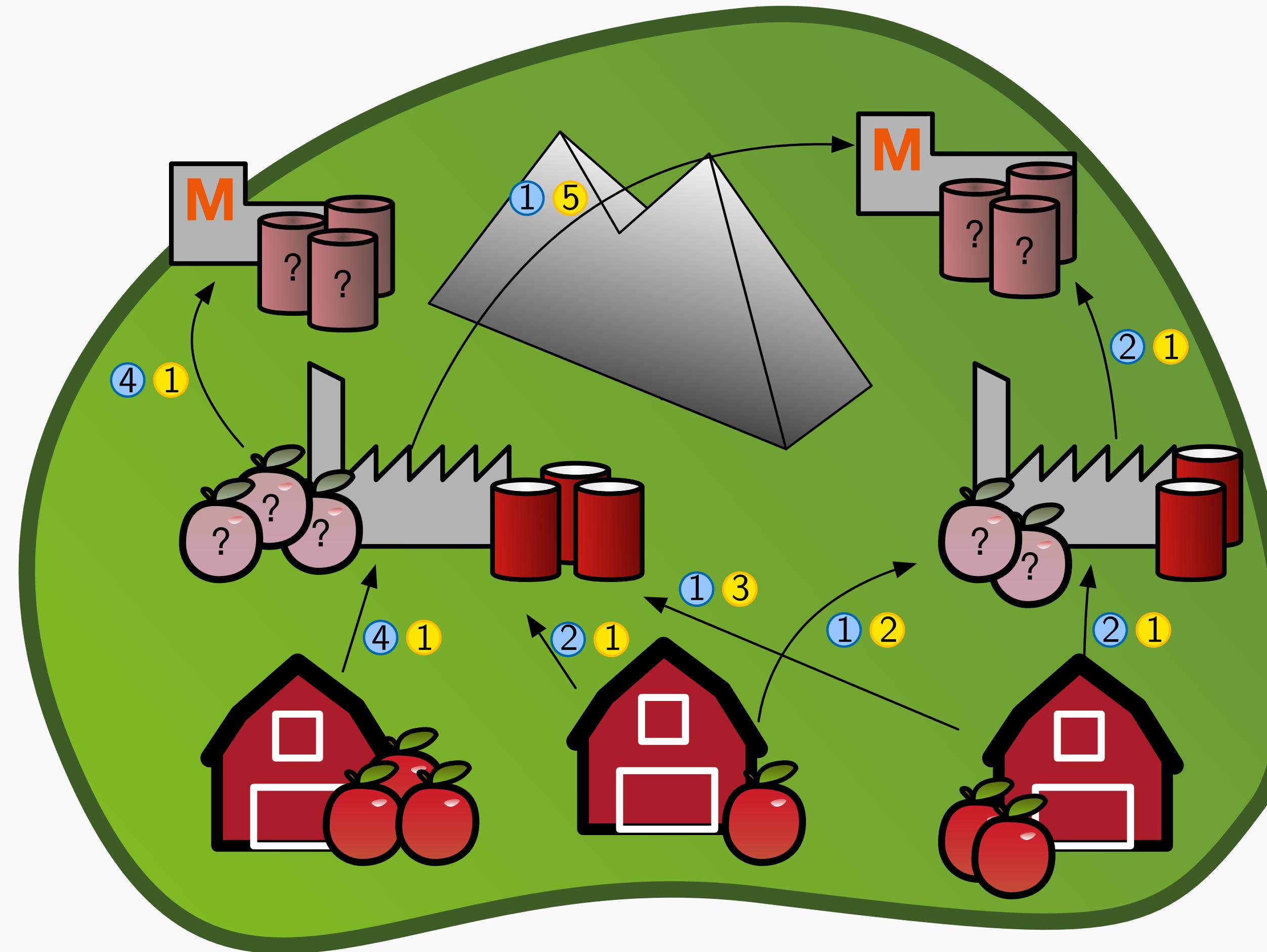
Cost: $1 \cdot 1 + 1 \cdot 5 + 1 \cdot 2 + 2 \cdot 2 = 12$

Example: Fruit Delivery



Extended Example: Canned Fruit Delivery

Extension: Canned fruit requires transportation to and from a canning factory.



Min Cost Max Flow with BGL

There are two algorithms available in BGL (available in BGL v1.55+):

- ▶ `cycle_canceling()`
 - ▶ slow, but can handle negative costs
 - ▶ needs a maximum flow to start with (call e.g. `push_relabel_max_flow` before)
 - ▶ runtime $\mathcal{O}(C \cdot (nm))$ where C is the cost of the initial flow
 - ▶ [\[BGL documentation\]](#), [\[BGL example\]](#).
- ▶ `successive_shortest_path_nonnegative_weights()`
 - ▶ faster, but works only for non-negative costs
 - ▶ sum up all residual capacities at the source to get the flow value
 - ▶ runtime $\mathcal{O}(|f| \cdot (m + n \log n))$
 - ▶ [\[BGL documentation\]](#), [\[BGL example\]](#).

Useful things to know:

- ▶ costs implemented as `edge_weight_t` property
- ▶ call `find_flow_cost()` to compute the cost of the flow

Min Cost Max Flow with BGL

Weights and capacities, just one more nesting level in the typedefs:

```
24 // Graph Type with nested interior edge properties for Cost Flow Algorithms
25 typedef boost::adjacency_list_traits<boost::vecS, boost::vecS, boost::directedS> Traits;
26 typedef boost::adjacency_list<boost::vecS, boost::vecS, boost::directedS, boost::no_property,
27     boost::property<boost::edge_capacity_t, long,
28         boost::property<boost::edge_residual_capacity_t, long,
29             boost::property<boost::edge_reverse_t, Traits::edge_descriptor,
30                 boost::property <boost::edge_weight_t, long> >>> Graph; // new!
31 // Interior Property Maps
32 typedef boost::property_map<Graph, boost::edge_capacity_t>::type EdgeCapacityMap;
33 typedef boost::property_map<Graph, boost::edge_weight_t>::type EdgeWeightMap; // new!
34 typedef boost::property_map<Graph, boost::edge_residual_capacity_t>::type ResidualCapacityMap;
35 typedef boost::property_map<Graph, boost::edge_reverse_t>::type ReverseEdgeMap;
36 typedef boost::graph_traits<Graph>::vertex_descriptor Vertex;
37 typedef boost::graph_traits<Graph>::edge_descriptor Edge;
38 typedef boost::graph_traits<Graph>::out_edge_iterator OutEdgeIt; // Iterator
```

Code file: → [tut9_bgl_mincostmaxflow.cpp](#)

Min Cost Max Flow with BGL

Extending the edge adder:

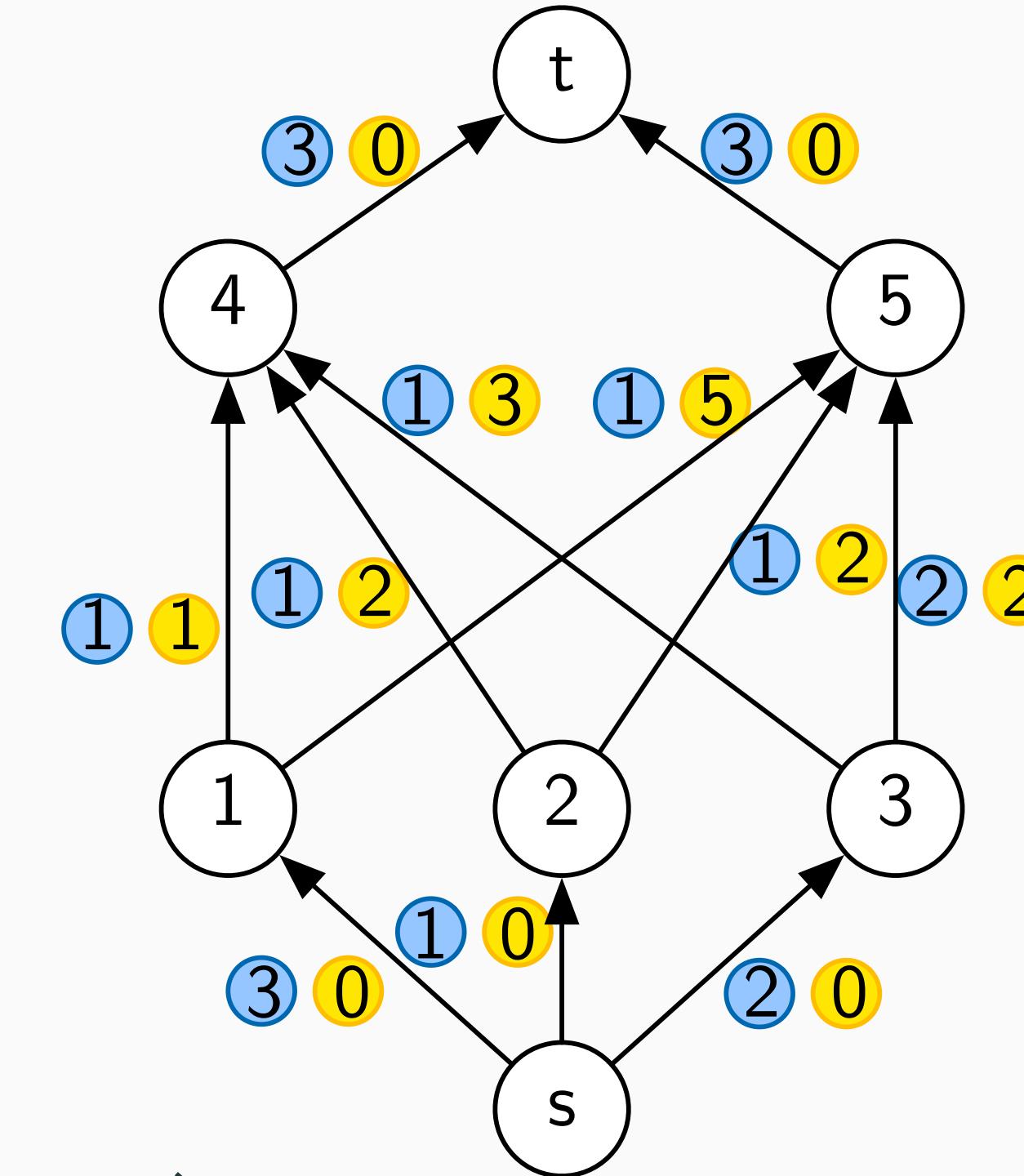
```
49 class EdgeAdder {
50     EdgeAdder(Graph &G, EdgeCapacityMap &capacitymap,
51               EdgeWeightMap &weightmap, ReverseEdgeMap &revedgemap)
52         : G(G), capacitymap(capacitymap), weightmap(weightmap),
53           revedgemap(revedgemap) {}
54     void addEdge(int u, int v, long c, long w) {
55         Edge e, rev_e;
56         boost::tie(e, boost::tuples::ignore) = boost::add_edge(u, v, G);
57         boost::tie(rev_e, boost::tuples::ignore) = boost::add_edge(v, u, G);
58         capacitymap[e] = c;
59         weightmap[e] = w; // new!
60         capacitymap[rev_e] = 0;
61         weightmap[rev_e] = -w; // new
62         revedgemap[e] = rev_e;
63         revedgemap[rev_e] = e;
64     }
65 };
```

Min Cost Max Flow with BGL

Building the graph

```
67 const int N=7;
68 const int v_source = 0;
69 const int v_farm1 = 1;
70 const int v_farm2 = 2;
71 const int v_farm3 = 3;
72 const int v_shop1 = 4;
73 const int v_shop2 = 5;
74 const int v_target = 6;

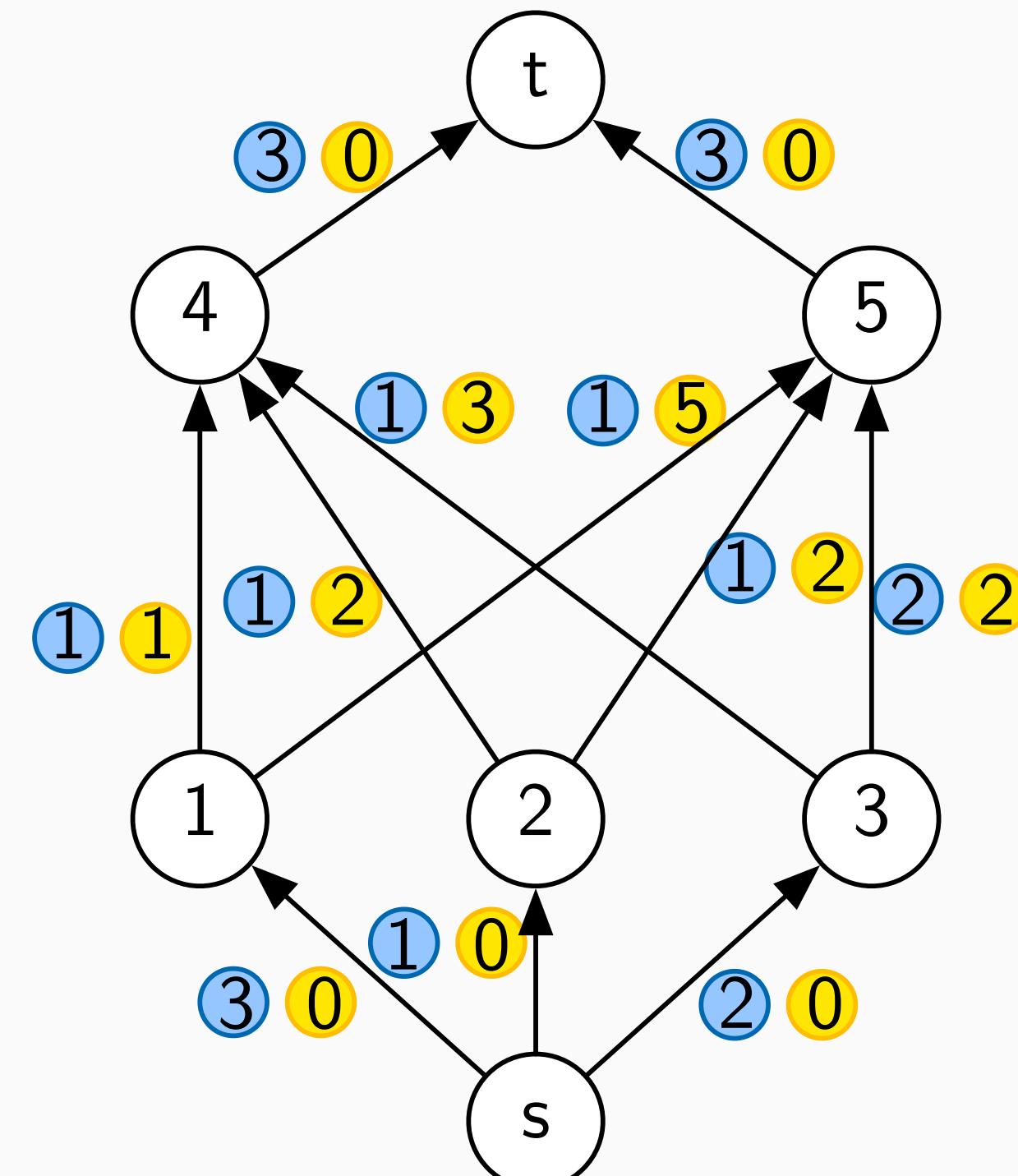
75 // Create Graph and Maps
76 Graph G(N);
77 EdgeCapacityMap capacitymap = boost::get(boost::edge_capacity, G);
78 EdgeWeightMap weightmap = boost::get(boost::edge_weight, G);
79 ReverseEdgeMap revedgemap = boost::get(boost::edge_reverse, G);
80 ResidualCapacityMap rescapacitymap = boost::get(boost::edge_residual_capacity, G);
81 EdgeAdder eag(G, capacitymap, weightmap, revedgemap);
```



Min Cost Max Flow with BGL

Add the edges:

```
85 eaG.addEdge(v_source, v_farm1, 3, 0);
86 eaG.addEdge(v_source, v_farm2, 1, 0);
87 eaG.addEdge(v_source, v_farm3, 2, 0);
88
89 eaG.addEdge(v_farm1, v_shop1, 1, 1);
90 eaG.addEdge(v_farm1, v_shop2, 1, 5);
91 eaG.addEdge(v_farm2, v_shop1, 1, 2);
92 eaG.addEdge(v_farm2, v_shop2, 1, 2);
93 eaG.addEdge(v_farm3, v_shop1, 1, 3);
94 eaG.addEdge(v_farm3, v_shop2, 2, 2);
95
96 eaG.addEdge(v_shop1, v_target, 3, 0);
97 eaG.addEdge(v_shop2, v_target, 3, 0);
```



Min Cost Max Flow with BGL

Running the algorithm:

```
101 // Option 1: Min Cost Max Flow with cycle_canceling
102 int flow = boost::push_relabel_max_flow(G, v_source, v_target);
103 boost::cycle_canceling(G);
104 int cost = boost::find_flow_cost(G);
105 std::cout << "flow" << flow << std::endl; // 5
106 std::cout << "cost" << cost << std::endl; // 12
```

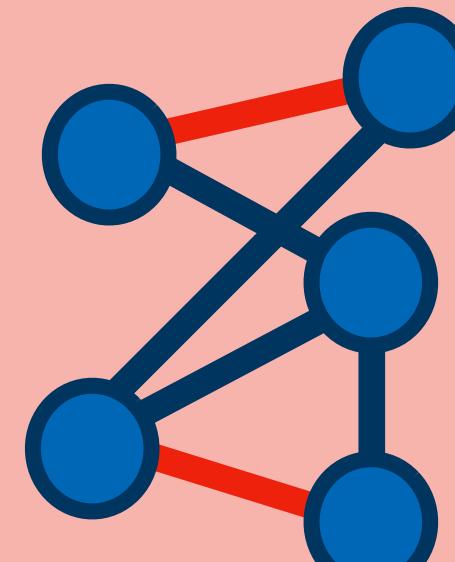
Min Cost Max Flow with BGL

Running the algorithm:

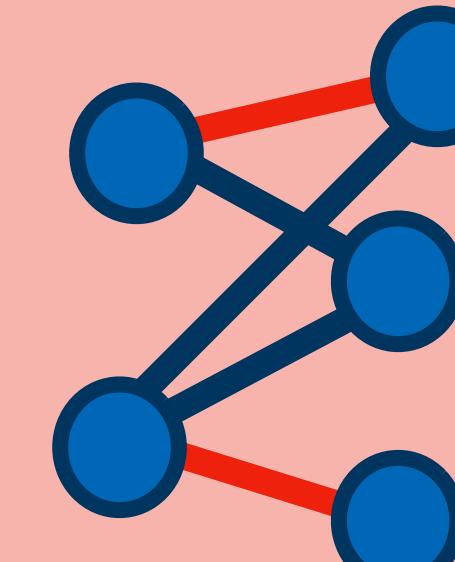
```
110 // Option 2: Min Cost Max Flow with successive_shortest_path_nonnegative_weights
111 boost::successive_shortest_path_nonnegative_weights(G, v_source, v_target);
112 int cost = boost::find_flow_cost(G);
113 // Iterate over all edges leaving the source to sum up the flow values.
114 int s_flow = 0;
115 OutEdgeIt e, eend;
116 for(boost::tie(e, eend) = boost::out_edges(boost::vertex(v_source,G), G); e != eend; ++e) {
117     s_flow += capacitymap[*e] - rescapacitymap[*e];
118 }
119 // Or equivalently, you can do the summation at the sink.
120 int t_flow = 0;
121 for(boost::tie(e, eend) = boost::out_edges(boost::vertex(v_target,G), G); e != eend; ++e) {
122     t_flow += rescapacitymap[*e] - capacitymap[*e]; // reverse for residual edges
123 }
124 std::cout << "s_outflow" << s_flow << std::endl; // 5
125 std::cout << "t-inflow" << t_flow << std::endl; // 5
126 std::cout << "cost" << cost << std::endl; // 12
```

Complexity Landscape

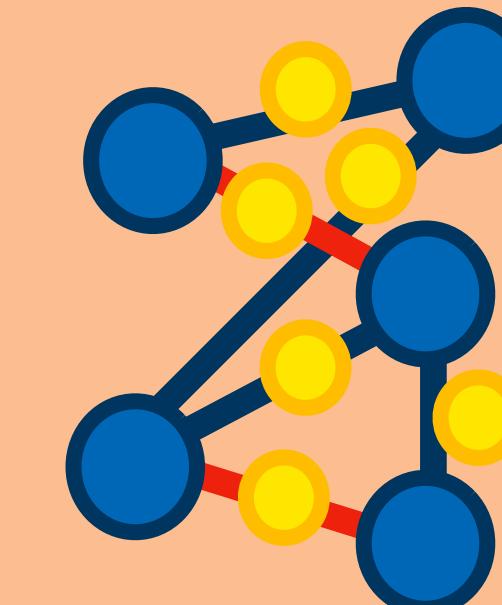
General Matching



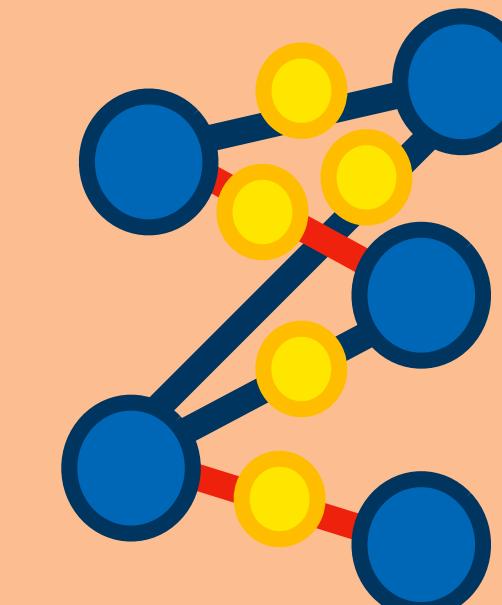
Bipartite Matching



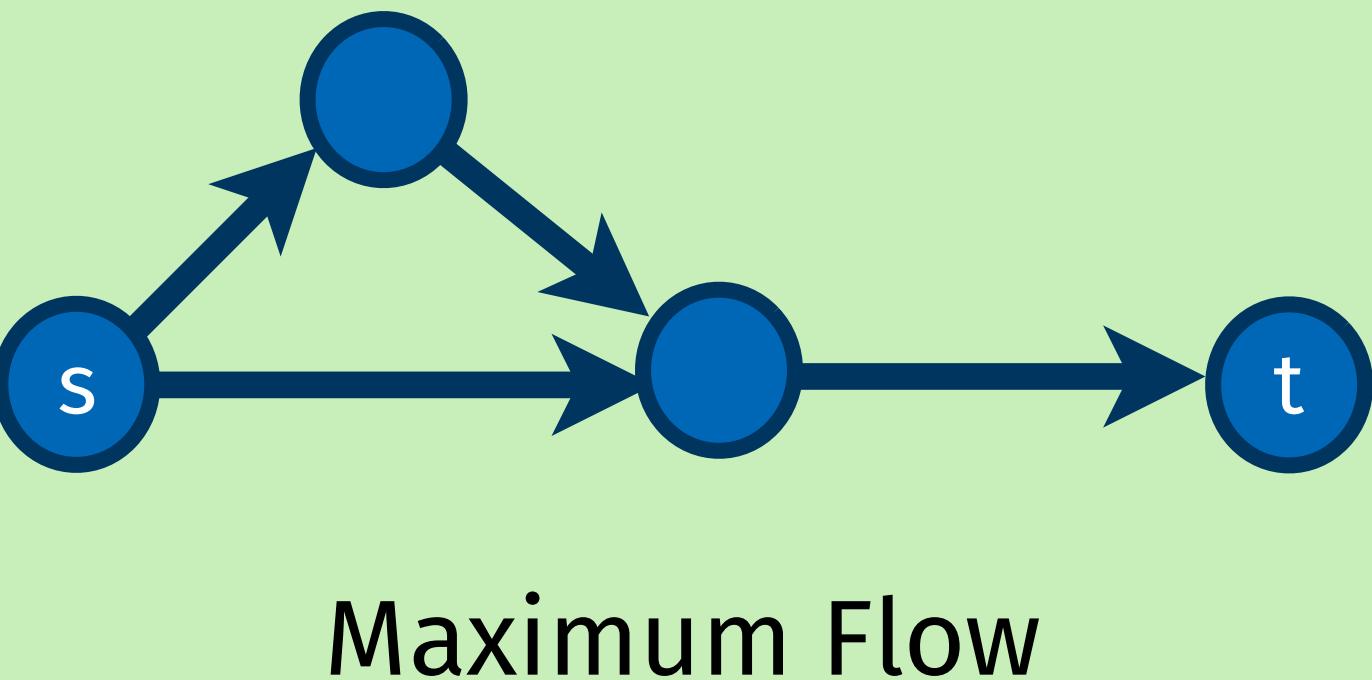
General Cost Matching



Bipartite Cost Matching



Edmonds
Maximum
Cardinality
Matching



???
not in
BGL

Min Cost Max Flow

Summary: Min Cost Max Flow with BGL

What you should remember from this part:

Minimum Cost Maximum Flow

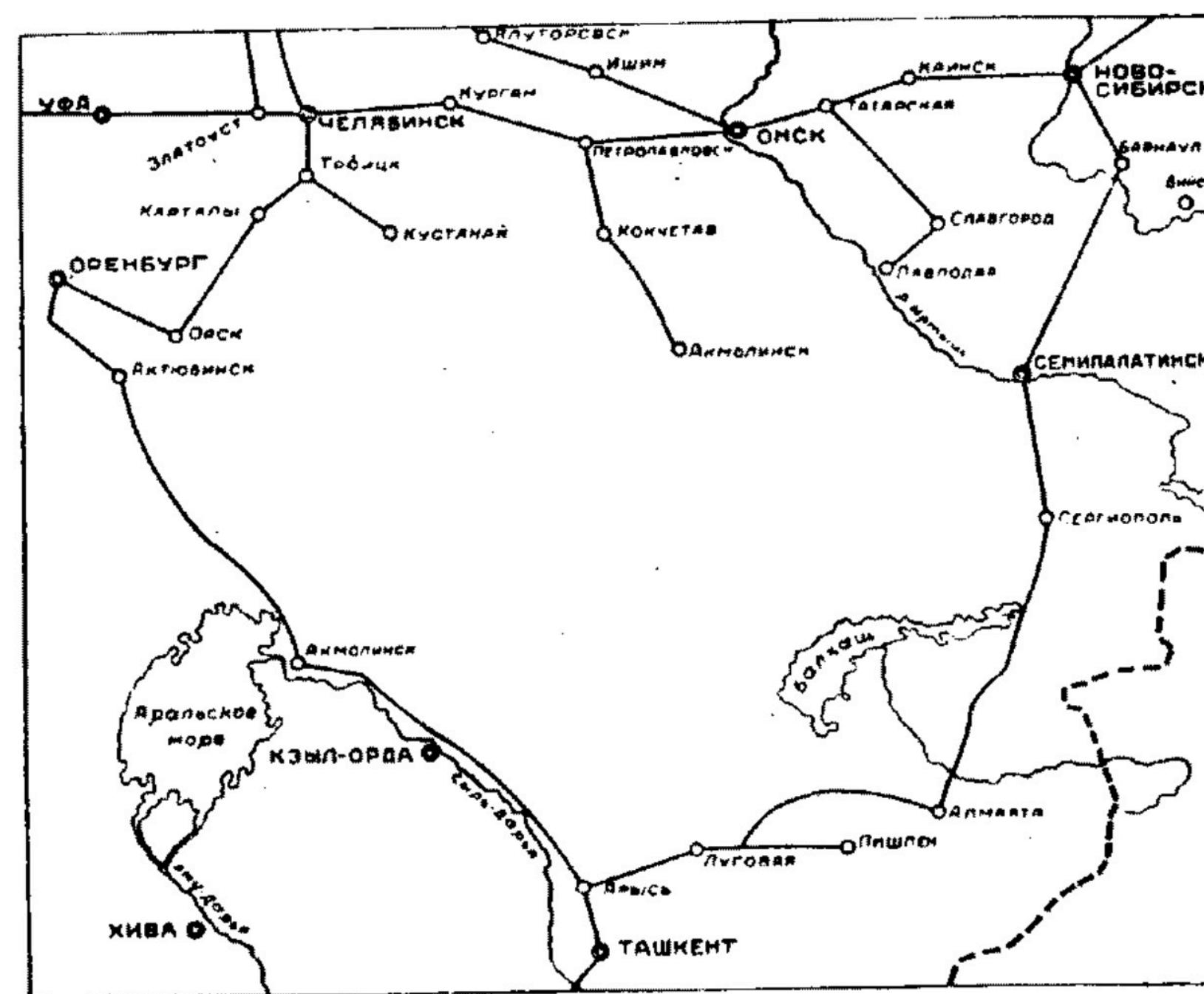
- ▶ is a powerful and versatile modeling tool.
- ▶ is a tiebreaker among several maximum flows (but might still not be unique).
- ▶ = *maximum cost maximum flow with negated costs*.
- ▶ can often be reformulated without negative costs which allows us to use a faster algorithm in BGL (key step in many problems).
- ▶ can easily be implemented when starting with [our template on the Judge](#).
- ▶ is not harder to use in BGL than the regular network flow algorithms, see [BGL Docs](#).

If you are interested in the theory behind these algorithms: (not needed for this course)

- ▶ Stanford CS 261 by Prof. Tim Roughgarden, Spring 2016, [full lectures on Youtube](#)

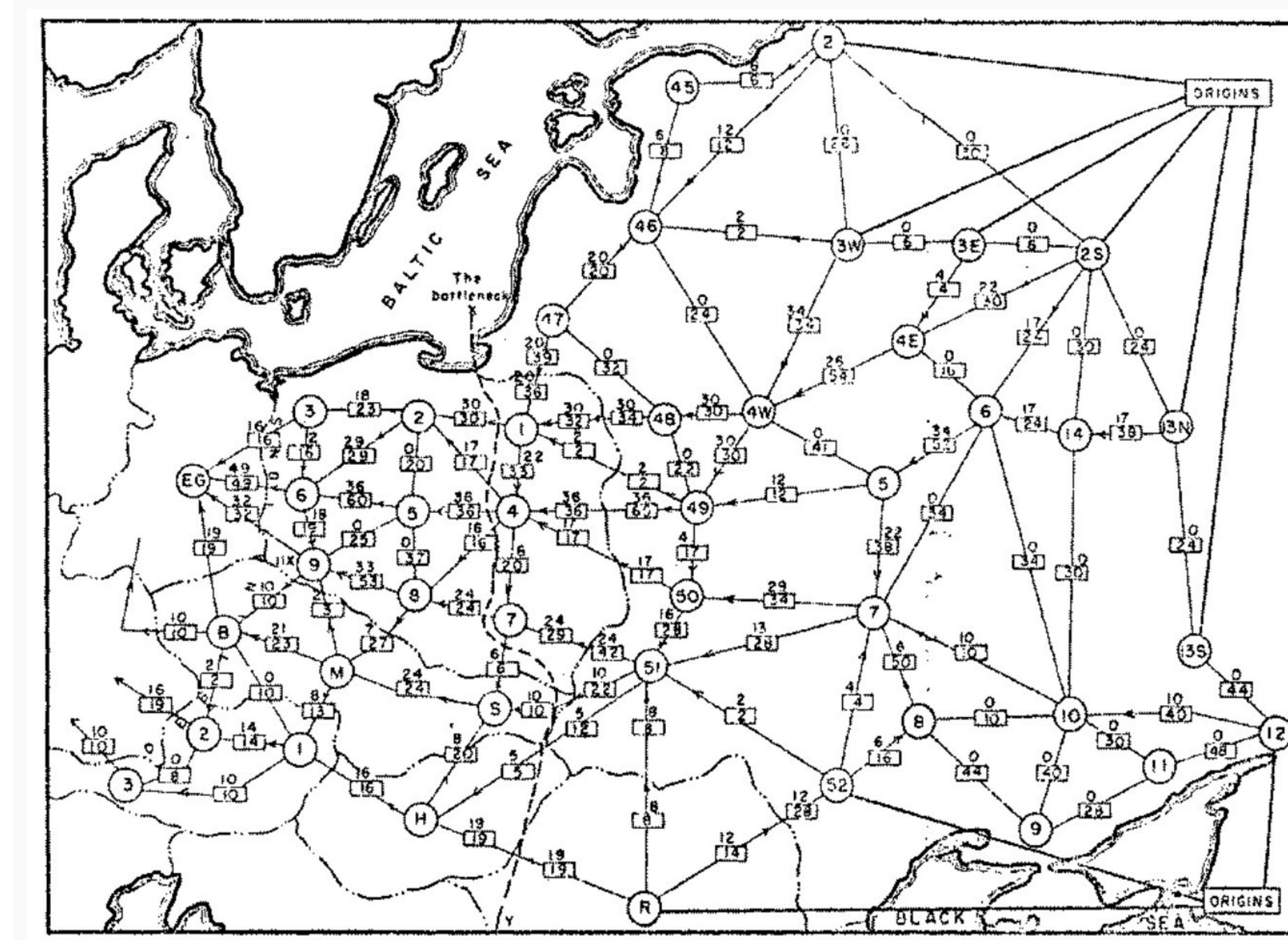
A little bit of history...

What we do in Algolab today was cutting edge research in 1955 ;-)
→ “On the history of the transportation and maximum flow problems”
by Alexander Schrijver, Mathematical Programming, 2002



Min Cost Flow, A. N. Tolstoi, 1930

salt & cement transportation railway network of Soviet Union



Russia/Europe-Min Cut, @ RAND 1955

block max flow of 163'000 tons by air strike

T. E. Harris, F. S. Ross, 1955

First Algorithms

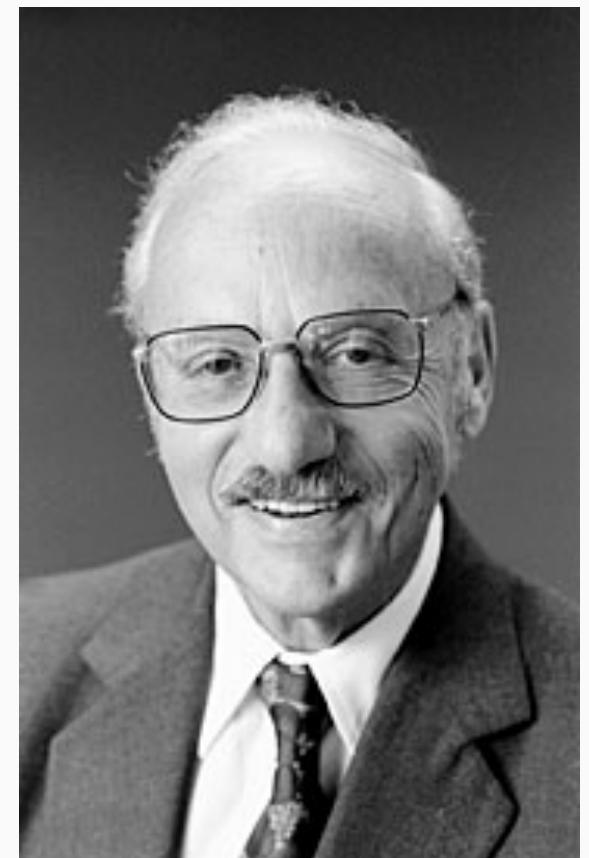
- Cycle canceling idea
A. N. Tolstoi, 1930
 - Greedy flooding
RAND, A. W.
Boldyreff, 1955
 - Augmenting paths
RAND, L. R. Ford Jr.,
D. R. Fulkerson, 1955

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RAND (Research ANd Development) ~1940-1960

Simplex Method



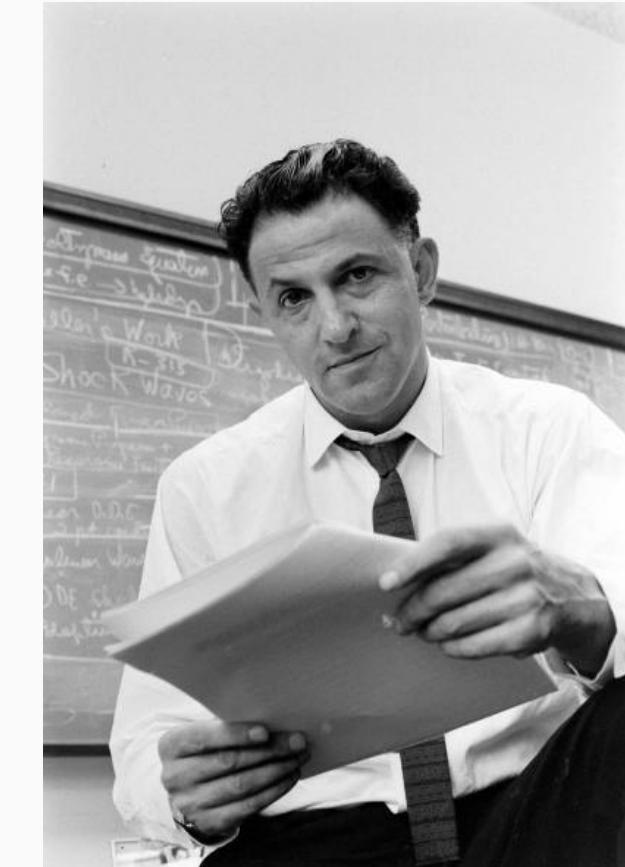
G. Dantzig
~1947

Network Flow



L. R. Ford Jr. D. R. Fulkerson
~1955

Dynamic Programming



R. Bellman
~1957

and so much more...



J. van Neumann J. Nash

A little bit of a conclusion...

No new theory and no new tools past this point

But be prepared to combine all your skills:

- ▶ On top of a flow problem, do binary search for the answer
- ▶ LP formulation vs. flow formulation?
- ▶ Some graph problems can be solved greedily (e.g. MST), others not (e.g. flow)
- ▶ Dijkstra and MinCostMaxFlows are just “special” dynamic programs
- ▶ Find a Min Cost Max Flow formulation where greedy fails for non-unit weights
- ▶ Do BFS on Delaunay triangulation or do Union-Find on Euclidean MST
- ▶ ...

Starting next week:

- ▶ How to balance reading, solving, coding, debugging under time constraints
- ▶ No more problems labeled by topic – figure it out yourself
- ▶ In-class exercises on Wednesdays