1 Proof of Theorem 3.1

We first consider the objective function for $\lambda > 0$:

$$J(\mathbf{s}) = \|\mathbf{s} - \mathbf{r}_g\|^2 + \lambda\, \mathbf{s}^\top \left((1-\alpha-\beta)L_u + \alpha L_g + \beta L_{\text{uni}}\right) \mathbf{s} \tag{1}$$

. To find the optimal $\mathbf{s}_g$ in the sense of minimizing $J(\mathbf{s})$, we take the gradient of $J(\mathbf{s})$ with respect to $\mathbf{s}$ and set it to zero:

$$\nabla_{\mathbf{s}} J(\mathbf{s}) = 2(\mathbf{s} - \mathbf{r}_g) + 2\lambda \left((1-\alpha-\beta)L_u + \alpha L_g + \beta L_{\text{uni}}\right) \mathbf{s} = \mathbf{0}, \tag{2}$$

where $\mathbf{0}$ is the all-zeros vector. Eq. (2) can be rewritten as

$$\left(I + \lambda \left((1-\alpha-\beta)L_u + \alpha L_g + \beta L_{\text{uni}}\right)\right) \mathbf{s} = \mathbf{r}_g. \tag{3}$$

Due to the fact that $L_u = I - \bar{P}_u$, $L_g = I - \bar{P}_g$, and $L_{\text{uni}} = I - \bar{P}_{\text{uni}}$, we have

$$\begin{aligned} A &\triangleq I + \lambda \left((1-\alpha-\beta)(I - \bar{P}_u) + \alpha(I - \bar{P}_g) + \beta(I - \bar{P}_{\text{uni}})\right) \\ &= (1+\lambda)I - \lambda \left((1-\alpha-\beta)\bar{P}_u + \alpha\bar{P}_g + \beta\bar{P}_{\text{uni}}\right). \end{aligned} \tag{4}$$

Let $H = (1-\alpha-\beta)\bar{P}_u + \alpha\bar{P}_g + \beta\bar{P}_{\text{uni}}$. Then, it follows that

$$A = (1+\lambda)I - \lambda H, \tag{5}$$

which results in

$$\begin{aligned} \mathbf{s} &= \left((1+\lambda)I - \lambda H\right)^{-1} \mathbf{r}_g \\ &= \frac{1}{1+\lambda} \left(I - \frac{\lambda}{1+\lambda} H\right)^{-1} \mathbf{r}_g. \end{aligned} \tag{6}$$

Letting $\mu = \frac{\lambda}{1+\lambda}$, we have

$$\mathbf{s} = (1-\mu)\left(I - \mu H\right)^{-1} \mathbf{r}_g. \tag{7}$$

Assuming $0 < \alpha, \beta < 1$, since $0 < \mu < 1$ and H is a combination of three matrices $\bar{P}_u$, $\bar{P}_g$, and $\bar{P}_{\text{uni}}$, the spectral radius of μH is less than 1. Thus, we can expand the inverse of $I - \mu H$ using the Neumann series:

$$\left(I - \mu H\right)^{-1} = \sum_{n=0}^{\infty} (\mu H)^n. \tag{8}$$

Thus, Eq. (7) can be rewritten as

$$\mathbf{s} = (1-\mu) \sum_{n=0}^{\infty} (\mu H)^n \mathbf{r}_g. \tag{9}$$

This infinite series represents a polynomial function of H applied to $\mathbf{r}_g$. Truncating the series at a finite K, we have

$$\mathbf{s} \approx (1-\mu) \sum_{n=0}^{K} (\mu H)^n \mathbf{r}_g, \tag{10}$$

which indicates that $\mathbf{s}$ can be approximated by applying a polynomial graph filter to $\mathbf{r}_g$, where the filter coefficients are determined by $1-\mu$ and μ^n.

Given that H is a weighted sum of $\bar{P}_u$, $\bar{P}_g$, and $\bar{P}_{\text{uni}}$, and each item similarity graph can be represented using their respective polynomial filters $f_1(\bar{P}_u)$, $f_2(\bar{P}_g)$, and $f_3(\bar{P}_{\text{uni}})$, Eq. (10) can be finally expressed as

$$\mathbf{s} \approx \mathbf{r}_g \left((1-\alpha-\beta)f_1(\bar{P}_u) + \alpha f_2(\bar{P}_g) + \beta f_3(\bar{P}_{\text{uni}})\right), \tag{11}$$

where the polynomial filters approximate the series expansion. Therefore, we conclude that Group-GF's prediction is an approximate solution to the optimization problem in Eq.(11) under the condition that the spectral radius of μH is less than 1, which holds when $0 < \alpha, \beta < 1$. This completes the proof of Theorem 3.1.