Math 104 Lecture Notes 22

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April 18, 2023

1 Fundamental Theorem of Calculus

1. $F(x)$ is differentiable and $F'(x)$ is integrable then $\int_a^b F'(x)\,dx = F(b) - F(a)$.
2. f(x) is continous. Then, $F(x) = \int_a^x f(t)\,dt$. $F'(x) = f(x)$.
3. $f(x)$ is integrable. Then $F(x) = \int_a^x f(t)\,dt$ is continuous. We cannot show that $F(x)$ is differentiable.

1.1 Applications of Fundamental Theorem of Calculus

1.1.1 Change of Variables

Theorem Given $g : [a,b] \to \mathbb{R}$ differentiable and $g'(x)$ is integrable. Given $f : I \to \mathbb{R}$ continuous. Then, $\int_{g(a)}^{g(b)} f(u)du = \int_a^b \Big(f(g(x)) - g'(x)\Big)dx$.

Proof

We will be using the second condition of FTC.

$$\text{Let } F(x) = \int_{g(a)}^{x} f(u)du \Rightarrow F'(x) = f(x)$$

$$G(x) = F(g(x)) = \int_{g(a)}^{g(x)} f(u)du$$

$$\Rightarrow G'(x) = F'(g(x))g'(x) = f(g(x)) * g'(x)$$

Now we apply the first condition of FTC.

$$\Rightarrow \int_a^b f(g(x)) * g'(x)dx = G(b) - G(a) = \int_{g(a)}^{g(b)} f(u)du$$

$G(a)$ is always equal to zero.

1.1.2 Integration by Parts

Theorem Given $f, g : [a,b] \to \mathbb{R}$. Assume that f, g are differentiable and f', g' are integrable.

$$\Rightarrow \int_a^b f(x)g'(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b g(x)f'(x)\,dx$$

Proof

$$\int_a^b (f(x)g'(x) + g(x)f'(x))\,dx = f(b)g(b) - f(a)g(a)$$

Notice that $\int_a^b (f(x)g'(x) + g(x)f'(x))\,dx$ is $(F(x) = f(x)g(x))'$. Then, we apply the first of FTC. Let $F(b) = f(b)g(b)$ and $F(a) = f(a)g(a)$. Thus, this equality holds.

2 Limit and Function Convergence

Question: $f_n(x)$ is integrable functions. Assume $f_n(x) \to f(x)$ pointwisely on $[a,b]$. Does this imply that $\lim_{n\to+\infty} \int_a^b f_n(X)dx = \int_a^b f(x)dx$

Answer: This is not true for pointwise convergence. However, if the convergence is uniformly convergence, then this holds!

Theorem: Suppose that $f_n : [a,b] \to \mathbb{R}$ is Riemann integrable for each $n \in \mathbb{N}$ and $f_n \to f$ uniformly on $[a,b]$. Then,

1. f is integrable on $[a,b]$.
2. $\lim_{n\to\infty} \int_a^b f_n dx = \int_a^b f(x)dx$

Proof: Recall the definition of uniform convergence.

$$\forall \epsilon > 0, \exists N s.t. \forall n > N, |f_n(x) - f(x)| < \epsilon, \forall x \in [a,b]$$

Recall the definition of cauchy condition.

Idea Sketch:

f is integrable $\Leftrightarrow \forall \epsilon > 0, \exists p$ s.t. $U(f;p) - L(f;p) < \epsilon$.

$\forall \epsilon > 0, \exists N > 0$, s.t. $\forall n > N |f_n(x) - f(x)| < \frac{\epsilon}{b-a}, \forall x \in [a,b]$

Then let $n = N+1$. Since f_{N+1} is integrable,

$\Rightarrow \exists P$ s.t. $U(f_{N+1};p) - L(f_{N+1};p) < \frac{\epsilon}{2}$

Proof Start Consider $U(f_{N+1};p)$ vs $U(f;p)$ $P = \{a = P_0 < P_1 < \cdots P_n = b\}$

$U(f_{N+1};p) = \sum_k (P_{k+1} - P_k) * sup_{[p_k,p_{k+1}]} f_k$

$L(f_{N+1};p) = \sum_k (P_{k+1} - P_k) * inf_{[p_k,p_{k+1}]} f_k$

Claim 1 : $\forall k, |sup_{[p_k,p_{k+1}]} f_{N+1} - sup_{[p_k,p_{k+1}]} f| < \frac{\epsilon}{2(b-a)}$

Prove claim 1:

$\forall x \in [P_k, P_{k+1}]$, it holds that $f_{N+1}(x) < \frac{\epsilon}{2(b-a)} + f(x)$.

$$\Rightarrow f_{N+1}(x) \leq \frac{\epsilon}{2(b-a)} + sup f$$

$$\Rightarrow sup f_{N+1}(x) \leq \frac{\epsilon}{2(b-a)} + sup f$$

$$\Rightarrow sup f_{N+1}(x) - sup f \leq \frac{\epsilon}{2(b-a)}$$

Now that we proved claim 1, we can show that for upper riemann sum that

$$|U(f_{N+1};p) - U(f;p)| \leq \sum (P_{k+1} - P_k) * \frac{\epsilon}{2(b-a)} \leq \frac{\epsilon}{3}$$

We can also show for lower riemann sum.

$$|L(f_{N+1};p) - Lf;p)| \leq \sum (P_{k+1} - P_k) * \frac{\epsilon}{2(b-a)} \leq \frac{\epsilon}{3}$$

Then,

$U(f;p) - L(f;p) = U(f;p) - U(f_{N+1};p) + U(f_{N+1};p) - L(f_{N+1};p) + L(f_{N+1};p) - L(f;p) < \epsilon$

$L(f_{N+1};p) \leq L(f;p) \leq U(f;p) \leq U(f_{N+1};p) + \frac{\epsilon}{3}$

(continued...)

$\Rightarrow L(f_{N+1};p) - \frac{\epsilon}{3} \leq \int_a^b f\,dx \leq U(f_{N+1};p) + \frac{\epsilon}{3}$ (true integral is between upper and lower riemman sum)
Since $U(f_{N+1};p) - L(f_{N+1};p) < \frac{\epsilon}{3}$, $\Rightarrow |U(f_{N+1};p) - \int_a^b f_{N+1}\,dx| < \frac{\epsilon}{3}$
Then, $L(f_{N+1};p) - \frac{\epsilon}{3} \leq \int_a^b f\,dx \leq U(f_{N+1};p) + \frac{\epsilon}{3}$,

$$\Rightarrow \int_a^b f_{N+1}\,dx < \int_a^b f\,dx \leq \int_a^b f_{N+1}\,dx + \frac{2\epsilon}{3}$$

$$\Rightarrow |\int_a^b f\,dx - \int_a^b f_{N+1}\,dx| < \frac{2\epsilon}{3}$$

(haven't finished the proof)

Consider an example $f_n(x) \begin{cases} 0 & x = 0 \\ n & 0 < x < 1/n \\ 0 & x \leq 1/n \end{cases}$