Math 104 Lecture Notes on Metric Space

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1 Metric space

Metric Space: 1) Metric, 2) Open, closed set, 3) Convergence of Sequence

Def Condition of Metric: Assume that there exists a set X. A metric d on X is a function $d(x,y): XxX \to \mathbb{R}$ such that for any $x, y, z \in X$

1. **Positivity:** $d(x,y) \geq 0$ and $d(x,y) = 0 \Leftrightarrow x = y$.
2. **Symmetry:** $d(x,y) = d(y,x)$
3. **Triangle Inequality:** $d(x,z) \leq d(x,y) + d(y,z)$

Def (X,D) is a **metric space**.

1.0.1 Examples of Metric Space

1. $(\mathbb{R}, d(x,y) = |x-y|)$
2. (X,$d_{trivial}$) where $d_{trivial} \begin{cases} 0 & \text{if } x = y \\ -1 & \text{if } x \neq y \end{cases}$
3. $X = C([0,1]) \forall f, g, \in X$ (elements in X are functions),
   $d_\infty(f,g) = sup_{x\in[0,1]}|f-g|(L_\infty-$ distance between functions)

Using the example 3, we will show that d_∞ is a metric system on X.

- Condition 1: Since $|f(x)-g(x)| \geq 0 \Rightarrow sup|f-g| \geq 0 \Rightarrow d(f,g) \geq 0$. $d(f,g) = 0 \Leftrightarrow sup_{x\in(0,1)}|f-g| = 0 \Leftrightarrow |f-g| = 0 \Leftrightarrow f = g$
- Condition 2: $|f-g| = |g-f|$

- Condition 3: $\forall f, g, h, d(f,g) \le d(f,h) + d(g,h). \forall |f(x) - g(x)| \le |f(x) - h(x)| + |h(x) - g(x)| \le d(f,h) + d(h,g)$

  Since sup is the smallest upper bound, $d(f,g) \le d(f,g) + d(h,g)$.

L_p **distance**: $d_p(f,g) = (\int_0^1 |f-g|^p dx)^{\frac{1}{p}}$

2 Open Ball

Def (Open Ball) Let (X,d) be a metric space. The open ball $B_r(x)$ with radius r and centered at x is defined as $B_r(x) = z|d(z,x) < r$.
Def Let (X,d) be a metric space. Then, a subset $U \subseteq X$ is open if $\forall x \in U$, $\exists r > 0$ such that $B_r(x) \subset U$.
V is a closed set if V^c is open.

Theorem

1. Empty set and X are both open and closed sets.
2. An arbitrary union of open set is open.
3. A finite intersection of open set is open.

   (infinite intersection is not ex: $(-1-\frac{1}{n}, 1)$)

Proofs

1. Check definition
2. $G = \cup_i U_i$. $\forall x \in G, \exists U_i s.t. x \in U_i \Rightarrow$Since U_i is open, $\exists r > 0$ such that $B_r(x) \subseteq G, B_r(x) \subseteq U_i$
3. $G = \cap_{i=1} U_i$ $\forall x \in G \Rightarrow \forall 1 \le i \le n, x \in U_i \Rightarrow \exists r_i > 0 s.t. B_r(x) \subseteq U_i \Rightarrow r = min_{1 \le i \le n} r_i > 0 \Rightarrow B_r(X) \subseteq B_{r_i}(x) \subseteq U_i, \forall i \Rightarrow \cap_{i=1}^n U_i = G$

   It should be finite intersections because r can become 0 if the min gets really small. IF there are infinitely many minimums, then we need to replace $r = min r_i$ by $r = inf r_i$. This is why the intersections should be finite.

Def Let (X,d) be a metric space, $A \subseteq X$.

1. A point $x \in A$ is an **interior point** if $\exists r > 0$s.t. $B_r(x) \subseteq A$
2. A point $x \in A$ is an **isolated point** if $\exists r > 0$ s.t. $B_r(x) \cap A = x$
3. A point $x \in X$ is called a **boundary point** of A if $\forall r > 0, B_r(x) \cap A \ne \phi, B_r(x) \cap A^c \ne \phi$
4. A point X is an accumulation point of A if for $r > 0, \exists y \in A \cap B_r(x) and y \ne x$

Def Let (X, d) be a metric space, $A \subseteq X$.

1. A°: set of **interior points** of A
2. ∂ A: set of boundary points of A
3. $\bar{A} = \partial A \cup A$: **closure of A**

Proposition

1. A° is the largest open set contained in A. Also, $A^\circ = \cup U \subseteq A$
2. $\bar{A}$ is the smallest closed set that contains A. $\bar{A} = \{A \subseteq F$, F is closed $\}$
3. $\partial A = \bar{A} / A = x \in \bar{A}, x \notin A$

Theorem V_i is a closed set.

1. Arbitrary intersection of V_i is closed.
2. Finite union of V_i is closed.

Proof using $(\cup A_i)^c = \cap (A_i)^c$