# Math 104 Homework 10

Chae Yeon Lee

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## 1 Problem 1

**Question**: Suppose f is a continuous function on $[a,b]$, and $f(x) \geq 0$ for all $x \in [a,b]$. Prove that if $\int_a^b f = 0$, then $f(x) = 0$ for all $x \in [a,b]$.
**Answer**:
Let's prove by contradiction. Assume that $\exists x \in [a,b]$ such that $f(x) > 0$. Since f is continuous, we can find $\delta > 0$ such that

$$\forall x \in (x_0 - \delta, x_0 + \delta), |f(x) - f(x_0)| < \frac{\epsilon}{2}$$

Then by triangular inequality, we get that

$$f(x) < |f(x_0)| - \frac{\epsilon}{2} > \frac{\epsilon}{2}$$

Since $(x_0 - \delta, x_0 + \delta) \subset [a,b]$,

$$\int_a^b f(x) \geq \int_{x_0-\delta}^{x_0+\delta} f(x) \geq \frac{\epsilon}{2} * 2\delta = \delta\epsilon > 0$$

This contradicts the given that $\int_a^b f = 0$. Thus, it must be true that $\forall x \in [a,b]$ it holds that $f(x) = 0$.

## 2 Problem 2

**Question:** Construct an example of a function where $f(x)^2$ is integrable on $[0,1]$ but $f(x)$ is not.
**Answer:**

$$f(x) = \begin{cases} 1 & \text{if } x \in [0, 0.5) \\ -1 & \text{if } x \in [0.5, 1] \end{cases} \tag{2.1}$$

# 3 PROBLEM 3

**Question:** Ross 33.7
**Answer for a):**
For any subset $S \subseteq [a, b]$ and $x_0, y_0 \in S$, it holds that

$$f(x_0)^2 - f(y_0)^2 = (f(x_0) + f(y_0))(f(x_0) - f(y_0))$$

Then,

$$f(x_0)^2 - f(y_0)^2 \leq |f(x_0) + f(y_0)||f(x_0) - f(y_0)| \leq 2B|f(x_0) - f(y_0)| \leq 2B|M(f,S) - m(f,S)|$$

Then it holds that for all partitions P of S,

$$U(f^2, p) - L(f^2, p) \leq 2B[U(f, P) - L(f, P)]$$

**Answer for b):**
If $f$ is integrable, there $\exists$ partition p such that $U(f, p) = L(f, p)$. Also, in part a, we showed that $U(f^2, p) - L(f^2, p) \leq 2B[U(f, p) - L(f, p)]$. Since $U(f, p) - L(f, p) < \frac{\epsilon}{2B}$, by the definition of integrability of $f$, we can see that $U(f^2, p) - L(f^2, p) \leq 2B\frac{\epsilon}{2B} \leq \epsilon$ for some partition p. Thus, $f^2$ is also integrable.

# 4 PROBLEM 4

**Question:** Ross 33.8
**Answer for a):**
We are given that f and g are integrable. $4fg = (f+g)^2 + (f-g)^2$. $f+g$ and $f-g$ is integrable because adding and subtracting two integrable functions results in integrable function. In problem 3 part b), we showed that if f is integrable, then $f^2$ is also integrable. Thus $(f+g)^2$ and $(f-g)^2$ are also integrable. Thus, adding these two functions, or $4fg$, results in integrable function. Since $4fg$ is integrable and dividing an integrable function by a constant results in an integrable function, we have that $fg$ is an integrable function.
**Answer for b):**

- min(f,g) is integrable on $[a, b]$:

  min(f,g) = $\frac{1}{2}(f+g)$ - $\frac{1}{2}|f-g|$. The sum of two integrable function is still integrable, so $f+g$ and $f-g$ are integrable. Also, taking an absolute value of an integrable function is still integrable. Thus, min(f,g) is integrable.

- max(f,g) is integrable on $[a, b]$:

  max(f,g) = $-min(-f,-g)$. Since f and g are integrable, $-f$ and $-g$ are also integrable. Since multiplying an integrable function (min(-f,-g)) by a constant produces another integrable function, max(f,g) is also integrable.

# 5 PROBLEM 5

**Question a:** For any two numbers $u, b \in \mathbb{R}$, prove that $uv \leq (u^2+v^2)/2$. Let f and g be two integrable funciton on $[a,b]$. Prove that if $\int_a^b f^2 = 1$ and $\int_a^b g^2 = 1$, then

$$\int_a^b fg \leq 1$$

**Answer for a):**
First, let's show that $uv \leq (u^2+v^2)/2$.

$$(u^2+v^2)/2 - uv \geq 0 \Leftrightarrow \frac{u^2+v^2-2uv}{2} \geq 0 \Leftrightarrow \frac{(u-v)^2}{2} \geq 0$$

If f and g are integrable function, then

$$\int_a^b f^2 + \int_a^b g^2 = \int_a^b f^2 + g^2 = 2$$

Then,

$$\int_a^b fg \leq \int_a^b \frac{f^2+g^2}{2} = 2/2 = 1$$

**Question b:** Prove the Schwarz inequality.
**Answer for b):**

$$|\int_a^b fg| \leq \int_a^b |fg| \leq \int_a^b |f||g| \leq \int_a^b |f| \int_a^b |g| = \int_a^b (f^2)^{1/2} \int_a^b (g^2)^{1/2} = (\int_a^b f^2)^{1/2} (\int_a^b g^2)^{1/2}$$