Integers:

$\mathbb{N} = \{1, 2, 3, \dots\}$ positive integers

(Assume n is an integer $\Rightarrow n \in \mathbb{N}$)

$\mathbb{Z} = \mathbb{N} \cup \{0\} \cup -\mathbb{N}$

Example of calculations: $1+1=2$, $2-3=-1$, $2\times 3 = 6$.
$2\times -3 = -6$,

Fundamental Theorem of proving by Induction.

Thm. Suppose that $A \subset \mathbb{N}$ is a set of positive integers
($\subset$: subset)

such that ① $1 \in A$

② If $n \in A$, then $n+1 \in A$.

$\Rightarrow A = \mathbb{N}$

---

① + ② $\Rightarrow 2 \in A$.

Let $n=1$, then $2 \in A$ ...

proposition: For every $n \in \mathbb{N}$, $\sum_{k=1}^{n} k^2 = \frac{1}{6} n(n+1)(2n+1)$ ($*$)

→ using proof by induction

let A be the set of positive integers. that ($*$) is true.

Then, $A \subset \mathbb{N}$ (we want to show that $A = \mathbb{N}$).

→ we check ① $1 \in A$, ② if $n \in A$, then $n+1 \in A$.

→ ① $1 \in A$??

$n=1$, left of $(\star) = 1$
right of $(\star) = 1$.

$\Rightarrow$ $1 \in A$.

② (If $m \in A$, then $m+1 \in A$.

left $= \sum_{k=1}^{m+1} k^2 = (m+1)^2 + \boxed{\sum_{k=1}^{m} k^2}$

$\Downarrow (\star)$ for $n=m$.

$= (m+1)^2 + \frac{1}{6} m(m+1)(2m+1)$

$= \frac{1}{6}(m+1)\left[6(m+1) + m(2m+1)\right]$

$= \frac{1}{6}(m+1)(2m^2+7m+6)$

$= \frac{1}{6}(m+1)(m+2)(2m+3)$.

right hand side.

2. Rational Numbers.

$\mathbb{Q} = \{ \frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0 \}$ + $\frac{p_1}{q_1} = \frac{p_2}{q_2}$ if $p_1 q_2 = q_1 p_2$.

Theorem: (zero point of polynomials)
consider the polynomial equations $(n \geq 1, n \in \mathbb{N})$

$$x^n + c_{n-1} x^{n-1} + c_{n-2} x^{n-2} + \dots + c_1 x + c_0 = 0. \quad (\star\star)$$

where $c_{n-1}, c_{n-2}, \dots, c_0 \in \mathbb{Z}$.

→ Any rational solutions of this equations must be an integer that divides c_0.

proof by contradiction

proposition: $\sqrt{2}$ is <u>not</u> a rational number.

proof: Define $\sqrt{2}$ as the solution for $x^2-2=0$ and $x>0$

use the theorem with $n=2$, $c_1=0$, $c_0=-2$.

assume x is a rational number. $\exists$ $x \in \mathbb{Z}$, $x \mid c_0=-2$

$x=1, 2, -1, -2$.

none of these satisfy the equation.

therefore, we cannot find a solution. therefore, x is not a rational number.

So, $\sqrt{2}$ is not a rational number.

Theorem 2.1 Consider the polynomial equation $(n \geq 1)$

$$x^n + C_{n-1}x^{n-1} + C_{n-2}x^{n-2} + \cdots + C_1 x + C_0 = 0$$

where $C_0, \ldots, C_{n-1} \in \mathbb{Z}$, $C_0 \neq 0$.

Then every $\boxed{\text{rational}}$ solution to this equation must be an $\boxed{\text{integer}}$ that divides C_0.

pf Assume $x = p/q$ $p, q \in \mathbb{Z}$. p, q are relatively prime

$$\left(\frac{p}{q}\right)^n + C_{n-1}\left(\frac{p}{q}\right)^{n-1} + \cdots + C_0 = 0$$

$$\overset{* q^n}{\Rightarrow} p^n + C_{n-1}p^{n-1} \cdot q + C_{n-2}p^{n-2}q^2 + \cdots + C_0 q^n = 0$$

$$p^n = -(C_{n-1}p^{n-1} \cdot q + \cdots) - C_0 q^n$$

$$\Rightarrow p^n = -p\,(C_{n-1}p^{n-2} \cdot q + \cdots + C_1 q^{n-1}) - C_0 q^n$$

$$\overset{1/p}{\Rightarrow} p^{n-1} = -(C_{n-1}p^{n-2} \cdot q + \cdots + C_1 q^{n-1}) - \frac{C_0 q^n}{p}$$

$\frac{C_0 q^n}{p} \in \mathbb{Z}$. $\Rightarrow \frac{C_0}{p} \in \mathbb{Z} \Rightarrow p \,|\, C_0$

$\because$ p and q^n are relatively prime.

move (II) to right side

$$C_0 q^n = -(p^n + C_{n-1}p^{n-1}q + \cdots + C_1 p q^{n-1})$$

$$C_0 q^n = -p^n - (C_{n-1}p^{n-1}q + \cdots + C_1 p q^{n-1})$$

$$= -p^n - q\,(C_{n-1}p^{n-1} + \cdots + C_1 p q^{n-2})$$

$$\overset{1/q}{\Rightarrow} \underbrace{C_0 q^{n-1}}_{\text{Int}} = \underbrace{-\frac{p^n}{q}} - \underbrace{(C_{n-1}p^{n-1} + \cdots)}_{\text{Integer}}$$

$$\Rightarrow \frac{p^n}{q} \in \mathbb{Z}.$$

$\Rightarrow$ since p, q are relatively prime, p^n, q are also relatively prime.

$\Rightarrow$ q has to be 1 because p^n and q are relatively prime.

Real Numbers

① algebraic property ($+, \times, -, \div$)
② order property ($>, <, \geq, \leq, =$)
③ metric property
④ sup, inf
⑤ completeness axiom

1) Algebraic property axiom

There exists binary operations: $a, m : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

call $a(x,y) = x+y$, $m(x,y) = x \cdot y = xy$

There exists a two elements $0, 1 \in \mathbb{R}$ s.t.

Identity a) $x+0 = x \Rightarrow a(x,0) = x$

Inverse b) For any $x \in \mathbb{R}$, we find $y \in \mathbb{R}$ s.t $x+y=0$ (y is called $-x$)
$\to$ there is an inverse

Associative c) $x+(y+z) = (x+y)+z$

Commutative d) $x+y = y+x$

multiplicative identity e) $x \cdot 1 = x$

multiplication inverse f) For any $x \in \mathbb{R} \setminus \{0\}$, $\exists$ y such that $xy = 1$

g) $x(yz) = (xy)z$

h) $xy = yx$

i) $z(x+y) = zx+zy$

2. order properties

There exists $\underbrace{3 \text{ relations}}_{}$ on $\mathbb{R}$ $\quad \forall x, y \in \mathbb{R}$,

$>, =, <$

either $x > y$
$x = y$
$x < y$

(a) if $x < y \Rightarrow x + z < y + z$
(b) $x < y$, $z > 0 \Rightarrow xz < yz$
(c) if $x < y \Rightarrow -x > -y$

($1 \le 1 \to$ always true.)

3. metric properties

↳ Distance on the real line.

$x, y \in \mathbb{R}$, distance between x, y.

Definition : 1. $A : \mathbb{R} \to \mathbb{R}$ $\quad (A(\cdot) = |\cdot|)$ is the absolute value

$$A(x) = \begin{cases} x & x \ge 0 \\ -x & x < 0 \end{cases}$$

$x, y \in \mathbb{R}$

2. $dis(x, y) = A(x-y) = |x-y|$

4.

$\boxed{\text{Theorem}}$ Given $x, y \in \mathbb{R}$, (1) $|x| \ge 0$
(2) $|xy| = |x||y|$
(3) $|x+y| \le |x| + |y|$ Triangle inequality

<u>proof</u> 1) $|x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$ $\quad$ if $x \ge 0$, then $|x| \ge 0$
if $x < 0$, then $\underline{-x > -0 = 0}$
$|x| = -x > 0$ $\quad$ From order property (3)

2) $|xy| = |x||y|$

• if $x \ge 0, y \ge 0 \Rightarrow xy \ge 0 \Rightarrow |x \cdot y| = xy = |x||y|$
$x = |x| \quad y = |y|$

· also check $x < 0, y \ge 0$

• if $x \ge 0$, $y < 0 \Rightarrow xy \le 0 \Rightarrow |xy| = -x \cdot y = |x||y|$
$|x| = x \quad |y| = -y$

• if $x < 0$, $y < 0 \Rightarrow xy \ge 0 \Rightarrow |xy| = xy = |x||y|$
$|x| = -x \quad |y| = -y$

3-3

3) $\underline{x \le |x| \, , \quad -x \le |x|}$

$$\left.\begin{aligned} &\Rightarrow x \le |x| \, , \quad y \le |y| \quad \Rightarrow \quad x+y \le |x|+|y| \\ &\Rightarrow -x \le |x| \, , \quad -y \le |y| \quad \Rightarrow \quad -(x+y) \le |x|+|y| \end{aligned}\right\} \Rightarrow |x+y| \le |x|+|y|$$

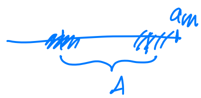
Notation for sets

$(-\infty, b) = \{x \in \mathbb{R} \mid x < b\}$

$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$

$(a, \infty) = \{x \in \mathbb{R} \mid x > a\}$

① Maximal element



~~A set~~ A has a maximal element $a_M \in A$ if $a \le a_M$ for every $a \in A$.

② minimal element

$a_m \in A$ if $a_m \le a$ for every $a \in A$.

③ upper bound <not unique>.

A set A is bounded from above if there exists some $M \in \mathbb{R}$ s.t. for any $a \in A$, $M \ge a$.

↳ difference between maximal element and upper bound:

maximal element $a_M \in A$ but upper bound doesn't need to belong to M.

④ Lower bound : " s.t. for any $a \in A$, $m \le a$.

$A = (0, 1] = \{x \mid 0 < x \le 1\}$

$\max A = 1$, $\min A =$ DNE

$A = (-\infty, 1]$, $\{x \mid x \le 1\}$

$\max A = 1$. $\min A =$ DNE

upperbound $A = \{u \in \mathbb{R} \mid u \ge 1\}$

lowerbound $A =$ DNE $(-\infty \notin \mathbb{R})$

$A = (0, 1]$

$\max = 1$, $\min A$, DNE, $\inf A = 0$

- M sup if ① upper bound of A. ② For $\forall$ upperbound M of A, $M_{sup} \le M$.

  smallest upperbound of A.

- M inf if ① lower bound of A. ② For $\forall$ lower bound m of A, $M_{inf} \ge m$.

properties of sup, inf

- (uniqueness) ① if $M_1 = \sup(A)$, $M_2 = \sup(A)$ then $M_1 = M_2$.

  ② $\sup A \ge \inf A$

  ③ $\max A$ exists then $\max A = \sup(A)$

completeness axiom

thm: Given a set $A \subset \mathbb{R}$, $A \ne \phi$

if the upper bound of A is not empty (exists)

then sup A exists $\sup A \in \mathbb{R}$

$A = \{q \in \mathbb{Q} \mid q^2 < 2\}$

$\sup A = \sqrt{2} \notin \mathbb{Q}$

$\in \mathbb{R}$ → real number doesn't have a gap.

corollary : Given a set $A \subset \mathbb{R}$, $A \ne \phi$

if A has a lower bound. the inf A exists, $\inf A \in \mathbb{R}$.

⇒ if a lower bound exists, then inf A exist.

proof = $\inf(A) = -(\sup(-A))$

$-A = \{-a \mid a \in A\}$

Since a has a lower bound, then, $-a$ has a upper bound.

Then, $\sup(-A)$ always exists by the theorem. Hence $-(\sup(-A))$ always exist.

constructing R from Q.

$S_r = \{q \in \mathbb{Q} \mid q < r\}$

Proposition: Assume $S \subset \mathbb{Q}$

① $S \ne \phi$

② If $q \in S$, then for any $q_1 < q$, $q_1 \in S$

③ $\sup S \notin S$.

then $\exists$ $r \in \mathbb{R}$ such that $S = \{q \in \mathbb{Q} \mid q < r\}$ ← dedekind cut in Q

$\sup_r S_r \longrightarrow r$

① $S_{r_1} \ne S_{r_2}$ then sup

Already $\mathbb{Q}$

$\mathbb{R} = \{\text{dedekind cuts in } \mathbb{Q}\}$

define "$r_1 + r_2$" $\rightarrow S_{r_1} + S_{r_2} = \{ q_1 + q_2 \mid q_1 \in S_{r_1},\ q_2 \in S_{r_2} \}$

A subset $A \subseteq \mathbb{R}$ is bounded if A is upper and lowerbounded.
$\Leftrightarrow$ There exists $m, M \in \mathbb{R}$ s.t. $m \leq a \leq M$ for every $a \in A$.

A sequence $(x_n)_{n=1}^{\infty}$ of real number is an ordered list of numbers $x_n \in \mathbb{R}$, indexed by the natural number $n \in \mathbb{N}$

$x_n = n^2 \rightarrow$ sequence

Def: A sequence $(x_n)_{n=1}^{\infty}$ of real number is a function $f: \mathbb{N} \rightarrow \mathbb{R}$

ex) $x_n = n^3$

$x_n = \frac{1}{n}$

sequence 's element can repeat.

convergence of a sequence

A sequence $(x_n)_{n=1}^{\infty}$ of real numbers converges to a limit $x \in \mathbb{R}$, if for any $\varepsilon > 0$, there exists $N > 0$ s.t. for any $n \in \mathbb{N}$, $|x_n - x| < \varepsilon$.

① $\lim_{n \to \infty} x_n = x$ ② $x_n \to x$ as $n \to \infty$

If $(x_n)_{n=1}^{\infty}$ converges, then the limit of $(x_n)_{n=1}^{\infty}$ is unique.

proof : $\lim_{n \to \infty} x_n = x$, $\lim_{n \to \infty} x_n = x'$, Goal : $x = x'$

- For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$, for any $n > N$
  $|x_n - x| < \varepsilon$
- For any $\varepsilon > 0$, there exists $N' \in \mathbb{N}$ for any $n > N'$
  $|x_n' - x| < \varepsilon$

$\Rightarrow$ when $n > \max\{N, N'\}$, then $|x_n - x| < \varepsilon$, $|x_n - x'| < \varepsilon$.

$\Rightarrow |x-x'| = \underbrace{|x - x_n + x_n - x'|}_{a+b} \le \underbrace{|x-x_n|}_{a} + \underbrace{|x_n - x'|}_{b} < 2\varepsilon.$ Triangle Inequality

$\therefore \left(|x-x'| < 2\varepsilon \text{ for any } \varepsilon > 0\right)$

$\boxed{\text{Hence, } x = x'.}$

if $x \neq x'$ then $|x-x'| > 0$,

let $p = |x-x'|$. $\varepsilon = p/2$.

we can choose ε that is supersmall (smaller than the distance). Hence, the $|x-x'| = 0$.

(ex) $x_n = \frac{1}{n}$

prove $\lim_{n\to\infty} \frac{1}{n} = 0$

For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ s.t. for any $n > N$, $|\frac{1}{n} - 0| < \varepsilon$.

think: $|x_n - 0| = \frac{1}{n} < \varepsilon \Leftrightarrow n > \frac{1}{\varepsilon}$

$\Rightarrow$ Then, let $n > \lceil \frac{1}{\varepsilon} \rceil$ smallest integer larger than $\frac{1}{\varepsilon}$.

$\therefore \boxed{|x_n - 0| < \varepsilon.}$

When a limit D.N.E.

$x_n = (-1)^n$

prove: this sequence doesn't converge

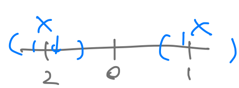
Assume that there exists

You just set this $\rightarrow \varepsilon = \frac{1}{2}$

$x_n = x$, $\varepsilon = \frac{1}{2}$, there exists $N \in \mathbb{N}$ s.t. for any $n > N$,

$|x_n - x| < \varepsilon = \frac{1}{2}$

$\Rightarrow |1 - x| < \frac{1}{2}$ and $|(-1) - x| \le \frac{1}{2}$



so, the two inequalities cannot happen together.

between these intersection

$\rightarrow$ There is no intersection

$|1-x| < \frac{1}{2}$, $|(-1)-x| \leq \frac{1}{2}$

$$2 = |1-(-1)| = |(1-x+x+1)| \leq |1-x| + |x-(-1)|$$

$$< \frac{1}{2} + \frac{1}{2} \neq 1$$

Hence there is a contradiction.

Def $\lim_{n\to\infty} x_n = \infty$ if for any $M \in \mathbb{R}$, then there exists $N \in \mathbb{N}$, for any $n > N$, $x_n > M$.

Theorem (boundedness) If $(x_n)_{n=1}^{\infty}$ converges, then $(x_n)_{n=1}^{\infty}$ is bounded, meaning that there exists $m, M \in \mathbb{R}$, $m < x_n < M$ for any $n \in \mathbb{N}$.

pf Assume $\lim_{n\to\infty} x_n = x$ (because the sequence converges)

Then $\varepsilon = 1$, there exist $n \in \mathbb{N}$, s.t. when $n > N$, $|x_n - x| < 1$.

Properties of abs values $\Rightarrow$ $x_n < x+1$ and $x_n > x-1$ when $n > N$.

Now we consider x_n when $n \leq N$.

$\tilde{m} = \min(x_1, x_2, \dots, x_N)$

since N is a finite #, $x_1, \dots, x_N$ is also finite.

Therefore, we can find minimum & maximum element.

$\tilde{M} = \max(x_1, \dots, x_N)$

$m = \min(\tilde{m}, x-1)$

$M = \min(\tilde{M}, x+1)$

claim: $m \leq x_n \leq M$.

① when my $n \leq N$, because of def of $\tilde{m}$, $x_n \geq \tilde{m} \geq m$.

② when my $n \geq N$, $x_n > x-1 \geq m$

$\therefore x_n \geq m$ for any $n \in \mathbb{N}$.

Theorem: suppose $(x_n)_{n=1}^{\infty}$, $(y_n)_{n=1}^{\infty}$ converge, $c \in \mathbb{R}$.

Then ① $\lim_{n\to\infty} c x_n = c \lim_{n\to\infty} x_n$

② $\lim_{n\to\infty} x_n + y_n = \lim x_n + \lim y_n$

$\Rightarrow$ ③ $\lim_{n\to\infty} x_n y_n = (\lim_{n\to\infty} x_n)(\lim_{n\to\infty} y_n)$

④ if $y_n \neq 0$, $\lim y_n \neq 0$. $\lim \left(\frac{x_n}{y_n}\right) = \frac{\lim x_n}{\lim y_n}$

For example, $x_n = (-1)^n$, $y_n = (-1)^{n+1}$.

proof ③

Show $\lim_{n\to\infty} |x_n y_n - xy|$ is very small.

$|\underbrace{x_n y_n - x \cdot y_n}_{a} + \underbrace{x \cdot y_n + xy}_{b}| \leq |x_n y_n - x y_n| + |x \cdot y_n + xy|$

$= |y_n||x_n - x| + |x||y_n - y|$

$\lim y_n = y$ ≈ 0 ≈ 0 for large n.

while y_n depends on n, it is okay because we have bound for n.

$= |x_n - x| M + |x||y - y_n|$

where M is chosen s.t. $|y_n| < M$.

Step 1: since $(y_n)_{n=1}$ converges, there exists $M\in\mathbb{R}$ s.t $|y_n|<M$ for any $n\in\mathbb{N}$.

Step 2: For any given $\varepsilon>0$,

For x_n: $\varepsilon_1 = \frac{\varepsilon}{2\cdot M}$, there exist $N_1\in\mathbb{N}$ s.t. for any $n>N_1$, $|x_n-x|<\varepsilon_1$.

For y_n: $\varepsilon_2 = \frac{\varepsilon}{2\cdot |x|}$, there exists $N_2\in\mathbb{N}$ s.t ... $|y_n-y|<\varepsilon_2$.

let $N=\max\{N_1, N_2\}$

$|x_ny_n - x\cdot y| \le |x_n-x|\cdot M + |y_n-y|\cdot|x| < \varepsilon_1\cdot M + \varepsilon_2|x| = \varepsilon$ ⇒ shown that

$(\lim x_n)(\lim y_n) = \lim(x_ny_n)$

Theorem (preserve ordering)

Assume $\lim x_n = x$, $\lim y_n = y$

If $x_n\le y_n$ for any n, then $x\le y$

(X) If $x_n<y_n$ for any n, then $x<y$

False.

counterexamples

$x_n = 1-\frac{1}{n}$ $\lim x_n = 1$

$y_n = 1+\frac{1}{n}$ $\lim y_n = 1$

proof: key: $\boxed{\text{For any } \varepsilon>0,\ x<y+\varepsilon. \Rightarrow x\le y}$

very similar $|x|\le\varepsilon$, for any $\varepsilon>0$, then $x=0$.

For x_n: $\varepsilon_1=\frac{\varepsilon}{2}$.

have N_1, st. $n>N_1$, $|x_n-x|<\varepsilon_1$

$x<x_n+\frac{\varepsilon}{2}$

For y_n: $\varepsilon_2=\frac{\varepsilon}{2}$

have N_2. st. $n>N_2$, $|y_n-y|<\varepsilon_2$

$n=\max\{N_1,N_2\}$

$x<x_n+\frac{\varepsilon}{2}<y_n+\frac{\varepsilon}{2}<y_n+\frac{\varepsilon}{2}+\frac{\varepsilon}{2}<y_n+\varepsilon$

since $y-\frac{\varepsilon}{2}<y_n<y+\frac{\varepsilon}{2}$

exam

IF $x>y$, then there exists $N>0$ s.t. if $n>N$, then $x_n>y_n$

monotone sequence

Def. (increasing) $x_{n+1}>x_n$ for any n.

(decreasing) $x_n<x_{n+1}$ for any n.

Theorem (x_n) is monotone sequence.

(1) $(x_n)_{n=1}$ is increasing & $\sup(x_n)\in\mathbb{R}$ exists, then $\lim x_n = \sup_{n\ge1}(x_n)$

(2) $(x_n)_{n=1}$ is decreasing & $\inf(x_n)\in\mathbb{R}$, then $\lim x_n = \inf_{n\ge1}\{x_n\}$.

(3) $(x_n)_{n=1}$ is increasing, $\sup(x_n)=+\infty$, then $\lim x_n = \infty$

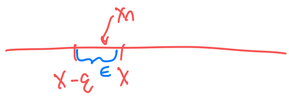
(4) $(x_n)_{n=1}$ is decreasing, $\inf(x_n)=-\infty$, then $\lim x_n=-\infty$

Prove ① given for any $\varepsilon > 0$, call $\sup x_n = x$

$\boxed{\text{Claim: Fin } n_\varepsilon \in \mathbb{N},\ x_{n_\varepsilon} > x - \varepsilon}$ exercise

$N = n_\varepsilon$, $n > N$ $\quad x \geq x_n \geq x_{n_\varepsilon} > x - \varepsilon \implies |x_n - x| \leq \varepsilon.$

by sup. increasing sequence



proof by contradict

Assume that the claim is wrong, "for any $n \in \mathbb{N}$, $x_n \leq x - \varepsilon$."

$\implies$ $x - \varepsilon$ is an upper bound of x_n

$\sup x_n = x$ $\quad\implies$ since x is the smallest upper bound of x_n,

$x - \varepsilon < x$

$\neq$

we notice contradiction.

$\lim_{n \to \infty} x_n = x$

Given $\forall \varepsilon > 0$, $\exists N > 0$, st $\forall n > N$ $|x_n - x| < \varepsilon$

lim sup, lim inf

- always exists
- characterize the existence of the limit
- characterize boundedness.

$$\underbrace{x_1, x_2, \ldots, \overset{|}{N}, x_n}$$

After $N \to \infty$, sup will become limsup.

Given a sequence $(x_n)_{n=1}^{\infty}$, $y_n = \sup\{x_k \mid k \geq n\}$ $\xrightarrow{n \text{ is large}}$ $\limsup_{n \to \infty} x_n = \lim_{n \to \infty} y_n$

$z_n = \inf\{x_k \mid k \geq n\}$ $\longrightarrow$ $\liminf_{n \to \infty} x_n = \lim_{n \to \infty} z_n$

first assume that $\lim_{n \to \infty} y_n$ and $\lim_{n \to \infty} z_n$ exist.

- $y_{n+1} \leq y_n \to$ monotonically decreasing sequence

Reason $\{x_k \mid k \geq n+1\} \subseteq \{x_k \mid k \geq n\}$

exercise: $A \subseteq B \Rightarrow \sup A \leq \sup B$

since sequence of limsup is a monotonically decreasing sequence, their limit always exist.

- $z_{n+1} \geq z_n \Rightarrow$ liminf is a monotonically decreasing sequence.

Ex $x_n = (-1)^{n+1}(1 + \frac{1}{n})$

$2 \quad -\frac{3}{2} \quad \frac{4}{3} \quad -\frac{5}{3}$

① $1 + \frac{1}{n}$ decreases to 1.

② if n = odd number. $\Rightarrow x_n > 0$.

$1 + \frac{1}{n}, -(1 + \frac{1}{n+1}), 1 + \frac{1}{n+2}, -(1 + \frac{1}{n+3})$

$y_n = \{\sup(x_k \mid k \geq n)\}$

even: $-(1 + \frac{1}{n}), 1 + \frac{1}{n+1}, -(1 + \frac{1}{n+2})$

$y_n = \begin{cases} 1 + \frac{1}{n}, & n = \text{odd} \\ 1 + \frac{1}{n+1}, & n = \text{even} \end{cases}$

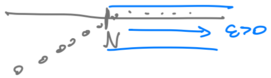
$z_n = \begin{cases} -(1 + \frac{1}{n+1}) & n = \text{odd} \\ -(1 + \frac{1}{n}), & n = \text{even} \end{cases}$

$$\boxed{\begin{aligned} \limsup_{n \to \infty} x_n &= \lim_{n \to \infty} y_n = 1 \\ \liminf_{n \to \infty} x_n &= \lim_{n \to \infty} z_n = -1. \end{aligned}}$$

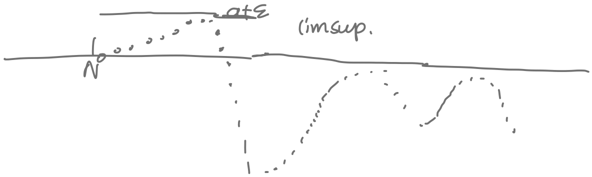
Theorem Given a sequence $(x_n)_{n=1}^{\infty}$, $a, b \in \mathbb{R}$

① $\lim_{n\to\infty} x_n = a$ iff $\forall \varepsilon > 0$ 1. $\exists N > 0$, s.t. $\forall n > N$, $x_n < a + \varepsilon$

2. $\forall N > 0$, $\exists n > N$, $x_n > a - \varepsilon$



Definition of limit.



limsup.

② $\lim_{n\to+\infty} \inf x_n = b$ iff $\forall \varepsilon > 0$

① $\exists N > 0$ st. $\forall n > N$, $x_n > a - \varepsilon$

② $\forall N > 0$, $\exists n > N$ s.t. $x_n < a + \varepsilon$.

For $\varepsilon_1 = \frac{\varepsilon}{2}$, $\exists N_1 > 0$. st. $\forall n > N_1$, $|y_n - a| < \varepsilon_1 = \frac{\varepsilon}{2}$

$\Rightarrow \frac{\varepsilon}{2} - a \le y_n \le \frac{\varepsilon}{2} + a$

1. $\forall n > N_1$ $x_n \le y_n < a + \frac{\varepsilon}{2} < a + \varepsilon$

$$\boxed{\begin{array}{l}\text{claim: Given } \forall \varepsilon > 0, \\ \text{if } y_{\tilde{n}} > a - \frac{\varepsilon}{2}. \\ \Rightarrow \exists n \ge \tilde{n}, \text{st. } x_n > a - \varepsilon\end{array}}$$

$\Rightarrow$ for $\forall N > 0$, find $\tilde{n} > \max\{N_1, N\}$

$\Rightarrow y_{\tilde{n}} > a - \frac{\varepsilon}{2} \Rightarrow \exists n \ge \tilde{n} > \max\{N_1, N\} \ge N$

$x_n > a - \varepsilon$.

" $\Leftarrow$ " Goal: prove $\lim_{n\to\infty} y_n = a$

Given $\varepsilon > 0$, use "1".

$\Rightarrow \exists N_1 > 0$ st. $n > N_1$, always have $x_n < a + \varepsilon$

$N_2 = N_1 + 1$. $\forall n \ge N_1 + 1$, $x_n < a + \varepsilon$ $\Rightarrow y_{N_1+1} \le a + \varepsilon$.

$\forall n > N_1 + 1 = N_2$

$y_n \le y_{N_1+1} \le a + \varepsilon$

② $\exists n^* > N_2$ s.t. $x_{n^*} > a - \frac{\varepsilon}{2}$

$\forall n > 0$

$y_n > a - \frac{\varepsilon}{2}$.

$\Rightarrow \underbrace{y_{n^*}}_{\text{sup}} > a - \frac{\varepsilon}{2}$ since y_{n^*} is sup, it should be greater than x_{n^*}.

let $N = N^*$, then $\forall N^* > 0$, $\exists n^{**} > n^*$. st. $x_{n^{**}} > a - \frac{\varepsilon}{2}$.

$\Rightarrow y_{n^*+1} > a - \frac{\varepsilon}{2}$

$\Rightarrow \forall n > n^*$, $\boxed{y_n > a - \frac{\varepsilon}{2}}$.

$\Rightarrow \quad \forall n > \max\{n^*, N_2\}, \quad a-\varepsilon \le y_N < a+\varepsilon.$

① "$\Rightarrow$" $\lim_{n\to\infty} y_n = a$

$\forall \varepsilon > 0 \qquad \varepsilon_1 = \frac{a}{2}$, $\exists N_1, \forall n > N_1, \ |y_n - a| < \frac{\varepsilon}{2}.$ $\Rightarrow$ (definition of limit)

i) choose $N = N_1$, Since $|y_n - a| < \frac{\varepsilon}{2}.$ $\Rightarrow y_n < \frac{\varepsilon}{2} + a$
when $n > N = N_1$

since y_n is sup of x_n, $y_n < a + \frac{\varepsilon}{2} \Rightarrow x_n \le y_n < a + \frac{\varepsilon}{2}.$

ii) claim: If $y_{\tilde{n}} > a - \frac{\varepsilon}{2}$, then there exists $n \ge \tilde{n}$, $x_n > a - \varepsilon$.

$y_{\tilde{n}} = \sup\{x_k \mid k \ge n\}$

$\to$ proof by contradiction.

For any $N > 0$, pick $\tilde{n} > \max\{N, N_1\}$.

$\Rightarrow y_{\tilde{n}} > a - \frac{\varepsilon}{2}$ (by $|y_n - a| < \frac{\varepsilon}{2}$)

$\Rightarrow \exists n > \tilde{n} > \max\{N, N_1\} \ge N$ s.t. $x_n > a - \varepsilon$

($\Leftarrow$) Goal show: $\lim y_n = a$.

Given $\varepsilon > 0$, want to find $N > 0$ s.t. $\forall n > N$, $|y_n - a| < \varepsilon$.

1. $\exists N_1 > 0$ s.t. $\forall n > N_1$, $x_n < a + \frac{\varepsilon}{2}$.
$\Rightarrow y_n \le a + \frac{\varepsilon}{2} < a + \varepsilon$

$y_n - a < \varepsilon$ ✓

$y_n - a > \varepsilon$

Fix $n > 0$, $y_n > a - \varepsilon$.

For any n, $\exists \tilde{n} > n$, $x_{\tilde{n}} > a - \varepsilon$ (condition 2)

$\Rightarrow y_n \ge x_{\tilde{n}} > a - \varepsilon$

Def Given (x_n)

$$y_n = \sup \underbrace{\{x_k \mid k \geq n\}}_{\text{tail of } x_n}, \limsup x_n = \lim y_n$$

→ y_n is monotonic decreasing sequence.

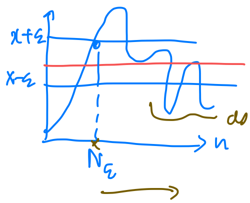
$$z_n = \inf \{x_k \mid k \geq n\}, \liminf x_n = \lim z_n$$

→ z_n is a monotonic increasing sequence.

Theorem

$\limsup x_n = x \iff$ (1) $\forall \varepsilon > 0, \exists N > 0$, s.t. $\forall n > N$ $|x_n - x| < \varepsilon$

(2) $\forall \varepsilon > 0, \forall N > 0$, s.t. $\exists n > N$ $x_n > x -$



up bound is sharp

do long as it comes back then limsup works.

(i) below N_ε, $x_n < x + \varepsilon$

why do we study limsup and liminf?

→ given a sequence (x_n), the sequence converges and $\lim x_n = x \iff \limsup x_n = \liminf x_n = x$.

proof

$(\Rightarrow)$ Assume sequence converges.

$x-\varepsilon$ x $x+\varepsilon$

$\forall \varepsilon > 0, \exists N > 0$ s.t. $\forall n > N$, $|x_n - x| < \varepsilon/2 \Rightarrow x - \varepsilon/2 < x_n < x + \varepsilon/2$

since k is greater than n.

$y_n = \sup\{x_k \mid k \geq n\}$

$\therefore y_n \leq x + \varepsilon/2$, $z_n \geq x - \varepsilon/2$

$$\Rightarrow \quad x - \varepsilon/2 \leq \underbrace{z_n \leq y_n}_{\inf \leq \sup} \leq x + \varepsilon/2$$

$$\Rightarrow |y_n - x| \leq \frac{\varepsilon}{2} < \varepsilon$$

$$\Rightarrow |z_n - x| \leq \frac{\varepsilon}{2} < \varepsilon$$

$\therefore \lim_{n\to\infty} y_n = x$

$\lim_{n\to\infty} z_n = x$

$(\Leftarrow)$ $y_n = \sup\{x_k \mid k \geq n\}$

$\boxed{y_n \geq x_n}$

$z_n = \inf\{x_k \mid k \geq n\}$

$\boxed{z_n \leq x_n}$

$\therefore y_n \geq x_n \geq z_n$

$$\underbrace{\limsup x_n = \liminf x_n = x}$$

$\iff \lim y_n = \lim z_n = x$

by sandwich theorem,

we have that

$\lim_{n\to\infty} x_n = x$

Cauchy sequence

→ Benefit: get rid of x dependence

→ can tell if sequence converges or not.

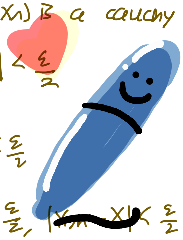
Sequence $\{x_n\}$ is a cauchy sequence if $\varepsilon>0$, $\exists N$ s.t. $\forall n,m>N$, $|x_n-x_m|<\varepsilon$.

Thm (x_n) converges $\Leftrightarrow$ (x_n) is a cauchy sequence.

($\Rightarrow$) $\forall \varepsilon>0$, $\lim |x_l - x| < \frac{\varepsilon}{2}$

$\exists N>0$

s.t. $l>N$ $\quad |x_l - x|<\frac{\varepsilon}{2}$



$\Rightarrow \forall n,m>N \quad |x_n-x|<\frac{\varepsilon}{2}, \; |x_m - x| < \frac{\varepsilon}{2}$

$\Rightarrow |x_n-x_m| < |\overbrace{x_n-x}^{a}| + |\overbrace{x_m-x}^{b}| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} < \varepsilon$

$|\underbrace{x_n-x}_{a} + \underbrace{x-x_m}_{b}|$

($\Leftarrow$) $x = \limsup x_n$ // other option $x = \liminf x_n$

for any sequence, this is true.

since limsup must always exist.

① WTS : x is finite

If x_n is a cauchy sequence, then x_n is bounded, meaning $\exists M>0$. s.t. $\forall n \; -M<x_n<M$

proof $|x_n|\le M$

set $\varepsilon=1$.

$|x_n-x_m|<1 \quad \forall n>N$

$|x_n - x_{N+1}|<1$

$x_{N+1}-1 < x_n < x_{N+1}+1$ always bounded.

set $\max(x_{N+1}+1, \sup(x_n ; n<N))$

$-M \le z_n \le y_n \le M$

elm y_n is bounded by 2 numbers ($-M$ and M).

see $-M$ and M as new sequence

$(-M,-M,-M,-M)$ $\quad (M,M,M,M)$..

$-M \le x_n \le M$

thus, x is finite.

② $\lim x_n = x$

Given $\varepsilon>0$.

$|x_n-x|<\varepsilon. \Leftrightarrow \overbrace{\varepsilon-x < x_n}^{\text{ii}} \overbrace{< \varepsilon+x}^{\text{i}}$

when n is large enough.

use (i) in thm of limsup.

since $\limsup x_n = x$, $\exists N>0$, st. $\forall n>N_1$, $x_n<x+\varepsilon$.

(ii) x_n is a cauchy sequence.

$\exists N_2$ s.t. when $n,m>N_2$, $|x_n-x_m|<\varepsilon/2$.

use ② of limsup thm, $N=\max\{N_1,N_2\}$

$\Rightarrow \tilde{n} > N$ s.t. $x_{\tilde{n}} > x-\varepsilon/2$

when $n>N=\max\{N_1,N_2\}$, $|x_n - x_{\tilde{n}}| < \frac{\varepsilon}{2}$. , $x_{\tilde{n}} > x-\varepsilon/2$

$\Downarrow$

$x_n > x_{\tilde{n}} - \frac{\varepsilon}{2} > x - \varepsilon/2 - \varepsilon/2 > x-\varepsilon.$

Subsequence

Given (x_n) $\quad x_1, x_2, x_3, x_4, \ldots$

* must keep the order of the sequence.

<u>Recall</u> Def: A sequence (x_n) is a function $f: \mathbb{N} \to \mathbb{R}$.

$$x_n = f(n)$$

Given a sequence x_n and f, a subsequence is a strictly increasing function $g: \mathbb{N} \to \mathbb{N}$

write $a_n = f(g(n)) = x_{g(n)}$

<u>Thm</u> Given (x_n), (x_n) converges iff every subsequence of x_n converges and also have the same limit.

→ want to prove that a sequence diverges.

(1) Find a subsequence that doesn't converge ⇒ not easy

(2) two subsequences of x_n where two subsequences converge to different member

ex) $x_n = (-1)^n$

$-1, 1, -1, 1, -1, \ldots$

⇒ $A_n = X_{2n+1}$

$B_n = X_{2n}$

then $A_n = 1$ for all $n \in 2n+1$ and each converge to different number.

$B_n = -1$.

Given $(x_n)_{n=1}^{\infty}$. assume $(x_n)_{n=1}^{\infty}$ is bounded.

convergent

must weaker condition due to this

Then, $\lim x_n = x \iff$ ① every subsequence of x_n has the same limit

$\iff$ ② every subsequence of x_n converges to x

① ② ③

proof ①

(x_n) is a subsequence of (x_n)

$n_1 < n_2 < n_3$

$a_1 = x_{n_1}$, $a_2 = x_{n_2}$, $a_3 = x_{n_3}$

proof ② know $\lim_{n\to\infty} x_n = x$, Assume we have a subsequence $a_k = x_{n_k}$ where $k \in \mathbb{N}$.

and $\lim_{k\to\infty} a_k$ exists.

Goal: prove $\lim_{k\to\infty} a_k = x$.

Since $\lim_{n\to\infty} x_n = x$, for $\varepsilon > 0$, $\exists N$, s.t. $\forall n > N$, $|x_n - x| < \varepsilon$ By definition of limit.

Notice that $n_k \geq k$, for any $k \in \mathbb{N}$.

why? since a_k is subsequence of x_n, x_1 x_2 x_3 x_4 . Thus $n_{k+1} > n_k$ and $n_1 \geq 1$.

$x_2 \to a_1$, $x_3 \to a_2$, ...

since the subsequence should preserve the order of sequence.

then the

How subsequence is formed from a sequence?

ex: (x_n)

$n_k = k^2$	$n = k \cdot 2$	$n = 10 - k$
Then $a_k = x_{k^2}$	Then $a_k = x_{2k}$	Then a_{10-k}
$\Rightarrow x_1, x_4, x_9, \ldots$	$x_1, x_2, x_4, \ldots$	$a_k = x_{10-k} \Rightarrow \underbrace{x_{10}, x_9, x_8}_{\text{changes the order.}}, \ldots$

Not possible because it changes the order of subsequence

$\Rightarrow$ For any $k > N$, we have $n_k \geq k > N \Rightarrow |x_{n_k} - x| < \varepsilon$ (let a_k)

$\Rightarrow$ For any $k > N$, we have $|a_k - x| < \varepsilon \Rightarrow \therefore \lim_{k\to\infty} a_k = x$

Thm Bolzano Weistrass.

every bounded sequence has a convergent subsequence.

ex: Given (x_n) doesn't converge but we can find one subsequence of (x_n) that converges.

using Bolzano Weistrass, let's prove ③

proof ③ Assume there exists a subsequence a_k that doesn't converge to x.

Fact of a_k: ① a_k is bounded. since x_k is bounded, and a_k is subsequence, a_k is bounded.

② when k is large, a_k is not close to x.

$\exists \varepsilon > 0$, s.t. $\forall k$, $\exists \kappa > k$, $|a_\kappa - x| \geq \varepsilon$. $\Rightarrow$ lim of $a_k \neq x$.

Idea: since subsequence A_k is bounded, we can view A_k as a new sequence.
By Bolzano theorem, we can see that A_k has a convergent subsequence.
$\Rightarrow$ Find a subsequence of a_k such that it converges to another point.
then, we find a subsequence of X_n that doesn't converge to x, which violates statement ②.

$k=1$, $\exists k_1 > 1$, s.t. $|a_{k_1} - x| \geq \varepsilon$.
$k=k_1$ $\exists k_2 > k_1$ s.t. $|a_{k_2} - x| \geq \varepsilon$
$k=k_3$ $\exists k_3 > k_2$ s.t. $|a_{k_3} - x| \geq \varepsilon$
⋮

Define $b_j = a_{k_j}$ is a subsequence of a_k $|b_j - x| \geq \varepsilon$.
$\rightarrow (b_j)$ is a subsequence of X_n.

b_j is not necessarily a convergent sequence. Thus, we use the Bolzano theorem to find a convergent sequence.

since b_j is a subsequence of x_n, it is bounded.

A subsequence C_ℓ of b_j s.t. (C_ℓ) converges.

$$|C_\ell - x| \geq \varepsilon$$

c_ℓ doesn't converge to x $\Rightarrow$ This contradicts ②

proof of B & W thm.

⊛ If bounded sequence, the limsup of the sequence exists and is a finite number.

goal: construct a subsequence of (x_n) s.t. this subsequence converges to $\limsup_{n\to\infty} x_n = z$

- $\varepsilon_1 = 1$, using the equivalence condition of limsup x_n,

  ① $\exists N_1$ s.t. $\forall n > N_1$, $x_n < z+1$

  ② $\exists n_1 > N_1$ s.t. $x_{n_1} > z-1$. $\Rightarrow$ $|x_{n_1} - z| < 1$. $\rightarrow a_1 = x_{n_1}$

- $\varepsilon_2 = 1/2$

  ① $\exists N_2$ s.t. $n > N_2$, $x_n < z + 1/2$

  ② $\exists N_2 > \underbrace{\max(n_1, n_2)}_{\text{to keep the order of original sequence.}}$ s.t. $x_{n_2} > z - 1/2$

$a_n = x_{n_2}$

• $\varepsilon_3 = 1/3$

$\Downarrow$

$a_3 = x_{n_3}$ s.t. $|x_{n_3} - z| < 1/3$

$\vdots$

• $\varepsilon_k \to a_k = x_{n_k}$ s.t. $|x_{n_k} - z| < \frac{1}{k}$

$\Downarrow$

(a_k) sequence satisfies $|a_k - z| < 1/k$

TOPIC Series (infinite sum of real numbers)

Def ① Given a sequence $(a_n)_{n=1}^{\infty}$. The series is the summation of $\sum_{k=1}^{\infty} a_k$.

② The series is convergent if the sequence $S_n = \sum_{k=1}^{n} a_k$ converges to some real number $S \in \mathbb{R}$ when $n \to \infty$.

Then, we write $\sum_{k=1}^{\infty} a_k = S$.

(ex) $a_n = (\frac{1}{2})^n$, $\sum_{k=1}^{\infty} (\frac{1}{2})^k = 1$.

$S_n = \sum_{k=1}^{n} (\frac{1}{2})^k = \frac{1}{2}\left(\frac{1-(\frac{1}{2})^n}{1-\frac{1}{2}}\right) = 1-(\frac{1}{2})^n$

$\lim_{n\to\infty} S_n = 1-0 = 1 \Rightarrow \therefore$ the series is convergent to 1

Theorem1: Given a non-negative sequence (a_n) $(a_n \geq 0)$, then the series $\sum_{k=1}^{\infty} a_k$ converges iff $S_n = \sum_{k=1}^{n} a_n$ is bounded.

proof: Notice that (S_n) is a monotonic sequence. (increasing) $\Rightarrow$ $(S_{n+1} \geq S_n)$

Definition: Given series $\sum_{n=1}^{\infty} a_k$ is a cauchy series if $\forall \varepsilon > 0$, $\exists N > 0$ s.t. $\forall n \geq m_\varepsilon$ N, $\left|\sum_{k=m}^{n} a_k\right| < \varepsilon$.

Theorem: the series $\sum_{k=1}^{\infty} a_k$ is a cauchy series $\Leftrightarrow$ The sequence $S_n = \sum_{k=1}^{n} a_k$ is a cauchy sequence.

$\Leftrightarrow \sum_{k=1}^{\infty} a_k$ converges.

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{k=1}^{\infty} a_k$ diverges.

proof proof by contradiction

Assume $\lim_{n\to\infty} a_n \neq 0$, but $\sum_{k=1}^{\infty} a_k$ converge.

$\Rightarrow \forall \varepsilon > 0$, $\exists N > 0$ s.t. $\forall n,m > N$ $\left|\sum_{k=m}^{n} a_k\right| < \varepsilon$ → by definition, this is a cauchy seq.

$\xRightarrow{n=m}$ $\forall n > N$, $|a_n| < \varepsilon$. $\Rightarrow \lim_{n\to\infty} a_n = 0$, which is a contradiction.

(!) Not true! $\lim a_n = 0 \not\Rightarrow \sum_{k=1}^{\infty} a_k$ converge.

(ex) $a_n = \frac{1}{n}$ $\lim a_n = 0$.

$\sum_{n=1}^{\infty} a_n \Rightarrow$ diverges. , we can show $\sum_{k=1}^{n} \frac{1}{k} \approx C \cdot \ln(n)$, when $n \geq 1$.

Non-negative sequence (a_n) $n \geq 0$.

Four tests to help us check the convergence of a series.

① comparison test

② ratio test

③ root test

④ integral test.

1. comparison test. Given a non-negative sequence a_n, assume $\exists$ another non-negative sequence (b_n) s.t. $b_n \geq a_n$ for $\forall n \in \mathbb{N}$.

Then if $\sum_{k=1}^{\infty} b_n$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.

proof: $F_n = \sum_{k=1}^{n} b_n$, $S_n = \sum_{k=1}^{n} a_n$

$\Rightarrow F_n \geq S_n$ for $\forall n \in \mathbb{N}$.

F_n is an increasing sequence. $F_n \leq \lim_{n\to\infty} F_n$ since F_n is an increasing seq.

$$= \sum_{n=1}^{\infty} b_k$$

$$\Rightarrow S_n \leq \sum_{k=1}^{\infty} b_k, \ \forall n$$

corollary: Given non-negative sequence (a_n), if $a_n \geq b_n \geq 0$ and $\sum_{k=1}^{\infty} b_k$ diverges, then $\sum_{k=1}^{\infty} a_k$ diverges.

ex) ① p-series: $\sum_{k=1}^{\infty} \left(\frac{1}{k}\right)^p$ converges for $p>1$ and diverges for $p \leq 1$.

② geometric: $\sum_{k=1}^{\infty} a^k$ converges for $|a|<1$ and diverges for $a \geq 1$.

③ $\sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{1}{\ln(k+1)}\right)^p$ converges when $p>1$
$p \leq 1$

general: $\frac{1}{k+a} \cdot \left(\frac{1}{\ln(k+b)}\right)^p$, $a, b > 0$

→ use one of these 3 examples to perform the comparison test.

ratio test: Given a nonzero sequence (a_n)

① If $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, then $\sum_{k=1}^{\infty} a_n$ converges (actually, we can show that $\sum_{k=1}^{\infty} |a_k|$ also converges.)

② If $\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$, then $\sum_{k=1}^{\infty} a_n$ diverges. ratio test only works if it's strictly less than 1.

Remark: If $\limsup \left|\frac{a_{n+1}}{a_n}\right| = 1 \Rightarrow$??

$\liminf \left|\frac{a_{n+1}}{a_n}\right| = 1 \Rightarrow$??

Thm (root test): Given any sequence (a_n), let $r = \limsup_{n\to\infty} \sqrt[n]{|a_n|} = \limsup_{n\to\infty} (a_n)^{1/n}$

① if $0 \le r \le 1$, then $\sum_{k=1}^{\infty} a_k$ converge.

② if $r > 1$, then $\sum_{k=1}^{\infty} a_k$ diverges.

Integral test

$\{x \ge 0 : x \in \mathbb{R}\}$

Given non-negative and ~~decreasing~~ sequence (a_n), then if there exists a decreasing $f: [0, +\infty) \to \mathbb{R}^+$ s.t. $f(n) = a_n$.

then $\sum_{k=1}^{\infty} a_k$ converges $\Longleftrightarrow \int_1^{\infty} f(x)\,dx$ converges.

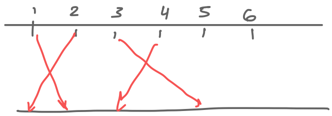
ex) $a_n = a^n$

$\frac{a_{n+1}}{a_n} = a \Rightarrow \sum a_n$ converge if $a < 1$

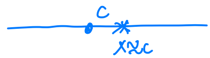
$\Rightarrow \sum a_n$ diverge if $a > 1$

Cauchy condtn

Function: $f: A \subseteq \mathbb{R} \to \mathbb{R}$



limit, continuity



Def (limit of function) Given a function $f: A \subseteq \mathbb{R} \to \mathbb{R}$ and suppose c is an accumulation point. Then we can find $\lim_{x \to c} f(x) = L$ (limit of f is L when $x \to c$).

If $\forall \varepsilon > 0$, $\exists \delta$ s.t. $\forall x \in A$ and $\underbrace{0 < |x-c|}_{\text{why } >0?} < \delta$, $|f(x) - L| < \varepsilon$.

$c \in A$ $\quad \lim_{x \to c} f(x) \neq f(c)$

What is an accumulation point?

Define c is an accumulation point of A for any $\delta > 0$, $\exists x \in A$ st. $|x - c| < \delta$.

ex $A = [0,1)$. 1 is an accumulation point of A. ⇒ Even if a point is not in the set, it can still be the accumulation point.

$A = [0,1] \cup \{2\}$ 2 is not an accumulation point.

$\Rightarrow$ Let $\delta = \frac{1}{2}$, then we cannot find an $x \in A$ and $x \neq 2$ s.t. $|x - 2| < \frac{1}{2}$.

example $\quad f(x) = \dfrac{x-9}{\sqrt{x} - 3}$, wts study $\lim_{x \to 9} f(x)$.

$A = \mathbb{R} \setminus \{9\} \to \mathbb{R}$

$\boxed{\text{Guess}}$ Notice: $(x-9) = (\sqrt{x}+3)(\sqrt{x}-3) \Rightarrow f(x) = \dfrac{(\sqrt{x}+3)\cancel{(\sqrt{x}-\sqrt{3})}}{\cancel{\sqrt{x}-\sqrt{3}}} = \sqrt{x}+3$, $x \neq 9$

$\lim_{x \to 9} f(x) = \sqrt{9} + 3 = 6$.

prove: $\lim_{x \to 9} f(x) = 6$

$\left[\text{WTS: } \forall \varepsilon > 0, \exists \delta > 0, \forall x \neq 9, |x - 9| < \delta \text{ we have } |f(x) - 6| < \varepsilon. \right]$

Since when $x \neq 9$, $f(x) = \sqrt{x} + 3$, $\Rightarrow f(x) - 6 = \sqrt{x} - 3$

$\Rightarrow$ want δ s.t. when $|x - 9| < \delta$,

$|\sqrt{x} - 3| < \varepsilon$.

$|\sqrt{x} - 3| = \left| \dfrac{(\sqrt{x}-3)(\sqrt{x}+3)}{\sqrt{x}+3} \right| = \dfrac{|x-9|}{|\sqrt{x}+3|} \leq \dfrac{1}{3}|x - 9|$ ⇒ It suffices to choose δ s.t. $\frac{1}{3}|x-9| < \varepsilon$.

$\Leftrightarrow |x - 9| < 3\varepsilon$.

$\therefore$ Let $\delta = 3\varepsilon$.

proposition limit of a function is always unique, meaning

$$\lim_{x\to c} f(x) = L_1 \text{ and } \lim_{x\to c} f(x) = L_2 \quad \text{then } L_1 = L_2$$

PF WTS. that $\forall \varepsilon > 0 \quad |L_1 - L_2| < \varepsilon$

$\Rightarrow$ since $\lim_{x\to c} f(x) = L_1$, $\exists\, \delta_1 > 0$ s.t. $\forall\ 0 < |x-c| < \delta_1$, $\underline{|f(x) - L_1| < \frac{\varepsilon}{2}}$

$\Rightarrow$ since $\lim_{x\to c} f(x) = L_2$, $\exists\, \delta_2 > 0$ s.t. $\forall\ 0 < |x-c| < \delta_2$, $\underline{|f(x) - L_2| < \frac{\varepsilon}{2}}$.

$\delta = \min(\delta_1, \delta_2)$

$\forall\, 0 < |x-c| < \delta$, $|f(x) - L_1| < \varepsilon/2$, $|f(x) - L_2| < \varepsilon/2$. triangle inequality

$$\Rightarrow |L_1 - L_2| = |L_1 - f(x) + f(x) - L_2| \le |L_1 - f(x)| + |f(x) - L_2|$$

$$= \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

$$\therefore \boxed{|L_1 - L_2| < \varepsilon}$$

proposition : Given $f, g: A \to \mathbb{R}$ c is an accumulation point of A.

then ① $\lim_{x\to c} (f(x) \pm g(x)) = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x)$

② $\lim_{x\to c} f(x) - g(x) = \lim f(x) - \lim g(x)$

③ $\lim \dfrac{f(x)}{g(x)} = \dfrac{\lim f(x)}{\lim g(x)}$ if

④ if $g(x) \ge f(x)$, $\forall x \in A$, then $\lim_{x\to c} g(x) \ge \lim_{x\to c} f(x)$.

Thm $\lim_{x\to c} f(x) = L \iff$

For any sequence $(x_n) \in A$, and $\lim_{n\to\infty} x_n = c$, we have $\lim_{n\to\infty} f(x_n) = L$

* relating limit and sequence.

corollary: lim $f(x)$ doesn't exist iff $\exists (x_n), (y_n) \subset A$, $\lim x_n = \lim y_n = c$.

but $\lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n)$.

→ how to prove that limit doesn't exist.

pf "⇒" we know $\lim_{x\to c} f(x) = L$.

① ⇒ $\forall \varepsilon>0$, $\exists \delta>0$, s.t. $\forall\, 0<|x-c|<\delta$, $|f(x)-L|<\varepsilon$

② ⟹ $\exists N>0$, s.t. $\forall n>N$, $0<|x_n-c|<\delta$.

③ $\forall \varepsilon>0$, $\exists N>0$, s.t. $\forall n>N$, $|f(x_n)-L|<\varepsilon$.

Feb 28

Recall: $f: A \to \mathbb{R}$, c is an accumulation point of A then $\lim_{x \to c} f(x) = L$
if $\forall \varepsilon > 0$, $\exists \delta$ s.t. $\forall$ $0 < |x-c| < \delta$

Def : Assume A is not bounded from above, then $\lim_{x \to \infty} f(x) = L$ if $\forall \varepsilon > 0$,
$\exists M > 0$ s.t. $\underline{\forall x > M}$ $|f(x) - L| < \varepsilon$.
since $x \to \infty$

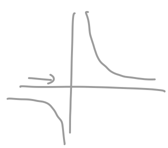
$\lim_{x \to -\infty} f(x) = L$: $\exists M < 0$, s.t. $\forall x < M$ $|f(x) - L| < \varepsilon$.

<u>Def</u> $\lim_{x \to c} f(x) = +\infty$ if $\forall M > 0$, $\exists \delta > 0$, s.t. $\forall$ $0 < |x-c| < \delta$.
→ if f(x) is very close to c, then f(x) will be very large.

$= -\infty$ if $\forall M < 0$, $f(x) < M$.

<u>ex</u> $f(x) = 1/x$

$\lim_{x \to 0} f(x) = +\infty$ ✗



$\lim_{x \to 0^-} f(x) = -\infty$

$\lim_{x \to 0^+} f(x) = +\infty$

Thus, $\lim_{x \to 0} f(x)$ D.N.E.

<u>continuous.</u> $f: A \to \mathbb{R}$, $c \in A$, we say f is continuous at c if $\forall \varepsilon > 0$, $\underline{\exists \delta > 0}$, s.t. $\forall |x-c| < \delta$, $x \in A$, $|f(x) - f(c)| < \varepsilon$.
choice of δ is dependent on ε and c.

(ex) $A = [0,1] \cup \{2\}$
since 2 is not an accumulation point of A, $\lim_{x \to 2} f(x)$ ✗.
→ if c is not an accumulation pt, then f is <u>always</u> continuous at c.
b/c
$\exists \delta > 0, \forall |x-c| < \delta \Rightarrow |f(x) - f(c)| = |f(c) - f(c)| = 0 < \varepsilon.$
↑
x has to equal to c if c is not A.C.

<u>Thm</u> If c is an accumulation point, then f is continuous at c iff $\lim_{x \to c} f(x) = f(c)$.

<u>Corollary</u> $f: (a,b) \to \mathbb{R}$, $\forall c \in (a,b)$
f is continuous at $c \iff \lim_{x \to c} f(x) = f(c)$.

ex Thomae function → Good counter example.

$f(x) = \begin{cases} \frac{1}{q}, & \text{if } x \in \mathbb{Q},\ x \neq 0,\ x = \frac{p}{q} \text{ where } p, q \text{ are relatively prime} \\ 0, & \text{if } x \notin \mathbb{Q} \text{ or } x = 0 \end{cases}$

Claim: $f(x)$ is continuous at $x=0$ or $x \notin \mathbb{Q}$.

and it is not continuous at $x = \mathbb{Q}$. $x \neq 0$

① prove $x = p/q$. $f(x)$ is not continuous at x.

$f(x) = \frac{1}{q}$, $\varepsilon = \frac{1}{q}$, $\forall \delta > 0$, $\exists z \notin \mathbb{Q}$ s.t. $|z - x| < \delta$

$\Rightarrow f(z) = 0$, $|f(z) - f(x)| = \frac{1}{q} \geq \varepsilon$.

Difficult

② f is continuous at $c \notin \mathbb{Q}$.

if $c \notin \mathbb{Q}$, then $f(c) = 0$ according to the definition.

$\forall \varepsilon > 0$, Question: How to find δ?

$|f(x) - 0| < \varepsilon$ need $|\frac{1}{q}| < \varepsilon$

$|f(x)| < \varepsilon$ $\Rightarrow q > \frac{1}{\varepsilon}$

Let q_ε be the smallest prime number that is larger than $1/\varepsilon$.

$Q_\varepsilon = \{ \frac{p}{q} \mid p, q \text{ are relatively prime}, q < q_\varepsilon \}$.

$Q_\varepsilon \cap [c-1, c+1]$ ⇒ contains finite number of rational #'s.

$\tilde{Q} = \{ \frac{p}{q} \mid q < 10 \}$, $\tilde{Q} \cap [-1, 1]$

$\delta = \min \{ |z - c| \mid z \in Q_\varepsilon \cap [-1, 1] \} > 0$.

$\forall |x - c| < \delta$.

ⓘ $x \notin \mathbb{Q} \Rightarrow |f(x)| < \varepsilon$.

② $x \in \mathbb{Q} \Rightarrow x \notin Q_\varepsilon$, $x = \frac{p}{q}$, $q > q_\varepsilon$

$|f(x)| = \frac{1}{q} \leq \frac{1}{q_\varepsilon} < \varepsilon$.

Theorem : $f, g : A \to \mathbb{R}$

f and g are continuous at c.

Then, $f \pm g$ is also continuous at c

$f * g$

$\frac{f}{g}$ $(g(c) \neq 0)$

f is continuous in A.

if f is continuous at every point of A.

Thm: $f: A \to \mathbb{R}$, $g: B \to \mathbb{R}$, $f(A) \subset B$

then f and g are continuous. $g \circ f(x) = g(f(x))$ is also continuous

Pf Pick any point $c \in A$.

check that $|g(f(x)) - g(f(c))| < \varepsilon$.

use that g is continuous.

① For any $\varepsilon > 0$, $\exists \delta_1 > 0$ s.t. $\forall |y - f(c)| < \delta_1$, s.t. $|g(y) - g(f(c))| < \varepsilon$.

② $\exists \delta > 0$, s.t. $\forall |x - c| < \delta$, $|f(x) - f(c)| < \delta_1$

①+② $\Rightarrow$ $\forall |x - c| < \delta$, $|f(x) - f(c)| < \delta_1 \Rightarrow |g(f(x)) - g(f(c))| < \varepsilon$

uniform continuity <uniform continuity implies continuity>

Def : $f : A \to \mathbb{R}$, f is uniform continuous on A if $\forall \varepsilon > 0$, $\boxed{\exists \delta > 0}$, s.t. $\forall |x-y| < \delta$, $|f(x) - f(y)| < \varepsilon$.

choice of δ is independent of c.

typical example .

$f(x) = x^2$ on $\mathbb{R}$

$f(x)$ is not uniform continuous on $\mathbb{R}$.

$|x^2 - y^2| = |x+y|\,|x-y|$ → choice of δ has to depend on x and y.

→ but this function is uniform continuous on $[0,1]$.

If fixed y, $\delta = \min\{1, \frac{\varepsilon}{1+2|y|}\}$

If $|x-y| < \delta$

$\Rightarrow |x| < 1 + |y|$

$\Rightarrow |x+y| < 1 + 2|y|$

then $|x+y| \cdot |x-y| \leq (1+2|y|) \cdot \delta \leq \varepsilon$.

March 2 **Uniform Continuity**

Uniform continuity

Def $f: A \to \mathbb{R}$ $x, y \in A$

f is uniformly continuous if $\forall \varepsilon > 0, \exists \delta > 0$. s.t. $\forall |x-y| < \delta$, $|f(x)-f(y)| < \varepsilon$.

* δ works for every y.

정의는 같은데 Former depends on y.

Continuity

Def Continuity

$f: A \to \mathbb{R}$ * δ depends on y. and ε.

f is continuous at y if $\forall \varepsilon > 0$, $\exists \delta$ s.t. $\forall x \in A$, $|x-y| < \delta$, $|f(x) - f(y)| < \varepsilon$.

Ex: $f(x) = x^2$ on $\mathbb{R}$

Fix $y \in \mathbb{R}$, $\forall \varepsilon > 0$, want to find $\delta > 0$. s.t. $\forall |x-y| < \delta$, $|f(x)-f(y)| < \varepsilon$

$$|f(x)-f(y)| = |x^2 - y^2| = |(x+y)(x-y)| = \underbrace{|x+y|}_{\text{Find } \delta \text{ that this term} = \varepsilon} \cdot \overbrace{|x-y|}^{<\delta}$$

- need to find a way we control $x+y$.
- when we prove continuity y is treated as constant.
- if $x \approx y$, then $|x+y| \cdot |x-y| \approx |2y| \cdot |x-y|$
  $< |2y| \cdot \delta \le |2y|\delta = 2|y|\delta$

  then $\delta = \frac{\varepsilon}{2|y|}$

$$\delta = \min\left\{1, \frac{\varepsilon}{2|y|+1}\right\}$$

if $y \to \infty$, then $\delta \to 0$

then $|x-y| < \delta$ where δ is very small.

So, it is not uniform in x and y.

when $|x-y| < \delta$, $|x-y| < \min\left(1, \frac{\varepsilon}{2|y|+1}\right) = 1$

$|x-y| < 1 \Rightarrow |x| < 1 + |y| \Rightarrow |x| + |y| < 1 + 2|y|$

triangle ineq.

$$|x^2 - y^2| = |x+y||x-y| \le (|x|+|y|)(|x-y|) < (\underbrace{1+2|y|}_{\text{fixed since } y \text{ is fixed}}) \cdot |x-y| < \min\left\{1+2|y| \cdot \frac{\varepsilon}{2|y|+1}\right\}$$

$\boxed{<\varepsilon}$

How do you check if a function is Not uniformly continuous?

prop $f: A \to \mathbb{R}$ is not uniformly continuous iff $\exists\, \varepsilon_0 > 0$ (x_n) $(y_n) \subset A$ s.t.

$\lim_{n\to\infty} |x_n - y_n| = 0$, $|f(x_n) - f(y_n)| > \varepsilon_0$

*on second midterm, he will ask us to construct a sequence.

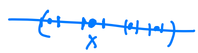
$(\Rightarrow)$ $\delta_n = \frac{1}{n}$, construct the sequence $(x_n), (y_n)$

$(\Leftarrow)$ proof by contradiction.

Continuous Function on Compact set

Def Compact set A set A is compact if A is close and bounded.

~~say ① A is closed if A^c is open~~
~~② A set U is open then~~ $\forall x \in U$, $\exists\, \delta > 0$, s.t. $(x-\delta, x+\delta) \subset U$



In our class, A is closed if any convergent sequence in A will converge to some $a \in A$.

(ex) $A = [0, 1)$ is not a closed set.

Because $x_n = 1 - \frac{1}{n}$ because x_n converges to 1 but 1 doesn't belong to A.

Thm of Compact set

A is compact iff for any sequence $(x_n) \subset A$, $\exists$ subsequence $(x_{n_k}) \subset A$

s.t. $\lim_{k\to\infty} x_{n_k} = a \in A$.

proof $\boxed{\text{A bounded sequence always have a convergent subsequence.}}$

$(\Rightarrow)$

Since compact, it's a bounded sequence. By using this definition we show that subsequence converges to an element in A.

(⇐)

Thm: if X is continuous on A, and A is compact, then f is uniform continuous on A.

Pf by contradiction.

Assume f is not uniformly continuous.

Then, there exist some $\varepsilon_0 > 0$, $\exists (x_n)(y_n)$ s.t. $\lim_{n\to\infty} |x_n - y_n| = 0$, $|f(x_n) - f(y_n)| > \varepsilon$.

I can find convergent subsequence of (x_n): (x_{n_k})

$\lim x_{n_k} \to x^* \in A$. (by Thm of compact set)

$k \gg 1$, $|x_{n_k} - x^*| \ll 1$.

$\Rightarrow f(x_{n_k}) \approx f(x^*)$ since f is a continuous function

Since $n_k \gg 1$, $|x_{n_k} - y_{n_k}| \ll 1$.

$\Downarrow$

$|y_{n_k} - x^*| \ll 1$.

$f(y_{n_k}) \approx f(x^*)$

$f(x_{n_k}) \approx f(x^*) \approx f(y_{n_k})$

Then $f(x_{n_k}) - f(y_{n_k}) \ll 1$.

But, this contradicts $|f(x_n) - f(y_n)| > \varepsilon_0$.

Hence, prove by contradiction!

Thm: f is continuous on A. A is compact. Then f(A) is compact

$$\boxed{\text{compact}} \xrightarrow{f} \boxed{\text{compact}}$$

Pf pick a sequence $(y_n) \subset f(A)$.

want to find a subsequence (y_{n_k}) s.t. $\lim_{k\to\infty} y_{n_k} = y^* \in f(A)$

$\exists (x_n) \subset A$ s.t. $y_n = f(x_n)$

A is compact, then we can find convergent sequence x_{n_k}

(x_{n_k}), $\lim_{k\to\infty} x_{n_k} = x^* \in A$

Define $y_{n_k} = f(x_{n_k})$

y_{n_k} is then subsequence of (y_n).

$\lim_{k\to\infty} f(x_{n_k}) = f(x^*) = \lim_{k\to\infty} y_{n_k}$. by f's continuity.

$f(x^*) = y^*$

Weistrass Extreme thm

f is continuous on A

A is compact

then f is bounded on A and can attain its max and min

$\max f = \sup f$, $\min f$, $\inf f$

☆ 만약 set이 compact 하고, function이 cont. 하다면, f(X)의 min 과 max 를 찾을 수 있다.

max 와 min 이 must exist

proof construct a sequence $(y_n) \in f(A)$ s.t. $\lim_{n\to\infty} y_n = \sup f(A) \xRightarrow{f(A) \text{ is compact}}$ $\sup f(A) \in f(A)$

$\sup f(A) = \max f(A)$

example of constructing sequence.

property of $\sup f(A)$:

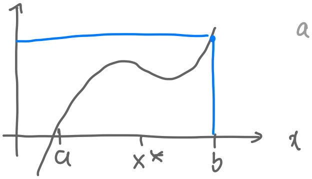
$\forall \delta > 0$, $\exists\, y \in f(A)$, s.t. $\sup f(A) - \delta < y \le \sup f(A)$

y is very close to $\sup f(A)$

$\boxed{\delta_n = \frac{1}{n} \rightarrow (y_n)}$ $\left(y_n \ge \sup f(A) - \frac{1}{n} \right)$

Today: Intermediate Value Theorem

Differentiable Function



$a < x^* < b$

$f(x^*) = c$

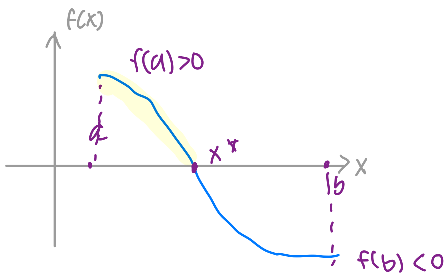
Intermediate Value Theorem

Theorem: $f: [a,b] \to \mathbb{R}$ continuous function

Assume $f(a)>0$, $f(b)<0$ (or $f(a)<0$, $f(b)>0$), then $\exists x^* \in (a,b)$ s.t. $f(x^*)=0$

proof Assume $f(a)>0$, $f(b)<0$.

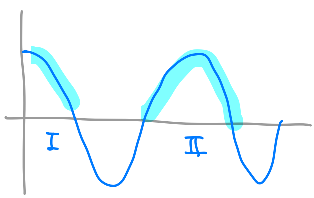
$\boxed{\text{Question}}$: How to find x^* s.t. $f(x^*)=0$?



$E = \{x \mid x \in [a,b],\ f(x) > 0\}$

$E = [a, x^*)$

claim $x^* = \sup(E)$ then $f(x^*) = 0$



$E = \mathrm{I} \cup \mathrm{II}$

pf

① since $f(b)<0$, by the continuity of f,

$\exists \delta > 0$ s.t. $\exists x \in [b-\delta, b]$, $f(x) < 0$. ①

According to ①, ↓ these are diff x.

$\forall x \in E$, $x < b-\delta$ $x \neq b$

$\Rightarrow x^* = \sup E < b - \delta$. because $f: [a,b]$.

pf by contradiction.

Assume $f(x^*) \neq 0$. Then either ① or ②.

① $f(x^*) > 0$

Given x^* is sup of E. but, if $f(x^*)>0$, x^* is not a sup of E.

use continuity of $f \Rightarrow \varepsilon = \frac{f(x^*)}{2}$, $\exists \delta > 0$ s.t. for any $|x-x^*|<\delta$,

$|f(x) - f(x^*)| > \varepsilon = \frac{f(x^*)}{2}$.

$\Rightarrow |f(x) - f(x^*)| < \frac{f(x^*)}{2}$

$\Rightarrow f(x) > f(x^*) - \frac{f(x^*)}{2} = \frac{f(x^*)}{2} > 0.$ This is true for any $x \in [x^*-\delta_1, x^*+\delta_1]$

implies that

$\Longrightarrow x^* + \delta_1 \in E$ → this contradicts the assumption that $x^* = \sup(E)$

② $f(x^*) < 0$. use continuity of f ... exact same argument as $f(x^*) > 0$ case but

$|f(x) - f(x^*)| < \frac{f(x^*)}{2}$ δ_2, $\varepsilon = \frac{|f(x^*)|}{2}$

$\Rightarrow f(x) < f(x^*) + \frac{|f(x^*)|}{2} = \frac{f(x^*)}{2} < 0$ when $x \in [x^* - \delta_2, x + \delta_2]$

$\Rightarrow x^* - \delta_1$ is an upper bound of E.

$x^* - \delta_1$ is smaller than x^*.

Hence x^* is no longer sup of E. Thus, we observe a contradiction

mathematically

since x^* is an upper bound of E,

$\Rightarrow \forall x \in E,\ x \le x^*$.

since $f(x) < 0$, when $x - \frac{\delta_1}{2} \le x \le x^*$,

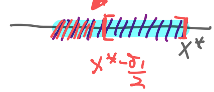
$\Rightarrow [x^* - \frac{\delta_1}{2}, x^*] \notin E$

if $x \in E$, then

① $x \le x^*$

② $x \notin [x^* - \frac{\delta_1}{2}, x^*]$

$\} \rightarrow x \le x^* - \frac{\delta_1}{2}$.



Hence $x^* - \frac{\delta_1}{2}$ is an upper bound of E. which is contradiction.

Thm (intermediate value thm) $f: [a,b] \rightarrow \mathbb{R}$ continuous function. Assume $f(a) < f(b)$

then $\forall c \in (f(a), f(b))$, $\exists x \in (a,b)$, $f(x^*) = c$.

pf. $g(x) = f(x) - c$

$\Rightarrow g(a) = f(a) - c < 0$, $g(b) = f(b) - c > 0 \Rightarrow \exists x^*\ g(x^*) = 0$

$\Rightarrow f(x^*) = c.$

Differentiable function $\subset$ Continuous function

Def. $f:(a,b) \to \mathbb{R}$ $(f(a,b))$ given $c \in (a,b)$

Define: $f'(c) = \lim_{h\to 0} \dfrac{f(c+h) - f(c)}{h}$

if the limit exists (say f is differentiable at pt c.)

ex $f(x) = \begin{cases} x^2, & x>0 \\ 0, & x<0 \end{cases} \Rightarrow f'(x) = \begin{cases} 2x & x>0 \\ 0 & x<0 \end{cases}$

$f'(0) \Rightarrow \lim_{h\to 0} \dfrac{f(h)-f(0)}{h} = \lim_{h\to 0} \dfrac{f(h)}{h}$

$\lim_{h\to 0^+} \dfrac{f(h)}{h} = \dfrac{h^2}{h} = h = 0$

$\lim_{h\to 0^-} \dfrac{f(t)}{h} = \dfrac{0}{h} = 0$

$f: (a,b) \in \mathbb{R}$, $c \in (a,b)$

Right derivative of f at point c. $f'(c^+) = \lim_{h\to 0^+} \dfrac{f(c+h)-f(c)}{h}$

The derivative of f at point c: $f'(c^-) = \lim_{h\to 0^-} \dfrac{f(c+h)-f(c)}{h}$

Thm: $f'(c)$ exists $\Leftrightarrow$ $f'(c^+)$, $f'(c^-)$ exist and $f'(c^+) = f'(c^-)$.

Thm f is differentiable at c.

$\Rightarrow$ f is continuous at c $\quad \left(\begin{array}{c} f:(a,b)\to \mathbb{R} \\ c\in(a,b) \end{array} \right)$

Given $\lim_{h\to 0} \dfrac{f(c+h)-f(c)}{h}$

proof $\lim_{h\to 0} f(c+h) - f(c) = \lim_{h\to 0} \left[\dfrac{f(c+h)-f(c)}{h} \right] [h] = \lim_{h\to 0} \left[\dfrac{f(c+h)-f(c)}{h} \right] \cdot \left[\lim_{h\to 0} h \right]^{\to 0}$

$= 0$

mean-value Theorem $f: [a,b] \to \mathbb{R}$ continuous differentiable on (a,b)

$\exists c \in (a,b)$ s.t. $f'(c) = \frac{f(b)-f(a)}{b-a}$.

General version : let $f, g : [a,b] \to \mathbb{R}$ continuous and differentiable on (a,b).

Then, $\exists c \in (a,b)$ s.t. $\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

let $g(x) = x$.

According to classical mean value thm,

$\exists c_f \in (a,b)$ s.t. $f'(c_f) = \frac{f(b)-f(a)}{b-a}$, $\exists c_g \in (a,b)$, $g'(c_g) = \frac{g(b)-g(a)}{b-a}$

$\Rightarrow \frac{f'(c_f)}{g'(c_g)} = \frac{f(b)-f(a)}{g(b)-g(a)}$ $\quad (c_f \neq c_g)$

pf $h(x) = (f(x)-f(a)) \cdot (g(b)-g(a)) - (f(b)-f(a)) \cdot (g(x)-g(a))$

$h(b) = (f(b)-f(a)) \cdot (g(b)-g(a)) - (f(b)-f(a)) \cdot (g(b)-g(a)) = 0$.

$h(a) = 0 - 0 = 0$

$h'(x) = f'(x) \cdot (g(b)-g(a)) - (f(b)-f(a)) \cdot g'(x) = 0$

$\Rightarrow \exists c \in (a,b)$ s.t. $h'(c) = \frac{h(b)-h(a)}{b-a} = 0 \Rightarrow \frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

(L'hospital's Rule)

Thm $f, g : [a,b] \to \mathbb{R}$ continuous function. differentiable on (a,b).

① Assume $\lim_{x \to a^+} f(x) = \lim_{x \to a^+} g(x) = 0$, $\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{f(x)}{g(x)}$

prof ① $\frac{f(x)}{g(x)} = \frac{f(x) - \overset{=0}{f(a)}}{g(x) - \underset{=0}{g(a)}} = \frac{f'(c)}{g'(c)}$, $a < c < x$,

$\Rightarrow$ if $x \to a^+ \Rightarrow c_x \to a^+$

$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{c \to a^+} \frac{f(c)}{g(c)}$

② Assume $\lim_{x \to a^+} |f(x)| = \lim_{x \to a^+} |g(x)| = +\infty$, $\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)}$

prof ② Given $a < x < y < b$, Calculate $\frac{f(x)-f(y)}{g(x)-g(y)}$

$\exists\, c_{xy}$ s.t. $\dfrac{f(x)-f(y)}{g(x)-g(y)} = \dfrac{f'(c_{xy})}{g'(c_{xy})}$ where $a<x<c_{x,y}<y<b$.

$$\frac{f(x)}{g(x)} = \frac{f(x)-f(y)}{g(x)} + \frac{f(y)}{g(x)} = \left(\frac{f(x)-f(y)}{g(x)-g(y)}\right)\cdot\left(\frac{g(x)-g(y)}{g(x)}\right) + \frac{f(y)}{g(x)}$$

$$= \lim_{x\to a_0^+} \left(\frac{f(x)-f(y)}{g(x)-g(y)}\right)\cdot\left(1-\frac{g(y)}{g(x)}\right) + \frac{f(y)}{g(x)}$$

let $\lim\limits_{x\to a^+} \dfrac{f(x)}{g'(x)} = L$. want to show $\lim\limits_{x\to a^+} \dfrac{f(x)}{g(x)} = L$.

First, $\left(\dfrac{f(x)}{g(x)} - L\right) = \left(\dfrac{f(x)-f(y)}{g(x)-g(y)} - L\right)\left(1-\dfrac{g(y)}{g(x)}\right) - L\cdot\dfrac{g(y)}{g(x)} + \dfrac{f(y)}{g(x)}$

$$\left|\frac{f(x)}{g(x)}\right| \le \left|\frac{f(x)-f(y)}{g(x)-g(y)} - L\right|\left|1-\frac{g(y)}{g(x)}\right| + L\cdot\left|\frac{g(y)}{g(x)}\right| + \left|\frac{f(y)}{g(x)}\right|.$$

$$\lim_{x\to\infty}\left|\frac{f(x)}{g(x)} - \right| \le \lim_{x\to+\infty}\left|\frac{f(x)-f(y)}{g(x)-g(y)} - L\right|\cdot\left|1-\frac{g(y)}{g(x)}\right| + L\cdot\left|\frac{g(y)}{g(x)}\right| + \left|\frac{f(y)}{f(x)}\right| = 0.$$

choose $y = a+\delta$

then $\dfrac{f(x)-f(y)}{g(x)-g(y)} = \dfrac{f'(c_{xy})}{g'(c_{xy})}$, $a<x<c_{xy}<y=a+\delta$.

$\exists$ choose $y=a+\delta$. Then $\forall\, a<x<y$

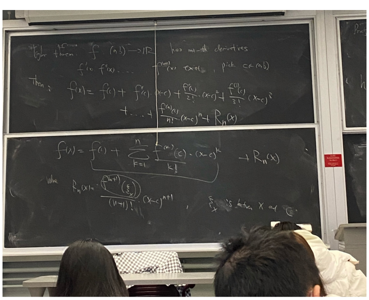
$$\left|\frac{f(x)}{f(y)} - L\right| \le \varepsilon\left|1-\frac{g(y)}{g(x)}\right| + L\cdot\left|\frac{g(y)}{g(x)}\right| + \left|\frac{f(y)}{g(x)}\right|$$

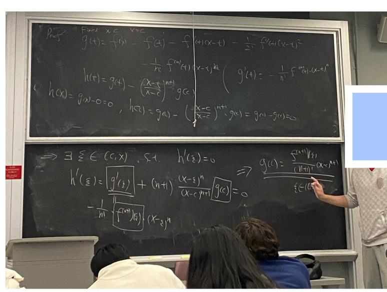
$\Downarrow$

$$\lim_{x\to a^+}\left|\frac{f(x)}{g(x)} - L\right| \le \varepsilon.$$

$$\forall \varepsilon>0 \Rightarrow \lim_{x\to a^+}\left(\frac{f(x)}{g(x)} - L\right) = 0.$$

Taylor.





Corollary:

if $|f^{(n+1)}(x)| \le M$,

$$\Rightarrow \left| f(x) - f(c) - \sum_{k=1}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k \right| \le \frac{M}{(n+1)!}|x-c|^{n+1} \le \frac{M R^{n+1}}{(n+1)!} \text{ if } |x-c|<R.$$

Can bound the error.

Definition uniform convergence

if $\forall \varepsilon>0, \exists N>0, \forall n>N \ |f_n(x)-f(x)|<\varepsilon, \ \forall x$

Thm ① f_n bounded / continuous

② $f_n \xrightarrow[n\to\infty]{uni} f$

$\Rightarrow$ f is bounded & continuous.

diff. with point-wise convergence: uniformly for all x, N doesn't depend on x.

We care about uniform convergence then f is bounded and continuous.

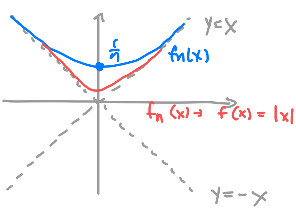
Differentiability

Question : ① f_n is bounded / continuous / differentiable.

② f_n is uniformly converged to f

$f_n \xrightarrow[n\to+\infty]{uni} f$

$\not\Rightarrow$ differentiable. * Differentiable condition doesn't preserve.



$f_n(x)$ uniformly converged to $f(x) = |x|$.

however, $|x|$ is not differentiable at point $x=0$.

Thus, the limit doesn't exist.

we need condition ③ for uniform convergence to preserve differentiability.

Thm: ① f_n is differentiable

② $f_n \xrightarrow[n\to\infty]{uni} f$

③ $f_n' \xrightarrow[n\to\infty]{uni} g$

$\Rightarrow$ f is differentiable and $f' = g$.

pf Fix any c.

WTS $\underbrace{\lim_{x\to c} \frac{f(x)-f(c)}{x-c}}_{f'(c)} = g(c).$

This is showing that for $\forall c$, $f'(c) = g(c)$.

$\Leftrightarrow \exists \delta>0, \forall \varepsilon>0$, s.t. $\forall |x-c|<\delta$

$$\left| \underbrace{\frac{f(x)-f(c)}{x-c}} - g(c) \right| < \varepsilon$$

insert some $f(n)$ to bound this inequality.

$$\underbrace{\left| \frac{f(x)-f(c)}{x-c} - g(c) \right|}_{\text{doesn't depend on } n} = \underbrace{\left| \frac{f(x)-f(c)}{x-c} - \frac{f_n(x)-f_n(c)}{x-c} \right|}_{① < \varepsilon/3} + \underbrace{\left| \frac{f_n(x)-f_n(c)}{x-c} - f_n'(c) \right|}_{② < \varepsilon/3} + \underbrace{\left| f_n'(c) - g(c) \right|}_{③ < \varepsilon/3}$$

depends on n.

① : choose large n but n depends on x.

② : N depends on n and x

Fix n.

③ : we simply have to choose large enough n.

we can choose an N that such inequality work.

step 1 $\exists N>0$, s.t. $\forall n>N$, $|①| < \frac{\varepsilon}{3}$, $\forall x$.

(using unif convergence)

step 2 $\exists N_1>0$, st. $\forall n>N_1$, $|③| < \frac{\varepsilon}{3}$

no matter x I pick, if N is large enough, ① and ③ $< \frac{\varepsilon}{3}$.

step 3 n: $\max\{N, N_1\}+1$

$\exists \delta>0$ s.t. $\forall |x-c|<\delta$, $\overbrace{\left| \frac{f_n(x)-f(c)}{x-c} - f_n'(c) \right|}^{②} < \frac{\varepsilon}{3}$.

$\Rightarrow \exists \delta>0$, $n = \max\{x_1, x_2\}+1$

$\forall |x-c|<\delta$, $\left| \frac{f(x)-f(c)}{x-c} - g(c) \right| < \varepsilon$.

$$\left| \frac{f(x)-f(c)}{x-c} - \frac{f_n(x)-f_n(c)}{x-c} \right| \le \left| \frac{f(x)-f(c)}{x-c} - \frac{f_n(x)-f_m(c)}{x-c} \right| + \left| \frac{f_n(x)-f_m(c)}{x-c} - \frac{f_n(x)-f_n(c)}{x-c} \right|$$

f_m

$h(z) = f_m(z) - f_n(z)$ $\quad$ ✱ $\frac{h(x)-h(c)}{x-c} = h'(\xi)$, where $\xi \in (x, c)$.

$= f_m'(\xi) - f_n'(\xi)$

$\exists N>0$, st. $\forall n>N$, $|f_n'(x) - g(x)| < \varepsilon/12$

$\Rightarrow \forall n, m > N$, $|f_n'(x) - f_m'(x)| < |f_n'(x) - g(x)| + |g(x) - f_m'(x)| < \varepsilon/6$, $\forall x$.

$\Rightarrow |\circledast| < \varepsilon/6$

$\Rightarrow \forall n, m > N$

we make m super large. then

$|①| < \frac{\varepsilon}{6} + \left| \underbrace{\frac{f(x)-f(c)}{x-c} - \frac{f_m(x)-f_m(c)}{x-c}}_{\text{same as } ①} \right| < \frac{\varepsilon}{6} < \frac{\varepsilon}{2}$

$\frac{f(x)-f(c)}{x-c}$ and $\frac{f_m(x)-f_m(c)}{x-c}$ are very close to each other.

①'s bound is independent of n.

Hence we just have to choose m that we can bound the term.

choose $m > M_x$.

Series of Functions

Suppose we have a sequence of functions $\{f_n\}$.

Define $S_n(x) = \sum_{k=1}^{n} S_k(x)$

The series $\sum_{k=1}^{\infty} S_k(x)$ converges uniformly / pointwise if $S_n(x) \to S(x)$ uniformly and point-wisely.

Thm (Cauchy condition)

$S_n(x)$ converges uniformly $\Leftrightarrow$ $\forall \varepsilon > 0, \exists N > 0,$ s.t. $\forall m, n > N, \left| \sum_{k=n}^{m} f_k(x) \right| < \varepsilon \ \forall x.$

proof $S_n(x) - S_{n-1}(x)$

Thm. If $\exists$ a positive sequence $\{M_n > 0\}$ s.t.

① $|f_n| < M_n$

② $\sum_{n=1}^{+\infty} M_n < +\infty \Rightarrow \sum_{n=1}^{+\infty} f_n$ converges uniformly.

Thm if a positive sequence