Problem 1

a)

1) show that $E(\hat{u}_K) = \mu_K$

$$E(\hat{u}_K) = E\left(\frac{1}{n}\sum_{i=1}^{n} x_i^K\right) = \frac{1}{n}\sum_{i=1}^{n} E(x_i^K) = \frac{1}{n}\cdot n\cdot \mu_K = \mu_K$$

Hence, it is an unbiased estimator.

b) consistency. By the Law of Large Numbers,

$$E\left(\sum_{i=1}^{n} x_i^K\right) \to E(X^K)$$

Therefore, $\hat{u}_K \xrightarrow{P} \mu_K$.

Problem 2

a) method of moment estimator for p.

$X_i \sim$ geometric (p)

$$\bar{x} = \frac{1}{p} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$\hat{p} = \frac{1}{\bar{x}} = \frac{1}{\frac{1}{n}\sum_{i=1}^{n} x_i} = \frac{n}{\sum_{i=1}^{n} x_i}$$

b) $\hat{p} = \frac{1}{\bar{x}} \Rightarrow g(\bar{x}_n) = \frac{1}{\bar{x}_n}$

let $g(\mu_1) = \frac{1}{\mu_1} = p$

$g'(\mu_1) = -\mu_1^{-2} = -($

by CLT, $\bar{x} \sim$ normal $\left(\frac{1}{p}, \frac{1-p}{p^2}\right)$

$$\sqrt{n}\,\frac{\bar{x}_n - \frac{1}{p}}{\sqrt{\frac{1-p}{p^2}}} \rightsquigarrow N(0,1)$$

by delta method, $\sqrt{n}\,\frac{\hat{g}(\bar{X}_n) - g(\mu_1)}{|g'(\mu_1)|\sigma}$

since we don't know μ, σ, we use the approximate (= estimates).

$$\hat{\sigma} = \sqrt{\frac{1-\hat{p}}{\hat{p}^2}} = \frac{\sqrt{1-\hat{p}}}{\hat{p}}$$

$$g'(\mu_1) = -\hat{p}^2$$

$$\therefore \sqrt{n}\,\frac{\hat{p}-p}{\hat{p}\,\cancel{\hat{p}^{3}}\cdot\frac{\sqrt{1-p}}{\cancel{\hat{p}}}} = \sqrt{n}\,\frac{\hat{p}-p}{\hat{p}(\sqrt{1-\hat{p}})} \to N(0,1)$$

problem 3

a) $E(X) = \int_0^\infty x\lambda e^{-\lambda x}\,dx$

let $y = x\lambda \quad dy = \lambda dx$

$= \int_0^\infty y\, e^{-y} \frac{dy}{\lambda}$

$= \frac{1}{\lambda}\int_0^\infty y e^{-y} dy = \frac{1}{\lambda}[-e^{-y} - y e^{-y}]_0^\infty = \frac{1}{\lambda}$

$u = y \qquad v = -e^{-y}$

$du = dy \qquad dv = e^{-y} dy$

$E(X) = \frac{1}{n}\sum_{i=1}^{n} x_i \Rightarrow \bar{X} = \frac{1}{\lambda}$

$$\hat{\lambda} = \frac{1}{\bar{X}} = \frac{1}{\frac{1}{n}\sum_{i=1}^{n} x_i} = \frac{n}{\sum x_i}$$

b) $E(\hat{\lambda}) = E\left(\frac{1}{\bar{X}}\right) = E\left[\frac{n}{\sum x_i}\right]$

sum of iid $\exp(\lambda)$ r.v is gamma (n, λ) dist.

$x_1, \dots, x_n \simeq \exp(\lambda) \Rightarrow \sum_{i=1}^{n} x_i = \Gamma(\lambda, n) \quad$ let $y \sim \Gamma(\lambda, n)$

$E\left(\frac{n}{\sum x_i}\right) = E\left(\frac{n}{y}\right) = \int_0^\infty \frac{n}{y} \frac{\lambda^n}{\Gamma(n)} y^{n-1} e^{-\lambda y} dy$

$= \int_0^\infty \frac{n}{y} \frac{\lambda^{n-1}}{(n-1)} \frac{\lambda}{\Gamma(n-1)} y^{n-1} e^{-\lambda y} dy$

$= \lambda \int_0^\infty \frac{n}{n-1} \frac{\lambda^{n-1}}{\Gamma(n-1)} y^{(n-1)-1} e^{-\lambda y} dy = \lambda \frac{n}{n-1} \int_0^\infty \frac{\lambda^{n-1}}{\Gamma(n-1)} y^{(n-1)-1} e^{-\lambda y} dy$

$= \boxed{\lambda \frac{n}{n-1}}$

since $E(\hat{\lambda}) \neq \lambda$, it is biased

$\boxed{E\left(\frac{n-1}{n}\hat{\lambda}\right) = \lambda} \quad \therefore \quad \frac{n-1}{n}\hat{\lambda} = \frac{n-1}{n}\left(\frac{1}{\bar{X}}\right)$

$= \frac{n-1}{n}\left(\frac{n}{\sum_{i=0}^{n} x_i}\right)$

$= \boxed{\frac{n-1}{\sum_{i=0}^{n} X_i}}$ unbiased est.

c) $P(X>1) = 1 - P(X \le 1)$

$= e^{-\hat{\lambda}}$

$\therefore = \boxed{e^{-\frac{n-1}{\sum x_i}}}$

divide denom & num. by $\bar{x}$.

$$= \boxed{\frac{n}{\sum(x_i^{-1} - \bar{x}^{-1})}}$$

Problem 4 $x_1, x_2, \ldots, x_n \sim$ inverse Gaussian dist

$$f(x; \mu, \lambda) = \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} \exp\left(-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right) \quad x > 0.$$

① $L: K(\mu, \lambda)$

a) $$= f(x \mid \mu, \lambda) = \prod_{i=1}^{n} \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} \exp\left(-\sum_{i=1}^{n} \frac{\lambda(x_i-\mu)^2}{2\mu^2 x_i}\right)$$

$$= \left(\frac{\lambda}{2\pi}\right)^{n/2} \prod_{i=1}^{n} x_i^{-3/2} \, e^{\sum \frac{-(\lambda(x_i-\mu)^2)}{2\mu^2 x_i}}$$

① Log likelihood:

$$\ell_x(\mu, \lambda) = \log(\mathrm{Lik}(\mu, \lambda))$$

$$= \cancel{\log}\ \frac{n}{2}(\ln\lambda - \ln 2\pi) - \frac{2}{3}\sum \ln x_i + \sum \frac{-\lambda(x_i-\mu)^2}{2\mu^2 x_i}$$

② take derivative of log likelihood function

$$\frac{d\ell}{d\mu} = \frac{d}{d\mu}\left(\frac{-\lambda}{2} \sum \frac{-(x_i-\mu)^2}{\mu^2 x_i}\right) = \frac{-\lambda}{2} \sum \frac{-2(x_i-\mu)\mu^2 x_i - (x_i-\hat{\mu})^2 2\cdot\mu x_i}{\mu^4 x_i^2}$$

$$= \sum \frac{\cancel{2(x_i-\hat{\mu})}\cdot 2\mu x_i (x_i - \hat{\mu})[\cancel{-\mu} - (x_i - \cancel{\hat{\mu}})]}{\mu^4 x_i^2}$$

$$= \sum \frac{-2\cancel{\hat{\mu}}\cancel{x_i}(x_i - \hat{\mu})(\cancel{x_i})}{\cancel{\mu^4}\cancel{x_i^2}}$$

$$= \sum \frac{-2(x_i - \hat{\mu})}{\mu^3 \cancel{x_i}} \Rightarrow \text{this should equal to } 0.$$

$$\frac{-2}{\mu^3}\sum(x_i - \hat{\mu}) = 0$$

when $\sum x_i - \hat{\mu} = 0$

$\therefore n\hat{\mu} = \sum x_i$

$$\boxed{\hat{\mu} = \frac{\sum x_i}{n}}$$

④ Find the MLE of λ. $\boxed{\text{answer: } \frac{n}{\sum_{i=1}^{n}(x_i - \bar{x}^{-1})}}$

$$\cancel{\frac{d\ell}{d\lambda} = \frac{d}{d\lambda}\left(\frac{-\lambda}{2}\sum\frac{-(x_i-\mu)^2}{\mu^2 x_i}\right)} = \frac{n}{2}$$

$$\frac{d\ell}{d\lambda} = \frac{d}{d\lambda}\left(\frac{n}{2}(\ln\lambda - \ln 2\pi) + \sum\frac{-\lambda(x_i-\hat{\mu})^2}{2\hat{\mu}^2 x_i^2}\right) = \frac{n}{2}\cdot\frac{1}{\lambda} - \frac{1}{2\hat{\mu}^2}\sum\frac{x_i(x_i-\hat{\mu})^2}{x_i^2}$$

$$\Rightarrow \frac{1}{2\hat{\mu}^2}\sum\frac{x_i(x_i-\hat{\mu})^2}{x_i^2} = \frac{n}{2}\cdot\frac{1}{\lambda}$$

$$\frac{1}{\hat{\mu}^2}\sum\frac{\cancel{x_i}(x_i-\hat{\mu})^2}{x_i^{\cancel{2}}} = \frac{n}{\lambda}$$

$$\hat{\lambda} = \frac{n\hat{\mu}^2}{\sum\frac{(x_i-\mu)^2}{x_i}} = \frac{n\bar{x}^2}{\sum\left(\frac{x_i^2 - 2x_i\bar{x} + \mu^2}{x_i}\right)} = \frac{n\bar{x}^2}{\sum\left(x_i - 2\bar{x} + \frac{\mu^2}{x_i}\right)}$$

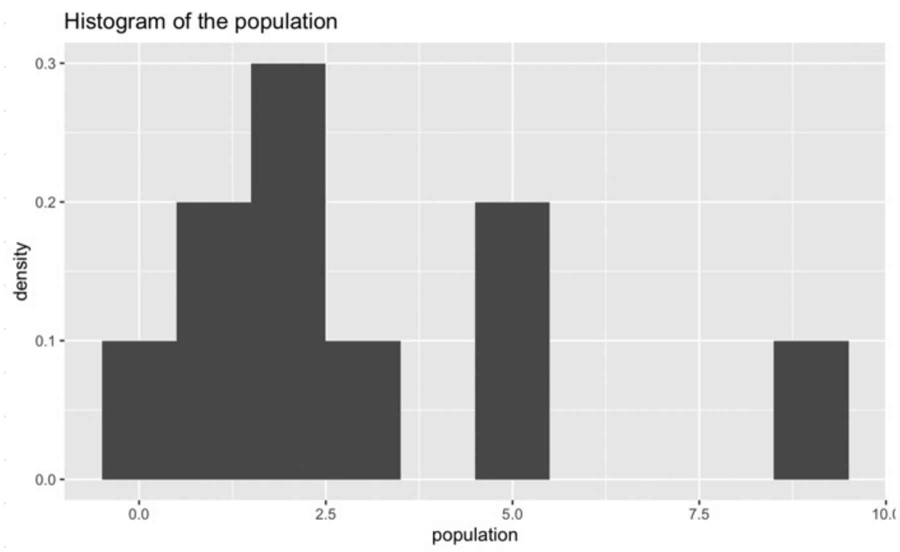
$$\hat{\lambda} = \frac{n\bar{x}^2}{n\bar{x} - 2n\bar{x} + \bar{x}^2\sum x_i^{-1}} = \frac{n\bar{x}^2}{\bar{x}^2\sum x_i^{-1} - n\bar{x}} = \frac{n\bar{x}}{\bar{x}\sum x_i^{-1} - n} \quad = \frac{n\bar{x}^2}{\sum x_i - 2n\bar{x} + \frac{\bar{x}^2}{\sum(x_i)}}$$

$\rightarrow$ check...

problem 5

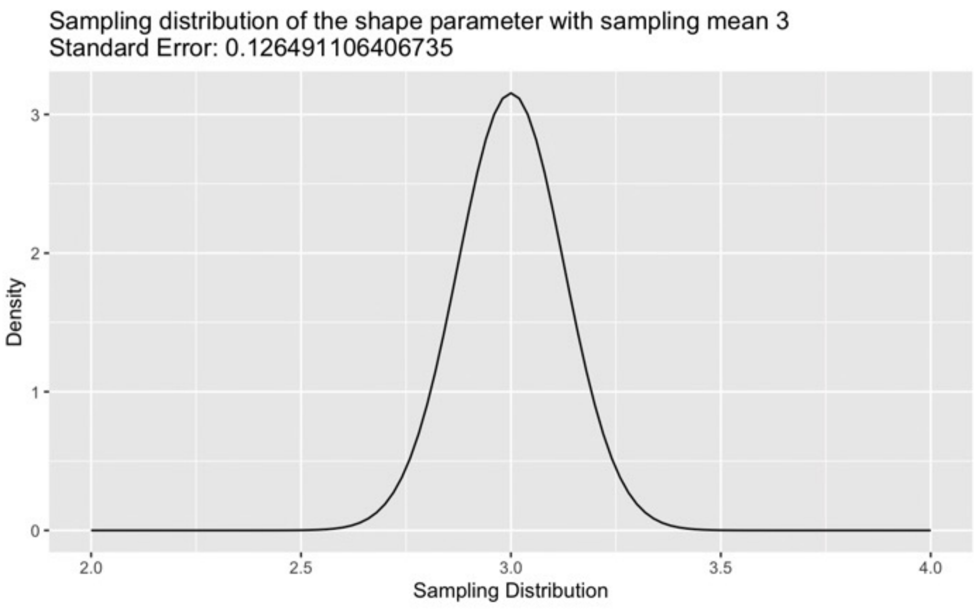
(a)

```{r}
library(ggplot2)
population <- c(2,0,2,2,9,1,5,1,3,5)
df <- as.data.frame(population)
ggplot(df) + geom_histogram(aes(x = population, y= ..density..), binwidth =1) +labs(title=paste0("Histogram of the population"))
mean <- mean(population)
SD <- sqrt(sd(population)**2 * 9/10)
```



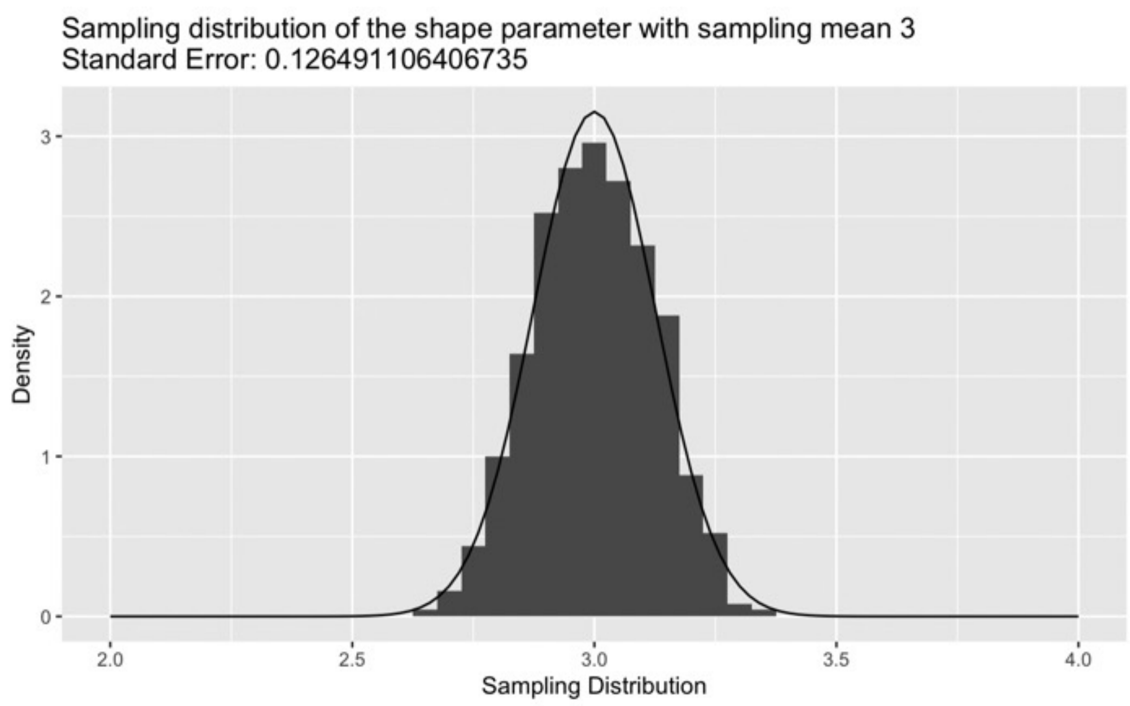
(b)

```{r}
set.seed(135)
sampling <- sample(population, 400, replace=TRUE)
std_error = sqrt(SD^2/400)

hist1 = ggplot(df)+stat_function(fun=dnorm, args=list(mean,std_error))+labs(x="Sampling Distribution", y="Density",
title=paste0("Sampling distribution of the shape parameter with sampling mean ", mean, "\n", "Standard Error: ", std_error))
+xlim(c(2,4))
hist1
```



(c)

```{r}
set.seed(135)
sample_mean <- rep(NA, 500)
for(i in 1:500){
  sampling <- sample(population, 400, replace=TRUE)
  sample_mean= mean(sampling)
  sampled_mean[i] = sample_mean
}

df_means <-  as.data.frame(sampled_mean)
ggplot(df_means) + geom_histogram(aes(x = sampled_mean, y= ..density..),binwidth = 0.05) +stat_function(fun=dnorm,
args=list(mean,std_error))+labs(x="Sampling Distribution", y="Density", title=paste0("Sampling distribution of the shape parameter
with sampling mean ", mean, "\n", "Standard Error: ", std_error)) +xlim(c(2,4))
```



(d)

````
```{r}
CI = rep(NA,500)
cnt = 0
for (i in 1:500){
  if (sample_means[i]-1.96*sample_std_error <= 3 && sample_means[i]+1.96*sample_std_error >= 3){
    cnt = cnt+1
  }
  CI[i]= paste0("(", round(sample_means[i]-1.96*sample_std_error,2), ",", round(sample_means[i]+1.96*sample_std_error,2), ")")

}
fraction = cnt/500
fraction
CI
```
````

```
[1] 0.958
  [1] "(2.76,3.24)" "(2.69,3.18)" "(2.9,3.39)"  "(2.72,3.21)" "(2.8,3.29)"  "(2.56,3.04)" "(2.8,3.28)"  "(3,3.49)"
  [9] "(2.69,3.17)" "(2.87,3.35)" "(2.76,3.24)" "(2.63,3.11)" "(2.65,3.14)" "(2.93,3.41)" "(2.58,3.07)" "(2.84,3.32)"
 [17] "(2.75,3.23)" "(2.57,3.05)" "(2.9,3.38)"  "(2.82,3.3)"  "(2.77,3.25)" "(2.67,3.16)" "(2.56,3.04)" "(2.76,3.25)"
 [25] "(2.69,3.18)" "(2.94,3.42)" "(2.84,3.32)" "(2.78,3.26)" "(2.87,3.35)" "(2.61,3.09)" "(2.85,3.33)" "(2.72,3.2)"
 [33] "(2.76,3.24)" "(2.85,3.33)" "(2.48,2.97)" "(3.09,3.57)" "(2.93,3.41)" "(2.84,3.33)" "(2.78,3.26)" "(2.75,3.23)"
 [41] "(2.7,3.18)"  "(3.02,3.5)"  "(2.66,3.15)" "(2.86,3.34)" "(2.95,3.43)" "(2.79,3.27)" "(2.69,3.17)" "(2.87,3.35)"
 [49] "(2.73,3.21)" "(2.82,3.3)"  "(2.7,3.18)"  "(2.48,2.97)" "(2.66,3.15)" "(2.89,3.37)" "(2.92,3.4)"  "(2.67,3.15)"
 [57] "(2.65,3.13)" "(2.67,3.16)" "(2.58,3.06)" "(2.76,3.24)" "(2.6,3.08)"  "(2.8,3.28)"  "(2.92,3.4)"  "(2.88,3.37)"
 [65] "(2.69,3.17)" "(2.58,3.07)" "(2.69,3.17)" "(2.8,3.28)"  "(2.78,3.26)" "(2.64,3.12)" "(2.8,3.29)"  "(2.9,3.38)"
 [73] "(2.65,3.13)" "(2.62,3.1)"  "(2.55,3.03)" "(2.81,3.29)" "(3.02,3.5)"  "(2.95,3.43)" "(2.74,3.22)" "(2.68,3.16)"
 [81] "(2.7,3.19)"  "(2.81,3.29)" "(2.77,3.25)" "(2.64,3.13)" "(2.86,3.34)" "(2.79,3.28)" "(2.83,3.31)" "(2.79,3.27)"
```