Notes on Curved Economics: A Geometric and Algebraic Re-foundation*

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Abstract—**This paper summarizes a series of research notes that propose a progressive re-foundation of economic theory on geometric and algebraic principles. We begin by re-embedding standard Euclidean decision spaces into Riemannian manifolds, demonstrating that optimality conditions are preserved while allowing behavioral anomalies to be interpreted as curvature-consistent phenomena. We then model strategy not as a point but as a smooth trajectory, optimized via a variational principle that penalizes velocity and acceleration. This dynamic framework is extended to include context-dependent activation thresholds, phase-dependent utility on the complex plane, endogenous misperception, strategic friction in Bayesian updating, and organizational vector fields. The inquiry culminates in the use of non-commutative (quaternion) and non-associative (octonion) algebras to model the structural origins of path dependence, identity fragmentation, and strategic collapse. This approach recasts economics as a theory of motion and existence within structured, often non-Euclidean, spaces, where inaction can be an ethical safeguard against ontological rupture.**

Index Terms—**Riemannian Manifolds, Calculus of Variations, Complex Analysis, Non-Associative Algebra, Quaternions, Octonions, Strategic Dynamics, Ontological Economics**

I. Introduction: From Euclidean Space to Curved Manifolds

Standard economic optimization occurs in Euclidean space, where the geometric structure is implicitly assumed to be flat and uniform. This paper argues that this assumption is a special case. We introduce a **re-embedding operator** $T : X \to \mathcal{M}_\kappa$ that maps a conventional decision space X onto a smooth Riemannian manifold $\mathcal{M}_\kappa$. While this transformation preserves the first-order optimality (KKT) conditions, it reframes the decision problem geometrically. Apparent behavioral anomalies in Euclidean space can be rationalized as normatively consistent behavior on a curved manifold, without altering fundamental utility or constraint structures.

II. Dynamic Strategy as Paths on Manifolds

We move from static choice to dynamic strategy by modeling an agent's behavior as a smooth path $S(t)$ on a manifold. Rationality is redefined as the selection of an optimal trajectory that maximizes a utility functional while penalizing the costs of motion.

A. The Strategic Response Functional

The agent's objective is to maximize a functional that balances utility with penalties on velocity and acceleration, a direct application of the calculus of variations:

$$\mathcal{R}(S) := \int_0^T [u(S(t)) - \lambda_1 \|\dot{S}(t)\|_g^2 - \lambda_2 \|\nabla_t \dot{S}(t)\|_g^2] dt \qquad (1)$$

Here, λ_1 and λ_2 represent costs of inertia and adjustment, respectively. The optimal path S^* is characterized by a second-order Euler-Lagrange equation, which governs strategic motion under curvature-aware frictions. **More fundamentally, the set of admissible strategies is often limited by a maximum tolerated complexity, framed as a curvature constraint on the path.** This framework is extended to include context-dependent **utility activation surfaces**, where alternatives become behaviorally admissible only after crossing an endogenous threshold, and **endogenous misperception**, where feedback loops between actual and perceived states induce curvature distortions. **This is further compounded by strategic friction in Bayesian updating, where lags in belief formation delay rational responses to new information.**

III. Organizational and Algebraic Structures

The framework is further generalized to account for external and internal structural constraints that transcend simple cost functions.

A. Organizational Vector Fields

We model institutional pressures or directives as a time-varying vector field $\vec{F}_{task}$ on the strategic manifold. The agent's cost is now a function of their deviation from this field, penalizing misalignment with organizational structure. Optimal behavior becomes a negotiation between internal utility gradients and external directional flows.

B. Non-Commutative and Non-Associative Dynamics

The most abstract layer of our analysis posits that the deep structure of strategic interaction may have algebraic origins.

- **Complex Plane ($\mathbb{C}$): As a first step, strategies are modeled in the complex plane, where utility activation is gated by the phase alignment between an agent's internal affective state and external signals. This introduces the concept of strategic resonance and phase-dependent policy effectiveness.**
- **Quaternions ($\mathbb{H}$):** The non-commutativity of quaternion multiplication ($ij \neq ji$) provides a formal basis for the path dependence and irreversibility of strategic identity. The order of events fundamentally alters the outcome.
- **Octonions ($\mathbb{O}$):** The non-associativity of octonion multiplication ($(ab)c \neq a(bc)$) is used to model the possibility of **strategic collapse**. Under cognitive load, an agent's identity can fragment, leading to semantically divergent yet input-consistent trajectories. Inaction or "burnout" is modeled as entrapment in these non-associative subspaces.

In this view, strategic dysfunction is not a psychological failure but a topological phase shift imposed by the algebraic structure of cognition.

IV. Conclusion: Towards an Ontological Economics

The final step reframes economics as a theory of **ontological constraints**. An agent is said to "exist strategically" only when an activation predicate, dependent on internal states and the manifold's curvature, is met. The model introduces an **ethical modulation**, where inaction is interpreted not as inefficiency but as a rational safeguard against "ontological rupture" or the collapse of coherent identity. Economics is thus recast as a topological structure of lived relations, where both existence and restraint are rationalized by the local geometry of ethical and strategic feasibility. The challenge is no longer just to model choice, but to understand the conditions for coherent action itself.

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