4.5 Homomorphic Filtering (1)

- The illumination-reflectance model from chap. 2

$$f(x, y) = i(x, y)r(x, y)$$

$i(x,y)$: illumination
$r(x,y)$: reflectance

- Characteristics of the illumination-reflectance model

 - Illumination → almost not varying → low frequency

 - Reflectance → varying → high frequency

- A method for enhancing an image

 - Separation of the illumination and reflectance components

 - Illumination : gray-level range compression

 - Reflectance : contrast enhancement

◇ Homomorphic Filtering (2)

- Fourier transform of the product of two functions is not separable

$$\Im\{f(x,y)\} \neq \Im\{i(x,y)\}\Im\{r(x,y)\}$$

- Suppose that

$$z(x,y) = \ln f(x,y) = \ln i(x,y) + \ln r(x,y)$$

- Then, Fourier transform becomes

$$\Im\{z(x,y)\} = \Im\{\ln f(x,y)\} = \Im\{\ln i(x,y)\} + \Im\{\ln r(x,y)\}$$

$$Z(u,v) = F_i(u,v) + F_r(u,v)$$

- Enhance image by multiplying filter function *H(u,v)*

$$S(u,v) = H(u,v)Z(u,v) = H(u,v)F_i(u,v) + H(u,v)F_r(u,v)$$

$$\begin{aligned} s(x,y) &= \Im^{-1}\{S(u,v)\} \\ &= \Im^{-1}\{H(u,v)F_i(u,v)\} + \Im^{-1}\{H(u,v)F_r(u,v)\} \end{aligned}$$

◇ Homomorphic Filtering (3)

- By letting

$$s(x,y) = i'(x,y) + r'(x,y)$$

$$where,\ i'(x,y) = \Im^{-1}\{H(u,v)F_i(u,v)\}\ and$$
$$r'(x,y) = \Im^{-1}\{H(u,v)F_r(u,v)\}$$

- Fourier inverse of the enhanced image becomes

$$g(x,y) = e^{S(x,y)} = e^{i'(x,y)}e^{r'(x,y)}$$
$$= i_0(x,y)r_0(x,y)$$

$$where, i_0(x,y) = e^{i'(x,y)}\ \ and\ \ r_0(x,y) = e^{r'(x,y)}$$

◇ Homomorphic Filtering (4)

- Homomorphic system concept
  - Dynamic range compression and contrast enhancement.

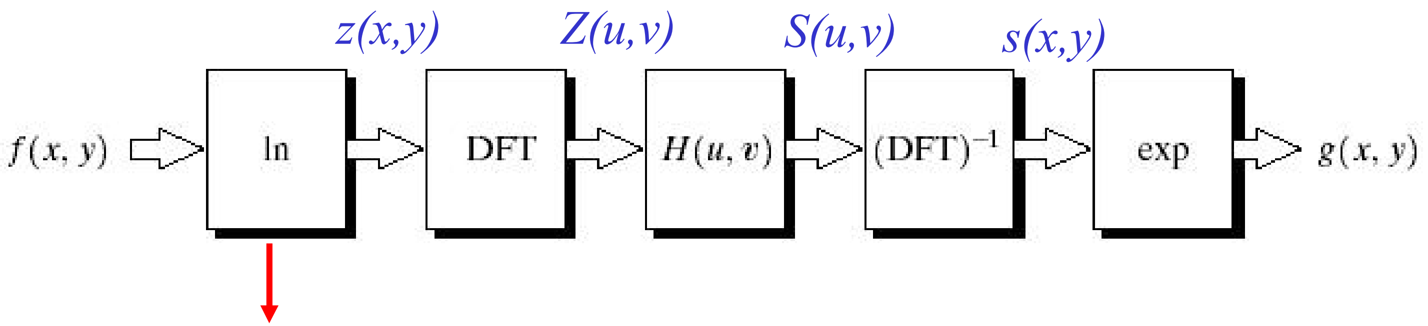
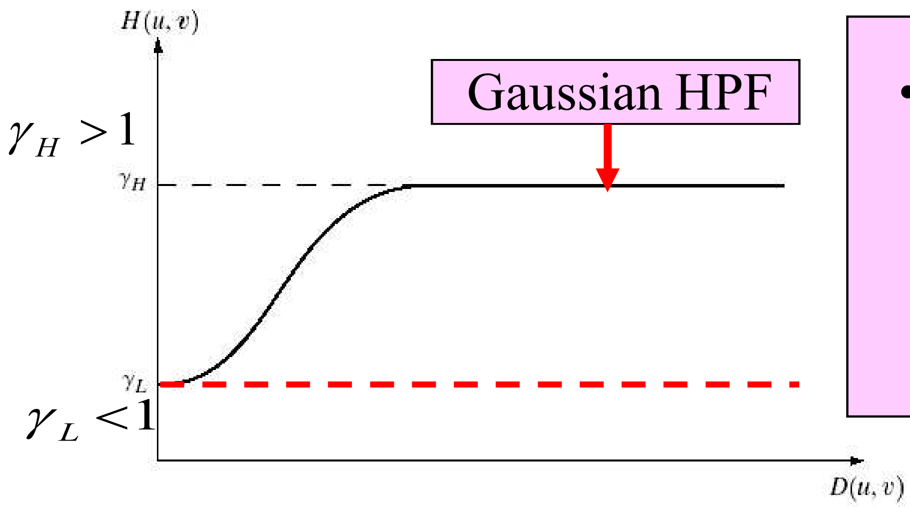


FIGURE 4.31 Homomorphic filtering approach for image enhancement.

$$\because \ \Im\{f(x, y)\} \neq \Im\{i(x, y)\}\, \Im\{r(x, y)\}$$

◇ Homomorphic Filtering (5)

- Homomorphic filter controls illumination and reflectance simultaneously.



- Characteristics of the filter
  - decrease the low frequency components (illumination)
  - amplify high frequency components (reflectance)

- Approximation of the homomorphic filiter using the Gaussian highpass filter

$$H(u,v) = \underbrace{(\gamma_H - \gamma_L)[1 - e^{-c(D^2(u,v)/D_0^{\,2})}]}_{\text{Gaussian HPF}} + \gamma_L$$

c : control factor for the sharpness of the slope of the filter

- This filter is similar to the high-frequency emphasis filter

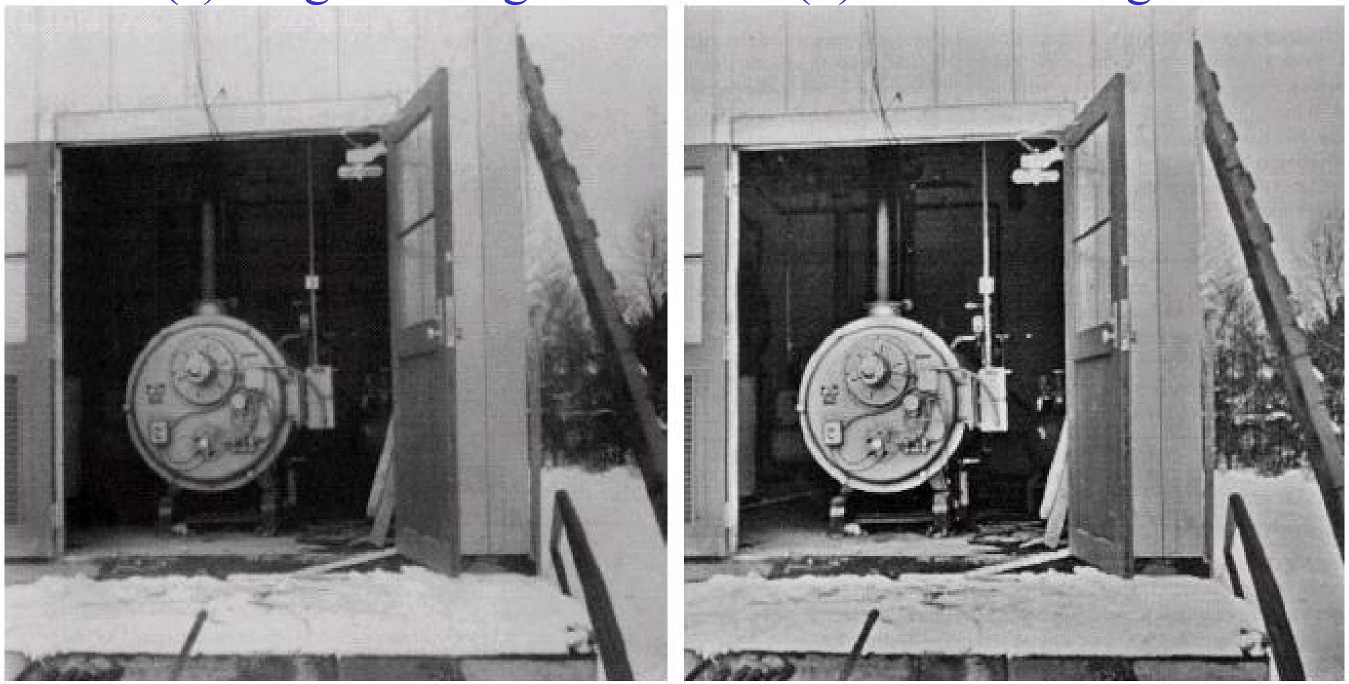
◇ Homomorphic Filtering (6)

- Homomorphic Filtering $with\ \gamma_L = 0.5\ and\ \gamma_H = 2.0$

(a) Original image (b) Filtered image

a b

FIGURE 4.33 (a) Original image. (b) Image processed by homomorphic filtering (note details inside shelter). (Stockham.)



- Fig.4.33(a) : The details inside the shelter are obscured by the glare from outside walls
- Fig.4.33(b) : Enhanced effects are clear in the details inside the shelter and balancing the gray levels of the outside walls

4.6 Implementation

- Properties of the 2-D Fourier transform
- Computing the Inverse Fourier transform
- More on periodicity : the need for padding
- The convolution and correlation theorems
- Summary of properties of the 2-D Fourier transform
- The Fast Fourier transform
- Some comments on filter design

4.6.1 Properties of the 2-D Fourier transform translation

- Translation
- Distributivity and scaling
- Rotation
- Periodic and conjugate symmetry
- Separability

◇ Translation

- The translation properties of the Fourier transform pair

$$f(x,y)e^{j2\pi(u_0x/M+v_0y/N)} \Leftrightarrow F(u-u_0, v-v_0)$$

$$f(x-x_0, y-y_0) \Leftrightarrow F(u,v)e^{-j2\pi(ux_0/M+vy_0/N)}$$

- $when\ \ u_0 = M/2\ and\ v_0 = N/2$

$$e^{j2\pi(u_0x/M+v_0y/N)} = e^{j\pi(x+y)} = (-1)^{x+y}$$

$$f(x,y)(-1)^{x+y} \Leftrightarrow F(u-M/2, v-N/2)$$

$$f(x-M/2, y-N/2) \Leftrightarrow F(u,v)(-1)^{u+v}$$

- A shift in *f(x,y)* does not affect the magnitude of its Fourier transform

$$|F(u,v)e^{-j2\pi(ux_0/M+vy_0/N)}| = |F(u,v)|$$

◇ Distributivity and Scaling

- The continuous or discrete transform pair,

$$\Im\{f_1(x,y)+f_2(x,y)\}=\Im\{f_1(x,y)\}+\Im\{f_2(x,y)\}$$

$$\Im\{f_1(x,y)\bullet f_2(x,y)\}\neq\Im\{f_1(x,y)\}\bullet\Im\{f_2(x,y)\}$$

- For two scales a and b,

$$af(x,y)\Leftrightarrow aF(u,v)$$

$$f(ax,\ by)\Leftrightarrow\frac{1}{|ab|}F(u/a,v/b)$$

◇ Rotation

- In the polar coordinates

$$f(x,y) = f(r,\theta) \qquad F(u,v) = F(\omega,\varphi)$$

$$where,\ x = r\cos\theta,\ y = r\sin\theta,\ u = \omega\cos\varphi\ and\ v = \omega\sin\varphi$$

- Rotating f(x,y) by an angle θ_0 rotates F(u,v) by the same angle. Similarly, rotating F(u,v) rotates f(x,y) by the same angle.

$$f(r,\ \theta + \theta_0) \Leftrightarrow F(\omega, \varphi + \theta_0)$$

◇ Periodicity and Conjugate Symmetry(1)

- The function with period M,N

$$F(u,v) = F(u+M,v) = F(u,v+N) = F(u+M,v+N)$$

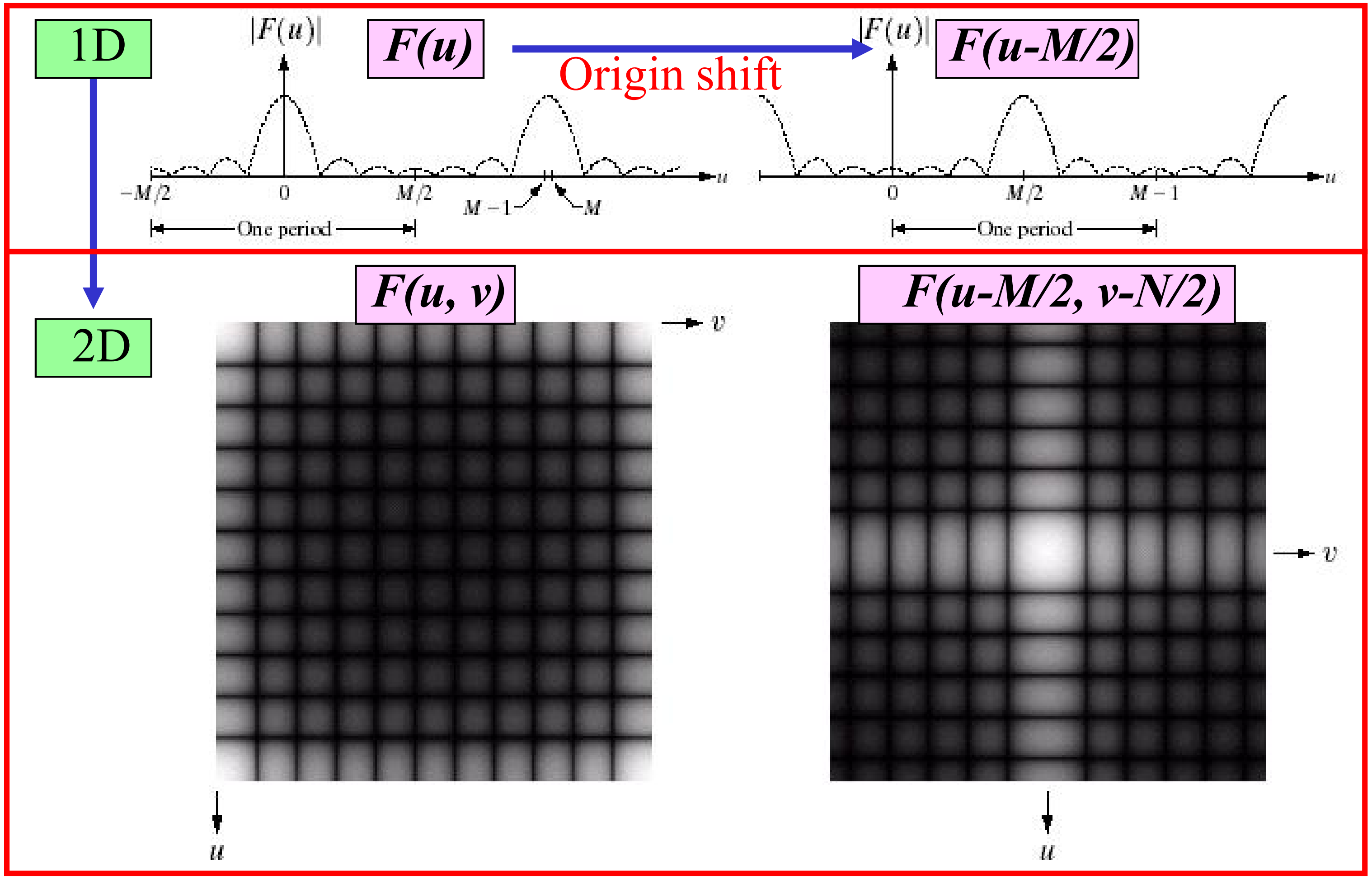
$$f(x,y) = f(x+M,y) = f(x,y+N) = f(x+M,y+N)$$

- If *f(x,y)* is real, the Fourier transform also exhibits conjugate symmetry from Section 4.2

$$F(u,v) = F^*(-u,-v)$$

$$|F(u,v)| = |F(-u,-v)|$$

◇ Periodicity and Conjugate Symmetry(2)



- Centering transform not only helps with visualization but also simplifies filtering

◇ Separability

- The separable forms in Fourier transform

$$F(u,v) = \frac{1}{M}\sum_{x=0}^{M-1} \exp[-j2\pi ux/M] \frac{1}{N}\sum_{y=0}^{N-1} f(x,y)\exp[-j2\pi vy/N]$$

$$= \frac{1}{M}\sum_{x=0}^{M-1} F(x,v)\exp[-j2\pi ux/M]$$

$$where,\ F(x,v) = \frac{1}{N}\sum_{y=0}^{N-1} f(x,y)\exp[-j2\pi vy/N]$$

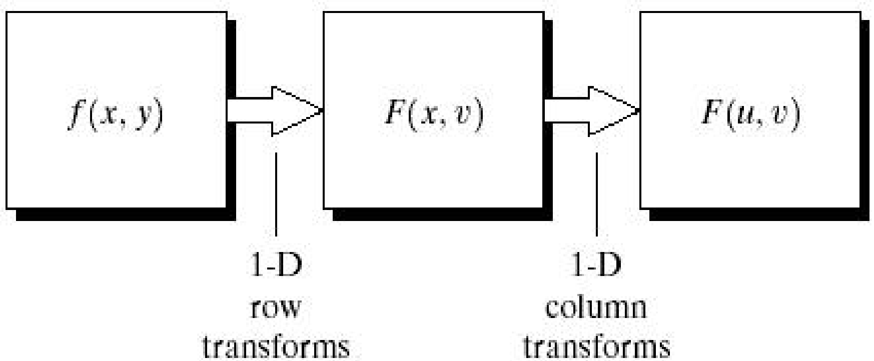


FIGURE 4.35 Computation of the 2-D Fourier transform as a series of 1-D transforms.

- We can compute the 2-D transform by first computing a 1-D transform along each row of the input image

4.6.2 Computing the Inverse Fourier transform using a Forward Transform algorithm

- The 1-D Fourier transform

$$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x)e^{-j2\pi ux/M}$$

Forward transform
for u = 0, 1, 2, …, M-1

$$f(x) = \sum_{u=0}^{M-1} F(u)e^{j2\pi ux/M}$$

Inverse transform
for x = 0, 1, 2, …, M-1

- Taking complex conjugate of *f(x)* and dividing both sides by M

$$\frac{1}{M} f^*(x) = \frac{1}{M}\sum_{u=0}^{M-1} F^*(u)e^{-j2\pi ux/M}$$

- Inputting F*(u) into an algorithm designed to compute the forward transform gives the quantity *f*(x)/M*

- A similar analysis for two variables

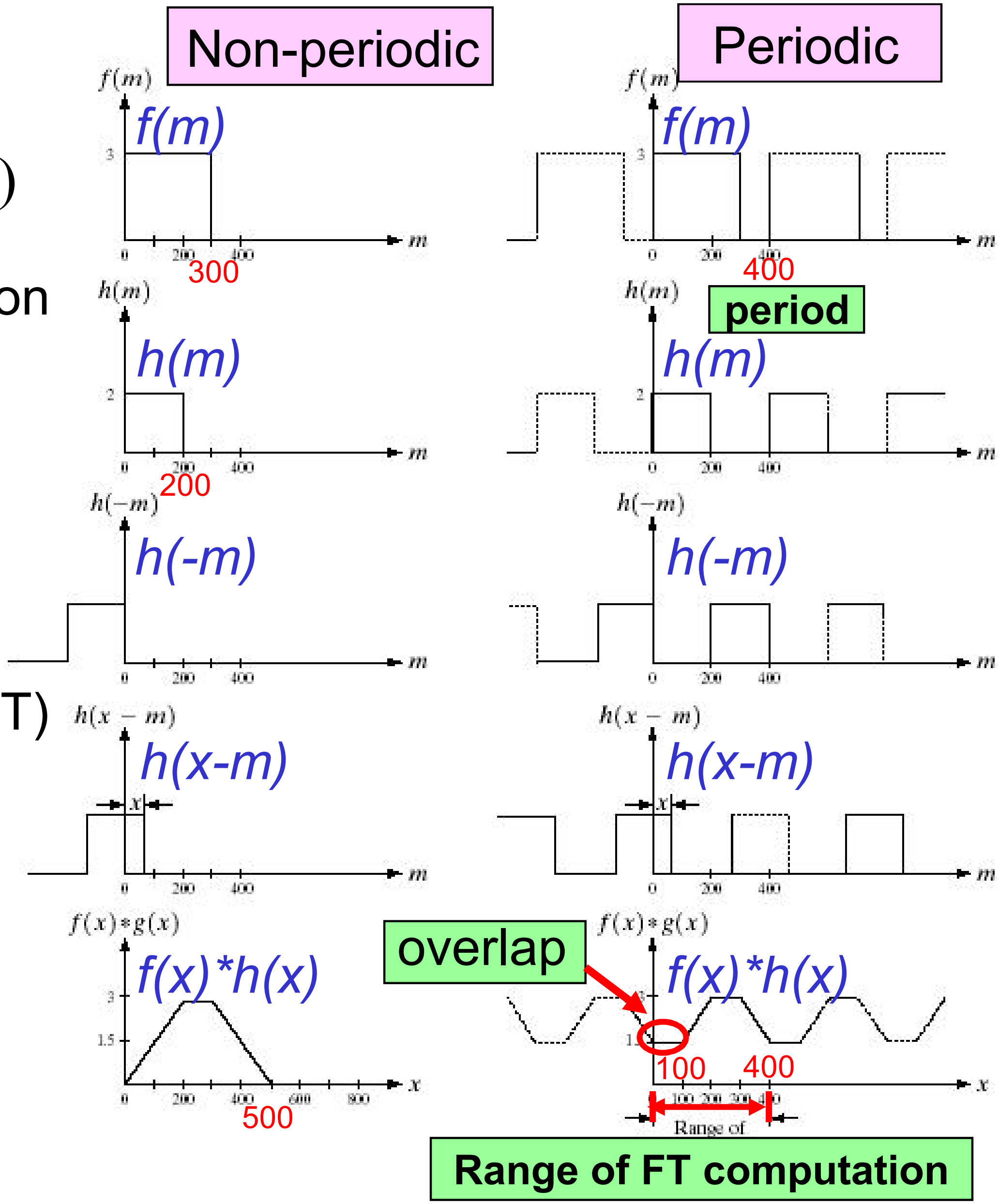
$$\frac{1}{MN} f^*(x,y) = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F^*(u,v)e^{-j2\pi(ux/M+vy/N)}$$

◇ 4.6.3 More on Periodicity (1)

- 1-D Convolution

$$f(x)*h(x)=\frac{1}{M}\sum_{m=0}^{M-1}f(m)h(x-m)$$

- Methods of convolution computation
  - spatial domain : $f(x)*h(x)$
  - freq. domain : $\Im^{-1}\{F(u)H(u)\}$
- In non-periodic function
  - obtain correct results
- In periodic function (treated as DFT)
  - in case with the period = 400
  - obtain incorrect (overlapped) results in the segment [0,100] because of periodicity



◇ More on Periodicity (2)

- Extended or padded functions *for avoiding convolution overlap*

$$f_e(x) = \begin{cases} f(x) & 0 \le \mathrm{x} \le \mathrm{A} - 1 \\ 0 & \mathrm{A} \le \mathrm{x} < \mathrm{P} \end{cases}$$ (Zero padding)

$$g_e(x) = \begin{cases} g(x) & 0 \le \mathrm{x} \le \mathrm{B} - 1 \\ 0 & \mathrm{B} \le \mathrm{x} < \mathrm{P} \end{cases}$$ (Zero padding)

Where, P is an identical period

- If P < A + B –1 : wraparound error (convolution overlap)
  P = A + B –1 : adjacent
  P > A + B –1 : separated

◇ More on Periodicity (3)

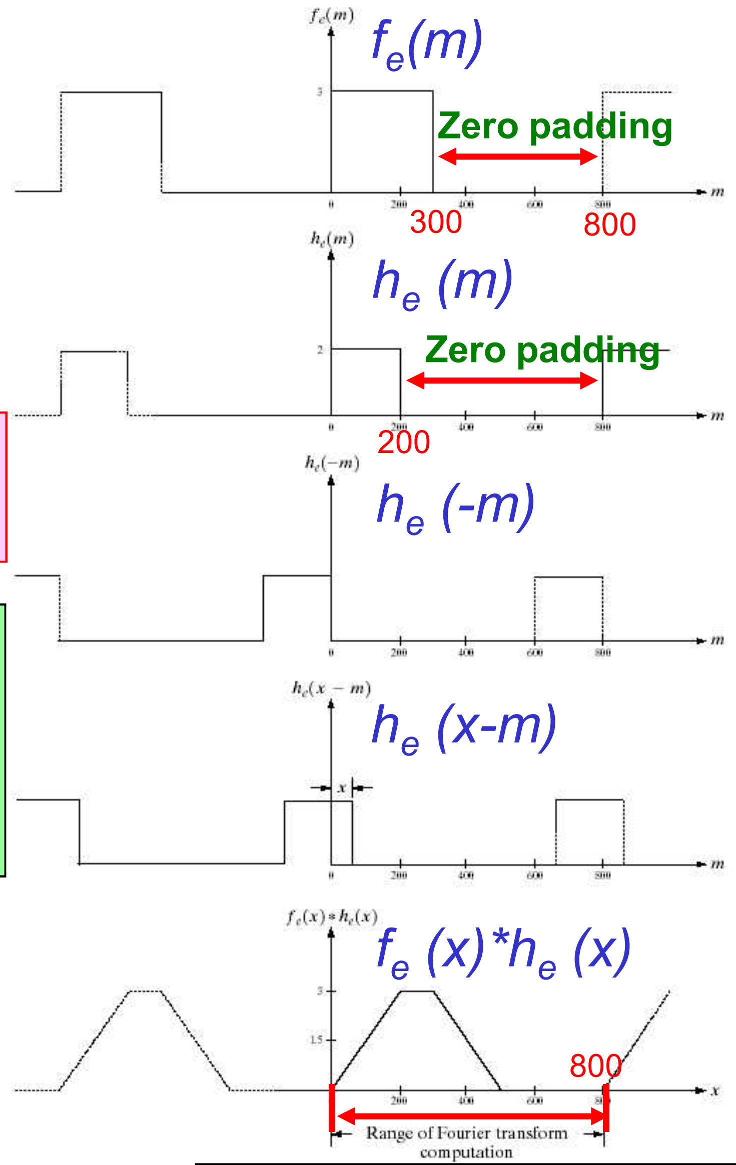


FIGURE 4.37 Result of performing convolution with extended functions. Compare Figs. 4.37(e) and 4.36(e).

A = 300, B = 200, P = 800

P > A + B –1 : separated

- Computation of convolution

$$\underbrace{f_e(x) * h_e(x)}_{\text{Spatial domain}} = \underbrace{\Im^{-1}\{F_e(u)H_e(u)\}}_{\text{Freq. domain}}$$

Range of FT computation

◇ More on Periodicity (4)

- Extension to the 2-D functions
- Two functions with zero padding

$$f_e(x,y)=\begin{cases} f(x,y) & 0 \le \mathrm{x} \le \mathrm{A}-1 \text{ and } 0 \le \mathrm{y} \le \mathrm{B}-1 \\ 0 & \mathrm{A} \le \mathrm{x} \le \mathrm{P} \quad \text{or} \quad \mathrm{B} \le \mathrm{y} \le \mathrm{Q} \end{cases}$$ (Zero padding)

$$h_e(x,y)=\begin{cases} h(x,y) & 0 \le \mathrm{x} \le C-1 \text{ and } 0 \le \mathrm{y} \le D-1 \\ 0 & \mathrm{C} \le \mathrm{x} \le \mathrm{P} \quad \text{or} \quad \mathrm{D} \le \mathrm{y} \le \mathrm{Q} \end{cases}$$ (Zero padding)

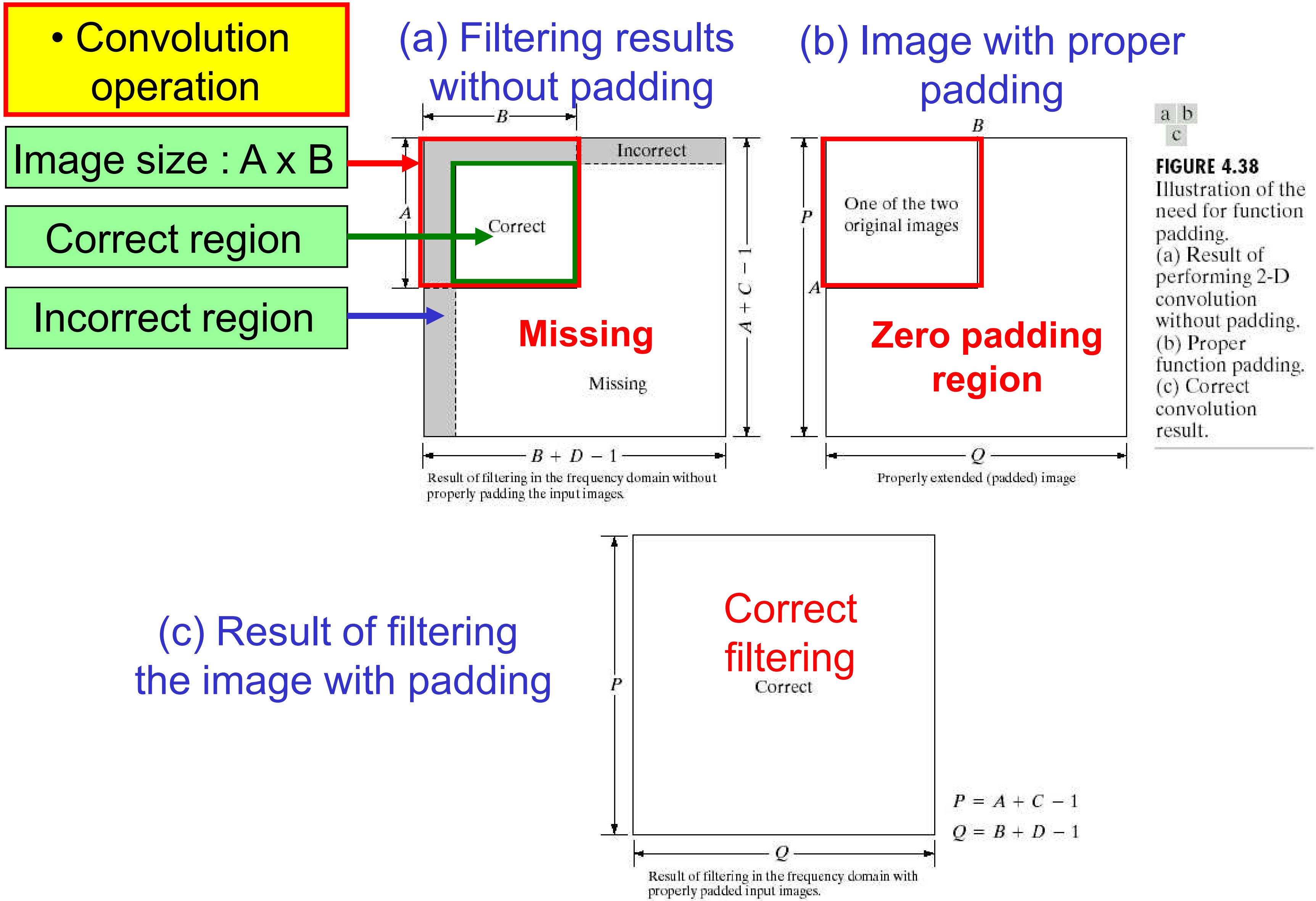
Where, f(x,y) is the original image and h(x,y) is filter function

- The conditions for avoiding wraparound error

$$P \ge A + C - 1$$ P : x-direction period

$$Q \ge B + D - 1$$ Q : y-direction period

◇ More on Periodicity (5)



◇ More on Periodicity (6)

• Convolution operation

Padded lowpass filter in the spatial domain

Results of Filtering with padding

4.6.4 Convolution and Correlation (1)

- The convolution of the 2-D functions, *f(x,y)* and *h(x,y)*

$$f(x,y)*h(x,y)=\frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}f(m,n)h(x-m,y-n)$$

$$f(x,y)*h(x,y)\Leftrightarrow F(u,v)H(u,v)$$

$$f(x,y)h(x,y)\Leftrightarrow F(u,v)*H(u,v)$$

- The correlation of the 2-D functions, *f(x,y)* and *h(x,y)*

$$f(x,y)\circ h(x,y)=\frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}f^*(m,n)h(x+m,y+n)$$

$$\textit{where, } f^* \textit{ denotes the complex conjugate of } f$$

$$f(x,y)\circ h(x,y)\Leftrightarrow F^*(u,v)H(u,v)$$

$$f^*(x,y)h(x,y)\Leftrightarrow F(u,v)\circ H(u,v)$$

By similarity of the convolution theorem

◇ Convolution and Correlation (2)

- The principal use of the convolution and correlation
  - convolution : filtering in the spatial and freq. domains
  - correlation : matching of a particular region or objects

- What is the matching ?
  - *f(x)* is an image containing objects or regions
  - *h(x)* is the image representing that object or region (called as template)
  - the correlation of the two functions will be maximum at the location where *h* finds a correspondence in *f*

◇ Convolution and Correlation (3)

- Classification of the correlation
  - Autocorrelation : $f(x,y)\circ f(x,y)$
  - (Cross) correlation : $f(x,y)\circ h(x,y)$
- Autocorrelation theorem

$$f(x,y)\circ f(x,y) \Leftrightarrow F^*(u,v)F(u,v) = \left|F(u,v)\right|^2$$

$$\left|f(x,y)\right|^2 = f^*(x,y)\circ f(x,y) \Leftrightarrow F(u,v)\circ F(u,v)$$

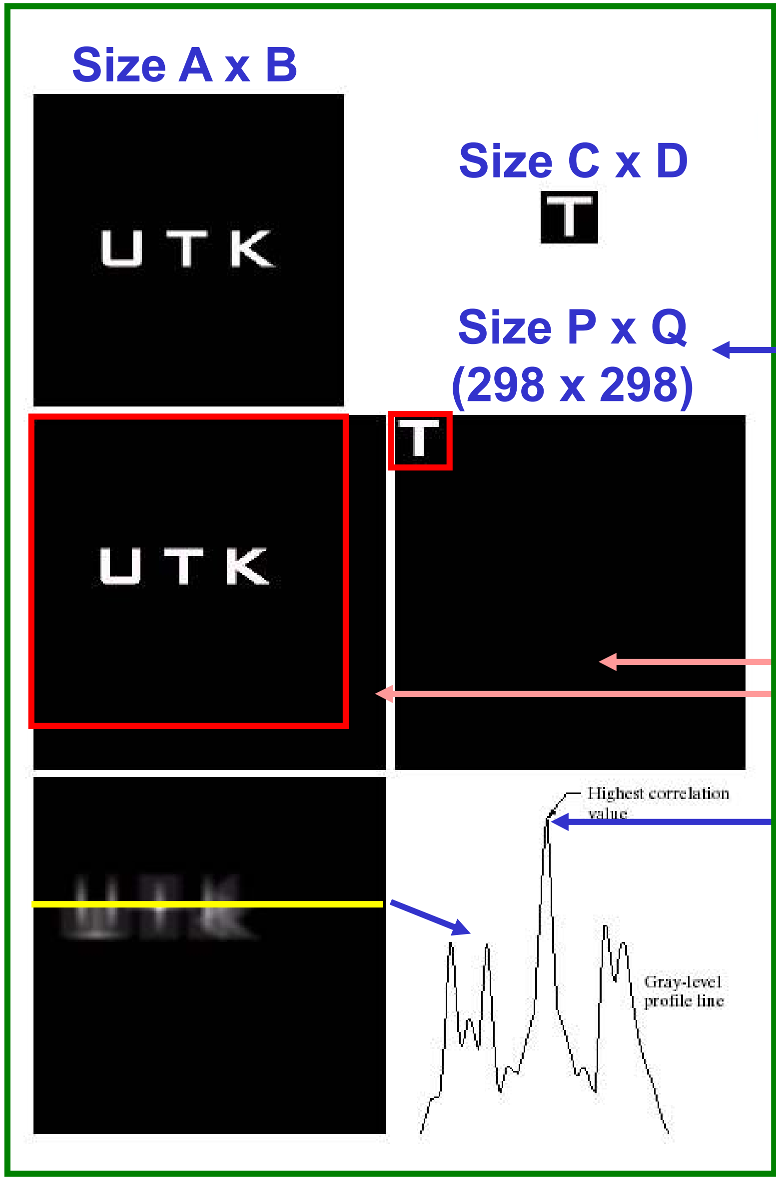
- Fourier transform of autocorrelation is the power spectrum

◇ Convolution and Correlation (4)

- Correlation operation

(a) Image

(b) Template



A= 256, B=256, C =38, D=42
P=A+C-1=293, Q=B+D-1=297
However, our choice is equal padding dimension (298x298)

(c) Padded image

(d) Padded template

Padded region

Highest value of correlation

(e) Correlation displayed as image

(f) Horizontal profile of (e)

4.6.5 Summary of properties of the 2-D Fourier transform (1)

TABLE 4.1
Summary of some important properties of the 2-D Fourier transform.

Property	Expression(s)
Fourier transform	$F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}$
Inverse Fourier transform	$f(x,y) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(ux/M+vy/N)}$
Polar representation	$F(u,v) = \lvert F(u,v)\rvert e^{-j\phi(u,v)}$
Spectrum	$\lvert F(u,v)\rvert = \left[R^2(u,v) + I^2(u,v)\right]^{1/2}$, $R = \text{Real}(F)$ and $I = \text{Imag}(F)$
Phase angle	$\phi(u,v) = \tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right]$
Power spectrum	$P(u,v) = \lvert F(u,v)\rvert^2$
Average value	$\bar{f}(x,y) = F(0,0) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)$
Translation	$f(x,y)e^{j2\pi(u_0x/M+v_0y/N)} \Leftrightarrow F(u-u_0, v-v_0)$ $f(x-x_0, y-y_0) \Leftrightarrow F(u,v)e^{-j2\pi(ux_0/M+vy_0/N)}$ When $x_0 = u_0 = M/2$ and $y_0 = v_0 = N/2$, then $f(x,y)(-1)^{x+y} \Leftrightarrow F(u-M/2, v-N/2)$ $f(x-M/2, y-N/2) \Leftrightarrow F(u,v)(-1)^{u+v}$

◇ Summary of properties of the 2-D Fourier transform (2)

Conjugate symmetry	$F(u,v) = F^*(-u,-v)$ $\lvert F(u,v)\rvert = \lvert F(-u,-v)\rvert$
Differentiation	$\dfrac{\partial^n f(x,y)}{\partial x^n} \Leftrightarrow (ju)^n F(u,v)$ $(-jx)^n f(x,y) \Leftrightarrow \dfrac{\partial^n F(u,v)}{\partial u^n}$
Laplacian	$\nabla^2 f(x,y) \Leftrightarrow -(u^2+v^2)F(u,v)$
Distributivity	$\Im[f_1(x,y)+f_2(x,y)] = \Im[f_1(x,y)] + \Im[f_2(x,y)]$ $\Im[f_1(x,y)\cdot f_2(x,y)] \neq \Im[f_1(x,y)]\cdot\Im[f_2(x,y)]$
Scaling	$af(x,y) \Leftrightarrow aF(u,v),\ f(ax,by) \Leftrightarrow \dfrac{1}{\lvert ab\rvert}F(u/a,v/b)$
Rotation	$x = r\cos\theta \quad y = r\sin\theta \quad u = \omega\cos\varphi \quad v = \omega\sin\varphi$ $f(r,\theta+\theta_0) \Leftrightarrow F(\omega,\varphi+\theta_0)$
Periodicity	$F(u,v) = F(u+M,v) = F(u,v+N) = F(u+M,v+N)$ $f(x,y) = f(x+M,y) = f(x,y+N) = f(x+M,y+N)$
Separability	See Eqs. (4.6-14) and (4.6-15). Separability implies that we can compute the 2-D transform of an image by first computing 1-D transforms along each row of the image, and then computing a 1-D transform along each column of this intermediate result. The reverse, columns and then rows, yields the same result.

◇ Summary of properties of the 2-D Fourier transform (3)

Property	**Expression(s)**
Computation of the inverse Fourier transform using a forward transform algorithm	$\frac{1}{MN} f^*(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F^*(u, v) e^{-j2\pi(ux/M + vy/N)}$ This equation indicates that inputting the function $F^*(u, v)$ into an algorithm designed to compute the forward transform (right side of the preceding equation) yields $f^*(x, y)/MN$. Taking the complex conjugate and multiplying this result by MN gives the desired inverse.
Convolution†	$f(x, y) * h(x, y) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m, n) h(x - m, y - n)$
Correlation†	$f(x, y) \circ h(x, y) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f^*(m, n) h(x + m, y + n)$
Convolution theorem†	$f(x, y) * h(x, y) \Leftrightarrow F(u, v) H(u, v)$; $f(x, y) h(x, y) \Leftrightarrow F(u, v) * H(u, v)$
Correlation theorem†	$f(x, y) \circ h(x, y) \Leftrightarrow F^*(u, v) H(u, v)$; $f^*(x, y) h(x, y) \Leftrightarrow F(u, v) \circ H(u, v)$

◇ Summary of properties of the 2-D Fourier transform (4)

Some useful FT pairs:

Impulse	$\delta(x, y) \Leftrightarrow 1$
Gaussian	$A\sqrt{2\pi}\sigma e^{-2\pi^2\sigma^2(x^2+y^2)} \Leftrightarrow Ae^{-(u^2+v^2)/2\sigma^2}$
Rectangle	$\text{rect}[a, b] \Leftrightarrow ab\dfrac{\sin(\pi ua)}{(\pi ua)}\dfrac{\sin(\pi vb)}{(\pi vb)}e^{-j\pi(ua+vb)}$
Cosine	$\cos(2\pi u_0 x + 2\pi v_0 y) \Leftrightarrow \frac{1}{2}\left[\delta(u + u_0, v + v_0) + \delta(u - u_0, v - v_0)\right]$
Sine	$\sin(2\pi u_0 x + 2\pi v_0 y) \Leftrightarrow j\frac{1}{2}\left[\delta(u + u_0, v + v_0) - \delta(u - u_0, v - v_0)\right]$

† Assumes that functions have been extended by zero padding.

4.6.6 The Fast Fourier Transform(1)

• Computation of the 1-D Fourier transform of M points

	DFT	FFT	DFT : FFT
Multiplications Additions	M^2	$M\log_2 M$	$M : \log_2 M$
Operations of M =1024	$\sim 10^6$	$\sim 10^4$	100:1
Operations of M =8192			600:1

◇ The Fast Fourier Transform(2)

- Definition of 1-D DFT

$$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x)e^{-j2\pi ux/M} \qquad u = 0,\ 1,\ \ldots,\ M\text{-}1$$

$$= \frac{1}{M}\sum_{x=0}^{M-1} f(x)W_M^{ux} \qquad where, \quad W_M = e^{-j2\pi/M}$$

- Assuming that $M = 2^n = 2K$

$$F(u) = \frac{1}{2K}\sum_{x=0}^{2K-1} f(x)W_{2K}^{ux}$$

$$= \frac{1}{2}\left[\frac{1}{K}\sum_{x=0}^{K-1} f(2x)W_{2K}^{u(2x)} + \frac{1}{K}\sum_{x=0}^{K-1} f(2x+1)W_{2K}^{u(2x+1)}\right]$$

- *Since* $W_{2K}^{2ux} = (e^{-j2\pi/2K})^{2ux} = (e^{-j2\pi/K})^{ux} = W_K^{ux}$,

$$F(u) = \frac{1}{2}\left[\underbrace{\frac{1}{K}\sum_{x=0}^{K-1} f(2x)W_K^{ux}}_{F_{even}(u)} + W_{2K}^{u}\underbrace{\frac{1}{K}\sum_{x=0}^{K-1} f(2x+1)W_K^{ux}}_{F_{odd}(u)}\right]$$

◇ The Fast Fourier Transform(3)

- Assuming that $F_{even}(u) = \frac{1}{K}\sum_{x=0}^{K-1} f(2x)W_K^{ux}$ and $F_{odd}(u) = \frac{1}{K}\sum_{x=0}^{K-1} f(2x+1)W_K^{ux}$

$$\boxed{F(u) = \frac{1}{2}\left[F_{even}(u) + W_{2K}^{u}F_{odd}(u)\right] \quad for\, u = 0,1,\ \ldots,\ K-1}$$

- $Spectrum\ \ for\ \ u+K = K, K+1,\ \ldots,\ M-1$

$$F(u+K) = \frac{1}{2}\left[F_{even}(u+K) + W_{2K}^{u+K}F_{odd}(u+K)\right]$$

$$F_{even}(u+K) = \frac{1}{K}\sum_{x=0}^{K-1} f(2x)W_K^{(u+K)x} \qquad = \frac{1}{K}\sum_{x=0}^{K-1} f(2x)W_K^{ux}$$

$$F_{odd}(u+K) = \frac{1}{K}\sum_{x=0}^{K-1} f(2x+1)W_K^{(u+K)x} = \frac{1}{K}\sum_{x=0}^{K-1} f(2x+1)W_K^{ux}$$

$$Since \quad W_K^{u+K} = (e^{-j2\pi/K})^{u+K} = (e^{-j2\pi/K})^{u} = W_K^{u} \qquad and$$

$$W_{2K}^{u+K} = (e^{-j2\pi/2K})^{u+K} = -(e^{-j2\pi/2K})^{u} = -W_{2K}^{u},$$

$$\boxed{F(u+K) = \frac{1}{2}\left[F_{even}(u) - W_{2K}^{u}F_{odd}(u)\right] \quad for\, u+K = K, K+1,\ \ldots,\ M-1}$$

◇ The Fast Fourier Transform(4)

- M-point DFT, $M = 2^n = 2K$

$$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x)e^{-j2\pi ux/M} = \frac{1}{M}\sum_{x=0}^{M-1} f(x)W_M^{ux} \qquad u = 0,\ 1,\ \ldots,\ M\text{-}1$$

$$= \frac{1}{2}\left[\underbrace{\frac{1}{K}\sum_{x=0}^{K-1} f(2x)W_{2K}^{u(2x)}}_{F_{even}(u)\ \text{K-point transform}} + \underbrace{\frac{1}{K}\sum_{x=0}^{K-1} f(2x+1)W_{2K}^{u(2x+1)}}_{F_{odd}(u)\ \text{K-point transform}}\right]$$

$$F(u) = \frac{1}{2}\left[F_{even}(u) + W_{2K}^{u}F_{odd}(u)\right] \qquad for\, u = 0, 1,\ \ldots,\ K-1$$ First half

$$F(u+K) = \frac{1}{2}\left[F_{even}(u) - W_{2K}^{u}F_{odd}(u)\right] \qquad for\, u+K = K, K+1,\ \ldots,\ M-1$$ Second half

- The $F(u)$ is an M-points transform
- The $F(u)$ can be divided into $F_{even}(u)$ and $F_{odd}(u)$
- $F_{even}(u)$ and $F_{odd}(u)$ are the M/2-points transforms
- The first half of $F(u)$ requires evaluation of the two (M/2)-point transforms
- The second half, $F(u+K)$ is obtained directly from the first half $F(u)$ without additional transform evaluations
- Continuing of division reduces largely computation quantities

◇ The Fast Fourier Transform(5)

- The DFT of 2-points, $M = 2^n = 2K, M = 2, n = 1, K = 1$

$$2F(u) = \sum_{x=0}^{2-1} f(x) e^{-j2\pi ux/2} \qquad u = 0,\ 1,$$

$$= f(0)e^{-j2\pi u0/2} + f(1)e^{-j2\pi u1/2}$$

$$= \underbrace{[f(0)]}_{F_{even}(u)} + e^{-j2\pi u/2} \bullet \underbrace{[f(1)]}_{F_{odd}(u)} = [f(0)] + e^{-j\pi u/1} \bullet [f(1)]$$

1 point transform 1 point transform

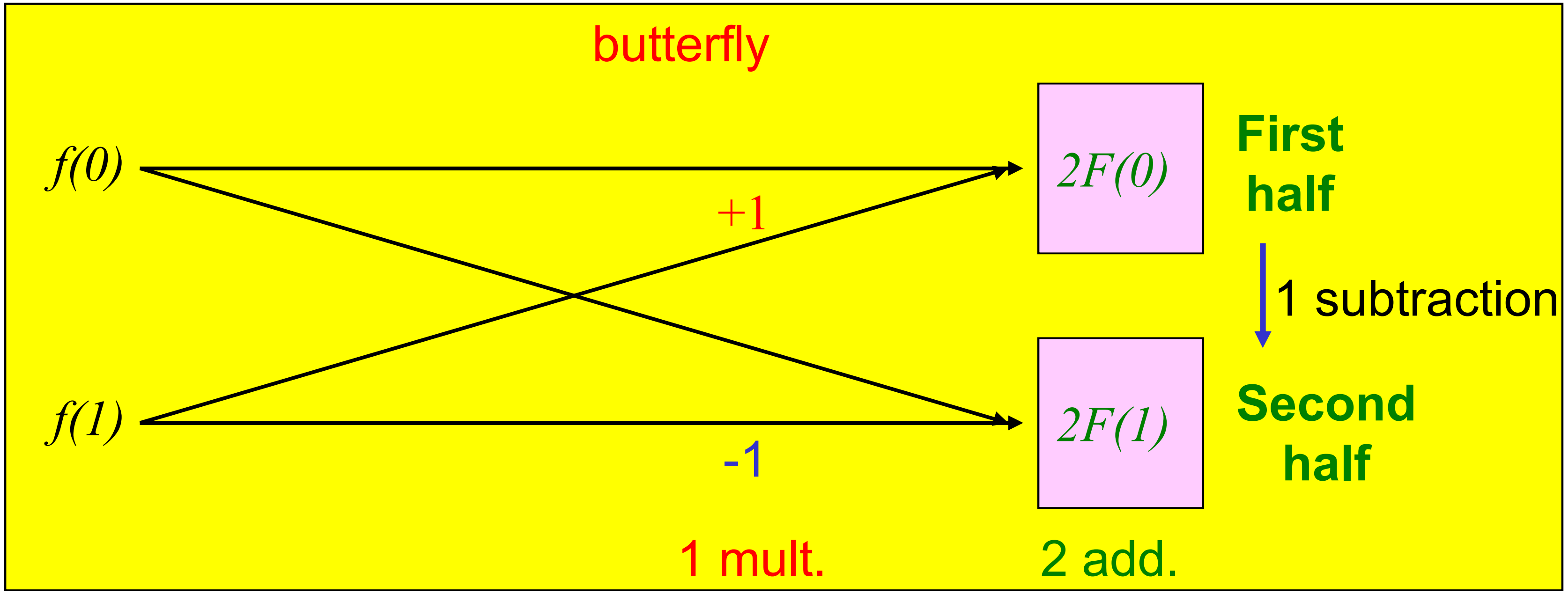
$$2F(u) = [f(0)] + e^{-j\pi u/1} \bullet [f(1)] \quad for\ u = 0$$ First half

$$2F(u+1) = [f(0)] - e^{-j\pi u/1}[f(1)] \quad for\ u+1 = 1$$ Second half

◇ The Fast Fourier Transform(6)

- The frequency spectrum

$$2F(0) = [f(0)] + 1 \bullet [f(1)]$$
$$2F(1) = [f(0)] - \quad 1[f(1)]$$



◇ The Fast Fourier Transform(7)

- The DFT of 4-points, $M = 2^n = 2K, M = 4, n = 2, K = 2$

$$4F(u) = \sum_{x=0}^{4-1} f(x)e^{-j2\pi ux/4} \qquad u = 0,\ 1,\ 2, 3$$

$$= \left[f(0)e^{-j2\pi u0/4} + f(1)e^{-j2\pi u1/4} + f(2)e^{-j2\pi u2/4} + f(3)e^{-j2\pi u3/4}\right]$$

$$= \left[f(0)e^{-j2\pi u0/4} + f(2)e^{-j2\pi u2/4}\right] + \left[f(1)e^{-j2\pi u1/4} + f(3)e^{-j2\pi u3/4}\right]$$

$$= \underbrace{\left[f(0)e^{-j2\pi u0/4} + f(2)e^{-j2\pi u2/4}\right]}_{F_{even}(u),\ \text{2 point transform}} + e^{-j2\pi u/4} \bullet \underbrace{\left[f(1)e^{-j2\pi u0/4} + f(3)e^{-j2\pi u2/4}\right]}_{F_{odd}(u),\ \text{2 point transform}}$$

$$4F(u) = [f(0) + e^{-j\pi u} \bullet f(2)] + e^{-j\pi u/2} \bullet [f(1) + e^{-j\pi u} \bullet f(3)] \quad for\ u = 0,1$$

First half

$$4F(u+2) = [f(0) + e^{-j\pi u} f(2)] - e^{-j\pi u/2} [f(1) + e^{-j\pi u} f(3)] \quad for\ u+2 = 2,3$$

Second half

- The frequency spectrum

$$4F(0) = [f(0) + 1 \bullet f(2)] + 1 \bullet [f(1) + 1 \bullet f(3)]$$

$$4F(1) = [f(0) - 1f(2)] - j \bullet [f(1) - 1f(3)]$$

$$4F(2) = [f(0) + 1f(2)] - 1[f(1) + 1f(3)]$$

$$4F(3) = [f(0) - 1f(2)] + j[f(1) - 1f(3)]$$

◇ The Fast Fourier Transform(8)

$$4F(0) = \left[f(0) + 1 \bullet f(2)\right] + 1 \bullet \left[f(1) + 1 \bullet f(3)\right]$$

$$4F(1) = \left[f(0) - 1 f(2)\right] - j \bullet \left[f(1) - 1 f(3)\right]$$

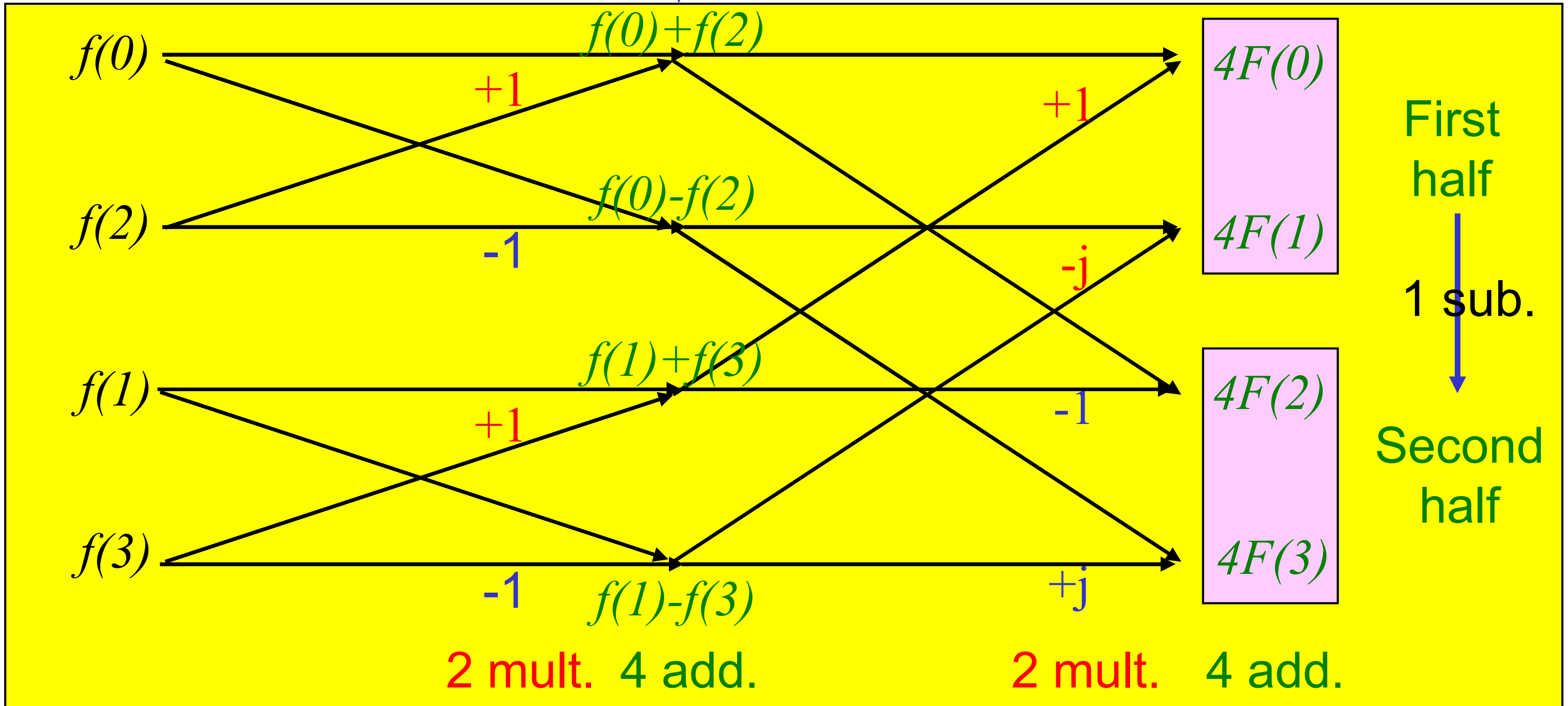
$$4F(2) = \left[f(0) + 1 f(2)\right] - 1 \left[f(1) + 1 f(3)\right]$$

$$4F(3) = \left[f(0) - 1 f(2)\right] + j \left[f(1) - 1 f(3)\right]$$

First half

1 subtraction

Second half



◇ The Fast Fourier Transform(9)

- The DFT of 8-points $M = 2^n = 2K, M = 8, n = 3, K = 4$

$$8F(u) = \sum_{x=0}^{8-1} f(x) e^{-j2\pi ux/8} \qquad u = 0,1,2,3,4,5,6,7$$

$$= f(0)e^{-j2\pi u0/8} + f(1)e^{-j2\pi u1/8} + f(2)e^{-j2\pi u2/8} + f(3)e^{-j2\pi u3/8}$$
$$+ f(4)e^{-j2\pi u4/8} + f(5)e^{-j2\pi u5/8} + f(6)e^{-j2\pi u6/8} + f(7)e^{-j2\pi u7/8}$$

$$= \left[f(0)e^{-j2\pi u0/8} + f(2)e^{-j2\pi u2/8} + f(4)e^{-j2\pi u4/8} + f(6)e^{-j2\pi u6/8}\right]$$
$$+ \left[f(1)e^{-j2\pi u1/8} + f(3)e^{-j2\pi u3/8} + f(5)e^{-j2\pi u5/8} + f(7)e^{-j2\pi u7/8}\right]$$

$$= \left[\underline{f(0)e^{-j2\pi u0/8} + f(2)e^{-j2\pi u2/8} + f(4)e^{-j2\pi u4/8} + f(6)e^{-j2\pi u6/8}}\right]$$

$F_{even}(u)$, **4 point transform**

$$+ e^{-j2\pi u/8}\left[\underline{f(1)e^{-j2\pi u0/8} + f(3)e^{-j2\pi u2/8} + f(5)e^{-j2\pi u4/8} + f(7)e^{-j2\pi u6/8}}\right]$$

$F_{odd}(u)$, **4 point transform**

◇ The Fast Fourier Transform(10)

- The second division of $F_{even}(u)$ and $F_{odd}(u)$

$$8F(u)=\Big[\underbrace{\Big(f(0)e^{-j2\pi u0/8}+f(4)e^{-j2\pi u4/8}\Big)}_{\boxed{F_{even}(u)'}\ \text{2 point transform}}+e^{-j2\pi u2/8}\underbrace{\Big(f(2)e^{-j2\pi u0/8}+f(6)e^{-j2\pi u4/8}\Big)}_{\boxed{F_{odd}(u)'}\ \text{2 point transform}}\Big]$$

$\boxed{F_{even}(u)}$ **4 point transform**

$$+e^{-j2\pi u/8}\Big[\underbrace{\Big(f(1)e^{-j2\pi u0/8}+f(5)e^{-j2\pi u4/8}\Big)}_{\boxed{F_{even}(u)'}\ \text{2 point transform}}+e^{-j2\pi u2/8}\underbrace{\Big(f(3)e^{-j2\pi u0/8}+f(7)e^{-j2\pi u4/8}\Big)}_{\boxed{F_{odd}(u)'}\ \text{2 point transform}}\Big]$$

$\boxed{F_{odd}(u)}$ **4 point transform**

$$=\Big[\Big(f(0)+e^{-j\pi u}\bullet f(4)\Big)+e^{-j(\pi/2)u}\bullet\Big(f(2)+e^{-j\pi u}\bullet f(6)\Big)\Big]$$
$$+e^{-j(\pi/4)u}\bullet\Big[\Big(f(1)+e^{-j\pi u}\bullet f(5)\Big)+e^{-j(\pi/2)u}\bullet\Big(f(3)+e^{-j\pi u}\bullet f(7)\Big)\Big]$$

$$8F(u)=\Big[\Big(f(0)+e^{-j\pi u}\bullet f(4)\Big)+e^{-j(\pi/2)u}\bullet\Big(f(2)+e^{-j\pi u}\bullet f(6)\Big)\Big]\quad for\ u=0,1,2,3$$
$$+e^{-j(\pi/4)u}\bullet\Big[\Big(f(1)+e^{-j\pi u}\bullet f(5)\Big)+e^{-j(\pi/2)u}\bullet\Big(f(3)+e^{-j\pi u}\bullet f(7)\Big)\Big]$$

First half

$$8F(u+4)=\Big[\Big(f(0)+e^{-j\pi u}f(4)\Big)+e^{-j(\pi/2)u}\Big(f(2)+e^{-j\pi u}f(6)\Big)\Big]\quad for\ u+4=4,5,6,7$$
$$-e^{-j(\pi/4)u}\Big[\Big(f(1)+e^{-j\pi u}f(5)\Big)+e^{-j(\pi/2)u}\Big(f(3)+e^{-j\pi u}f(7)\Big)\Big]$$

Second half

◇ The Fast Fourier Transform(11)

- The frequency spectrum

$$8F(0)=[(f(0)+1\bullet f(4))+1\bullet(f(2)+1\bullet f(6))]+1\bullet[(f(1)+1\bullet f(5))+1\bullet(f(3)+1\bullet f(7))]$$
$$8F(1)=[(f(0)-1f(4))-j\bullet(f(2)-1f(6))]+(1-j)/\sqrt{2}\bullet[(f(1)-1f(5))-j\bullet(f(3)-1f(7))]$$
$$8F(2)=[(f(0)+1f(4))-1(f(2)+1f(6))]-j\bullet[(f(1)+1f(5))-1(f(3)+1f(7))]$$
$$8F(3)=[(f(0)-1f(4))+j(f(2)-1f(6))]-(1+j)/\sqrt{2}\bullet[(f(1)-1f(5))+j(f(3)-1f(7))]$$
$$8F(4)=[(f(0)+1f(4))+1(f(2)+1f(6))]-1[(f(1)+1f(5))+1(f(3)+1f(7))]$$
$$8F(5)=[(f(0)-1f(4))-j(f(2)-1f(6))]-(1-j)/\sqrt{2}[(f(1)-1f(5))-j(f(3)-1f(7))]$$
$$8F(6)=[(f(0)+1f(4))-1(f(2)+1f(6))]+j[(f(1)+1f(5))-1(f(3)+1f(7))]$$
$$8F(7)=[(f(0)-1f(4))+j(f(2)-1f(6))]+(1+j)/\sqrt{2}[(f(1)-1f(5))+j(f(3)-1f(7))]$$

$$8F(0)=[(f(0)+1\bullet f(4))+1\bullet(f(2)+1\bullet f(6))]+1\bullet[(f(1)+1\bullet f(5))+1\bullet(f(3)+1\bullet f(7))]$$
$$8F(1)=[(f(0)-1f(4))-j\bullet(f(2)-1f(6))]+(1-j)/\sqrt{2}\bullet[(f(1)-1f(5))-j\bullet(f(3)-1f(7))]$$
$$8F(2)=[(f(0)+1f(4))-1(f(2)+1f(6))]-j\bullet[(f(1)+1f(5))-1(f(3)+1f(7))]$$
$$8F(3)=[(f(0)-1f(4))+j(f(2)-1f(6))]-(1+j)/\sqrt{2}\bullet[(f(1)-1f(5))+j(f(3)-1f(7))]$$

First half

1 sub.

$$8F(4)=[(f(0)+1f(4))+1(f(2)+1f(6))]-1[(f(1)+1f(5))+1(f(3)+1f(7))]$$
$$8F(5)=[(f(0)-1f(4))-j(f(2)-1f(6))]-(1-j)/\sqrt{2}[(f(1)-1f(5))-j(f(3)-1f(7))]$$
$$8F(6)=[(f(0)+1f(4))-1(f(2)+1f(6))]+j[(f(1)+1f(5))-1(f(3)+1f(7))]$$
$$8F(7)=[(f(0)-1f(4))+j(f(2)-1f(6))]+(1+j)/\sqrt{2}[(f(1)-1f(5))+j(f(3)-1f(7))]$$

Second half

◇ The Fast Fourier Transform(12)

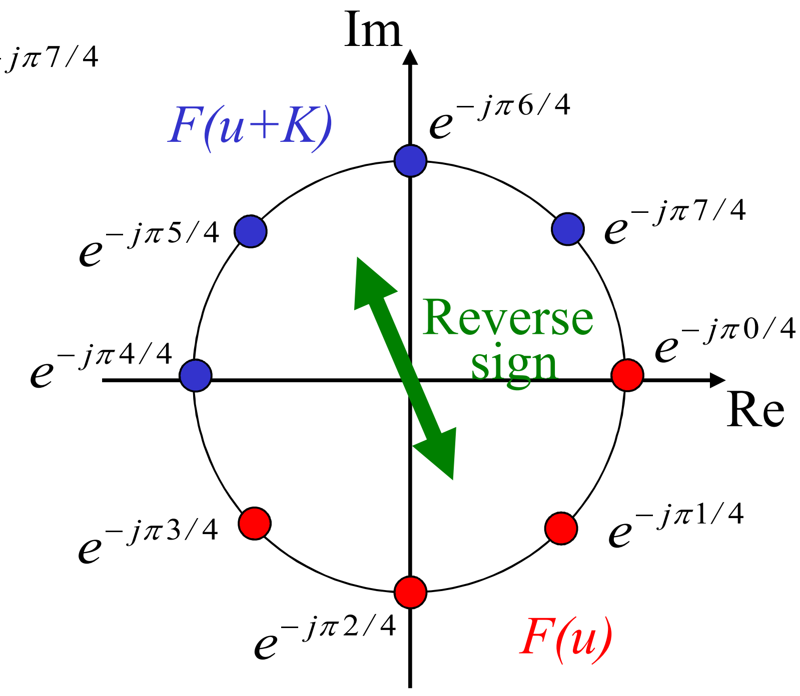
$$W_{2K}^{u} = (e^{-j2\pi/2K})^{u} = e^{-j\pi u/K},\ u = 0, 1, 2, 3$$

$$W_{2K}^{u+K} = (e^{-j\pi/K})^{u+K} = -(e^{-j\pi/K})^{u} = -W_{2K}^{u},\ u + 4 = 5, 6, 7, 8$$

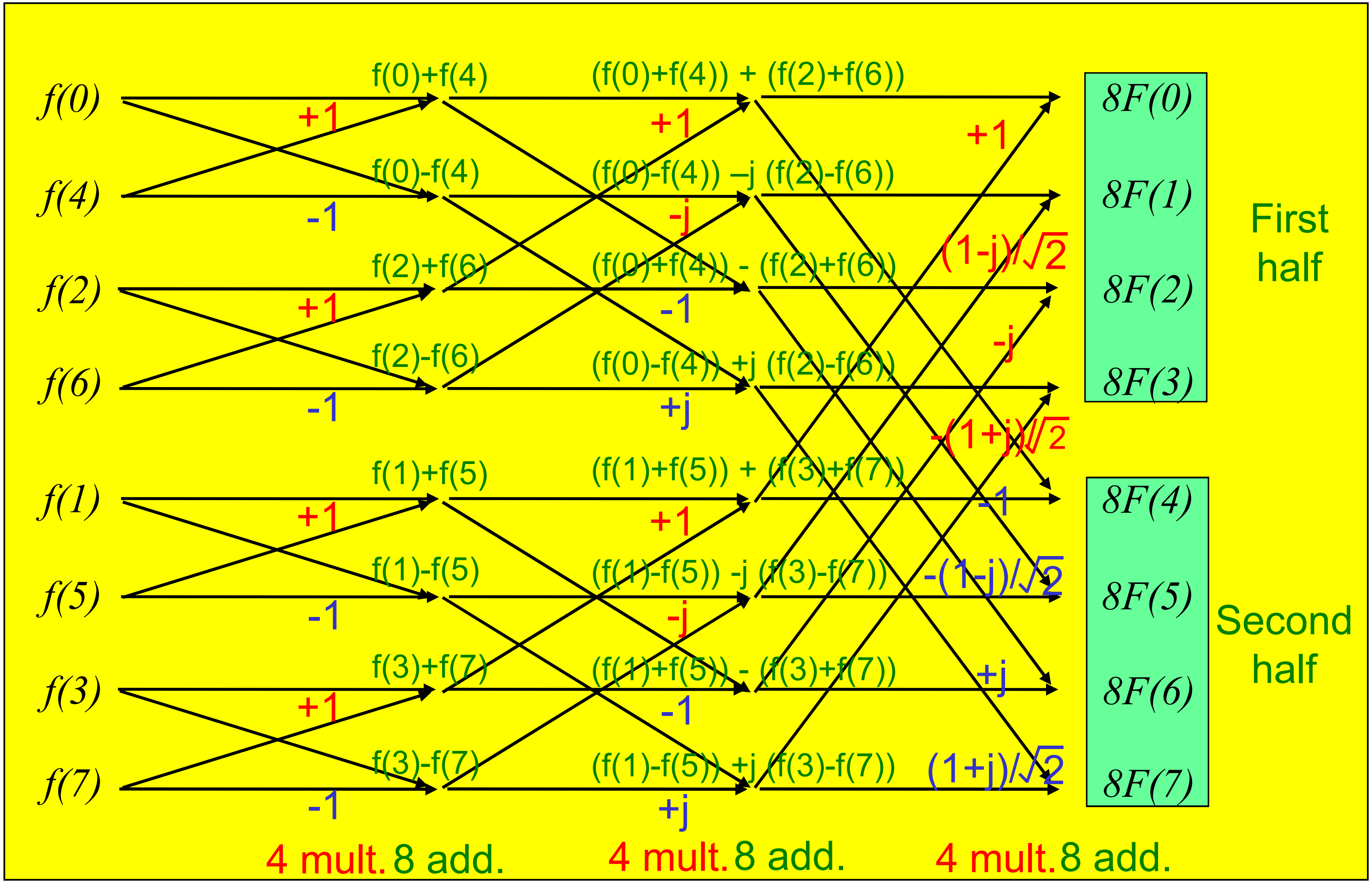
- M=8, K=4

$$W_{2K}^{u} : e^{-j\pi 0/4}, e^{-j\pi 1/4}, e^{-j\pi 2/4}, e^{-j\pi 3/4}$$

$$W_{2K}^{u+K} : e^{-j\pi 4/4}, e^{-j\pi 5/4}, e^{-j\pi 6/4}, e^{-j\pi 7/4}$$



◇ The Fast Fourier Transform(13)



◇ The Fast Fourier Transform(14)

- In case of M=8, ordering of the input sample, *f(x)*

- f
  - f(0),f(2),f(4),f(6)
    - f(0),f(4)
    - f(2),f(6)
  - f(1),f(3),f(5),f(7)
    - f(1),f(5)
    - f(3),f(7)

- In case of M=16 ,

- f
  - f(0),f(2),f(4),f(6),f(8),f(10),f(12),f(14)
    - f(0),f(4),f(8),f(12)
      - f(0),f(8)
      - f(4),f(12)
    - f(2),f(6),f(10),f(14)
      - f(2),f(10)
      - f(6),f(14)
  - f(1),f(3),f(5),f(7),f(9),f(11),f(13),f(15)
    - f(1),f(5),f(9),f(13)
      - f(1),f(9)
      - f(5),f(13)
    - f(3),f(7),f(11),f(15)
      - f(3),f(11)
      - f(7),f(15)

- How? Use **bit reversal**.

◇ The Fast Fourier Transform(15)

- **Ordering of Input sample : Bit reversal example**

Original index	bit pattern	bit reversed	new index
0	000	000	0
1	001	100	4
2	010	010	2
3	011	110	6
4	100	001	1
5	101	101	5
6	110	011	3
7	111	111	7

◇ The Fast Fourier Transform(16)

- FFT algorithm in $M = 2^n = 2K$

$$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x)e^{-j2\pi ux/M} = \frac{1}{M}\sum_{x=0}^{M-1} f(x)W_M^{ux} \qquad u = 0,\ 1,\ \ldots,\ M\text{-}1$$

$$F(u) = \frac{1}{2}\left[F_{even}(u) + W_{2K}^{u} F_{odd}(u)\right] \qquad for\ \ u = 0, 1,\ \ldots, K-1$$

$$F(u+K) = \frac{1}{2}\left[F_{even}(u) - W_{2K}^{u} F_{odd}(u)\right] \qquad for\ \ u+K = K, K+1,\ \ldots,\ M-1$$

- Two-point transform, $M=2,\ n=1,\ K=1,$

$$2F(u) = \sum_{x=0}^{2-1} f(x)e^{-j2\pi ux/2} = f(0)W_2^{u0} + f(1)W_2^{u1} \qquad u = 0,\ 1$$

$$2F(u) = F_{even}(u) + W_2^{u} \bullet F_{odd}(u) = f(0) + W_2^{u} \bullet f(1) \qquad for\ \ u = 0$$

$$2F(u+1) = F_{even}(u) - W_2^{u} F_{odd}(u) = f(0) - W_2^{u} f(1) \qquad for\ u+1 = 1$$

- In computation of *F(u)*, one multiplication of $F_{odd}(u)$ by W_2^u and one addition are required
- In computation of *F(u+1)*, one addition are required from *F(u)*
- Assume that *m(n)* is # of multiplication and *a(n)* is # of addition *m(1) = 1, a(1) = 2*

◇ The Fast Fourier Transform(17)

- 4-point transform, *M=4, n=2, K=2,*

$$4F(u) = \sum_{x=0}^{4-1} f(x) e^{-j2\pi ux/4} \qquad u = 0,\ 1,\ 2, 3$$

$$= \underbrace{\left[f(0) + f(2) \bullet e^{-j\pi u} \right]}_{F_{even}(u)\text{, 2 point transform}} + e^{-j\pi u/2} \bullet \underbrace{\left[f(1) + f(3) \bullet e^{-j\pi u} \right]}_{F_{odd}(u)\text{, 2 point transform}}$$

- The frequency spectrum

$$4F(0) = \left[f(0) + 1 \bullet f(2) \right] + 1 \bullet \left[f(1) + 1 \bullet f(3) \right]$$

$$4F(1) = \left[f(0) - 1 f(2) \right] - j \bullet \left[f(1) - 1 f(3) \right]$$

$$4F(2) = \left[f(0) + 1 f(2) \right] - 1 \left[f(1) + 1 f(3) \right]$$

$$4F(3) = \left[f(0) - 1 f(2) \right] + j \left[f(1) - 1 f(3) \right]$$

First half

1 subtraction

Second half

- The numbers of computation

m(2) = 2 m(1) + 2 = 4
a(2) = 2 a(1) + 4 = 8

◇ The Fast Fourier Transform(18)

- 8-point transform, $M=8,\ n=3,\ K=4,$

$$8F(u)=\sum_{x=0}^{8-1} f(x)e^{-j2\pi ux/8} \qquad u=0,1,2,3,4,5,6,7$$

$$=\left[\left(f(0)+1\bullet f(4)e^{-j\pi u}\right)+e^{-j(\pi/2)u}\bullet\left(f(2)+1\bullet f(6)e^{-j\pi u}\right)\right] \rightarrow F_{even}(u)\ \textbf{4 point transform}$$

$$+e^{-j(\pi/4)u}\bullet\left[\left(f(1)+1\bullet f(5)e^{-j\pi u}\right)+e^{-j(\pi/2)u}\bullet\left(f(3)+1\bullet f(7)e^{-j\pi u}\right)\right] \rightarrow F_{odd}(u)\ \textbf{4 point transform}$$

- The frequency spectrum

First half

$$8F(0)=\left[\left(f(0)+1\bullet f(4)\right)+1\bullet\left(f(2)+1\bullet f(6)\right)\right]+1\bullet\left[\left(f(1)+1\bullet f(5)\right)+1\bullet\left(f(3)+1\bullet f(7)\right)\right]$$

$$8F(1)=\left[\left(f(0)-1f(4)\right)-j\bullet\left(f(2)-1f(6)\right)\right]+(1-j)/\sqrt{2}\bullet\left[\left(f(1)-1f(5)\right)-j\bullet\left(f(3)-1f(7)\right)\right]$$

$$8F(2)=\left[\left(f(0)+1f(4)\right)-1\left(f(2)+1f(6)\right)\right]-j\bullet\left[\left(f(1)+1f(5)\right)-1\left(f(3)+1f(7)\right)\right]$$

$$8F(3)=\left[\left(f(0)-1f(4)\right)+j\left(f(2)-1f(6)\right)\right]-(1+j)/\sqrt{2}\bullet\left[\left(f(1)-1f(5)\right)+j\left(f(3)-1f(7)\right)\right]$$

1 sub.

$$8F(4)=\left[\left(f(0)+1f(4)\right)+1\left(f(2)+1f(6)\right)\right]-1\left[\left(f(1)+1f(5)\right)+1\left(f(3)+1f(7)\right)\right]$$

$$8F(5)=\left[\left(f(0)-1f(4)\right)-j\left(f(2)-1f(6)\right)\right]-(1-j)/\sqrt{2}\left[\left(f(1)-1f(5)\right)-j\left(f(3)-1f(7)\right)\right]$$

$$8F(6)=\left[\left(f(0)+1f(4)\right)-1\left(f(2)+1f(6)\right)\right]+j\left[\left(f(1)+1f(5)\right)-1\left(f(3)+1f(7)\right)\right]$$

$$8F(7)=\left[\left(f(0)-1f(4)\right)+j\left(f(2)-1f(6)\right)\right]+(1+j)/\sqrt{2}\left[\left(f(1)-1f(5)\right)+j\left(f(3)-1f(7)\right)\right]$$

Second half

- The numbers of computation

m(3) = 2 m(2) + 4 = 12
a(3) = 2 a(2) + 8 = 24

◇ The Fast Fourier Transform(19)

• The number of multiplications and additions required in FFT

- M = 2, n=1

$$m(1) = 1 = 2^0 \bullet 1$$

$$a(1) = 2 = 2^1 \bullet 1$$

- M = 4, n=2

$$m(2) = 2m(1) + 2^1 = 2^1 + 2^1 = 2^1 \bullet 2 = 4$$

$$a(2) = 2a(1) + 2^2 = 2^2 + 2^2 = 2^2 \bullet 2 = 8$$

- M = 8, n=3

$$m(3) = 2m(2) + 2^2 = 2^2 \bullet 2 + 2^2 = 2^2 \bullet 3 = 12$$

$$a(3) = 2a(2) + 2^3 = 2^3 \bullet 2 + 2^3 = 2^3 \bullet 3 = 24$$

◇ The Fast Fourier Transform(20)

- M = 16, n=4

$$m(4) = 2m(3) + 2^3 = 2^3 \bullet 3 + 2^3 = 2^3 \bullet 4$$

$$a(4) = 2a(3) + 2^4 = 2^4 \bullet 3 + 2^4 = 2^4 \bullet 4$$

- Generally, M = M, n=n

$$m(n) = 2m(n-1) + 2^{n-1} = 2^{n-1} \bullet (n-1) + 2^{n-1} = 2^{n-1} \bullet n$$

$$a(n) = 2a(n-1) + 2^n = 2^n \bullet (n-1) + 2^n = 2^n \bullet n$$

$$\therefore\ m(n) = \frac{1}{2} Mn = \frac{1}{2} M \log_2 M$$

$$a(n) = Mn = M \log_2 M$$

◇ The Fast Fourier Transform(21)

- The computational advantage of the FFT over DFT

  $C(M) = M^2/(M\log_2 M) = M/\log_2 M$

- Assuming that $M = 2^n$

  $C(n) = 2^n/n$

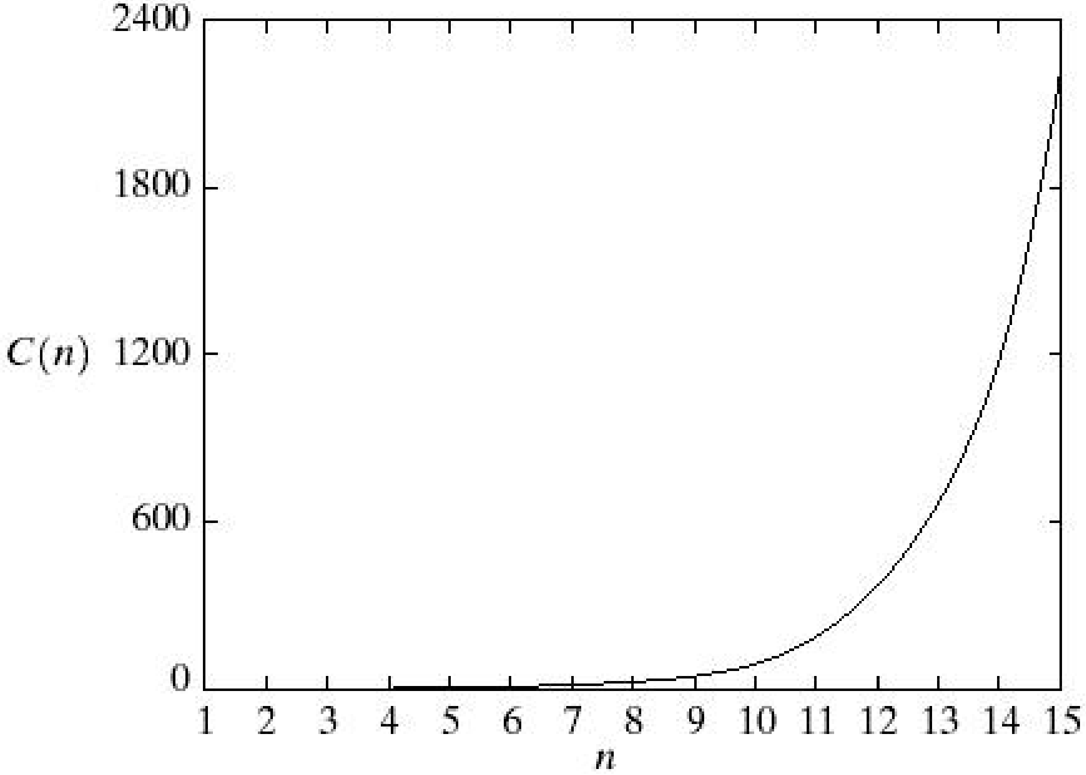


FIGURE 4.42 Computational advantage of the FFT over a direct implementation of the 1-D DFT. Note that the advantage increases rapidly as a function of n.

Homework

- 배포한 숙제 샘플에 다음의 기능을 추가한 프로그램을 제출하여라
- 4.4.4 Laplacian filter(P.185)를 프로그램으로 구현하여라
  - filter의 전달함수는 $H(u,v)= -(u^2+v^2)$ *scale이다.
  - Fig. 4.28(a)를 사용하여, (b), (c), (d)의 기능을 갖는 메뉴를 구성하여라
  - Fig. 4.28(b)의 메뉴에서는 scale = 20.0 / (256.0 * 256.0)을 사용
  - Fig. 4.28(c)의 메뉴에서는 scale = 40.0 / (256.0 * 256.0)을 사용 필터링된 이미지에 128을 더한 이미지이다.
  - Fig. 4.28(d)의 메뉴에서는 scale = 20.0 / (256.0 * 256.0)을 사용
- 다른 사람의 프로그램을 복사한 경우는, 보여준 사람과 복사한 사람 모두 숙제를 제출하지 않은 것으로 간주한다.
- 실행 가능한 프로그램을 제출하여라.