ELECENG 3FK4

Assignment 2

Set 4 - Maxwell's Equations

Authored By:

Ahmad Choudhry - chouda27 - 400312026

Instructor:

Dr. Shiva Kumar

McMaster University

Analytical Solution

Given:

- Permeability: $\mu = 1.0 \times 10^{-5}\,\mathrm{H/m}$
- Permittivity: $\epsilon = 1.0 \times 10^{-10}\,\mathrm{F/m}$
- Conductivity: $\sigma = 0$
- Electric Field: $\mathbf{E} = 1000\cos\left(10^{10}t - \beta x\right)\hat{a}_y\,\mathrm{V/m}$

Objective: Use Maxwell's equations to find expressions for β and $\mathbf{B}$.

Step 1: Use Maxwell's Equations

We'll follow the methodology used in the sample solution, starting with Maxwell's equations in differential form.

Maxwell's Equations (in the absence of free charges and currents):

1. **Faraday's Law:**
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
2. **Ampère's Law (with Maxwell's addition):**
$$\nabla \times \mathbf{H} = \epsilon\frac{\partial \mathbf{E}}{\partial t}$$
(Since $\mathbf{J} = \sigma\mathbf{E} = 0$ because $\sigma = 0$)
3. **Constitutive Relations:**
$$\mathbf{B} = \mu\mathbf{H}$$
$$\mathbf{D} = \epsilon\mathbf{E}$$

Step 2: Calculate $\nabla \times \mathbf{E}$

Given that $\mathbf{E}$ has only a y-component:

$$\mathbf{E} = E_y\hat{a}_y = 1000\cos(10^{10}t - \beta x)\hat{a}_y$$

Compute the curl of E:

$$\nabla \times \mathbf{E} = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\hat{a}_x + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\hat{a}_y + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\hat{a}_z$$

Since $E_x = E_z = 0$, the curl simplifies to:

$$\nabla \times \mathbf{E} = \left(\frac{\partial E_y}{\partial x} \right) \hat{a}_z$$

Compute $\frac{\partial E_y}{\partial x}$:

$$\frac{\partial E_y}{\partial x} = -1000\beta \sin\left(10^{10}t - \beta x\right)$$

Therefore:

$$\nabla \times \mathbf{E} = -1000\beta \sin\left(10^{10}t - \beta x\right) \hat{a}_z$$

Step 3: Use Faraday's Law to Find B

According to Faraday's Law:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Substitute the expression for $\nabla \times \mathbf{E}$:

$$-1000\beta \sin\left(10^{10}t - \beta x\right) \hat{a}_z = -\frac{\partial \mathbf{B}}{\partial t}$$

Simplify:

$$\frac{\partial \mathbf{B}}{\partial t} = 1000\beta \sin\left(10^{10}t - \beta x\right) \hat{a}_z$$

Integrate both sides with respect to time t:

$$\mathbf{B} = \int \frac{\partial \mathbf{B}}{\partial t}\, dt = \int 1000\beta \sin\left(10^{10}t - \beta x\right)\, dt$$

Integrate the sine function:

$$\int \sin\left(10^{10}t - \beta x\right)\, dt = -\frac{1}{10^{10}} \cos\left(10^{10}t - \beta x\right)$$

Therefore:

$$\mathbf{B} = -\frac{1000\beta}{10^{10}} \cos\left(10^{10}t - \beta x\right) \hat{a}_z + \mathbf{C}(x)$$

Note: $\mathbf{C}(x)$ is a function of x only. However, since the magnetic field should not have a static component (as there are no static currents or charges), we can set $\mathbf{C}(x) = 0$.

Simplify the expression:

$$\mathbf{B} = -\frac{1000\beta}{10^{10}} \cos\left(10^{10}t - \beta x\right) \hat{a}_z$$

Step 4: Use the Constitutive Relation $\mathbf{B} = \mu \mathbf{H}$

Compute the magnetic field intensity $\mathbf{H}$:

$$\mathbf{H} = \frac{\mathbf{B}}{\mu} = -\frac{1000\beta}{10^{10}\mu} \cos\left(10^{10}t - \beta x\right) \hat{a}_z$$

Step 5: Compute $\nabla \times \mathbf{H}$

Since $\mathbf{H}$ has only a z-component:

$$\mathbf{H} = H_z \hat{a}_z = -\frac{1000\beta}{10^{10}\mu} \cos\left(10^{10}t - \beta x\right) \hat{a}_z$$

Compute the curl:

$$\nabla \times \mathbf{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) \hat{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right) \hat{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) \hat{a}_z$$

Again, since $H_x = H_y = 0$, the non-zero component is:

$$(\nabla \times \mathbf{H})_y = -\frac{\partial H_z}{\partial x}$$

Compute $\frac{\partial H_z}{\partial x}$:

$$\frac{\partial H_z}{\partial x} = \frac{1000\beta^2}{10^{10}\mu} \sin\left(10^{10}t - \beta x\right)$$

Therefore:

$$(\nabla \times \mathbf{H})_y = -\frac{1000\beta^2}{10^{10}\mu} \sin\left(10^{10}t - \beta x\right)$$

$$\nabla \times \mathbf{H} = -\frac{1000\beta^2}{10^{10}\mu} \sin\left(10^{10}t - \beta x\right) \hat{a}_y$$

Step 6: Use Ampère's Law to Find β

According to Ampère's Law:

$$\nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t}$$

Compute $\frac{\partial \mathbf{E}}{\partial t}$:

$$\frac{\partial \mathbf{E}}{\partial t} = -1000 \times 10^{10} \sin\left(10^{10}t - \beta x\right) \hat{a}_y$$

Multiply by ϵ:

$$\epsilon\frac{\partial \mathbf{E}}{\partial t} = -1000\epsilon \times 10^{10} \sin\left(10^{10}t - \beta x\right) \hat{a}_y$$

Set the expressions equal:

$$-\frac{1000\beta^2}{10^{10}\mu} \sin\left(10^{10}t - \beta x\right) = -1000\epsilon \times 10^{10} \sin\left(10^{10}t - \beta x\right)$$

Simplify:

$$\frac{1000\beta^2}{10^{10}\mu} = 1000\epsilon \times 10^{10}$$

Divide both sides by 1000:

$$\frac{\beta^2}{10^{10}\mu} = \epsilon \times 10^{10}$$

Multiply both sides by $10^{10}\mu$:

$$\beta^2 = \epsilon\mu\left(10^{10}\right)^2$$

Therefore:

$$\beta = 10^{10}\sqrt{\mu\epsilon}$$

Step 7: Compute β

Given $\mu = 1.0 \times 10^{-5}\,\mathrm{H/m}$ and $\epsilon = 1.0 \times 10^{-10}\,\mathrm{F/m}$:

Compute $\sqrt{\mu\epsilon}$:

$$\sqrt{\mu\epsilon} = \sqrt{(1.0 \times 10^{-5})(1.0 \times 10^{-10})} = \sqrt{1.0 \times 10^{-15}} = 1.0 \times 10^{-7.5}$$

Note that $10^{-7.5} = 10^{-7} \times 10^{-0.5} = (1 \times 10^{-7}) \times 0.316227766 = 3.16227766 \times 10^{-8}$
So:

$$\sqrt{\mu\epsilon} = 3.16227766 \times 10^{-8}$$

Now compute β:

$$\beta = 10^{10} \times 3.16227766 \times 10^{-8} = 316.227766\,\mathrm{rad/m}$$

Step 8: Final Expression for B

Substitute β back into the expression for $\mathbf{B}$:

$$\mathbf{B} = -\frac{1000\beta}{10^{10}} \cos\left(10^{10}t - \beta x\right) \hat{a}_z$$

Compute the coefficient:

$$\frac{1000\beta}{10^{10}} = \frac{1000 \times 316.227766}{10^{10}} = \frac{316227.766}{10^{10}} = 3.16227766 \times 10^{-5}$$

Therefore:

$$\mathbf{B} = -3.16227766 \times 10^{-5} \cos\left(10^{10}t - 316.227766x\right) \hat{a}_z \, \mathrm{T}$$

Verification and Alignment with Sample Solution

In the sample solution, the steps involved:

1. Using Maxwell's equations to relate the curl of the magnetic field to the time derivative of the electric field.
2. Calculating the curl of the given field.
3. Integrating over time to find the displacement field **D** or **B**.
4. Using constitutive relations to find **E** or **H**.
5. Solving for β by equating expressions from different Maxwell's equations.

We followed similar steps:

- **Calculated** $\nabla \times \mathbf{E}$ to find **B** via Faraday's Law.
- **Used Ampère's Law** to relate $\nabla \times \mathbf{H}$ to $\frac{\partial \mathbf{E}}{\partial t}$, allowing us to solve for β.
- **Applied constitutive relations** to relate **B** and **H**.
- **Integrated over time** where necessary.

Final Expressions

Wave Number β:

$$\beta = 316.227766 \, \mathrm{rad/m}$$

Magnetic Field B:

$$\mathbf{B} = -3.16227766 \times 10^{-5} \cos\left(10^{10}t - 316.227766x\right) \hat{a}_z \, \mathrm{T}$$

MATLAB Simulation

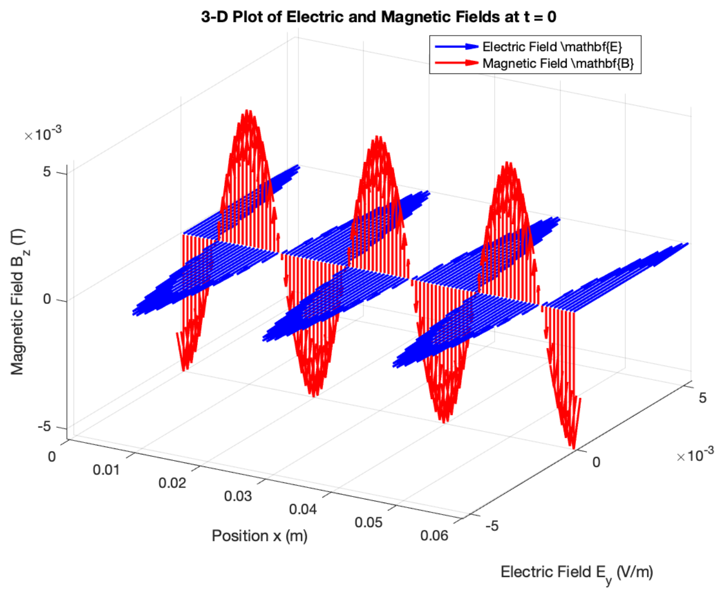

Figure 1

Listing 1: Assign2.m MATLAB Code

```
% Parameters
beta = 316.227766; % rad/m
E0 = 1000; % V/m
B0 = 3.16227766e-5; % T

% Wavelength
lambda = 2 * pi / beta;

% Spatial range (3 wavelengths)
x = linspace(0, 3 * lambda, 100); % From 0 to 3 wavelengths

% Electric field components at t = 0
E_x = zeros(size(x)); % E has no x-component
E_y = E0 * cos(beta * x); % V/m (along y-axis)
E_z = zeros(size(x)); % E has no z-component

% Magnetic field components at t = 0
```

```
B_x = zeros(size(x)); % B has no x-component
B_y = zeros(size(x)); % B has no y-component
B_z = -B0 * cos(beta * x); % T (along z-axis)

% Create a new figure
figure;

% Plot E field vectors
quiver3(x, E_x, E_z, zeros(size(x)), E_y, zeros(size(x)), 'b', 'LineWidth
hold on;

% Plot B field vectors
quiver3(x, B_x, B_z, zeros(size(x)), zeros(size(x)), B_z, 'r', 'LineWidth

% Labels and title
xlabel('Position x (m)');
ylabel('Electric Field E_y (V/m)');
zlabel('Magnetic Field B_z (T)');
title('3-D Plot of Electric and Magnetic Fields at t = 0');

% Set view angle for better visualization
view(30, 30);

% Add legend
legend({'Electric Field \mathbf{E}', 'Magnetic Field \mathbf{B}'}, 'Locat

% Grid for better visualization
grid on;

% Adjust axes limits for clarity
axis tight;

saveas(gcf, 'graph.png'); % Save the figure as a PNG file
```