题 1.

对于生成函数 $G(x)$，它的三阶累积量为 $\kappa = \alpha_3 + 3\alpha_2 + \alpha_1 - 3\alpha_2\alpha_1 - 3\alpha_1^2 + 2\alpha_1^3$，其中 $\alpha_m = G^{(m)}(1)$

由题意，令 $G(z) = H(z)/F(z)$，由于 $G(1) = F(1) = 1$，我们有 $\kappa_i(H) = \kappa_i(G) + \kappa_i(F), i \geq 1$

对于 $H(z) = z^3$

$$\begin{cases} H'(1) & = 3 \\ H''(1) & = 6 \\ H'''(1) & = 6 \end{cases} \tag{1}$$

带入三阶累积量公式可得 $\kappa_3(H) = 0$

对于 $F(z) = z^3 - 8(z-1)$，我们有

$$\begin{cases} F'(1) & = -5 \\ F''(1) & = 6 \\ F'''(1) & = 6 \end{cases} \tag{2}$$

带入公式得 $\kappa_3(F) = -216$，而 $\kappa_3(G) = \kappa_3(H) - \kappa_3(F) = 216$

题 2.

由题意，设 N 是所有不以 HTHTH 结束的模式，S 是所有以 HTHTH 结束的模式，则我们有

$$\begin{cases} 1 + N(\mathrm{H} + \mathrm{T}) = N + S \\ N\mathrm{HTHTH} = S(1 + \mathrm{TH} + \mathrm{THTH}) \end{cases} \tag{3}$$

解得

$$G(z) = \frac{p^3(1-p)^2z^5}{p^3(1-p)^2z^5 + (1-z)(1 + p(1-p)z^2 + p^2(1-p)^2z^4)} \tag{4}$$

令 $G(z) = H(z)/F(z)$，其中 $H(X) = z^5$，则

$$\begin{cases} H'(1) = 5 \\ H''(1) = 20 \\ F'(1) = 5 - \dfrac{1 + p(1-p) + p^2(1-p)^2}{p^3(1-p)^2} \\ F''(1) = 20 - 4\dfrac{p(1-p) + 2p^2(1-p)^2}{p^3(1-p)^2} \end{cases} \tag{5}$$

$$\Longrightarrow \begin{cases} \mathrm{Mean}(G) = \dfrac{1}{p^3(1-p)^2} + \dfrac{1}{p^2(1-p)} + \dfrac{1}{p} \\ \mathrm{Var}(G) = (\mathrm{Mean}(G))^2 - \dfrac{9}{p^3(1-p)^2} - \dfrac{5}{p^2(1-p)} - \dfrac{1}{p} \end{cases} \tag{6}$$

题 3.

由题意

$$
\begin{aligned}
&\left(n+2+\frac{3}{n+1}\right)^n \\
=&\left(\frac{n}{n+1}\right)^n\left(n+2+\frac{5}{n}\right)^n \\
=&\frac{n^{2n}}{(n+1)^n}\exp\left\{n\ln\left(1+\frac{3}{n}+\frac{5}{n^2}\right)\right\} \\
=&\frac{n^{2n}}{(n+1)^n}\exp\left\{n\left(\frac{3}{n}+\frac{5}{n^2}-\frac{1}{2}\frac{9}{n^2}+O\Big(n^{-2}\Big)\right)\right\} \\
=&\frac{n^{2n}}{(n+1)^n}\exp\left\{n\left(\frac{3}{n}+\frac{5}{n^2}-\frac{1}{2}\frac{9}{n^2}+O\Big(n^{-2}\Big)\right)\right\} \\
=&e^3\frac{n^{2n}}{(n+1)^n}\exp\Big\{\frac{1}{2n}+O(n^{-2})\Big\} \\
=&e^3\frac{n^{2n}}{(n+1)^n}\left(1+\frac{1}{2n}+O\left(n^{-2}\right)\right)
\end{aligned}
\tag{7}
$$

题 4.

由题意，$f(x)=1/(n^2+x^2)$

$$
\begin{aligned}
\sum_{k=1}^{n}\frac{1}{n^2+k^2}&=\sum_{1\le k<n}\frac{1}{n^2+k^2}+\frac{1}{2n^2} \\
&=\int_1^n\frac{1}{n^2+x^2}dx+\frac{1}{2n^2}+\sum_{k=1}^{m}\frac{B_k}{k!}f^{(k-1)}(x)\Big|_1^n+R_m \\
&=\frac{\pi}{4n}-\frac{1}{n}\arctan\left(1/n\right)-\frac{1}{2(n^2+x^2)}\Big|_1^n-\frac{x}{6(n^2+x^2)^2}\Big|_1^n+R_2 \\
&=\frac{\pi}{4n}-\frac{1}{n}\left(\frac{1}{n}-\frac{1}{3n^3}+O(\frac{1}{n^4})\right)-\frac{1}{4n^2}-\frac{1}{24n^3} \\
&+\frac{1}{2}(n^2+1)^{-1}+\frac{1}{6}(n^2+1)^{-2}+R_2 \\
&=\frac{\pi}{4n}-\frac{25}{12n^2}-\frac{1}{24n^3}+\frac{1}{n^4}+O\Big(\frac{1}{n^5}\Big)
\end{aligned}
\tag{8}
$$

题 5.

由于

$$
\ln\binom{2n}{n+k}=(2n+\frac{1}{2})\ln 2-\sigma-\frac{1}{2}\ln n-\frac{k^2}{n}+\mathcal{O}(n^{-\frac{1}{2}+3\varepsilon})
$$

从而

$$
\begin{aligned}
a_k(n)&=\binom{2n}{n+k}^3 \\
&=\frac{2^{3(2n+1/2)}}{e^{3\sigma}\sqrt{n^3}}e^{-3k^2/n}\Big(1+\mathcal{O}(n^{-1/2+3\varepsilon})\Big) \\
&=\frac{2^{3(2n+1/2)}}{(2\pi n)^{3/2}}e^{-3k^2/n}\Big(1+\mathcal{O}(n^{-1/2+3\varepsilon})\Big)
\end{aligned}
$$

从而

$$
\begin{aligned}
b_k(n) &= \frac{2^{3(2n+1/2)}}{(2\pi n)^{3/2}} e^{-3k^2/n} \\
c_k(n) &= 2^{6n} n^{-2+3\varepsilon} e^{-3k^2/n}
\end{aligned}
$$

$$
\begin{aligned}
\sum_k b_k(n) &= \frac{2^{3(2n+1/2)}}{(2\pi n)^{3/2}} \sum_k e^{-\frac{3}{n}k^2} \\
&= 2^{6n}(\pi n)^{-3/2} \cdot \sqrt{\pi \frac{n}{3}}(1+\mathcal{O}(n^{-M})) \\
&= 2^{6n}(\pi n)^{-1}\sqrt{3}(1+\mathcal{O}(n^{-M}))
\end{aligned}
$$

$$
\sum_c(n) = \sum_{|k|\le n^{1/2+\varepsilon}} 2^{6n} n^{-2+3\varepsilon} e^{-3k^2/n} \le 2^{6n} n^{-2+3\varepsilon}\Theta_{n/3} = \mathcal{O}(2^{6n} n^{-3/2+3\varepsilon}) \sum_{k>n^{1/2+\varepsilon}} e^{-3k^2/n} = \mathcal{O}(n^{-M})
$$

并且 $\sum_{k>n^{1/2+\varepsilon}} \binom{2n}{n+k}$ 足够小
从而我们有

$$
A_n = 2^{6n}(\pi n)^{-1}\sqrt{3}(1+\mathcal{O}(n^{-1/2+3\varepsilon}))
$$