题一

类比课本，有

$$r^{\underline{k}}\left(r-\frac{1}{3}\right)^{\underline{k}}\left(r-\frac{2}{3}\right)^{\underline{k}}=\frac{(3r)^{\underline{3k}}}{3^{3k}} \tag{1}$$

左边等于

$$\begin{aligned}&r\left(r-\frac{1}{3}\right)\left(r-\frac{2}{3}\right)\cdots(r-k+1)\left(r-k+\frac{1}{3}\right)\left(r-k+\frac{2}{3}\right)\\=&(k!)^3\binom{r}{k}\binom{r-\frac{1}{3}}{k}\binom{r-\frac{2}{3}}{k}\end{aligned} \tag{2}$$

右边等于

$$\begin{aligned}&\frac{3r(3r-1)\cdots(3r-3k+1)}{3\cdot 3\cdots 3}\\=&(k!)^3\binom{3r}{3k}\binom{3k}{2k}\binom{2k}{k}\frac{1}{3^{3k}}\end{aligned} \tag{3}$$

从而

$$\binom{r}{k}\binom{r-\frac{1}{3}}{k}\binom{r-\frac{2}{3}}{k}=\binom{3r}{3k}\binom{3k}{2k}\binom{2k}{k}\frac{1}{3^{3k}} \tag{4}$$

令 $r=k=n$，得

$$\binom{n-\frac{1}{3}}{n}\binom{n-\frac{2}{3}}{n}=\binom{3n}{2n}\binom{2n}{n}\frac{1}{3^{3k}} \tag{5}$$

对左边使用上指标反转得到

$$(-1)^n\binom{n-(n-\frac{1}{3})-1}{n}(-1)^n\binom{n-(n-\frac{2}{3})-1}{n}=\binom{-\frac{1}{3}}{n}\binom{-\frac{2}{3}}{n}=\binom{3n}{2n}\binom{2n}{n}\frac{1}{3^{3k}} \tag{6}$$

题二

对于

$$\sum_k t_k=\sum_k\binom{m}{n+k}\binom{n+k}{2k}4^k \tag{7}$$

有

$$\frac{t^{k+1}}{t^k}=\frac{(k+n-m)(k-n)(1)}{(k+\frac{1}{2})(k+1)} \tag{8}$$

所以

$$
\begin{aligned}
\sum_k t_k &= t_0 F(n-m,-n;\frac{1}{2};1)\\
&= \binom{m}{n} F(n-m,-n;\frac{1}{2};1)\\
&= \frac{m^{\underline{n}}}{n!}\frac{(m-\frac{1}{2})^{\underline{n}}}{(2n)!/n!}2^{2n}\\
&= \binom{m}{n}\binom{m-\frac{1}{2}}{n}/\binom{2n}{n}\cdot 2^{2n}\\
&= \binom{2m}{2n}
\end{aligned} \tag{9}
$$

所以

$$
\sum \frac{\delta k}{k^3-k} = \binom{2m}{2n} \tag{10}
$$

题三

(1)

令 $t(k)=1/(k^3-k)$，那么

$$
\frac{t(k+1)}{t(k)} = \frac{k-1}{k+2} = \frac{p(k+1)}{p(k)}\frac{q(k)}{r(k+1)} \tag{11}
$$

令 $p(k)=1, q(k)=k-1, r(k)=k+1$，满足 $(k+\alpha)\backslash q(k), (k+\beta)\backslash r(k) \Rightarrow \alpha-\beta$ 不是正整数

令 $Q(k)=q(k)-r(k)=-2, R(k)=q(k)+r(k)=2k$，所以有

$$
2 = Q(k)(s(k+1)+s(k)) + R(k)(s(k+1)-s(k)) \tag{12}
$$

因为 $d=\deg(p)-\deg(R)+1=0$，设 $s(k)=\alpha$，带入，解得 $s(k)=-\frac{1}{2}$，所以 $T(k)=r(k)s(k)t(k)/p(k)=(k+1)(-1/2)(1/(k^3-k))=-1/(2k(k-1))$，所以

$$
\sum \frac{\delta k}{k^3-k} = -\frac{1}{2k(k-1)} + C \tag{13}
$$

(2)

令 $t(k)=\binom{-3}{2k}2^k$

$$
\frac{t(k+1)}{t(k)} = \frac{k+2}{k+1}\frac{4k+6}{2k+1} = \frac{p(k+1)}{p(k)}\frac{q(k)}{r(k+1)} \tag{14}
$$

令 $p(k)=(2k+1)(k+1), q(k)=2, r(k)=1$，可得 $2s(k+1)-s(k)=(2k+1)(k+1), Q(k)=1, R(k)=3$，因为 $\deg(Q)=\deg(R)=0$，所以 $d=\deg(p)=2$，设 $s(k)=ak^2+bk+c$，带入解得 $s(k)=2k^2-5k+7$，所以

$$
\begin{aligned}
T(k) &= \frac{r(k)s(k)t(k)}{p(k)}\\
&= \frac{(-3)^{\underline{2k}}}{(2k+2)!}2^{k+1}(2k^2-5k+7)\\
&= 2^k(2k^2-5k+7)
\end{aligned} \tag{15}
$$

所以 $\sum \binom{-3}{2k}2^k\delta k = 2^k(2k^2-5k+7)$

题四

$$\begin{aligned}
\frac{t(n+1,k)}{t(n,k)} &= \frac{n+1}{n-2k+1} \\
p(n,k) &= \hat{p}(n,k) = (n-2k+1)\beta_0 + (n+1)\beta_1 \\
\hat{p}(n,k) &= q(n,k)s(n,k+1) - r(n,k)s(n,k) \\
\bar{t}(n,k) = \frac{t(n,k)}{n-2k+1} &= \frac{n!}{(2k)!(n-2k+1)!} \\
\frac{\bar{t}(n,k+1)}{\bar{t}(n,k)} &= \frac{(n-2k)(n-2k+1)}{(2k+1)(2k+2)} \\
r(n,k) &= (2k-1)2k \\
q(n,k) &= (n-2k)(n-2k+1)
\end{aligned} \tag{16}$$

$Q(k) = q(n,k) - r(n,k) = n^2 - (4k-1)n, R(k) = 4k(2k-1) - (4k-1)n + n^2$，因为 $\deg(Q) = 1 < \deg(R) = 2$，所以 $d = \deg(\hat{p}) - \deg(R) + 1 = 0$，设 $s(n,k) = \alpha$，带入可以得到 $((4k+3)n - n^2)\alpha = (n-2k+1)\beta_0 + (n+1)\beta_1$，比较系数可以得到

$$\begin{cases} -4n\alpha & = -2\beta_0 \\ (n^2+n)\alpha & = (n+1)(\beta_0+\beta_1) \end{cases} \tag{17}$$

解得 $\beta_0 = 2n, \beta_1 = -n, \alpha = 1$，所以 $\hat{t}(n,k) = 2nt(n,k) - nt(n+1,k)$ 是可求和的

$$T(n,k) = \frac{r(n,k)s(n,k)\hat{t}(n,k)}{\hat{p}(n,k)} = (2k-1)\binom{n}{2k-1} \tag{18}$$

所以有

$$\begin{aligned}
\hat{t}(n,k) &= T(n,k+1) - T(n,k) \\
\Leftrightarrow 2n(t,k) - nt(n+1,k) &= T(n,k+1) - T(n,k)
\end{aligned} \tag{19}$$

两边从 $0 \le k \le n+1$ 求和，得

$$\begin{aligned}
2nS_n + 2nt(n,n+1) - nS_{n+1} &= T(n,n+2) - T(n,0) \\
\Rightarrow 2nS_n + 2n\binom{n}{2n+2} - nS_{n+1} &= (2k+3)\binom{n}{2k+3} - (-1)\binom{n}{-1} \\
\Rightarrow &\begin{cases} S_{n+1} = 2S_n \ , n \ge 1 \\ S_1 = S_0 + 1 \ , n = 0 \end{cases}
\end{aligned} \tag{20}$$