A classical introduction to modern number theory

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A simple notes template. Inspired by Tufte-LaTeXclass and beautiful notes by

`https://github.com/abrandenberger/course-notes`

1 Groups

1.1 Laws of Composition

Problem 1. *Let $a, b \in S$, assume operation of S is associative, and its identity is e. If a is left inverse of b, does this imply that a is right inverse of b?*

Proof. Suppose b has left inverse a and right inverse c: $ab = e, bc = e$ but $a \neq c$. Then $ae = a = a(bc) = (ab)c = c$, which is a contradiction. □

1. If $la = e, ar = e$ (it imply that a has both left and right inverse), then $l = r$.
2. If a is invertible, its inverse is unique.
3. Inverse multipy in the opposite order: $(ab)^{-1} = b^{-1}a^{-1}$
4. An element a may have a left inverse or a right inverse, though it is not invertible.

The last statement is unique and interesting.

Lemma 1.1. *Every nonzero integer can be written as a product of primes.*

Consider how to prove this lemma.

Lemma 1.2. *If $a, b \in \mathbb{Z}$ and $b > 0$, there exist $q, r \in \mathbb{Z}$ such that $a = qb + r$ with $0 \leq r < b$.*

Easy to prove.

Definition 1.1. *Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis.*

We often see $(a, b) = d$, it means $(a, b) = (d)$ in fact.

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> **Definition 1.2.** *Here's is the beautiful Schrödinger equation*
>
> $$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x,t)\right]\Psi(x,t)$$

1.2 Groups and Subgroups

A group is a set G together with a law of composition that has the following properties:

1. associative, $(ab)c = a(bc)$ for all $a, b, c \in G$
2. identity element e, $ea = ae = a$ for all $a \in G$
3. for all $a \in G$, a has a inverse b, such that $ab = ba = 1$

An $abeliangroup$ is a group whose law of composition is commutative. For example, the set of nonzero real numbers forms an abelian group under multiplication, and the set of all real numbers forms a abelian group under addition.

> **Proposition 1.1 (Cancellation Law).** *Let a, b, c be elements of a group G whose law of composition is written multiplicatively. If $ab = ac$ or if $ba = ca$, then $b = c$. If $ab = a$ or if $ba = a$, then b=1.*

Proof. Multipy both sides of $ab = ac$ on the left by a^{-1} to obtain $b = c$. The other proofs are analogous. □

1. The $n \times n$ general linear group is the group of all invertible $n \times n$ matrices. It is denoted by $GL_n = n \times n$ invertiblematricesA. $GL_n(\mathbb{R}), GL_n(\mathbb{C})$ indicate matrices units are real or complex number. If all matrices of the group have determinant 1, then it's called the special linear group, it's a subgroup of GL_n, it's denoted by SL_n.
2. S_n is the group of permutations of $\{1, 2, \cdots, n\}$, sometimes it's called the symmetric group. The symmetric group S_n is a finite group of order $n!$.

The permutations of a set a, b of two elements are the identity and the transposition. It's a group of order two. Notice the difference between this set and S_2, especially definition of S_n.

Every group G has two obvious subgroups: the group G itself, and the trivial subgroup that consists of the identity element alone.

1.3 Subgroups of the Additive Group of Integers

Let a be an integer different from 0. We denote the subset of $\mathbb{Z}$ that consists of all multiples of a by $\mathbb{Z}a$:

$$\mathbb{Z}a = \{n \in Z \mid n = ka \text{ for some k in } \mathbb{Z}\}. \tag{1}$$

Theroem 1.1. *Let S be a subgroup of additive group $\mathbb{Z}^+$(or $(\mathbb{Z}, +)$). Either S is the trivial subgroup 0, or else it has the form $\mathbb{Z}a$, where a is the smallest positive integer in S.*

$Za \cap Zb = Zm, m = \operatorname{lcm}(a, b)$, and $Za + Zb = Za \cup Zb = Zn, n = \gcd(a, b)$

1.4 Cycle Groups

$\langle x \rangle$

1.5 Headings

Theroem 1.2. *Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris.*

Proof (Theorem 1.1). Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. □

Lemma 1.3. *Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris.*

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Corollary 1.1. *Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris.*

Proposition 1.2. *Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris.*

Problem 2. *Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis.*

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