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Problem 1.

(1)

```
class Solution {
public:
    void qsort(vector<int>&A, int k, int l, int r){
        if (l >= r) return ;
        int p = A[l];
        int i = l, j = r;
        for (; i < j;) {
            while (j > i && A[j] >= p)
                --j;
            swap(A[i], A[j]);
            while (j > i && A[i] <= p)
                ++i;
            swap(A[i], A[j]);
        }
        if (k <= i + 1)
            qsort(A, k, l, i - 1);
        else
            qsort(A, k, i + 1, r);
        return;
    }
    int findKthLargest(vector<int>& nums, int k) {
        qsort(nums, nums.size() - k + 1, 0, nums.size() - 1);
        return nums[nums.size() - k];
    }
};
```

(2) In the first partition, we need to scan all n elements, in the second partition, we only need to scan $n/2$ elements. Analogyly, $n/4, \cdots, 1$. So, in total we need to scan $n+n/2+\cdots+1 = 2n-1$ times. So time complexity is $O(n)$

Problem 2.

Assume $m \leq n$, so $\gcd(m,n) = \gcd(n, m \mod n)$.

The algorithm can not recur indefinitely, scince the second argument strictly decreases in each recursive call. Therefore, the algorithm always terminates with the correct answer. We need only consider how many times modulo operation happens.

In every recursive call, the algorithm uses the remainder of last round to factorize the divisor of last round. In above equation, the result of m modulo n is at least half of m, so this modulo operation happens $\log\ m$ at most. So its time complexity is $O(log\ m)$.

If $n \leq m$, we have time complexity $O(log\ n)$.

So time complexity is $O(log\ m + log\ n)$ or $O(max(log\ m, log\ n))$.

Problem 3.

Excise 14.

We have

$$\sum_{k=1}^{n} k2^k = \sum_{k=1}^{n}\sum_{j=1}^{k} 2^k = \sum_{1\le j\le k\le n} 2^k = \sum_{j=1}^{n}\sum_{k=j}^{n} 2^k$$

Note that $\sum^n a_1 q^n = a_1(1-q^n)/(1-q)$, So

$$\sum_{j=1}^{n}\sum_{k=j}^{n} 2^k = \sum_{j=1}^{n} 2^j\left(2^{n-j+1}-1\right) = \sum_{j=1}^{n} 2^{n+1} - \sum_{j=1}^{n} 2^j = (n-1)2^{n+1}+2$$

Excise 15.

We have

$$\sum_{1\le j\le k\le n} a_j a_k = \frac{1}{2}\left(\left(\sum_{k=1}^{n} a_k\right)^2 + \sum_{k=1}^{n} a_k^2\right) \tag{1}$$

So

$$\begin{aligned}
\cube_n &= 2\sum_{1\le j\le k\le n} jk - \square_n \\
&= \left(\sum_{k=1}^{n} k\right)^2 + \sum_{k=1}^{n} k^2 - \square_n \\
&= \frac{n^2(n+1)^2}{4}
\end{aligned}$$

Problem 4.

Excise 10.

Note that $\lceil (2x+1)/4\rceil - \lfloor (2x+1)/4\rfloor$ is either 0 or 1.

If it equals 0, it means (2x+1)/4 is integer. Assume $(2x+1)/4 = k$, k is integer, so we have $x = (4k-1)/2$, replace it with x, we have

$$\left\lceil \frac{2x+1}{2}\right\rceil + 0 = \lceil 2k\rceil = 2k$$

Scince x equals $2k - 1/2$ and k is integer, $2k$ equals to $\lceil x\rceil$.

If that part is 1, it means $(2x+1)/4$ is integer(x plus or minus 2 each time, we will have a such point on x axis). Then the original equation equals $\lceil x - 1/2\rceil = \lceil \lfloor x\rfloor + \{x\} - 1/2\rceil$. On x axis, we find that, x is equal to the integer which it is closest to. When $\{x\} \le 1/2$, $\lceil \lfloor x\rfloor + \{x\} - 1/2\rceil = \lfloor x\rfloor$. While if $x > 1/2$, $\lceil x - 1/2\rceil = \lceil \lfloor x\rfloor + \{x\} - 1/2\rceil$. On x axis, we find that, x is equal to the integer which it is closest to. When $\{x\} \le 1/2$, $\lceil \lfloor x\rfloor + \{x\} - 1/2\rceil = \lceil x\rceil$.

So

$$\left\lceil \frac{2x+1}{2}\right\rceil - \left\lceil \frac{2x+1}{4}\right\rceil + \left\lfloor \frac{2x+1}{4}\right\rfloor = \begin{cases} \lceil x\rceil, & \text{if } 0 \le \{x\} \le \dfrac{1}{2} \text{ and } x \ne \dfrac{4k-1}{2}, k\in\mathbb{Z} \\ \lfloor x\rfloor, & \text{if } \{x\} > \dfrac{1}{2} \text{ or } x = \dfrac{4k-1}{2}, k\in\mathbb{Z} \end{cases}$$

Excise 23.

Assume p is the n-th number of the sequence. Obviously

$$
\begin{aligned}
&\frac{p(p-1)}{2} < n \leq \frac{p(p+1)}{2} \\
\Leftrightarrow & p(p-1) < 2n \leq p(p+1)
\end{aligned} \tag{2}
$$

For interval $\left[\sqrt{2n+1/4}-1/2, \sqrt{2n+1/4}+1/2\right)$, it contains at least one integer, and we have $p = \left\lceil \sqrt{2n+1/4}-1/2 \right\rceil = \left\lfloor \sqrt{2n+1/4}+1/2-\epsilon \right\rfloor$. Then prove $\left\lfloor \sqrt{2n+1/4}+1/2-\epsilon \right\rfloor = \left\lfloor \sqrt{2n}+1/2 \right\rfloor$

$$
\begin{aligned}
\left\lfloor \sqrt{2n+\frac{1}{4}}+\frac{1}{2} \right\rfloor &= \left\lfloor \left\lfloor \sqrt{2n+\frac{1}{4}} \right\rfloor + \left\{ \sqrt{2n+\frac{1}{4}} \right\} + \frac{1}{2} \right\rfloor \\
\left\lfloor \sqrt{2n}+\frac{1}{2} \right\rfloor &= \left\lfloor \left\lfloor \sqrt{2n} \right\rfloor + \left\{ \sqrt{2n} \right\} + \frac{1}{2} \right\rfloor
\end{aligned}
$$

Note that the maximum difference between $\sqrt{2n+1/4}$ and $\sqrt{2n}$ is approximately equal to 0.085, so it is hardly possible that the floor of the two number having little difference is different.

Actually, according to (2), $p(p-1) < p(p-1)+1/4 < 2n \leq p(p+1) < p(p+1)+1/4$, then we have $p \in \left(\sqrt{2n}-1/2, \sqrt{2n}+1/2\right)$. In such an interval of length 1, we have $p = \left\lceil \sqrt{2n}-1/2+\epsilon \right\rceil = \left\lfloor \sqrt{2n}+1/2-\epsilon \right\rfloor$

Problem 5.

It is hard to get rid of $\lfloor\sqrt{}\rfloor$, but if a_{n-1} is square number and let $a_{n-1} = m^2$, it will be easier. If a_{n-1} is not square and let $a_{n-1} = m^2 + r$, r is a little number. We can see that $m^2+r \to m^2+m+r \to m^2+2m+r > (m+1)^2$, so it spends two steps at most from m^2 to $(m+1)^2$.

a_{n-1}	$\lfloor\sqrt{a_{n-1}}\rfloor$	a_n
m^2	m	m^2+m
$m^2+m = m^2+m-0$	m	m^2+2m
$m^2+2m = m^2+m+m$	m	m^2+3m
$m^2+3m = (m+1)^2+m-1$	$m+1$	m^2+4m+1
$m^2+4m+1 = (m+1)^2+2m$	$m+1$	m^2+5m+1
$m^2+5m+2 = (m+2)^2+m-2$	$m+2$	m^2+6m+4
$m^2+6m+4 = (m+2)^2+2m$	$m+2$	m^2+7m+6
$m^2+7m+6 = (m+3)^2+m-3$	$m+3$	m^2+8m+9
$m^2+8m+9 = (m+3)^2+2m$	$m+3$	$m^2+9m+12$

We need $k \leq m$, because $a_{n+2m+1} = (2m)^2$, and it will get in another loop in next step.

So we induce that, if $a_n = m^2$, we have

$$
\begin{cases}
a_{n+2k+1} = (m+k)^2+m-k, k \in [0,m] \text{ and } k \text{ is a integer} \\
a_{n+2k+2} = (m+k)^2+2m, k \in [0,m] \text{ and } k \text{ is a integer}
\end{cases} \tag{3}
$$