Problem 1.

(a)

$$
\begin{aligned}
4^n &= (1+1)^{2n} \\
&= \sum_{i=0}^{2n} \binom{2n}{i} \\
&= 1 + \sum_{i=0}^{2n} \binom{2n}{i} \leq 2n\binom{2n}{n}
\end{aligned}
$$

Since

$$
\begin{aligned}
\binom{2n}{i} &= \frac{2n!}{i!(2n-i)!} \\
&= \frac{2n!}{(i! \times (n+1)(n+2)\cdots(2n-i))n!} \leq \frac{2n!}{n!n!} = \binom{2n}{n}
\end{aligned}
$$

Therefore

$$
\binom{2n}{n} \geq \frac{2^{2n}}{2n}
$$

Since $\binom{2n}{n} \geq 2^{2n}/2n$, we have

$$
\begin{aligned}
&\frac{2^{2n}}{2n} \leq \prod_{p \leq 2n} p^{\ell_p} \leq (2n)^{\pi(2n)} \\
\Leftrightarrow & 2n\log 2 \leq (\pi(2n)+1)\log 2n \\
\Leftrightarrow & \pi(2n) \geq \frac{2n}{\log 2n}\log 2 - 1
\end{aligned}
$$

(b)

Note that

$$
\begin{aligned}
& n^{\pi(2n)-\pi(n)} \leq n^{\pi(n)} \leq 2^{2n} \\
\Leftrightarrow & \pi(2n) \leq \frac{2n\log 2}{\log n} + \pi(n)
\end{aligned}
$$

Actually, $\pi(n) \leq 2n\log 2/\log n$, so $\pi(2n) \leq 4n\log 2/\log 2n$.

(c)

We have $\pi(2n) \in [2n\log 2/\log 2n - 1, 4n\log 2/\log n]$, so $\pi(6n) \in [6n\log 2/\log 6n - 1, 12n\log 2/\log 3n]$

$$
\begin{aligned}
\pi(6n)_{\min} - \pi(n)_{\max} &= \frac{6n\log 2}{\log 6n} - 1 - \frac{2n\log 2}{\log n} \\
&\geq 1(\text{when } n \geq 6)
\end{aligned}
$$

When $n < 6$, it's apparent that there is a prime between $[n, 6n]$. So there is at least a prime between $[n, 6n]$.

(d)

(1)

We have $C_{2n}^{n} = (2n)!/n!n!$, if $p > 2n/3$, $2n!$ has factor $p, 2p$. if $p \leq n$, $n!n!$ has factor p^2, so $\ell_p = 0$.

(2)

Problem 2.

Oberserve that

$$\sum_{1\le m\le n}\sum_{d|m}\varphi(d)=\sum_{d\ge 1}\varphi(d)\lfloor n/d\rfloor \tag{1}$$

For the left of above equation

$$\begin{aligned}\left(\sum_{1\le k\le m+n}-\sum_{1\le k\le m}-\sum_{1\le k\le n}\right)\sum_{d|k}\varphi(d)&=\frac{(m+n)^2+(m+n)}{2}-\frac{m^2+m}{2}-\frac{n^2+n}{2}\\&=mn\end{aligned}$$

For the right

$$\sum_{d\ge 1}\varphi(d)\left(\left\lfloor\frac{n+m}{d}\right\rfloor-\left\lfloor\frac{n}{d}\right\rfloor-\left\lfloor\frac{m}{d}\right\rfloor\right)=\sum_{k\in S(m,n)}\varphi(k)$$

Note that

$$\left\lfloor\frac{n+m}{d}\right\rfloor-\left\lfloor\frac{n}{d}\right\rfloor-\left\lfloor\frac{m}{d}\right\rfloor=\begin{cases}1, & \text{If }\left\{\frac{m}{d}\right\}+\left\{\frac{n}{d}\right\}\ge 1\\ 0, & \text{If }\left\{\frac{m}{d}\right\}+\left\{\frac{n}{d}\right\}< 1\end{cases}$$

And $\{m/d\}+\{n/d\}\ge 1$ means $m \pmod d + n \pmod d \ge d$, $\{m/d\}+\{n/d\}<1$ means $m \pmod d + n \pmod d < d$,

Then we need to prove (1).

$$\begin{aligned}\sum_{1\le m\le n}\sum_{d|m}\varphi(m)&=\sum_{1\le m\le n}\sum_{k\ge 1}\sum_{d\ge 1}\varphi(d)[m=kd]\\&=\sum_{d\ge 1}\varphi(d)\sum_{d\le m\le n}\sum_{1\le k\le m}[m=kd]\\&=\sum_{d\ge 1}\varphi(d)\left\lfloor\frac{n}{d}\right\rfloor\end{aligned}$$

Problem 3.

(a)

$$
\begin{aligned}
z^m - 1 &= \prod_{0\le k\le m} (z-\omega^k)\times 1 \\
&= \prod_{0\le k\le m} (z-\omega^k) \prod_{d|m} [d = gcd(k,m)] \\
&= \prod_{d|m} \prod_{0\le k\le m} (z-\omega^k)\, [d = gcd(k,m)] \\
&= \prod_{d|m} \prod_{0\le k\le m} (z-\omega^k) \left[\left(\frac{k}{d},\frac{m}{d}\right) = 1\right] \\
&= \prod_{d|m} \prod_{0\le p\le m/d} (z-\omega^{pd}) \left[\left(p,\frac{m}{d}\right) = 1\right] \\
&= \prod_{d|m} \prod_{0\le p\le d} (z-\omega^{pm/d})\, [(p,d) = 1] \left(\text{note: Smilar to} \sum_{d|m} f(d) = \sum_{d|m} f\left(\frac{m}{d}\right)\right) \\
&= \prod_{d|m} \prod_{0\le p\le d} \left(z - (\omega^{(m/d)})^k\right) [(p,d) = 1] \left(\omega^{(m/d)} \text{ is d-th root of unity}\right) \\
&= \prod_{d|m} \Psi_d(z)
\end{aligned}
\tag{2}
$$

(b)

For (2), we have

$$\log(z^m - 1) = \sum_{d|m} \log \Psi_d(z)$$

By Möbius inversion

$$
\begin{aligned}
\log \Psi_m(z) &= \sum_{d|m} \mu\left(\frac{m}{d}\right) \log(z^d - 1) \\
\Leftrightarrow \Psi_m(z) &= \prod_{d|m} \left(z^d - 1\right)^{\mu(m/d)}
\end{aligned}
$$

Problem 4.

By brute force, we find that

m	$f(m)$
3	2^2
7	2^3
3×7	2^5
31	2^5
3×31	2^7
127	2^7
7×31	2^8
3×127	2^9
$3\times 7\times 31$	2^{10}
$\cdots$	$\cdots$

If m has a form of $2^k - 1$ and m is a prime, in other words m is a Mersenne prime, $f(m)$ is apparently a power of 2.

Again, above listing tells us m may be a product of distinct Mersenne primes.

$$\begin{aligned}
f(m_1 m_2) &= \sum_{d|m_1 m_2} d \\
&= \sum_{d_1 d_2|m_1 m_2} d_1 d_2 \\
&= \sum_{d_1|m_1} d_1 \sum_{d_2|m_2} d_2 \\
&= f(m_1) f(m_2)
\end{aligned}$$

So $f(m)$ is multiplicative.

If m is a product of distinct Mersenne primes, so m can be written as $m_1 m_2 \cdots m_k$, $m_{1,2\cdots}$ are distinct Mersenne primes. Since $f(m) = f(m_1) f(m_2) \cdots f(m_k)$ and $f(m_{1,2\cdots})$ is a power of 2, $f(m)$ is a power of 2.

For necessarity, $f(m) = f\left(\prod_i p_i^{r_i}\right) = \prod f(p_i^{r_i})$, so $f(m)$ is a power of 2 if and only if each of $f(p_i^{r_i})$ is a power of 2. For $p_i^{r_i}$

$$f(p_i^{r_i}) = 1 + p_i + p_i^2 + \cdots + p_i^{r_i}$$

If $f(p_i^{r_i})$ is power of 2, p_i and r_i are both odd, so let $r_i = 2m + 1$we can factor $f(p_i^{r_i})$ as

$$f(p_i^{r_i}) = (1 + p_i)(1 + p_i^2 + p_i^4 + \cdots + p_i^{2m})$$

So if $f(p_i^{r_i})$ is a power of 2, the above two factoried parts should be power of 2 both, and so $1 + p_i = 2^k$, it means m is product of some Mersenne primes. Then we need to explain ”distinct”.

$$1 + p_i^2 + p_i^4 + \cdots + p_i^{2m} = \frac{(2^k - 1)^m - 1}{2^k - 2}$$

This result is not always power of 2, it's power of 2 only and if only $m = 0$. So each of r_i equals 1.

So for sufficiency, if $f(m)$ is a power of 2, m is a product of distinct Mersenne primes.

So the necessary and sufficient condition of that $f(m)$ is a power of 2 is that m is a product of distinct Mersenne primes.