题 1.

$$\begin{aligned}\sum_{k=1}^{n}(\sqrt{k+1}-\sqrt{k}) &= \sum_{k=1}^{n}\frac{1}{\sqrt{k+1}+\sqrt{k}} \\ &\leq \sum_{k=1}^{n}\frac{1}{2\sqrt{k}}\end{aligned} \tag{1}$$

从而

$$\sqrt{n+1}-1 \leq \sum_{k=1}^{n}\frac{1}{2\sqrt{k}} \tag{2}$$

而

$$\begin{aligned}\sum_{k=1}^{n}(\sqrt{k+1}-\sqrt{k}) &= \sum_{k=1}^{n}\frac{1}{\sqrt{k+1}+\sqrt{k}} \\ &\geq \sum_{k=1}^{n}\frac{1}{2\sqrt{k+1}} = \sum_{k=2}^{n+1}\frac{1}{2\sqrt{k}}\end{aligned} \tag{3}$$

从而

$$\begin{aligned}\sqrt{n+1}-1 &\geq \sum_{k=2}^{n+1}\frac{1}{2\sqrt{k}} \\ \Leftrightarrow \sqrt{n}-1 &\geq \sum_{k=2}^{n}\frac{1}{2\sqrt{k}} \\ \Leftrightarrow \sqrt{n}-\frac{1}{2} &\geq \sum_{k=1}^{n}\frac{1}{2\sqrt{k}}\end{aligned} \tag{4}$$

所以

$$2\sqrt{n+1}-2 \leq \sum_{k=1}^{n}\frac{1}{\sqrt{k}} \leq 2\sqrt{n}-1 \tag{5}$$

题 2.

(1).

由题意

$$f_n = 2f_{n-1} - f_{n-2} + (-2)^n + 2[n=1] \tag{6}$$

那么

$$\begin{aligned}F(x) &= \sum_{n} f_n x^n \\ &= 2\sum_{n} f_{n-1}x^n - \sum_{n} f_{n-2}x^n + \sum_{n\geq 0}(-2)^n x^n + 2\sum_{n=1} x^n \\ &= 2\sum_{n} f_n x^{n+1} - \sum_{n} f_n x^{n+2} + \frac{1}{1+2x} + 2x \\ &= 2xF(x) - x^2F(x) + \frac{1}{1+2x} + 2x\end{aligned} \tag{7}$$

从而

$$F(x) = \frac{2x + 4x^2 + 1}{(1 + 2x)(1 - x)^2} \tag{8}$$

所以

$$f_n = a_1(-2)^n + (a_2 n + c)(1)^n \tag{9}$$

$$\begin{cases} a_1 & = \frac{4}{9} \\ a_2 & = \frac{7}{3} \end{cases} \tag{10}$$

$f_0 = 1$ 带入得 $c = \frac{5}{9}$
所以

$$f_n = \frac{(-2)^{n+2}}{9} + \frac{7}{3}n + \frac{5}{9} \tag{11}$$

(2).

由题意

$$\begin{cases} g_n & = 2h_{n-1} - g_{n-2} \\ h_n & = g_{n-1} - h_{n-2} + [n = 0] + 2[n = 1] \end{cases} \tag{12}$$

可得

$$\begin{cases} G(x) & = 2xH(x) - x^2 G(x) \\ H(x) & = xG(x) - x^2 H(x) + 1 + 2x \end{cases} \tag{13}$$

解得

$$\begin{cases} H(x) & = \frac{(1 + 2x)(x^2 + 1)}{x^4 + 1} = (2x^3 + x^2 + 2x + 1)U(x) \\ G(x) & = \frac{2x(1 + 2x)}{x^4 + 1} = (4x^2 + 2x)U(x) \end{cases} \tag{14}$$

其中 $U(x) = 1/(x^4 + 1)$，由于

$$U(x) = \frac{1}{x^4 + 1} = \sum_n (-1)^n x^{4n} \tag{15}$$

所以

$$\begin{cases} H(x) & = \sum_n \left((-1)^n 2x^{4n+3} + (-1)^n x^{4n+2} + (-1)^n 2x^{4n+1} + (-1)^n x^{4n}\right) \\ G(x) & = \sum_n \left((-1)^n 4x^{4n+2} + (-1)^n 2x^{4n+1}\right) \end{cases} \tag{16}$$

综上

$$
\begin{cases}
h_{4n} & = (-1)^n \\
h_{4n+1} & = 2(-1)^n \\
h_{4n+2} & = (-1)^n \\
h_{4n+3} & = 2(-1)^n \\
g_{4n+3} & = g_{4n} = 0 \\
g_{4n+1} & = 2(-1)^n \\
g_{4n+2} & = 4(-1)^n
\end{cases} \tag{17}
$$

题 4.

由于 X_1, X_2 独立，所以 $G_{X_1+X_2}(x) = G_{X_1}(x)G_{X_2}(x)$，且 $G_{X_1}(x) = G_{X_2}(x)$

$$
\begin{aligned}
G(x) &= \sum_{k\geq 0} \Pr(X = k) z^k \\
&= \sum_{k\geq 0} \frac{e^{-\lambda}\lambda^k}{k!} z^k \\
&= e^{\lambda z - \lambda}
\end{aligned} \tag{18}
$$

因为 $D(X) = E(X^2) - E^2(X)$

$$
\begin{aligned}
E(X) &= \sum_{k\geq 0} \Pr(X = k) x \\
&= \sum_{k\geq 0} \frac{e^{-\lambda}\lambda^k}{k!} k \\
&= \lambda e^{-\lambda} \sum_{k\geq 1} \frac{\lambda^{k-1}}{(k-1)!} \\
&= \lambda
\end{aligned} \tag{19}
$$

$$
\begin{aligned}
E(X^2) &= \sum_{k\geq 0} \Pr(X = k) k^2 \\
&= e^{-\lambda}\Big(\lambda^2 \sum_{k\geq 1} \frac{\lambda^{k-2}}{k!} k(k-1) + \lambda \sum_{k\geq 1} \frac{\lambda^{k-1}}{k!} k\Big) \\
&= \lambda^2 + \lambda
\end{aligned} \tag{20}
$$

所以 $D(X) = E(X^2) - E^2(X) = \lambda$，因为 X_1, X_2 方差都为 t，所以 $\lambda = t$

$$
\begin{aligned}
\Pr(X_1 + X_2 = n) &= \sum_{k=0}^{n} \Pr(X_1 = k)\Pr(X_2 = n - k)(X_1, X_2\text{相互独立}) \\
&= e^{-2t} \sum_{k=0}^{n} \frac{t^k}{k!} \frac{t^{n-k}}{(n-k)!} \\
&= e^{-2t} \frac{t^n}{n!} \sum_{k=0}^{n} \binom{n}{k} \\
&= e^{-2t} \frac{(2t)^n}{n!} \Big(n \geq 0\Big)
\end{aligned} \tag{21}
$$

题 5.

令 X_i 为第 i 次抛硬币的结果，那么设 $X_i = 1$ 表示抛出正面，令 $X = \sum_i^n X_i$，设 $X_i \sim B(1, p)$，则 $\hat{p} = X/n$

$$\begin{aligned} \Pr(|\hat{p} - p| \geq \epsilon) &\leq \delta \\ \Pr(|X/n - p| \geq \epsilon) &\leq \delta \\ \Pr(|X - np| \geq n\epsilon) &\leq \delta \end{aligned} \tag{22}$$

由于

$$\Pr(|X - \mu| \geq \mu\delta) \leq 2e^{-\delta^2\mu/3}, (0 \leq \delta \leq 1) \tag{23}$$

$\mu = E(X) = \sum_i^n p = np$，对于一开始得到的不等式的左边，有

$$\begin{aligned} \Pr(|X - np| \geq n\epsilon) &= \Pr\Big(|X - np| \geq np\Big(\frac{\epsilon}{p}\Big)\Big) \\ &\leq 2e^{-(\frac{\epsilon}{p})^2 np/3} \\ &= 2e^{-\epsilon^2 n/(3p)} \end{aligned} \tag{24}$$

因为 $\Pr \leq \delta$，只需要

$$\begin{aligned} 2e^{-\epsilon^2 n/(3p)} &\leq \delta \\ \Leftrightarrow n &\geq -\frac{3p}{\epsilon^2}\ln\frac{\delta}{2} \end{aligned} \tag{25}$$

所以

$$n = \left\lceil -\frac{3p}{\epsilon^2}\ln\frac{\delta}{2} \right\rceil \tag{26}$$