Assignment 7

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Hypothetical solar system

Assumptions:

- Orbits of the planets are assumed to be circular instead of elliptical. In reality, the orbit of planets is elliptical with sun at one focal point.
- Initially, eccentricity of the path is taken 0.02.
- Motion is assumed to be planar.
- Characteristic time scale is taken as 1 year (3.2*10^7 s) and length scale as 1 AU (1.5*10^11 m). Hence velocity of earth is 2*pi.
- Centripetal force of the planet balances the gravitational force between planet and Sun.
- Sun's mass is sufficiently large to neglect motion of the sun.

  Mathematical Model:

$$v_{x,i+1} = v_{x,i} - \frac{4\pi^2 x_i}{r_i^3} \Delta t$$

$$x_{i+1} = x_i + v_{x,i+1} \Delta t$$

$$v_{y,i+1} = v_{y,i} - \frac{4\pi^2 y_i}{r_i^3} \Delta t$$

$$y_{i+1} = y_i + v_{y,i+1} \Delta t ,$$

Computational Model:

Euler-Cromer method is used to compute this problem as using normal Euler method would not work since errors will keep accumulating with each iteration and the total energy of the planet will keep on increasing. This is similar to the problem faced in a driven pendulum.

At each time step *i*, calculate the position (x,y) and the velocity (v_x ,v_y) for time step

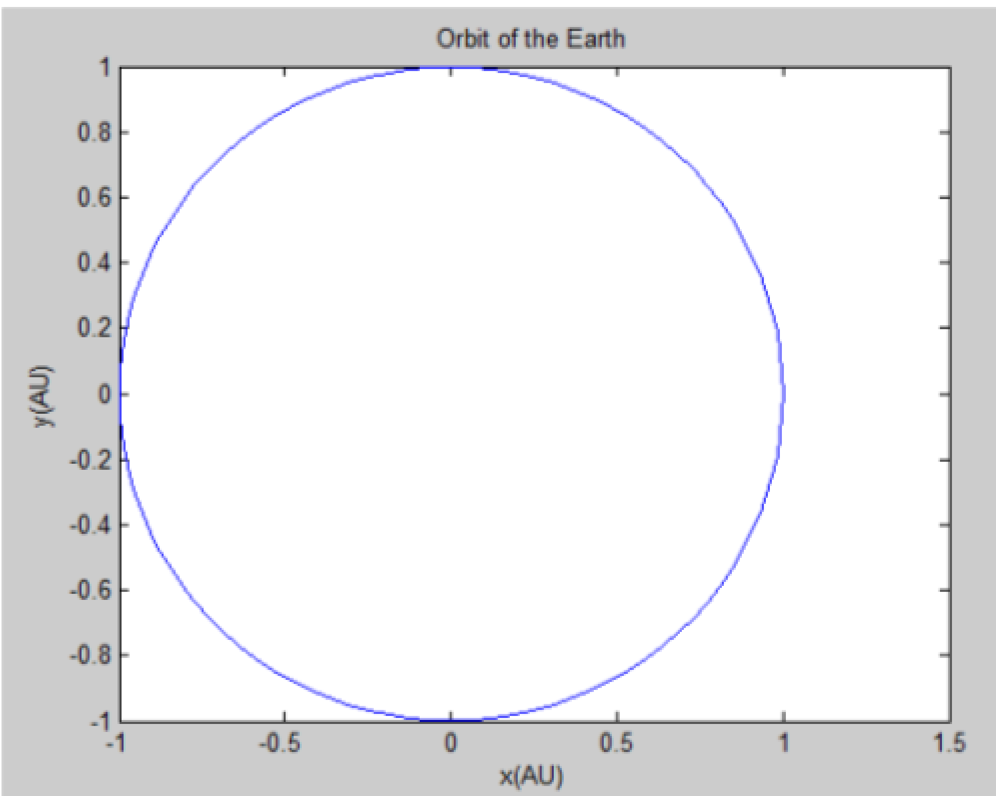
i+1 using Euler-Cromer method.

- Calculate the distance r_i from the sun, $r_i = (x_i^2+y_i^2)^{1/2}$
- Compute $v_{x,i+1}$ and $v_{y,i+1}$ using the above formulae

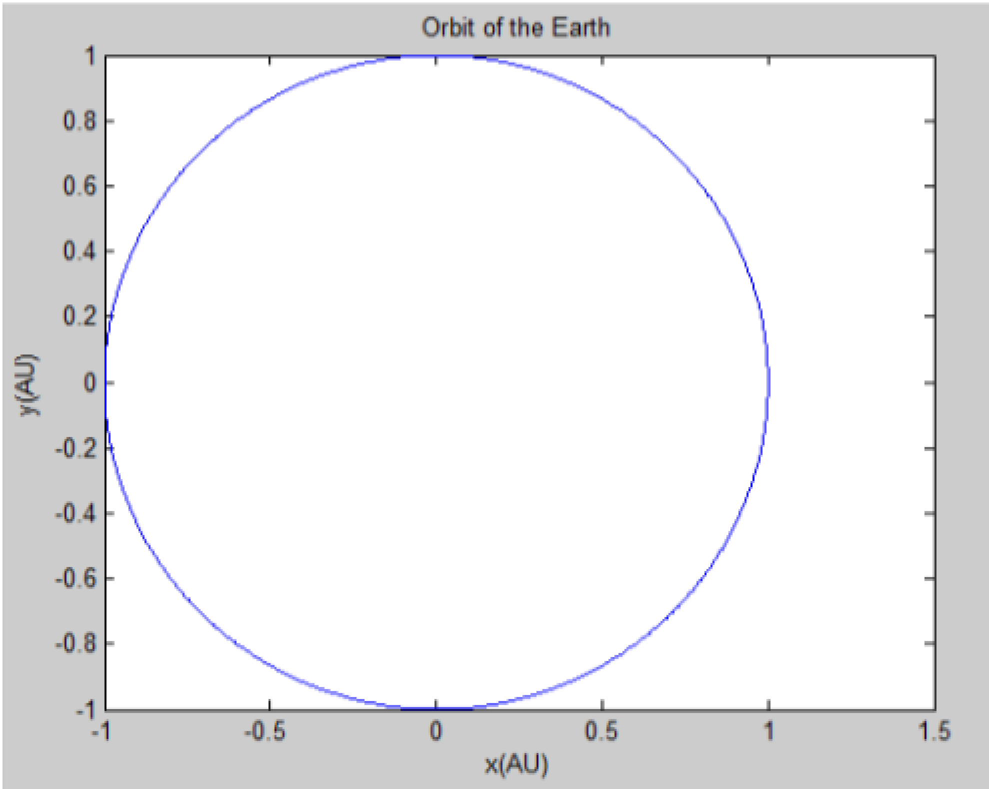
Planet	**Mass (kg)**	**Orbit (distance from sun in AU)**	**Time period (years)**	**Eccentricity (Actual)**
Venus	4.87e24	0.72	0.62	0.007

Earth	5.97e24	1	1	0.017
Mars	0.64e24	1.52	1.88	0.094
Jupiter	1898e24	5.20	12	0.049
Saturn	568e24	9.54	29	0.057
Uranus	86.8e24	19.18	84	0.046
Neptune	102e24	30.06	165	0.011

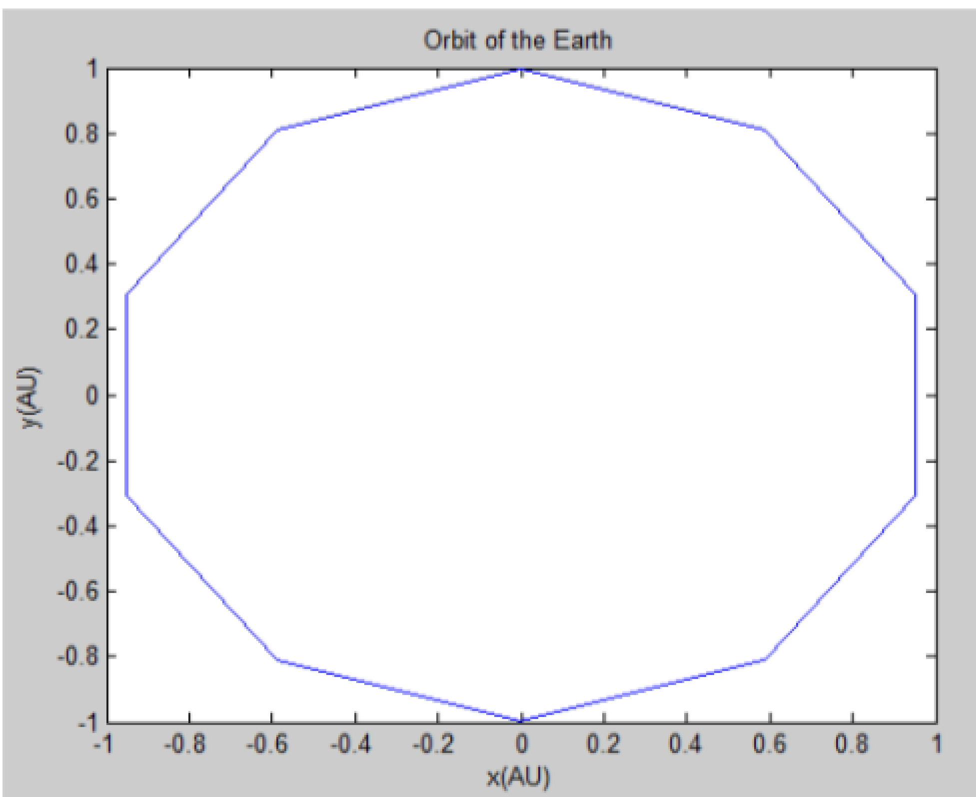
Plots:



dt = 0.01


Orbit of the Earth
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
y(AU)
-1
-0.5
0
0.5
1
1.5
x(AU)

dt = 0.01


Orbit of the Earth
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
y(AU)
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
x(AU)

dt = 0.1

Results and Conclusions:

Confirmation of the Kepler's Third Law

Planet	**T^2 / a^3 ($year^2/AU^3$)**
Venus	0.997
Earth	0.998
Mars	1.005
Jupiter	1.010
Saturn	0.988

Table-2

The table consists of values calculated of T^2 /a^3 for all planets except Mercury and Pluto. All the orbits are nearly circular, so a is nearly equal to the orbital radius mentioned in table 1. According to Kepler's third law, T^2 / a^3 should be constant (1 for Astronomical units). The given values are, hence, in accordance with Kepler's Third Law of Planetary Motion.

2) <u>Orbits:</u>

Assumptions:

- Motion is assumed to be happening in 2D-plane.
- Initial velocity magnitude - this value must be less than or greater than that of a circular orbit, up to the limiting value of the velocity of a parabolic orbit, where the kinetic energy of motion is equal to the gravitational energy of gravity
  KE=PE ==> $mv^2/2$ = GMm/R ==> v(parabolic)=$(2GM/R)^{1/2}$.
- Sun's mass is sufficiently large to neglect motion of the sun.
- We assume that the Sun is stationary at one focus of the elliptical orbit.

The direction of the gravitational force between planet and the sun is along the line joining the two bodies.

Velocity and Acceleration of the planet:

For Circular Orbit:

- **Direction of velocity:** Tangential
- **Magnitude of the velocity:** $v = \sqrt{(\frac{GM}{r})}$
- **Direction of acceleration:** Radial
- **Magnitude of the acceleration:** $a = GM/r^2$

For Elliptical Orbit:

The velocity will no longer be constant but will change with radius from earth. Kepler's Laws still control the motion.

- **Direction of velocity:** Tangential

- **Magnitude of the velocity:** changes at every instant

- **Direction of acceleration:** Radial

- **Magnitude of the acceleration:** changes at every instant

Circular Orbits -

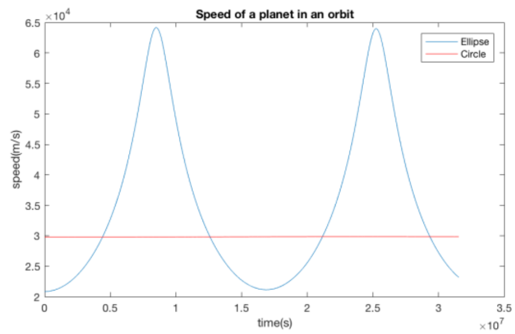
On absence of external torque, the planet undergoes uniform circular motion if its velocity is $\sqrt{(\mathrm{GMsun}/\mathrm{R})}$. We can see that in the plot of the potential energy vs time. We notice that Kinetic Energy remains a constant. The observation can be made for the potential energy because the object remains at a constant distance from the center of mass i.e. Sun.

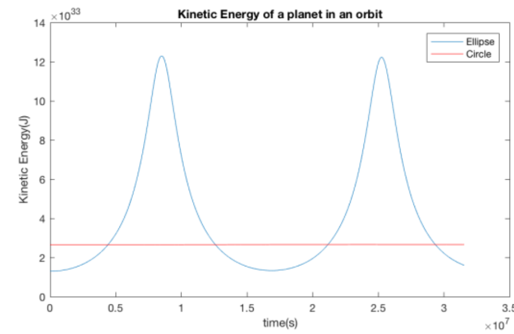
As KE and PE are both constants, we can conclude that the Total energy is also a constant on absence of an external torque.

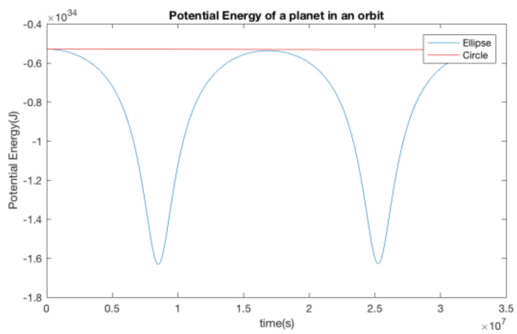
Elliptical Orbits -

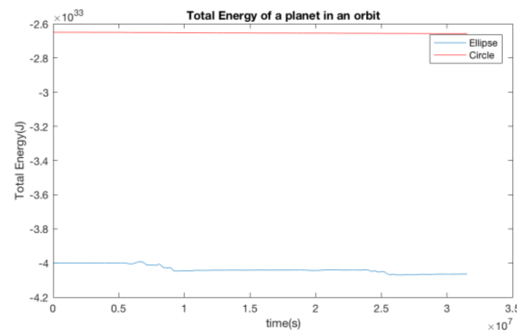
For a lower velocity or a slightly higher velocity, we see that the planet undergoes an elliptical motion, with the Sun lying at one of the foci of the elliptical orbit. Unlike in the case of the circular orbit, the velocity in this case constantly changes. Thus, the Kinetic energy also varies in a periodic manner. But at the same time whenever the velocity increases, the radius of orbit decreases, thereby lowering the Potential energy i.e. Making it more negative. Therefore, when KE increases, PE decreases accordingly so that the value of total energy remains constant.

Therefore, we see that both the orbits follow the law of conservation of energy.









Notice that between some points (along the orbit) the planet is speeding up, and between some other points the planet is slowing down. Report on it.

According to Kepler's Second Law, the radius vector sweeps equal area in equal intervals of time. Hence, we can infer from this that when the planet is nearer to the Sun, its velocity is less than when it is farther. Also, from Kepler's Second Law,

$$\frac{dA}{dt} = \frac{l^2}{2m}$$

For small displacement, $dA = rdx$ where r is the radius at that point.

$\therefore \frac{rdx}{dt} = \frac{l^2}{2m}$

$$\therefore \frac{dx}{dt} = \frac{l^2}{2mr}$$

$$\therefore v = \frac{l^2}{2mr}$$

$$\therefore v \propto \frac{1}{r}$$

This equation is in accordance with our claim that the velocity of the planet is inversely proportional to its distance from Sun.

In case of circular orbit, planet's distance from the Sun remains constant and hence its velocity also remains constant.

During equal time intervals, the radius vector from the sun to a planet sweeps out equal areas. What does this tell you about the angular momentum of the planets? What does this tell you about the motion of the planets/ planet's orbit?

Extending the above argument, let the position of the planet at any instant be r1 and velocity be v1. Let the position and velocity be r2 and v2 respectively at another instant of time. Then, from the previous relation between position and velocity,

$$r_1 v_1 = r_2 v_2$$

$$\therefore m r_1 v_1 = m r_2 v_2$$

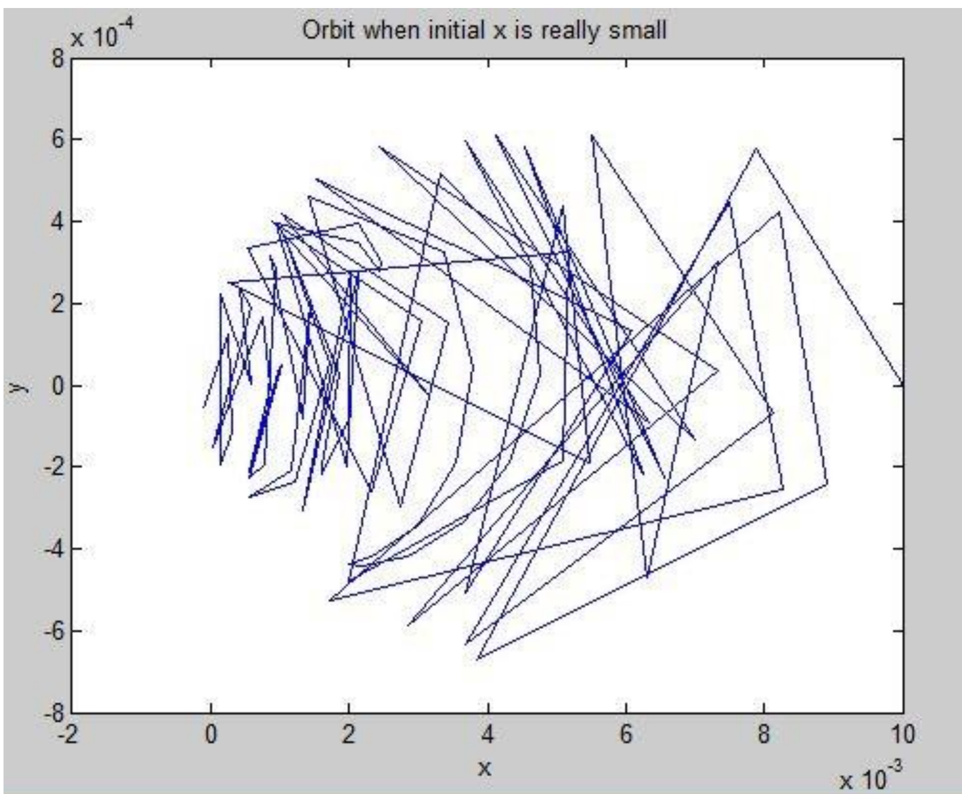
$$\therefore \ l_1 = l_2$$

$\therefore$ *Angular momentum of the planet remains constant*.

Here, the net force acting on the planet is the gravitation force and since the gravitation force and position vector are always in the same direction, the net torque due to gravitational force is zero and hence the angular momentum remains conserved which is in accordance with our observations.

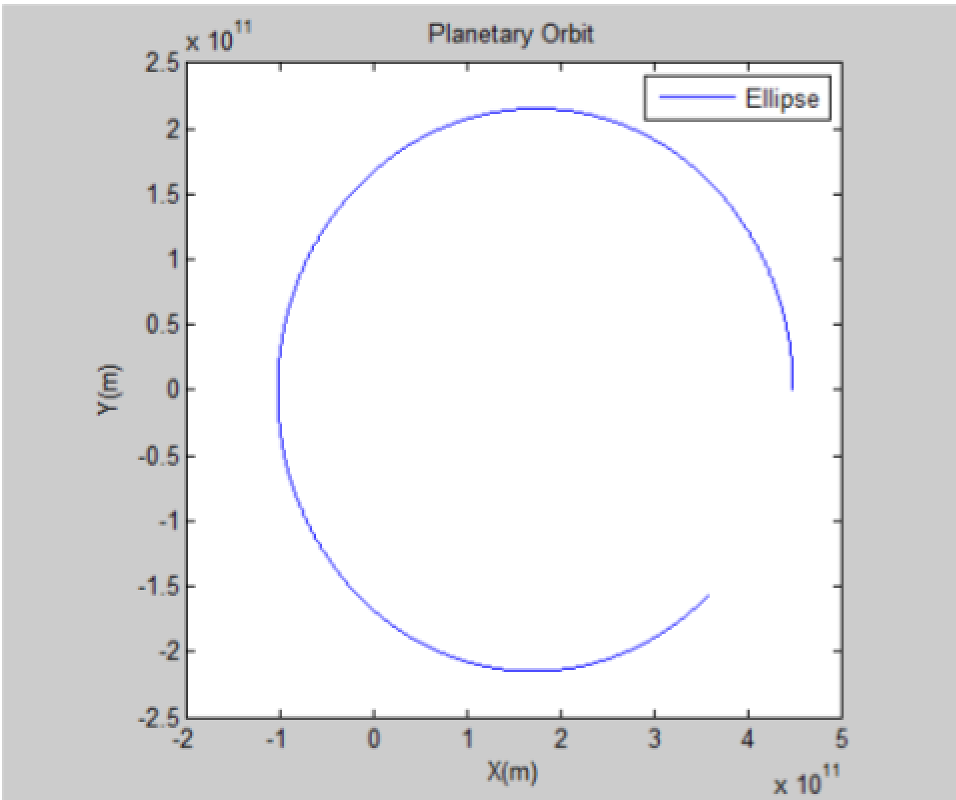
- **Summarizing the observations :**

- **What happens to the orbit when x gets really small?**



As seen in the figure above, the orbit of the body gets distorted when x gets really small.

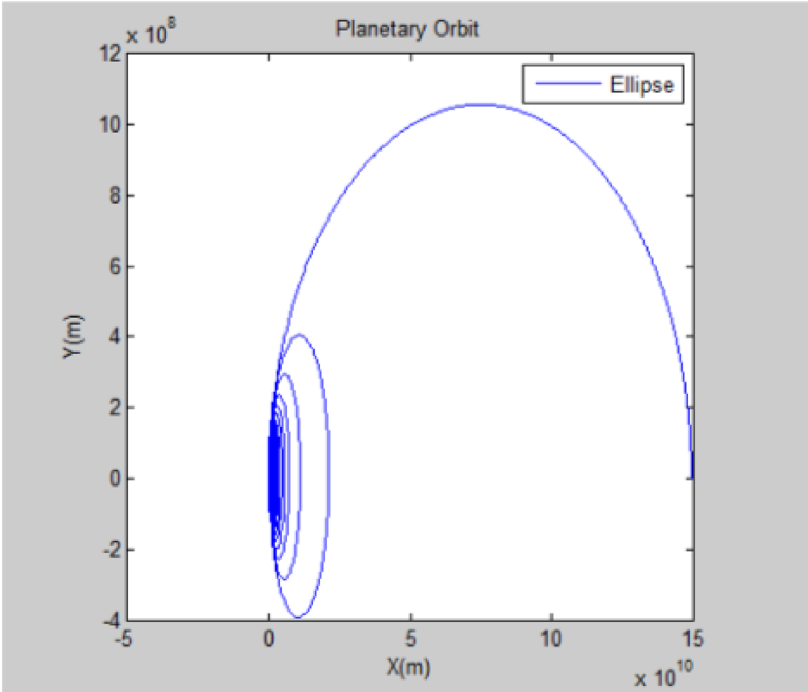
- **What happens to the orbit when x gets really large?**



As seen in the figure above, if x is very large, time taken to complete one revolution will become very high. Also, the orbit becomes more elliptical (for above image, eccentricity ≅ 0.8771)

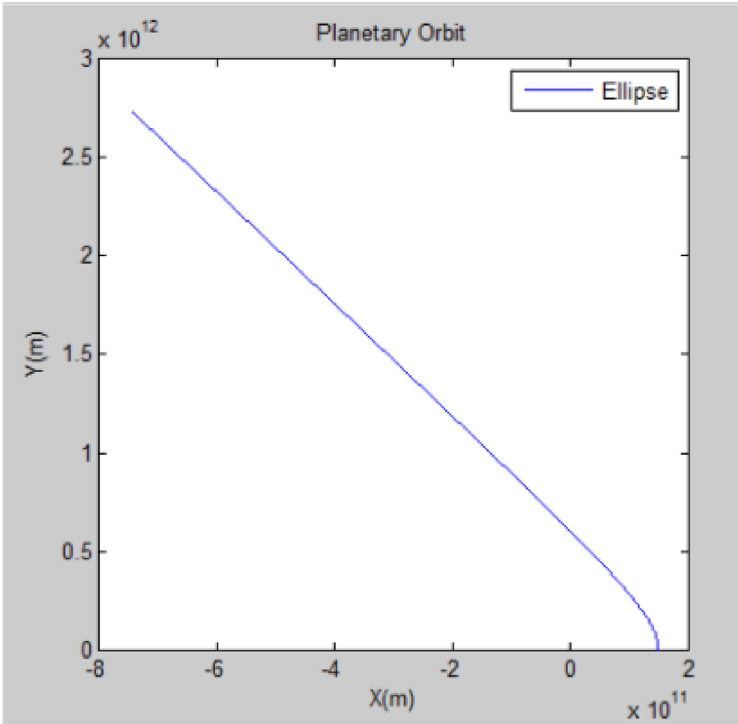
Now, as you vary the initial velocities of the planets, how do the orbital trajectories change?

When v is really small



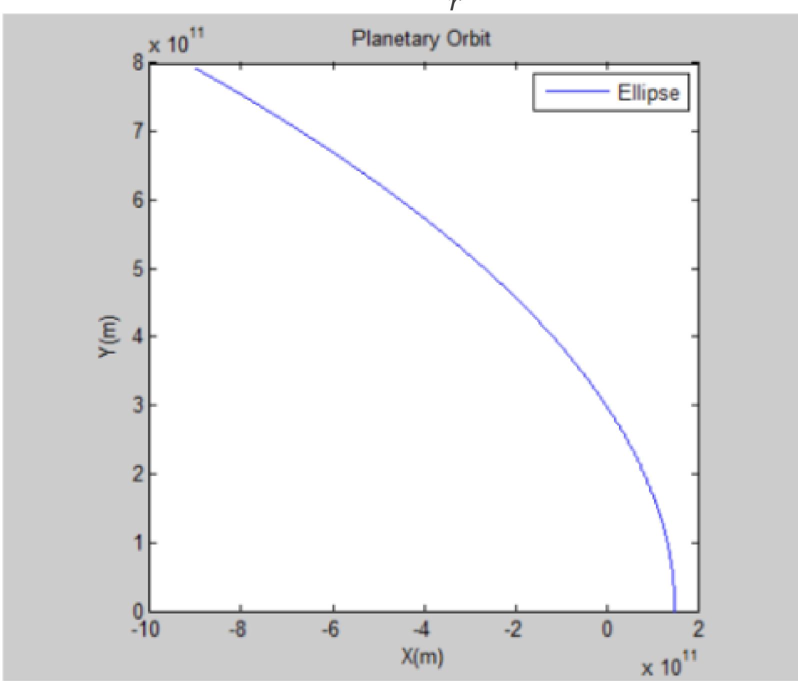
For the above graph, the initial velocity is very less. Hence, we can see that the planet will spiral into the Sun since it doesn't have the required energy to continue its motion.

When v is really large



For the above graph, the initial velocity is very high ($2\sqrt{(\frac{GM}{r})}$) and hence the orbit has eccentricity greater than 1.

For velocity v= $\sqrt{(\frac{2GM}{r})}$, the orbit is parabolic (e=1)



How do the values for total energy change when the type of orbit is changed?

From all the above graphs, we can see that the energy of the planet is the highest when the orbit is circular and when we change the orbit to elliptical (by decreasing the velocity), the total energy of the planet decreases.