Stranger Things

Amarnath Karthi (201501005) and Chahak Mehta (201501422)
Dhirubhai Ambani Institute of Information & Communication Technology, Gandhinagar, Gujarat 382007, India
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Certain complex non linear systems are highly sensitive to initial conditions. This causes an apparently random and an unpredictable behavior in the system dynamics. This is known as chaos. Amidst this perceived randomness are a special class of 3 Dimensional phase portrait structures known as strange attractors. This project aims to provide a brief understanding of chaos, strange attractors and the relationships between them.

I. INTRODUCTION

A chaotic system is a dynamical deterministic system that is highly sensitive to the initial conditions. This means that even a small perturbation from the initial conditions of an experiment can lead to extremely unpredictable outcomes. A chaotic system doesn't show fixed points like traditional nonlinear systems do but they often have a set of invariant values that are called Strange attractors. Strange attractors are fractal like structures to which a chaotic system tends to reach irrespective of the initial conditions. This project tries to understand chaos by looking at the Lorenz system which was the first chaotic system to be studied extensively.

II. LORENZ SYSTEM

The Lorenz system represents a simplified version of atmospheric convection. It is governed by a set of 3 ordinary differential equations [2] [1]:

$$\dot{x} = \sigma(y - x) \tag{1}$$
$$\dot{y} = x(r - z) - y \tag{2}$$
$$\dot{z} = xy - bz \tag{3}$$

The three parameters being σ(Prandtl number), ρ is the Rayleigh number and b is some geometrical descriptor of the system. This system of equations also arises in various other domains of science. The Lorenz system is a 3 dimensional system.

III. ANALYSIS

Notice that there are non linearities in the system, namely in xz and xy in equations 2 and 3. Therefore for a primitive analysis we try to find and classify the fixed points.

A. Fixed Points

Finding the fixed points by setting $\dot{x} = \dot{y} = \dot{z} = 0$, we get three possible fixed points, namely

$$\begin{array}{ll} O & (0,0,0) \\ C^+ & (\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1) \\ C^- & (-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1) \end{array}$$

The parameters r, σ and b are physical measurements and are greater than 0 by definition.Therefore C^+ and C^- are defined only for $r > 1$. Therefore the point $r = 1$ is a bifurcation.

B. Linear Stability Analysis

We calculate the Jacobian of the Lorenz system as:

$$\begin{bmatrix} -\sigma & \sigma & 0 \\ r-z & -1 & -x \\ y & x & -b \end{bmatrix}$$

For analyzing the nature of fixed point O, notice that the linearization about origin gives us the following set of equations:

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = rx - y$$
$$\dot{z} = -bz$$

It is clear that the the equations for $\dot{x}$ and $\dot{y}$ are decoupled from z. Therefore we can apply a stability analysis on x and y without considering z.

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -\sigma & \sigma \\ r & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

τ and Δ being the trace and determinant of the 2D Jacobi, we get

$$\tau^2 - 4\Delta = (\sigma - 1)^2 + \sigma r$$

Because of this, we can say that if $\Delta > 0$ then O is stable witrespect to (x, y). $\Delta > 0$ only for $r < 1$. Otherwise O

is a saddle point with respect to (x, y). Also since $b > 0$, $\dot{z} = -bz$ yields us the fact that $z = 0$ is a stable fixed point. Combining the results for the decoupled (x, y) and z, we conclude that O is locally stable for $r < 1$ and a saddle point for $r > 1$. This proves the local stability of O.

But we still are not guaranteed global stability. To prove this, a potential like function is used. The Lorenz system happens to be a non conservational system. We cannot find a conserved quantity $V(x, y, z)$. Therefore, instead its dissipative counterpart, the Lyapunov function is used [2]. Define:

$$V(x, y, z) = \frac{x^2}{\sigma} + y^2 + z^2 \tag{4}$$

It can be shown that moving along the flow lines, $\dot{V} < 0$ for $r < 1$. This means that for $r < 0$, V keeps on decreasing along the trajectories. But by V being a sum of squares, cannot become negative and is 0 only at O. Thus along the trajectories, V will eventually have to become equal to 0 therefore, eventually every point will come to O and settle. This proves the global stability of O for $r < 1$.

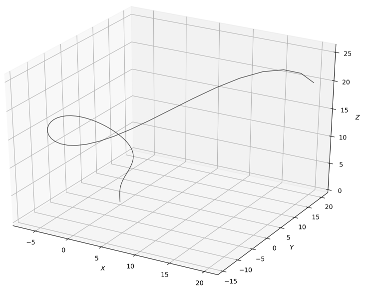

FIG. 1: Phase portrait for $r = 0.5, \sigma = 10, b = \frac{8}{3}$. O is a global stable fixed point.

C. Chaos

On finding the eigen-vectors of the Jacobian of the Lorenz system, a characteristic equation for the eigenvalues can be found out as:

$$\lambda^3 + (\sigma + b + 1)\lambda^2 + b(r + \sigma)\lambda + 2b\sigma(r - 1) = 0 \tag{5}$$

On solving equation (5), we can see that there occurs a Hopf's bifurcation for

$$r = \sigma \frac{(\sigma + b + 3)}{(\sigma - b - 1)} \tag{6}$$

This bifurcation is a subcritical Hopf's bifurcation *(Marsden and McCracken, 1976)* and hence it is seen that for $r > r_H$, an "unstable" surface surrounds the fixed points C^+, C^-. This means that there is no stable fixed point or a limit cycle for $r > r_H$ which implies that there is no attractor exists at $r > r_H$. Therefore a particle has no attractor to go to. This can mean 2 things. Either all the points are diverging to infinity or

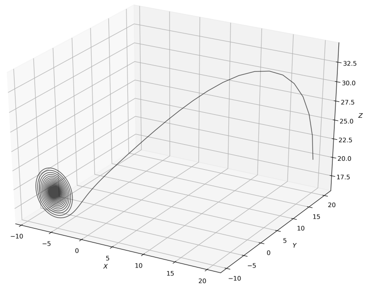

FIG. 2: Phase portrait for $r = 20, \sigma = 10, b = \frac{8}{3}$.

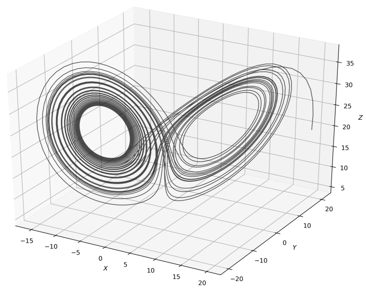

FIG. 3: Phase portrait for $r = 24, \sigma = 10, b = \frac{8}{3}$.On the verge of chaos.

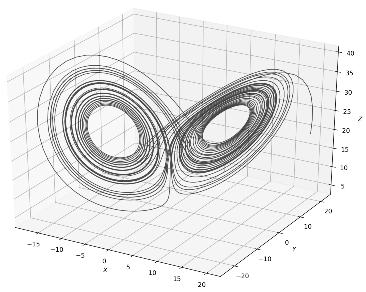

FIG. 4: Phase portrait for $r = 25, \sigma = 10, b = \frac{8}{3}$. Chaos begins.

something strange is happening. It can also be shown that in the Lorenz System, particles are bounded in a finite region R, which is a function of x, y, z. Therefore the particles are even not going to infinity.

To solve this problem, the concept of strange attractors was introduced by Lorenz. On one hand, particles are repelled from 1 fixed point to another, but at the same time, they are constrained into a finite region. Also by definition of streamlines, no 2 particle trajectories should intersect each other. Such constraints give rise to a strange attractor.

D. Strange Attractors

We take an example of parameters such that $r > r_H$. Let $r = 28$, $\sigma = 10$ and $b = \frac{8}{3}$. (These are the same parameters as those taken initially by Lorenz.). We plot the variation of $x(t)$ with time for 2 initial conditions $I_1(5, 5, 5)$ and $I_2(5.1, 5, 5)$, refer to Figure(5). Corresponding to these parameters, the phase plot is shown in Figure(6) Notice how even a change of 0.1 in the initial conditions, leads to a drastic change in the final and the intermediate values and also the position in the phase space. This is a special property of chaos and also many strange attractors. It is known as a sensitive dependence on initial conditions.

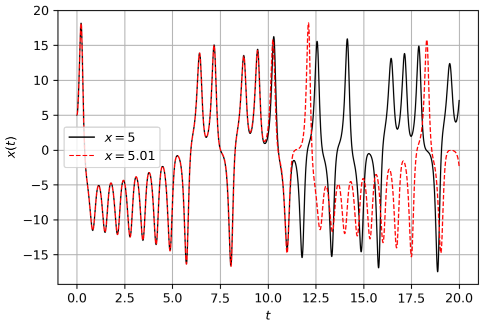

FIG. 5: Variation of $x(t)$ for I_1 and I_2, $\rho = 28, \sigma = 10, b = \frac{8}{3}$.

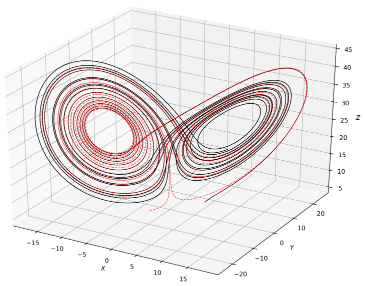

FIG. 6: Variation of trajectories for I_1 and I_2, $\rho = 28, \sigma = 10, b = \frac{8}{3}$.

Also the phase plot reveals the an interesting shape of the strange attractor. This structure is known as a Lorenz attractor. It is like a curved "8" shape, with 2 spirals. These are infact fractals. They may seem like periodic orbits, but we know that this requires the presence of a traditional attractors (whose presence has been ruled out). According to Lorenz, it seems initially like 2 surfaces. Then on the completion of 1 circuit around C^+ and C^-, there appears to be 4 surfaces. This can be done recursively, and one concludes that there are infinite surfaces, infinitesimally close to each other. Such a geometry is known as a fractal.

E. Lyapunov Exponents

Let there be 2 points p_1 and p_2 in the phase space under the the presence of the Lorenz attractor. p_1 and p_2 are infinitesimally close to each other and are inside the strange attractor. Let $\delta(t)$ be the distance between them. As time proceeds, we observe that p_1 and p_2 go further and further apart and $\delta(t)$ increases with time. When calculated theoretically, it is seen that δ varies as :

$$\delta(t) \sim \delta_0 e^{\lambda t} \tag{7}$$

where $\lambda = 0.9$. Our experimental plots agree with this result. Refer to Figure(7) This result justifies the observations seen in 6, which in a nutshell can be described as sensitivity to initial conditions. The parameter λ is defined as the Lyapunov Exponent. A positive Lyapunov exponent indicates a high sensitivity to minor perturbations in the initial conditions, whereas non chaotic systems traditionally have negative or very low Lyapunov exponents.

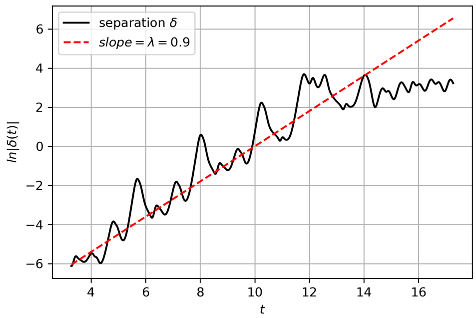

FIG. 7: $\ln \delta(t)$ has a slope of $\lambda = 0.9$ for parameters same as those in **??**

$t_{horizon}$ is defined a the time till which we can make a reasonable prediction of the future position ignoring the perturbation or a loss of information δ_0.

$$t_{horizon} \sim O(\frac{1}{\lambda} ln \frac{a}{\delta_0}) \tag{8}$$

The above equation shows us the difficulty and unpredictability underlying in chaotic system. To increase the prediction time $t_{horizon}$ by a factor of say 2, we will have to reduce the error in measurement by a factor of 100. This in itself is a difficult task. This lead Lorenz to a conclusion that long term weather prediction is very difficult if not impossible.

IV. LORENZ MAP

Till now, we have seen no predictability in any aspect of the Lorenz attractor. Let us look at the zy plot of the attractor. Notice in Figure(8) how the Z value changes

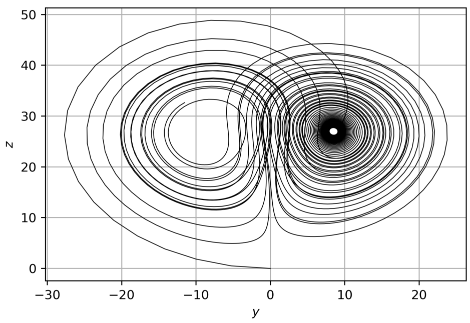

FIG. 8: ZY view of a typical Lorenz Attractor

periodically with time. A particle begins at approximately at the center of a spiral, and spirals outwards. After performing a critical number of spins, it changes its spiral and moves to the other spiral. It keeps on doing this shifting of spirals. In each spiral it keeps on spiralling outwards till it reaches a critical distance from center or in other words a maximum Z value.

Let z_n indicate the n^{th} local maxima of z. We create a maf : $z_n \to z$. An interesting pattern emerges when we plot this map. See Figure(9) Using the Lorenz map, we can say the next maximum Z value from the current value and so on. This give us some predictability.(Though not a lot). Also Lorenz Map can be used to prove that the Lorenz Attractor is infact a strange attractor and not just a very long limit cycle.

V. CONCLUSION

In this project, we took a brief look at chaos and strange attractors using the Lorenz system as an example. We also took a look at how a chaotic system is deterministic and yet unpredictable by understanding the Lyapunov exponents. We saw that Lyapunov exponents

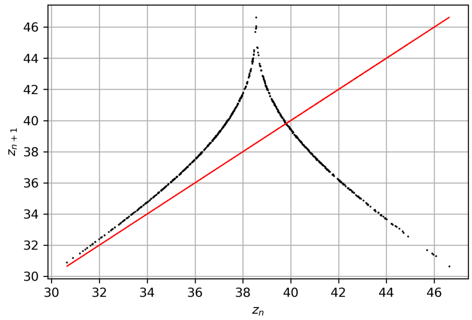

FIG. 9: Lorenz Map

help us to predict the future state upto a certain factor. Furthermore, we also saw how the Lorenz system has a hidden pattern that rules out the possibility of a limit cycle with the help of Lorenz Map. All the above observations show that a chaotic system is not a stochastic system whos' output cannot be determined but they are systems who have a unique output but it is just very difficult to predict the output.

[1] Edward Lorenz, *Deterministic Nonperiodic Flow*, http://eaps4.mit.edu/research/Lorenz/Deterministic_63.pdf

[2] Steven Strogatz, *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering*, Sarat Book House.

[3] William Boyce, *Elementary differential equations*