Stranger Things

An Analysis of the Lorenz system

CS 304 – Nonlinear Science

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The Lorenz system represents a simplified version of atmospheric convection. It is governed by a set of 3 ordinary differential equations [2] [1]:

$$\begin{aligned}\dot{x} &= \sigma(y - x)\\ \dot{y} &= x(r - z) - y\\ \dot{z} &= xy - bz\end{aligned}$$

σ(Prandtl number), ρ is the Rayleigh number and b is some geometrical descriptor of the system

Fixed points calculated by $\dot{x} = \dot{y} = \dot{z} = 0$:

$$\begin{array}{ll} O & (0,0,0) \\ C^{+} & (\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1) \\ C^{-} & (-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1) \end{array}$$

The Jacobian for this system is: $\begin{bmatrix} -\sigma & \sigma & 0 \\ r-z & -1 & -x \\ y & x & -b \end{bmatrix}$

and on linearization about the origin it becomes: $\begin{bmatrix} -\sigma & \sigma \\ r & -1 \end{bmatrix}$

- For origin O:

$$\begin{aligned} r < 1 &\implies \textit{Stable fixed point} \\ r > 1 &\implies \textit{Saddle point} \end{aligned}$$

- Subcritical Hopf's bifurcation *(Marsden and McCracken, 1976)* for value of $r > r_H$ where $r_H = \sigma\frac{(\sigma+b+3)}{(\sigma-b-1)}$ at C^+ and C^-.
- Hence, no stable fixed point or a limit cycle for $r > r_H$ which implies that there is no attractor exists at $r > r_H$.

Lorenz system

Analysis

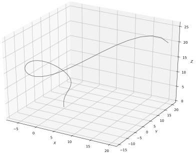

Figure: Phase portrait for $r = 0.5, \sigma = 10, b = \frac{8}{3}$. O is a global stable fixed point.

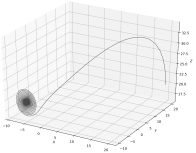

Figure: Phase portrait for $r = 20, \sigma = 10, b = \frac{8}{3}$.

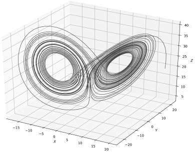

Figure: Phase portrait for $r = 25, \sigma = 10, b = \frac{8}{3}$. Chaos begins.

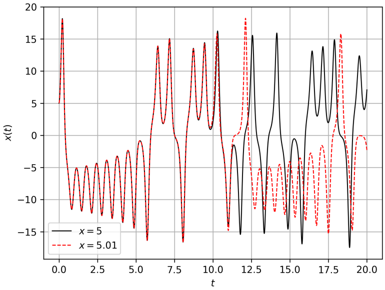

Figure: Variation of $x(t)$ for I_1 and I_2, $\rho = 28, \sigma = 10, b = \frac{8}{3}$.

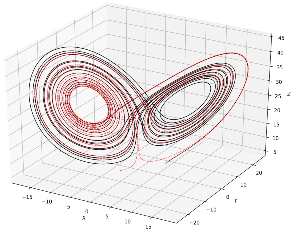

Figure: Variation of trajectories for I_1 and I_2, $\rho = 28, \sigma = 10, b = \frac{8}{3}$.

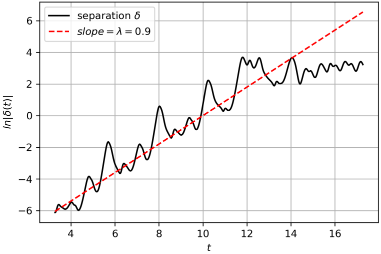

Figure: $\ln \delta(t)$ has a slope of $\lambda = 0.9$ for parameters same as those in 5

$$\delta(t) \sim \delta_0 e^{\lambda t}$$

$$t_{horizon} \sim O(\frac{1}{\lambda} ln \frac{a}{\delta_0})$$

$$z_{n+1} = f(z_n)$$

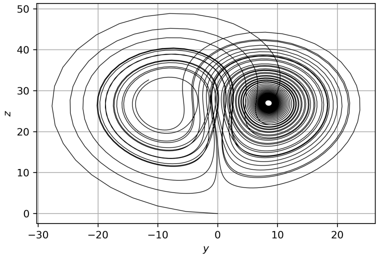

Figure: *ZY* view of a typical Lorenz Attractor

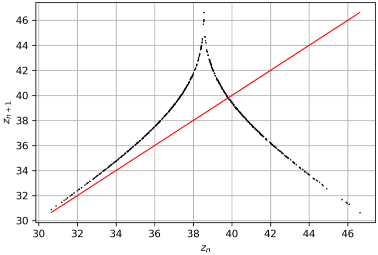

Figure: Lorenz Map

- Edward Lorenz, *Deterministic Nonperiodic Flow*, http://eaps4.mit.edu/research/Lorenz/Deterministic_63.pdf
- Steven Strogatz, *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering*, Sarat Book House.
- William Boyce, *Elementary differential equations*