

SC-374
Computational and Numerical Methods
Assignment 1

Amarnath Karthi 201501005
Chahak Mehta 201501422

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1 Exponential and Logarithmic functions

With the help of a single code, plot the following functions:

1. $y = e^x$
2. $y = x$
3. $y = \ln(x)$

Use suitable ranges of x for each of the functions and judge their properties on various scales of x . Extending this exercise plot $e^{\pm x}$ on the same graph and compare them.

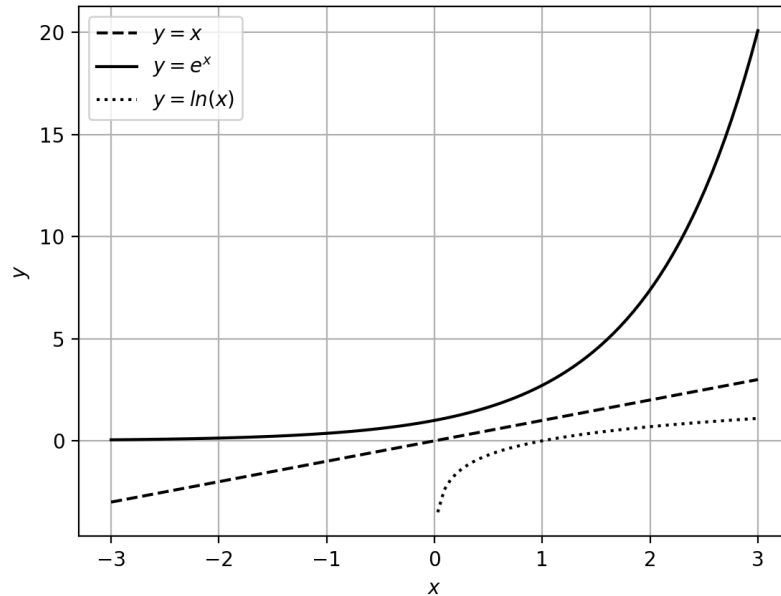


Figure 1: Plots of $y = x$, $y = e^x$ and $y = \ln(x)$

In **figure 1**, we can see the behavior of the three functions over different scales of x .

Some important observations :

- The function $y = \ln(x)$ is defined only of positive real numbers, while $y = x$ and $y = e^x$ are.
- The function $y = \ln(x)$ is the inverse of $y = e^x$. We can interpret this visually by noting that both are mirror images of each other about the line $y = x$.

- All the three functions are strictly increasing. This is because their derivatives are always positive and therefore have no extrema.

$$\lim_{x \rightarrow \infty} x = \lim_{x \rightarrow \infty} e^x = \lim_{x \rightarrow \infty} \ln(x) = \infty \quad (1)$$

$$\frac{d}{dx}(e^x) = e^x \quad \frac{d}{dx}(x) = 1 \quad \frac{d}{dx}(\ln(x)) = \frac{1}{x} \quad (2)$$

- For large positive values of x , $y = e^x$ dominates over $y = x$ which in turn dominates over $y = \ln(x)$. This is due to the fact that for large values of x , the derivative of $y = e^x$ is larger than the derivative of $y = x$ which is in turn larger than derivative of $y = \ln(x)$
- For very small values of x (< 1), the situation becomes vice-versa. $\ln(x)$ dominates over x and e^x . The reasoning is the same as above.
- For negative values of x , the function $y = x$ dominates over $y = e^x$.
- The function $y = e^x$ has the line $y = 0$ as its asymptote. This is because as x approaches $-\infty$, e^x tends to 0.

$$\lim_{x \rightarrow -\infty} e^x = 0 \quad (3)$$

- The function $y = \ln(x)$ has the line $x = 0$ as its asymptote. This is because as x approaches 0, $\ln(x)$ becomes more and more negative. Also its derivative tends to $-\infty$.

$$\lim_{x \rightarrow 0} e^x = -\infty \quad (4)$$

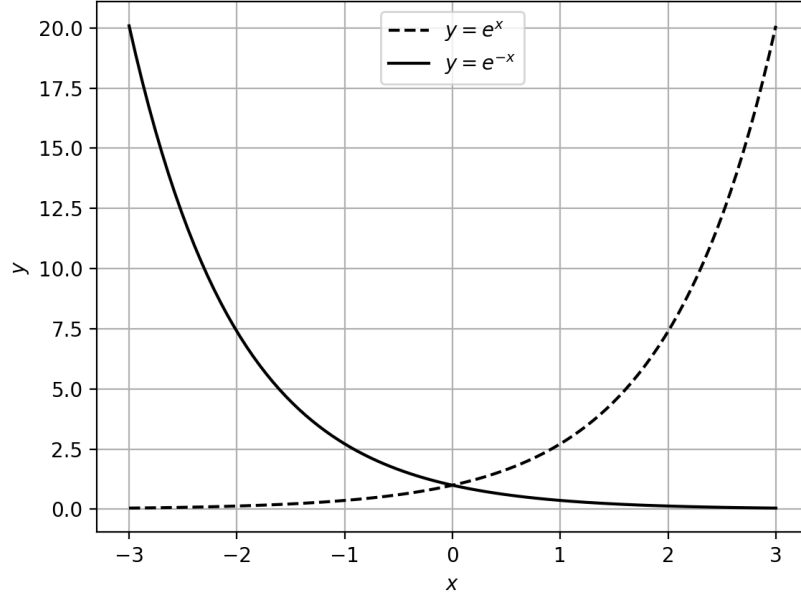


Figure 2: Plots of $y = e^x$ and $y = e^{-x}$

In **figure 2**, we can see the behavior of e^x and e^{-x} functions over different scales of x . As e^{-x} is just the function e^x for negative values of x , we can see that both are mirror images of each other about the line $x = 0$. Also the 2 functions intersect at $x = 0$ where their value is 1. Both the functions have a common asymptote $y = 0$ but while e^x converges to 0 for large positive values of x , e^{-x} converges to 0 for large negative values of x . Also while e^x is a strictly increasing function, e^{-x} is a strictly decreasing function, its derivative being always negative. Mathematically these observations can be summarized by the following equations :

$$\lim_{x \rightarrow -\infty} e^x = \lim_{x \rightarrow -\infty} e^{-x} = 0 \quad (5)$$

$$\frac{d}{dx}(e^x) = e^x \quad \frac{d}{dx}(e^{-x}) = -e^{-x} \quad (6)$$

2 Sinusoidal functions

For a fixed parameter k , plot the function $y = \sin(k \cdot x)$ for a few suitably chosen values of k . What is the role of k in determining the profile of the function? Thereafter, for $k = 1$, plot $\sin(x)$ and $\sin^2(x)$ on the same graph within $-\pi < x < \pi$. Compare both.

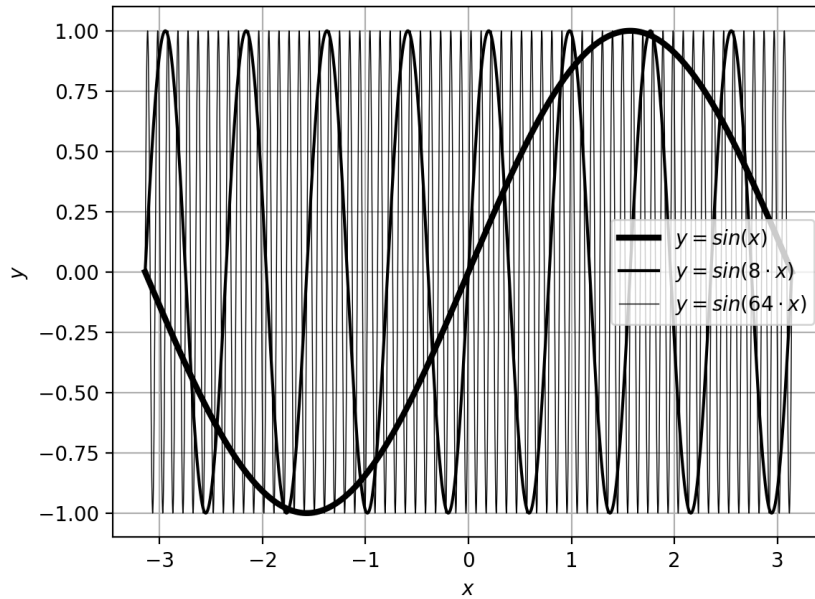


Figure 3: Plots of $y = \sin(k \cdot x)$ for different values of k

In **figure 3**, we can see how the fixed parameter k affects the behaviour of the function $y = \sin(k \cdot x)$. We see that on increasing the value of k , the sinusoids become more "dense". k is the number of cycles undergone by the sinusoid in 2π radians. When $k = 1$, from $-\pi$ to π the sinusoid has only 1 cycle. When $k = 8$, from $-\pi$ to π the sinusoid has 8 cycles.

Formally, k is defined as the **angular wave number**, which is the number of radians covered by the sinusoid per unit length.

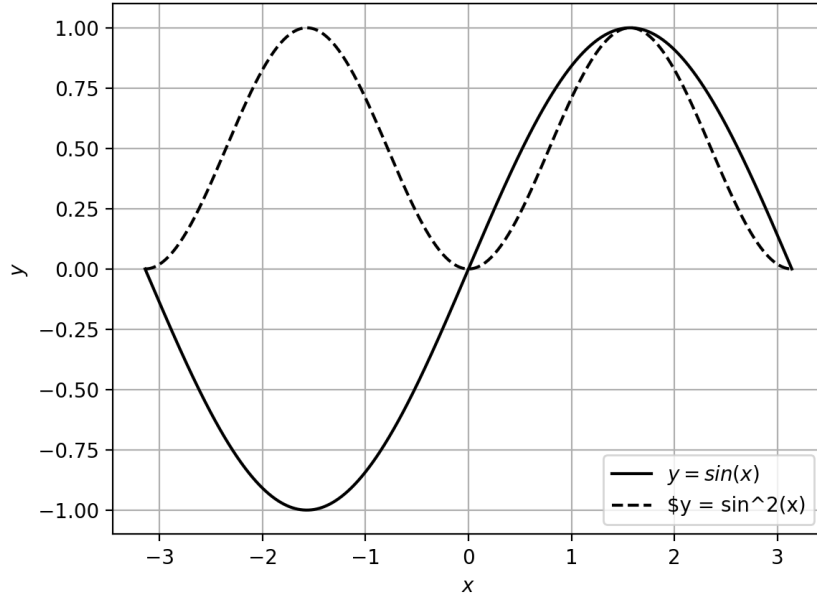


Figure 4: Plots of $y = \sin^2(x)$ and $y = \sin(x)$

In **figure 4** we can see the behavior of $y = \sin(x)$ and $y = \sin^2(x)$. Both the functions have the maxima and zeros at the same points. We have :

$$\begin{aligned}
 & -1 \leq \sin(x) \leq 1 \\
 \implies & |\sin(x)| \leq 1 \\
 \implies & \sin^2(x) \leq 1
 \end{aligned}$$

Also from the above equations, it is clear that :

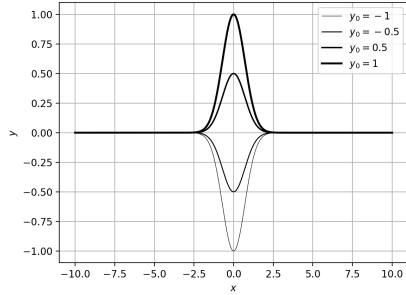
$$\sin^2(x) \leq |\sin(x)|$$

Therefore $\sin(x)$ is always greater in magnitude than $\sin^2(x)$

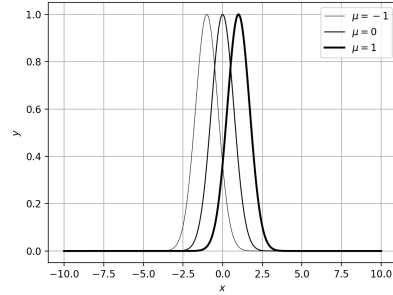
3 Gaussian and the Lorentz functions

Plot the Gaussian function $y = y_0 e^{-a(x-\mu)^2}$ for a few suitably chosen values of the fixed parameters y_0 , a and μ . Examine the shifting profile of the function with changes in the parameters ($\mu = v \cdot t$ simulates a single wave pulse like a tsunami, travelling with velocity v). Then for $y_0 = a = 1$ and $\mu = 0$, consider a first order expansion of the Gaussian function to obtain the Lorentz function. Plot both of them together and compare their behavior. For every value of x , take the difference between the two functions, and plot it against x over $0 < x < 10$.

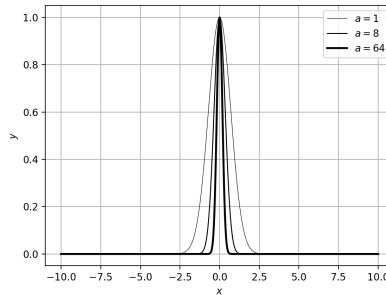
$$y = y_0 \cdot e^{-a(x-\mu)^2}$$



(a) Variation in y_0



(b) Variation in μ



(c) Variation in a

Figure 5: Variation of the Gaussian function with changes in y_0 , a , and μ

- **Dependence on y_0 :** This parameter controls the peak value of the function. It is the value of the Gaussian function at $x = \mu$, and is the maxima or minima, depending on whether y_0 is positive or negative respectively.
- **Dependence on μ :** This parameter controls the peak value of the function will occur. It is the value of x for which the function takes the

value y_0 . Therefore if μ changes with time as $\mu = v \cdot t$ then this waveform will move along the positive x-axis at the rate of v units per second.

- **Dependence on a :** This parameter controls how fast the function "decays" to 0. The more the value of a , the faster it converges to 0 about $x = \mu$.

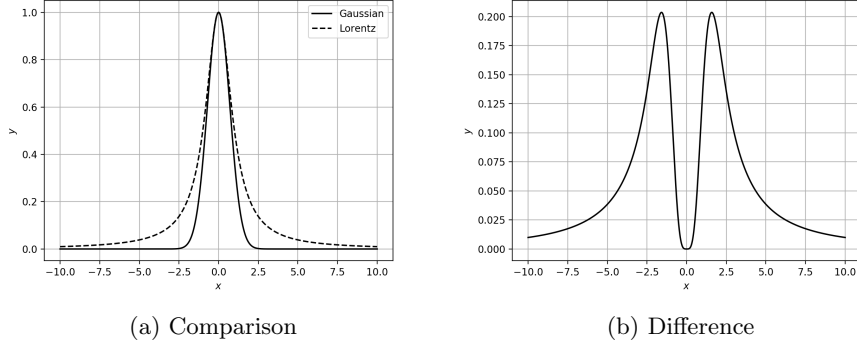


Figure 6: The Gaussian and the Lorentz function comparison and difference

For the parameters $y_0 = 1$, $a = 1$ and $\mu = 0$ we can write the following :

$$\begin{aligned} y &= e^{-x^2} \\ &= \frac{1}{e^{x^2}} \end{aligned}$$

Replacing e^{x^2} with its Taylor series expansion in the above equation, we get :

$$y = \frac{1}{1 + x^2 + \frac{x^4}{2!} + \dots} \quad (7)$$

Comparing the above equation with the Lorentz function :

$$y = \frac{1}{1 + x^2} \quad (8)$$

we can clearly see that it is the first order expansion of the Gaussian function. Also as the denominator in the Gaussian function is larger than that in the Lorentz function, it will always be smaller than the Lorentz function in magnitude. This fact is shown in the plots in **figure 6**.

4 Function $x \cdot \ln(x)$

Plot the function $y = x \cdot \ln(x)$ for $0 < x < 2$. Provide an analytical justification for what you observe. Also note the growth of the function for very large x .

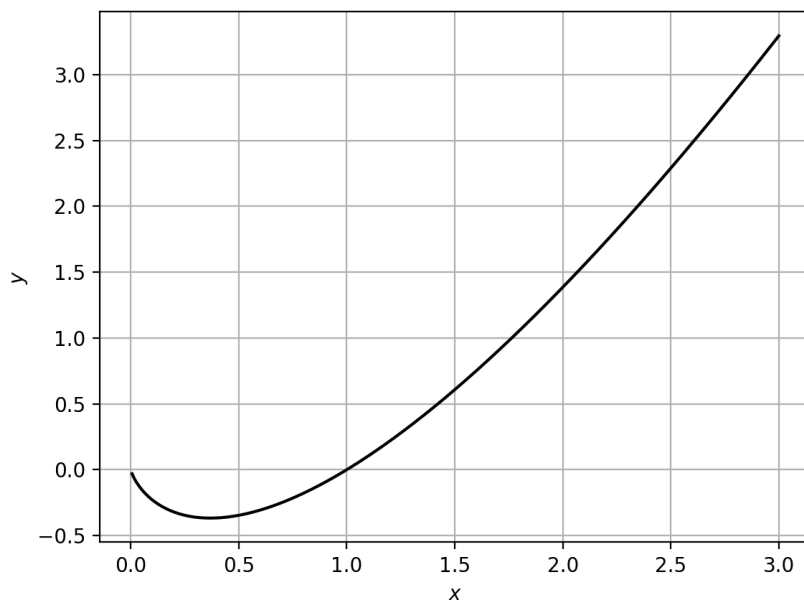


Figure 7: Plot of the function $y = x \cdot \ln(x)$

Some observations:

- This function has only one zero, at $x = 1$.
- Let us analyze if it has any extrema. We have:

$$\begin{aligned} \frac{d}{dx}(x \cdot \ln(x)) &= 0 \\ \implies 1 + \ln(x) &= 0 \\ \implies x &= \frac{1}{e} \end{aligned}$$

Also,

$$\left(\frac{d^2}{dx^2}(x \cdot \ln(x)) \right)_{x=\frac{1}{e}} = e > 0$$

Therefore, we have a minima at $x = \frac{1}{e}$.

- This function is not defined at $x = 0$ because $\ln(0)$ is undefined, but it is still continuous to the right of $x = 0$. We use the L'Hopital's rule to prove this fact:

$$\lim_{x \rightarrow 0} x \cdot \ln(x) = \lim_{x \rightarrow 0} \frac{\ln(x)}{\frac{1}{x}} = \lim_{x \rightarrow 0} \frac{\frac{d}{dx} \ln(x)}{\frac{d}{dx} \frac{1}{x}} = \lim_{x \rightarrow 0} -x = 0 \quad (9)$$