Applications of Laplace Transform in Physics

C040, C048, C049, C051

01

Modelling Radioactive Decay

- The decay of radioactive isotopes can be described by differential equations.
- The Laplace transform can be used to solve these equations and find the activity (amount of radioactive material) remaining at any given time.

Consider a radioactive isotope with:

- **$N(t)$**: Number of radioactive nuclei at time t
- **λ**: Decay constant (positive value, probability of decay per unit time)

The rate of change of $N(t)$ is proportional to the number of nuclei present and the decay constant (λ).

This can be expressed by the differential equation:

$$dN(t)/dt = -\lambda N(t)$$

Using Laplace Transform:

Take the Laplace transform of both sides of the equation:

$$L\{dN(t)/dt\} = L\{-\lambda N(t)\}$$

Applying the property of Laplace transform for derivatives (assuming N(0) is the initial number of nuclei):

$$s * N(s) - N(0) = -\lambda * N(s)$$

Solving for N(s):

Rearrange the equation:

$$N(s)(s + \lambda) = N(0)$$

$$N(s) = N(0) / (s + \lambda)$$

Finding N(t):

This is the solution in the s-domain. To get N(t) (number of nuclei in the time domain), we need to perform the inverse Laplace transform.

$$N(t) = L^{-1}\{ N(0) / (s + \lambda) \}$$

The inverse Laplace transform of $1/(s + \lambda)$ is $e^{(-\lambda t)}$.

Therefore:

$$N(t) = N(0) * e^{(-\lambda t)}$$

This equation shows that the number of radioactive nuclei (N(t)) decays exponentially with time (t) with a rate constant of λ. N(0) represents the initial number of nuclei present.

02

Solving Heat Transfer Problems

- Heat flow is often governed by partial differential equations.
- The Laplace transform can be applied to convert these equations into a more manageable form, allowing for the analysis of temperature distribution within a material or system.

Heat transfer within a material
can be described by the *one-dimensional heat equation*:

$$\partial u(x,t) / \partial t = \alpha \, \partial^2 u(x,t) / \partial x^2$$

where:

- $u(x,t)$ is the temperature at position x (along a rod or material) and time t
- α is the thermal diffusivity (material property related to heat conduction)

Using Laplace Transform:

$$L \{ \partial u(x,t) / \partial t \} = L \{ \alpha \, \partial^2 u(x,t) / \partial x^2 \}$$

The Laplace transform property for derivatives gives: (assuming initial temperature distribution u(x,0) is known)

$$s * U(x,s) - u(x,0) = \alpha * d^2U(x,s) / dx^2$$

where:

- U(x,s) is the Laplace transform of u(x,t) with respect to time
- (t)s is the Laplace transform variable

Solving for U(x,s):

We define boundary conditions, which are the fixed temperatures at both ends of the rod (x=0 and x=L). These are also transformed to the Laplace domain (U(0,s) and U(L,s)).

Finding u(x,t):

By solving the transformed equation with these boundary conditions, we will get U(x,s).

To find temperature distribution in the time domain (u(x,t)) we will finally perform Laplace inverse, which gives us the actual temperature distribution within the material over time (u(x,t)).

The Laplace transform takes a complex equation describing heat flow over time and transforms it into a series of simpler steps. This allows us to solve for the temperature distribution within the material and understand how it changes over time.

03

Studying Mechanical Systems

- Motion of objects and vibrations of structures are also described by differential equations.
- The Laplace transform can be used to analyze the behavior of these systems, such as finding the displacement of a mass on a spring or the natural frequencies of a bridge.

Consider a mass (m) connected to a spring with spring constant (k) and a damper with damping coefficient (b). The system is subjected to an external force (F(t)).

The equation governing the motion of the mass (x(t)) is:

$$m * \frac{d^2x(t)}{dt^2} + b * \frac{dx(t)}{dt} + k * x(t) = F(t)$$

This represents the balance of forces:

Left side:
Inertia ($m * d^2x/dt^2$) + Damping ($b * dx/dt$) + Spring force ($k * x$)

Right side:
External force (F(t))

Using Laplace Transform:

Apply Laplace transform to both sides:

$L\{ m * d^2x(t) / dt^2 \} + L\{ b * dx(t) / dt \} + L\{ k * x(t) \} = L\{ F(t)\}$

Utilize the properties of Laplace transform for derivatives (assuming initial conditions x(0) and dx(t)/dt at t=0 are known):

$ms^2 * X(s) + bs * X(s) + kx(s) = F(s) + mx(0) + bsx(0)$

where:

- X(s) is the Laplace transform of x(t)
- s is the Laplace transform variable

Solving for X(s):

This becomes an algebraic equation in terms of s, which can be easily solved for X(s):

$X(s) = (F(s) + mx(0) + bsx(0)) / (ms^2 + bs + k)$

Finding x(t):

This gives us X(s), the solution in the s-domain. To find the actual displacement of the mass (x(t)), we need to perform the inverse Laplace transform of X(s).

The Laplace transform converts the complex differential equation of motion into a simpler algebraic equation. This allows for easier analysis of the system's response to the external force, including: Displacement (x(t)), Natural Frequencies, Stability.

04

Fluid Mechanics

- Fluid flow can be described by partial differential equations involving pressure, velocity, and density. The Laplace transform can be used to solve these equations for specific scenarios.
- For example, analyzing the transient response of fluid flow in a pipe after a valve is suddenly opened.

Example: Transient Pipe Flow

Consider a pipe initially filled with stagnant fluid. A valve at one end is suddenly opened, allowing fluid to flow into the pipe. We want to analyze the transient pressure and velocity within the pipe.

Linearized Equation (assuming small pressure variations):

$$\partial p(x,t) / \partial t = - c * \partial v(x,t) / \partial x$$

(where c is a constant related to fluid properties)

Step 1:
Apply Laplace transform with respect to time (t):

$$s * P(x,s) - p(x,0) = - c * dV(x,s) / dx$$

(where P is the Laplace transform of p)

Step 2:
Solve for the transformed pressure **(P(x,s))** considering appropriate boundary conditions (pressure at the open end and initial conditions).

Step 3:
Apply the inverse Laplace transform to find the pressure distribution **p(x,t)** within the pipe over time.

By solving this simplified equation, we can understand how the pressure within the pipe builds up as the flow gets established after the valve is opened.

This approach can be extended to more complex scenarios in fluid mechanics using the Laplace transform after appropriate linearization and consideration of relevant physical properties.

THANK YOU!

Complex Variables and Transformations
Application of Laplace Transform in Physics

B.Tech Integrated (Computers)
Batch B2

C040 – Rahil Jethwa
C048 – Pratham Vasa
C049 – Chahel Gupta
C051 – Siddh Patel

- "Nuclear Physics: Principles and Applications" by SN Ghoshal [1]
- "Fundamentals of Heat and Mass Transfer" by Theodore L. Bergman et al. [2]
- "Mechanical Vibrations" by Singiresu S. Rao [3]
- "Fundamentals of Fluid Mechanics" by Bruce R. Munson et al. [4]