Model Order Reduction Techniques
Problem Set 1: RB for Linear Affine Elliptic Problems
Design of a Thermal Fin

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1 Introduction

We consider the problem of designing a thermal fin to effectively remove heat from a surface. The two-dimensional fin, shown in Figure 1, consists of a vertical central "post" and four horizontal "subfins"; the fin conducts heat from a prescribed uniform flux "source" at the root, Γ_{root}, through the large-surface-area subfins to surrounding flowing air. The fin is characterized by a five-component parameter vector, or "input," $\mu = (\mu_1, \mu_2, \ldots, \mu_5)$, where $\mu_i = k_i$, $i = 1, \ldots, 4$, and $\mu_5 = \text{Bi}$; μ may take on any value in a specified design set $D \subset \mathbf{R}^5$.

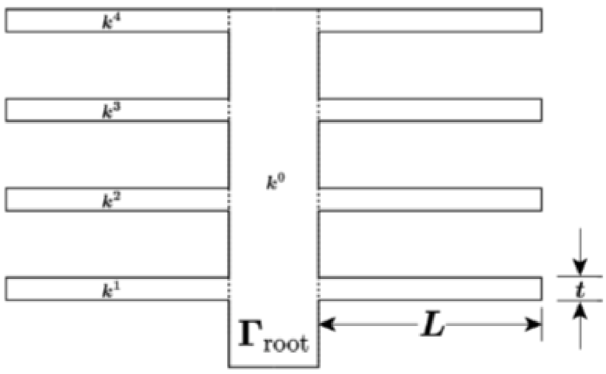

Figure 1: Thermal fin

Here k_i is the thermal conductivity of the ith subfin (normalized relative to the post conductivity $k_0 \equiv 1$); and Bi is the Biot number, a nondimensional heat transfer coefficient reflecting convective transport to the air at the fin surfaces (larger Bi means better heat transfer). For example, suppose we choose a thermal fin with $k_1 = 0.4$, $k_2 = 0.6$, $k_3 = 0.8$, $k_4 = 1.2$, and $\text{Bi} = 0.1$; for this particular configuration $\mu = \{0.4, 0.6, 0.8, 1.2, 0.1\}$, which corresponds to a single point in the set of all possible configurations D (the parameter or design set). The post is of width unity and height four; the subfins are of fixed thickness $t = 0.25$ and length $L = 2.5$.

2 Part 1 - Finite Element Approximation

We saw in class that the reduced basis approximation is based on a "truth" finite element approximation of the exact (or analytic) problem statement. To begin, we have to show that the exact problem described above does indeed satisfy the affine parameter dependence and thus fits into the framework shown in class.

2.1 a)

Show that $u_e(\mu) \in X_e \equiv H^1(\Omega)$ satisfies the weak form

$$a(u_e(\mu), v; \mu) = \ell(v), \quad \forall v \in X_e, \tag{1}$$

with

$$a(w, v; \mu) = \sum_{i=0}^{4} k_i \int_{\Omega_i} \nabla w \cdot \nabla v \, dA + \mathrm{Bi} \int_{\Gamma \backslash \Gamma_{\mathrm{root}}} wv \, dS, \tag{2}$$

$$\ell(v) = \int_{\Gamma_{\mathrm{root}}} v \tag{3}$$

2.2 b)

Show that $u_e(\mu)$ is the argument that minimizes

$$J(w) = \frac{1}{2} \sum_{i=0}^{4} k_i \int_{\Omega_i} \nabla w \cdot \nabla w \, dA + \frac{\mathrm{Bi}}{2} \int_{\Gamma \backslash \Gamma_{\mathrm{root}}} w^2 \, dS - \int_{\Gamma_{\mathrm{root}}} w \, dS \tag{4}$$

over all functions w in X_e.

We now consider the linear finite element space

$$X_N = \{ v \in H^1(\Omega) \mid v|_{T_h} \in P_1(T_h), \forall T_h \in T_h \}, \tag{5}$$

and look for $u_N(\mu) \in X_N$ such that

$$a(u_N(\mu), v; \mu) = \ell(v), \quad \forall v \in X_N; \tag{6}$$

our output of interest is then given by

$$T_N^{\mathrm{root}}(\mu) = \ell_O(u_N(\mu)). \tag{7}$$

Applying our usual nodal basis, we arrive at the matrix equations

$$\mathbf{A}_N(\mu) \mathbf{u}_N(\mu) = \mathbf{F}_N, \tag{8}$$

$$T_N^{\mathrm{root}}(\mu) = (\mathbf{L}_N)^T \mathbf{u}_N(\mu), \tag{9}$$

where $\mathbf{A}_N \in \mathbf{R}^{N \times N}$, $\mathbf{u}_N \in \mathbf{R}^N$, $\mathbf{F}_N \in \mathbf{R}^N$, and $\mathbf{L}_N \in \mathbf{R}^N$; here N is the dimension of the finite element space X_N, which (given our natural boundary conditions) is equal to the number of nodes in T_h.

3 Part 2 - Reduced-Basis Approximation

In general, the dimension of the finite element space, $\dim X = N$, will be quite large (in particular if we were to treat the more realistic three-dimensional fin problem), and thus the solution of $\mathbf{A}_N \mathbf{u}_N(\mu) = \mathbf{F}_N$ can be quite expensive. We thus investigate the reduced-basis methods that allow us to accurately and very rapidly predict $T_{\mathrm{root}}(\mu)$ in the limit of many evaluations — that is, at many different values of μ — which is precisely the "limit of interest" in design and optimization studies. To derive the reduced-basis approximation we shall exploit the energy principle,

$$u(\mu) = \arg\min_{w \in X} J(w), \tag{10}$$

where $J(w)$ is given by (8).

To begin, we introduce a sample in parameter space, $S_N = \{\mu_1, \mu_2, \ldots, \mu_N\}$ with $N \ll \mathcal{N}$. Each μ_i, $i = 1, \ldots, N$, belongs in the parameter set $\mathcal{D}$. For our parameter set we choose $\mathcal{D} = [0.1, 10.0]^4 \times [0.01, 1.0]$, that is, $0.1 \le k_i \le 10.0$, $i = 1, \ldots, 4$ for the conductivities, and $0.01 \le \mathrm{Bi} \le 1.0$ for the Biot number. We then introduce the reduced-basis space as

$$W_N = \mathrm{span}\{u_{\mathcal{N}}(\mu_1), u_{\mathcal{N}}(\mu_2), \ldots, u_{\mathcal{N}}(\mu_N)\} \tag{11}$$

where $u_{\mathcal{N}}(\mu_i)$ is the finite-element solution for $\mu = \mu_i$.

To simplify the notation, we define $\xi_i \in X$ as $\xi_i = u_{\mathcal{N}}(\mu_i)$, $i = 1, \ldots, N$; we can then write $W_N = \mathrm{span}\{\xi_i, i = 1, \ldots, N\}$. Recall that $W_N = \mathrm{span}\{\xi_i, i = 1, \ldots, N\}$ means that W_N consists of all functions in X that can be expressed as a linear combination of the ξ_i; that is, any member v_N of W_N can be represented as

$$v_N = \sum_{i=1}^{N} \beta_j \xi_j, \tag{12}$$

for some unique choice of $\beta_j \in \mathbf{R}$, $j = 1, \ldots, N$. (We implicitly assume that the ξ_i, $i = 1, \ldots, N$, are linearly independent; it follows that W_N is an N-dimensional subspace of $X_{\mathcal{N}}$.) In the reduced-basis approach we look for an approximation $u_N(\mu)$ to $u_{\mathcal{N}}(\mu)$ (which for our purposes here we presume is arbitrarily close to $u_e(\mu)$) in W_N; in particular, we express $u_N(\mu)$ as

$$u_N(\mu) = \sum_{i=1}^{N} u_j^N \xi_j; \tag{13}$$

we denote by $\mathbf{u}_N(\mu) \in \mathbf{R}^N$ the coefficient vector $(u_1^N, \ldots, u_N^N)^T$. The premise — or hope — is that we should be able to accurately represent the solution at some new point in parameter space, μ, as an appropriate linear combination of solutions previously computed at a small number of points in parameter space (the μ_i, $i = 1, \ldots, N$). But how do we find this appropriate linear combination? And how good is it? And how do we compute our approximation efficiently? The energy principle is crucial here (though more generally the weak form would suffice). To wit, we apply the classical Rayleigh-Ritz procedure to define

$$u_N(\mu) = \arg \min_{w_N \in W_N} J(w_N); \tag{14}$$

alternatively we can apply Galerkin projection to obtain the equivalent statement

$$a(u_N(\mu), v; \mu) = \ell(v), \quad \forall v \in W_N. \tag{15}$$

The output can then be calculated from

$$T_{\mathrm{root}}^N(\mu) = \ell_O(u_N(\mu)). \tag{16}$$

We now study this approximation in more detail.

3.1 a)

Prove that, in the energy norm $\|\cdot\| \equiv (a(,;\mu))^{1/2}$,

$$\|u(\mu) - u_N(\mu)\| \le \|u(\mu) - w_N\|, \quad \forall w_N \in W_N. \tag{17}$$

This inequality indicates that out of all the possible choices of w_N in the space W_N, the reduced basis method defined above will choose the "best one" (in the energy norm). Equivalently, we can say that even if we knew the solution $u(\mu)$, we would not be able to find a better approximation to $u(\mu)$ in W_N — in the energy norm — than $u_N(\mu)$.

3.2 b)

Prove that

$$T_{\text{root}}(\mu) - T^N_{\text{root}}(\mu) = \|u(\mu) - u_N(\mu)\|^2. \tag{18}$$

3.3 c)

Show that $u_N(\mu)$ as defined in (17)-(19) satisfies a set of $N \times N$ linear equations,

$$\mathbf{A}_N(\mu)\mathbf{u}_N(\mu) = \mathbf{F}_N; \tag{19}$$

and that

$$T^N_{\text{root}}(\mu) = \mathbf{L}_N^T \mathbf{u}_N(\mu). \tag{20}$$

Give expressions for $\mathbf{A}_N(\mu) \in \mathbf{R}^{N\times N}$ in terms of $\mathbf{A}_N(\mu)$ and Z, $\mathbf{F}_N \in \mathbf{R}^N$ in terms of $\mathbf{F}_N$ and Z, and $\mathbf{L}_N \in \mathbf{R}^N$ in terms of $\mathbf{L}_N$ and Z; here Z is an $N \times N$ matrix, the jth column of which is $\mathbf{u}_N(\mu_j)$ (the nodal values of $\mathbf{u}_N(\mu_j)$).

3.4 d)

Show that the bilinear form $a(w, v; \mu)$ can be decomposed as

$$a(w, v; \mu) = \sum_{q=0}^{Q} \theta_q(\mu) a_q(w, v), \quad \forall w, v \in X, \forall \mu \in D, \tag{21}$$

for $Q = 6$ and give expressions for the $\theta_q(\mu)$ and the $a_q(w, v)$. Notice that the $a_q(w, v)$ are not dependent on μ; the parameter dependence enters only through the functions $\theta_q(\mu)$, $q = 1, \ldots, Q$. Further show that

$$\mathbf{A}_N(\mu) = \sum_{q=1}^{Q} \theta_q(\mu) \mathbf{A}_{Nq}, \tag{22}$$

and

$$\mathbf{A}_N(\mu) = \sum_{q=1}^{Q} \theta_q(\mu) \mathbf{A}_q^N, \tag{23}$$

Give an expression for the $\mathbf{A}_{Nq}$ in terms of the nodal basis functions; and develop a formula for the $\mathbf{A}_q^N$ in terms of the $\mathbf{A}_{Nq}$ and Z.

3.5 e)

The coercivity and continuity constants of the bilinear form for the continuous problem are denoted by $\alpha_e(\mu)$ and $\gamma_e(\mu)$, respectively. We now assume that the basis function ξ_i, $i = 1, \ldots, N$, are orthonormalized. Show that the condition number of $\mathbf{A}_N(\mu)$ is then bounded from above by $\gamma_e(\mu)/\alpha_e(\mu)$.

3.6 f)

Take into account the parameters L and t as parameters: the geometric transformation must be taken into account in the affine decomposition procedure.

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4 Finite Element

4.1 a)

Show that $u_e(\mu) \in X_e \equiv H^1(\Omega)$ satisfies the weak form

$$a(u_e(\mu), v; \mu) = \ell(v), \quad \forall v \in X_e, \tag{24}$$

with

$$a(w, v; \mu) = \sum_{i=0}^{4} k_i \int_{\Omega_i} \nabla w \cdot \nabla v \, dA + \mathrm{Bi} \int_{\Gamma \backslash \Gamma_{\text{root}}} wv \, dS, \tag{25}$$

$$\ell(v) = \int_{\Gamma_{\text{root}}} v \tag{26}$$

The first we can rewrite the problem on this system of equation

$$\begin{cases} -k_i \Delta u_i = 0 & \text{in } \Omega_i, \quad i = 0, \ldots, 4, \\ u_0 = u_i & \text{on } \Gamma_i^{\text{int}}, \quad i = 1, \ldots, 4, \\ -(\nabla u_0 \cdot n_i) = -k_i(\nabla u_i \cdot n_i) & \text{on } \Gamma_i^{\text{int}}, \quad i = 1, \ldots, 4, \\ -(\nabla u_0 \cdot n_0) = -1 & \text{on } \Gamma_{\text{root}}, \\ -k_i(\nabla u_i \cdot n_i) = \mathrm{Bi} u_i & \text{on } \Gamma_i^{\text{ext}}, \quad i = 0, \ldots, 4, \end{cases} \tag{27}$$

To answewrs that, we choose the work space $X^e = H^1$ and let $v \in X^e$ a test function. For $i = 0$, the variational formulation is obtained by multiplying the differential equation by a test function $v \in X_e$ and integrating over the domain Ω_0. Using the Green's theorem, we get:

$$\int_{\Omega_0} k_0 \nabla u_0 \cdot \nabla v \, dA - \int_{\partial \Omega_0} k_0 (\nabla u_0 \cdot n) v \, dS = 0. \tag{28}$$

Applying the boundary conditions, we have:

$$\int_{\Omega_0} k_0 \nabla u_0 \cdot \nabla v \, dA - \int_{\Gamma_i^{\text{int}}} k_0 (\nabla u_0 \cdot n_i) v \, dS - \int_{\Gamma_{\text{root}}} (\nabla u_0 \cdot n_0) v \, dS - \int_{\Gamma_0^{\text{ext}}} k_0 (\nabla u_0 \cdot n_0) v \, dS = 0. \tag{29}$$

Using the boundary conditions and the fact that Ω_0 not disjoint with Ω_i for $i = 1, 2, 3, 4$, we have:

$$\begin{aligned} \int_{\Omega_0} k_0 \nabla u_0 \cdot \nabla v \, dA - \sum_{i=1}^{4} \int_{\Gamma_i^{\text{int}}} k_i (\nabla u_i \cdot n_i) v \, dS - \int_{\Gamma_{\text{root}}} v \, dS \\ - \int_{\Gamma_0^{\text{ext}}} \mathrm{Bi} u_0 v \, dS = 0. \end{aligned} \tag{30}$$

Simplifying, we obtain the variational formulation for $i = 0$:

$$\int_{\Omega_0} k_0 \nabla u_0 \cdot \nabla v \, dA - \sum_{i=1}^{4} \int_{\Gamma_i^{\text{int}}} k_i (\nabla u_i \cdot n_i) v \, dS - \int_{\Gamma_{\text{root}}} v \, dS + \int_{\Gamma_0^{\text{ext}}} \mathrm{Bi} u_0 v \, dS = 0. \tag{31}$$

For $i = 1, 2, 3, 4$, the variational formulation is obtained similarly by multiplying the differential equation by a test function $v \in X_e$ and integrating over the domain Ω_i. Using Green's theorem, we get:

$$\int_{\Omega_i} k_i \nabla u_i \cdot \nabla v \, dA - \int_{\partial\Omega_i} k_i (\nabla u_i \cdot n) v \, dS = 0. \tag{32}$$

Applying the boundary conditions, we have:

$$\int_{\Omega_i} k_i \nabla u_i \cdot \nabla v \, dA - \int_{\Gamma_i^{\text{int}}} k_i (\nabla u_i \cdot n_i) v \, dS - \int_{\Gamma_i^{\text{ext}}} k_i (\nabla u_i \cdot n_i) v \, dS = 0. \tag{33}$$

Using the boundary conditions:

$$\int_{\Omega_i} k_i \nabla u_i \cdot \nabla v \, dA - \int_{\Gamma_i^{\text{int}}} k_i (\nabla u_i \cdot n_i) v \, dS + \int_{\Gamma_i^{\text{ext}}} \text{Bi} u_i v \, dS = 0. \tag{34}$$

Simplifying, we obtain the variational formulation for $i = 1, 2, 3, 4$:

$$\int_{\Omega_i} k_i \nabla u_i \cdot \nabla v \, dA - \int_{\Gamma_i^{\text{int}}} k_i (\nabla u_i \cdot n_i) v \, dS + \int_{\Gamma_i^{\text{ext}}} \text{Bi} u_i v \, dS = 0. \tag{35}$$

Summing the variational formulations for $i = 0$ and $i = 1, 2, 3, 4$, we obtain the overall variational formulation:

$$\sum_{i=0}^{4} \int_{\Omega_i} k_i \nabla u_i \cdot \nabla v \, dA + \underbrace{\left[\sum_{i=1}^{4} \int_{\Gamma_i^{\text{int}}} k_i (\nabla u_i \cdot n_i) v \, dS - \sum_{i=1}^{4} \int_{\Gamma_i^{\text{int}}} (\nabla u_0 \cdot n_i) v \, dS \right]}_{\text{The normal is opposite to each other}} + \sum_{i=0}^{4} \int_{\Gamma_i^{\text{ext}}} \text{Bi} u_i v \, dS = \int_{\Gamma_{\text{root}}} v \, dS. \tag{36}$$

Since the internal boundaries Γ_i^{int} are shared between adjacent subdomains, the contributions from these boundaries will cancel out. Therefore, the final variational formulation is:

$$\sum_{i=0}^{4} k_i \int_{\Omega_i} \nabla u_i \cdot \nabla v \, dA + \text{Bi} \int_{\Gamma \setminus \Gamma_{\text{root}}} uv \, dS = \int_{\Gamma_{\text{root}}} v \, dS. \tag{37}$$

This shows that $u_e(\mu) \in X_e \equiv H^1(\Omega)$ satisfies the weak form:

$$a(u_e(\mu), v; \mu) = \ell(v), \quad \forall v \in X_e, \tag{38}$$

with

$$a(w, v; \mu) = \sum_{i=0}^{4} k_i \int_{\Omega_i} \nabla w \cdot \nabla v \, dA + \text{Bi} \int_{\Gamma \setminus \Gamma_{\text{root}}} wv \, dS, \tag{39}$$

$$\ell(v) = \int_{\Gamma_{\text{root}}} v \, dS. \tag{40}$$

4.2 b)

Show that $u_e(\mu)$ is the argument that minimizes

$$J(w) = \frac{1}{2} \sum_{i=0}^{4} k_i \int_{\Omega_i} \nabla w \cdot \nabla w \, dA + \frac{\text{Bi}}{2} \int_{\Gamma \setminus \Gamma_{\text{root}}} w^2 \, dS - \int_{\Gamma_{\text{root}}} w \, dS \tag{41}$$

over all functions w in X_e. Our objective is to determine the function $u_e(\mu) \in X^e$ such that

$$J(u_e(\mu)) = \min_{w \in X^e} J(w).$$

The first we have X_e is a Hilbert space, so is convex and nonempty set. Now, we prove that J is continuous for that :

$$\begin{aligned}
|J(w)| &= |\frac{1}{2}\sum_{i=0}^{4} k_i \int_{\Omega_i} \nabla w \cdot \nabla w \, dA + \frac{\text{Bi}}{2}\int_{\Gamma\backslash\Gamma_{\text{root}}} w^2 \, dS - \int_{\Gamma_{\text{root}}} w \, dS| \\
&\leq \frac{1}{2}\sum_{i=0}^{4} k_i \int_{\Omega_i} |\nabla w|^2 dA + \frac{\text{Bi}}{2}\int_{\Gamma\backslash\Gamma_{\text{root}}} |w|^2 dS + \int_{\Gamma_{\text{root}}} |w| dS \\
&\leq \frac{1}{2}\sum_{i=0}^{4} k_i ||\nabla w||^2_{L^2(\Omega_i)} + \frac{\text{Bi}}{2}||w||^2_{L^2(\Gamma\backslash\Gamma_{\text{root}})} + ||w||_{L^2(\Gamma_{\text{root}})} \qquad (42) \\
&\leq \frac{1}{2}\sum_{i=0}^{4} k_i ||w||^2_{H^1(\Omega_i)} + \frac{\text{Bi}}{2}||w||^2_{H^1(\Gamma\backslash\Gamma_{\text{root}})} + ||w||_{H^1(\Gamma_{\text{root}})} \\
&\leq (\frac{1}{2}\sum_{i=0}^{4} k_i ||w||_{H^1(\Omega_i)} + \frac{C_1\text{Bi}}{2}||w||_{H^1(\Gamma\backslash\Gamma_{\text{root}})} + C_2)||w||_{H^1(\Omega)}
\end{aligned}$$

Thus, J is continuous. It rest to prove that J is coercive, for that :

$$\begin{aligned}
|J(w)| &= |\frac{1}{2}\sum_{i=0}^{4} k_i \int_{\Omega_i} \nabla w \cdot \nabla w \, dA + \frac{\text{Bi}}{2}\int_{\Gamma\backslash\Gamma_{\text{root}}} w^2 \, dS - \int_{\Gamma_{\text{root}}} w \, dS| \\
&\geq \frac{1}{2}\sum_{i=0}^{4} k_i ||\nabla w||^2_{L^2(\Omega_i)} + \frac{\text{Bi}}{2}||w||^2_{L^2(\Gamma\backslash\Gamma_{\text{root}})} - ||w||_{H^1(\Gamma_{\text{root}})} \quad (||w||_{L^2} \leq ||w||_{H^2}) \qquad (43) \\
&\geq \frac{1}{2}k_{min}||w||^2_{L^2(\Omega_i)} + \frac{\text{Bi}}{2}||w||^2_{L^2(\Gamma\backslash\Gamma_{\text{root}})} - ||w||_{H^1(\Gamma_{\text{root}})} \\
&\geq M||w||^2_{H^1(\Omega)} - ||w||_{H^1(\Omega)} \quad \text{(using inequality of poincaré moyen)}
\end{aligned}$$

Thus, J is coercive.
Therefore, J is continuous and coercive, then J has a minimum in X^e. We can characterized them by Euler, finally $u_e(\mu)$ is the argument that minimizes $J(w)$ over all functions w in X^e.

5 Part 2 - Reduced-Basis Approximation

5.1 a)

Prove that, in the energy norm $|||\cdot||| \equiv (a(\cdot,\cdot;\mu))^{1/2}$,

$$|||u(\mu) - u_N(\mu)||| \leq |||u(\mu) - w_N|||, \forall w_N \in W_N. \qquad (44)$$

To prove that, we use the Galerkin orthogonality and the bilinearity of a. Given $w_N \in W_N$, we can write that

$$\begin{aligned}
|||u(\mu) - u_N(\mu)|||^2 &= a(u(\mu) - u_N(\mu), u(\mu) - u_N(\mu); \mu) \\
&= a(u(\mu) - u_N(\mu), u(\mu) - w_N + w_N - u_N(\mu); \mu) \\
&= a(u(\mu) - u_N(\mu), u(\mu) - w_N; \mu) - a(u(\mu) - \underbrace{u_N(\mu)}_{\in W_N}, \underbrace{u_N(\mu) - w_N}_{\in W_N}; \mu)
\end{aligned}$$

By applying the Galerkin orthogonality in W_N, we have that $a(u-u_N, u_N-w_N;\mu)=0$. Therefore, Or, we have that

$$\begin{aligned} a(u(\mu)-u_N(\mu), u(\mu)-w_N;\mu) &= a(u(\mu), u(\mu)-w_N;\mu) - a(u_N(\mu), u(\mu)-w_N;\mu) \\ &= \ell(u(\mu)-w_N) - \ell(u(\mu)-w_N) = 0 \end{aligned}$$

Thus, we obtain that

$$\begin{aligned} |||u(\mu)-u_N(\mu)|||^2 &= a(u(\mu)-u_N(\mu), u(\mu)-w_N;\mu) \\ &\underbrace{\leq}_{Cauchy-Schwarz} \quad |||u(\mu)-u_N(\mu)|||\cdot|||u(\mu)-w_N||| \end{aligned}$$

Finally, we the result,

$$|||u(\mu)-u_N(\mu)||| \leq |||u(\mu)-w_N||| \quad \forall w_N \in W_N \tag{45}$$

5.2 b)

Prove that

$$T_{\text{root}}(\mu) - T_{\text{root},N}(\mu) = |||u(\mu)-u_N(\mu)|||^2. \tag{21}$$

We have that, $T_{\text{root}}(\mu)=\ell^O(u(\mu))$ and $T_{\text{root},N}(\mu)=\ell^O(u_N(\mu))$. So, we can write that

$$\begin{aligned} T_{\text{root}}(\mu) - T_{\text{root},N}(\mu) &= \ell^O(u(\mu)) - \ell^O(u_N(\mu)) \\ &= \ell(u(\mu)) - \ell(u_N(\mu)) \\ &= a(u(\mu), u(\mu);\mu) - a(u(\mu), u_N(\mu);\mu) \\ &= a(u(\mu), u(\mu)-u_N(\mu);\mu) \quad (*) \\ &= a(u(\mu)-u_N(\mu), u(\mu)-u_N(\mu);\mu) \quad (**) \\ &= |||u(\mu)-u_N(\mu)|||^2 \quad \text{(by definition of the energy norm)} \end{aligned}$$

In $(*)$ to $(**)$, we used that for all $u\in X$, there exists $v\in X$ such that $a(u,v)=\ell(v)$, then in particular we can take $u=u(\mu)-u_N(\mu)$. Then we have the result.

5.3 c)

Show that $u_N(\mu)$ as defined in (17)-(19) satisfies a set of $N\times N$ linear equations,

$$\mathbf{A}_N(\mu)\mathbf{u}_N(\mu) = \mathbf{F}_N; \tag{46}$$

and that

$$T_{\text{root},N}(\mu) = \mathbf{L}_N^T\mathbf{u}_N(\mu). \tag{47}$$

Give expressions for $\mathbf{A}_N(\mu)\in\mathbf{R}^{N\times N}$ in terms of $\mathbf{A}^{\mathcal{N}}(\mu)$ and Z, $\mathbf{F}^N\in\mathbf{R}^N$ in terms of $\mathbf{F}^{\mathcal{N}}$ and Z, and $\mathbf{L}^N\in\mathbf{R}^N$ in terms of $\mathbf{L}^{\mathcal{N}}$ and Z; here Z is an $\mathcal{N}\times N$ matrix, the jth column of which is $\mathbf{u}_N(\mu^j)$ (the nodal values of $\mathbf{u}^{\mathcal{N}}(\mu^j)$).
We give the two relations (17) and (18) :

$$u(\mu) = \arg\min_{w\in X} J(w), \tag{48}$$

$$T_{\text{root}N}(\mu) = \ell^O(u_N(\mu)). \tag{49}$$

To show that, we have that $u_N(\mu)$ is the solution of the variational problem

$$a(u_N(\mu), v;\mu) = \ell(v) \quad \forall v\in W_N$$

and, we have the decomposition of $u_N(\mu)$ in the basis of W_N:

$$u_N(\mu) = \sum_{j=1}^{N} u_j(\mu)\xi^j$$

where ξ_j is the nodal basis function of W_N. Then, we replace $u_N(\mu)$ in the variational problem, we have that

$$\begin{aligned} a(u_N(\mu), v; \mu) &= \ell(v) \quad \forall v \in W_N \\ a(\sum_{i=1}^{N} u_i(\mu)\xi^i, v; \mu) &= \ell(v) \quad \forall v \in W_N \\ \sum_{i=1}^{N} u_i(\mu) a(\xi^i, \xi^j; \mu) &= \ell(\xi^j) \quad \forall j = 1, \dots, N \quad (*) \end{aligned}$$

It is a system of $N \times N$ linear of this form:

$$\begin{pmatrix} a(\xi^1, \xi^1; \mu) & \cdots & a(\xi^1, \xi^N; \mu) \\ \vdots & \ddots & \vdots \\ a(\xi^N, \xi^1; \mu) & \cdots & a(\xi^N, \xi^N; \mu) \end{pmatrix} \begin{pmatrix} u_1(\mu) \\ \vdots \\ u_N(\mu) \end{pmatrix} = \begin{pmatrix} \ell(\xi^1) \\ \vdots \\ \ell(\xi^N) \end{pmatrix}$$

Samplify, we have that

$$\mathbf{A}_N(\mu)\mathbf{u}_N(\mu) = \mathbf{F}_N$$

Or for the output, we have that

$$\begin{aligned} T_{\text{root},N}(\mu) &= \ell^O(u_N(\mu)) \\ &= \ell(u_N(\mu)) \\ &= \ell(\sum_{j=1}^{N} u_j(\mu)\xi^j) \\ &= \sum_{j=1}^{N} u_j(\mu)\ell(\xi^j) \quad (**) \\ &= \mathbf{L}_N^T \mathbf{u}_N(\mu) \end{aligned}$$

with $\mathbf{L}_N = (\ell(\xi^1), \dots, \ell(\xi^N))^T$ In $(*)$ and $(**)$, we used the linearity of the bilinear form a and the linear form ℓ. Don't forget that each ξ^j is a nodal basis function of $X^{\mathcal{N}}$, it decomposed in the basis of $X^{\mathcal{N}}$:

$$\xi^j = \sum_{i=1}^{\mathcal{N}} Z_i \xi_i^j(\mu^i)$$

Or, we have that $\mathbf{A}^{\mathcal{N}} u^{\mathcal{N}} = \mathbf{F}^{\mathcal{N}}$ and $\mathbf{T}_{\text{root}\mathcal{N}} = \mathbf{L}^{\mathcal{N}} u^{\mathcal{N}}(\mu)$. Then, we can write that

$$\begin{cases} \mathbf{A}^{\mathcal{N}}(\mu)_{i,j} = a(Z_i, Z_j; \mu) \quad \forall i, j = 1, \dots, \mathcal{N} \\ u^{\mathcal{N}}(\mu) = (u_1^{\mathcal{N}}(\mu), \dots, u_{\mathcal{N}}^{\mathcal{N}}(\mu))^T \\ \mathbf{F}^{\mathcal{N}} = (\ell(Z_1), \dots, \ell(Z_{\mathcal{N}}))^T \\ \mathbf{L}^{\mathcal{N}} = (\ell(Z_1), \dots, \ell(Z_{\mathcal{N}}))^T \end{cases} \tag{50}$$

Now, we can computing the matrix $\mathbf{A}_N(\mu)$, the vector $\mathbf{F}_N$ and the vector $\mathbf{L}_N$.

$$a(\xi^i, \xi^j; \mu) = a(\sum_{k=1}^{\mathcal{N}} Z_k \xi_k^i(\mu^k); \sum_{l=1}^{\mathcal{N}} Z_l \xi_l^j(\mu^l); \mu) = \sum_{k=1}^{\mathcal{N}} \sum_{l=1}^{\mathcal{N}} Z_k Z_l a(\xi_k^i(\mu^k), \xi_l^j(\mu^l); \mu) = (Z^T \mathbf{A}^{\mathcal{N}} Z)_{i,j}$$

Then we have that $\mathbf{A}_N(\mu) = Z^T\mathbf{A}^{\mathcal{N}}Z$. Where Z is an $\mathcal{N} \times N$ matrix, the jth column of which is $\mathbf{u}_N(\mu^j)$ (the nodal values of $\mathbf{u}_N(\mu^j)$). The same for the vector $\mathbf{F}_N$.

$$\ell(\xi^i) = \ell(\sum_{k=1}^{\mathcal{N}} Z_k\xi_k^i(\mu^k)) = \sum_{k=1}^{\mathcal{N}} Z_k\ell(\xi_k^i(\mu^k)) = (Z^T\mathbf{F}^{\mathcal{N}})_i$$

Then we have that $\mathbf{F}_N = Z^T\mathbf{F}^{\mathcal{N}}$. Finally, for the vector $\mathbf{L}_N$.

$$\ell(\xi^i) = \ell(\sum_{k=1}^{\mathcal{N}} Z_k\xi_k^i(\mu^k)) = \sum_{k=1}^{\mathcal{N}} Z_k\ell(\xi_k^i(\mu^k)) = (Z^T\mathbf{L}^{\mathcal{N}})_i$$

Then we have that $\mathbf{L}_N = Z^T\mathbf{L}^{\mathcal{N}}$.

5.4 d)

Show that the bilinear form $a(w, v; \mu)$ can be decomposed as

$$a(w, v; \mu) = \sum_{q=0}^{Q} \theta_q(\mu)a_q(w, v), \quad \forall w, v \in X, \forall\mu \in D, \tag{51}$$

for $Q = 6$ and give expressions for the $\theta_q(\mu)$ and the $a_q(w, v)$. Notice that the $a_q(w, v)$ are not dependent on μ; the parameter dependence enters only through the functions $\theta_q(\mu)$, $q = 1, \ldots, Q$. Further show that

$$\mathbf{A}^{\mathcal{N}}(\mu) = \sum_{q=1}^{Q} \theta^q(\mu)\mathbf{A}^{\mathcal{N}q}, \tag{52}$$

and

$$\mathbf{A}^{N}(\mu) = \sum_{q=1}^{Q} \theta^q(\mu)\mathbf{A}_N^q, \tag{53}$$

Give an expression for the $\mathbf{A}_{\mathcal{N}q}$ in terms of the nodal basis functions; and develop a formula for the $\mathbf{A}_q^N$ in terms of the $\mathbf{A}_{\mathcal{N}q}$ and Z.
We have that the bilinear form

$$a(w, v; \mu) = \sum_{i=0}^{4} k_i \int_{\Omega_i} \nabla w \cdot \nabla v \, dA + \mathrm{Bi} \int_{\Gamma\setminus\Gamma_{\mathrm{root}}} wv \, dS$$

can be decomposed as

$$\theta^1(\mu) = k^0, \qquad a_1(w, v) = \int_{\Omega_0} \nabla w \cdot \nabla v \, dA$$

$$\theta^2(\mu) = k^1, \qquad a_2(w, v) = \int_{\Omega_1} \nabla w \cdot \nabla v \, dA$$

$$\theta^3(\mu) = k^2, \qquad a_3(w, v) = \int_{\Omega_2} \nabla w \cdot \nabla v \, dA$$

$$\theta^4(\mu) = k^3, \qquad a_4(w, v) = \int_{\Omega_3} \nabla w \cdot \nabla v \, dA$$

$$\theta^5(\mu) = k^4, \qquad a_5(w, v) = \int_{\Omega_4} \nabla w \cdot \nabla v \, dA$$

$$\theta^6(\mu) = \mathrm{Bi}, \qquad a_6(w,v) = \int_{\Gamma\setminus\Gamma_{\mathrm{root}}} wv\, dS$$

Then, we have that

$$a(w,v;\mu) = \sum_{q=1}^{Q} \theta^q(\mu) a_q(w,v)$$

Now, we compute the matrix $\mathbf{A}^{\mathcal{N}}(\mu)$, and $\mathbf{A}^N(\mu)$. By 50, we have that

$$\mathbf{A}^{\mathcal{N}}(\mu)_{i,j} = a(Z_i, Z_j; \mu) = \sum_{q=1}^{Q} \theta^q(\mu) a_q(Z_i, Z_j)$$

Or we know that $a_q(w,v)$ is bilinear form, symetric and positive definite, then we have that

$$\mathbf{A}^{\mathcal{N}}(\mu) = \sum_{q=1}^{Q} \theta^q(\mu) \mathbf{A}^{\mathcal{N}q}$$

Where $\mathbf{A}^{Nq}$ is the matrix of the bilinear form $a_q(w,v)$ in the basis of W_N given by

$$\mathbf{A}^{Nq} = \begin{pmatrix} a_q(Z_1, Z_1) & \cdots & a_q(Z_1, Z_{\mathcal{N}}) \\ \vdots & \ddots & \vdots \\ a_q(Z_{\mathcal{N}}, Z_1) & \cdots & a_q(Z_{\mathcal{N}}, Z_{\mathcal{N}}) \end{pmatrix}$$

Now, we compute the matrix $\mathbf{A}^N(\mu)$.

$$\mathbf{A}^N(\mu)_{i,j} = Z^T \mathbf{A}^{\mathcal{N}}(\mu) Z = Z^T \sum_{q=1}^{Q} \theta^q(\mu) \mathbf{A}^{\mathcal{N}q} Z = \sum_{q=1}^{Q} \theta^q(\mu) Z^T \mathbf{A}^{\mathcal{N}q} Z$$

In the end, we have this result

$$\mathbf{A}^N(\mu) = \sum_{i=1}^{Q} \theta^q(\mu) \mathbf{A}_N^q$$

Where $\mathbf{A}_N^q(\mu) = Z^T \mathbf{A}^{\mathcal{N}q} Z$

5.5 e)

The coercivity and continuity constants of the bilinear form for the continuous problem are denoted by $\alpha_e(\mu)$ and $\gamma_e(\mu)$, respectively. We now assume that the basis function ξ_i, $i = 1, \ldots, N$, are orthonormalized. Show that the condition number of $\bar{A}_N(\mu)$ is then bounded from above by $\gamma_e(\mu)/\alpha_e(\mu)$.
We have that the condition number of a matrix $\mathbf{A}$ is defined by

$$\kappa(\mathbf{A_N}(\mu)) = \frac{\lambda_{\max}(\mathbf{A_N}(\mu))}{\lambda_{\min}(\mathbf{A_N}(\mu))} \tag{54}$$

We have that the bilinear form $a(w,w;\mu)$ is continuous, coercive and symmetric. Then,

$$a(w,w;\mu) \leq \gamma_e(\mu) |||w|||^2$$

and

$$a(w,w;\mu) \geq \alpha_e(\mu) |||w|||^2$$

a is coercive it means that the bilinear form is positive definite, then we have that

$$\alpha_e(\mu)|||w|||^2 \leq a(w,w;\mu) \leq \gamma_e(\mu)|||w|||^2$$

Or, we write the bilinear form $a(w,w;\mu)$ in matrix form, in the basis of W_N:

$$\mathbf{A}_N(\mu) = \begin{pmatrix} a(\xi^1,\xi^1;\mu) & \cdots & a(\xi^1,\xi^N;\mu) \\ \vdots & \ddots & \vdots \\ a(\xi^N,\xi^1;\mu) & \cdots & a(\xi^N,\xi^N;\mu) \end{pmatrix}$$

So, by the orthonormalization of the basis function ξ_i, then the eigenvalues of the matrix $\mathbf{A}_N(\mu)$ are corresponds to the values of the bilinear form $a(\xi^i,\xi^i;\mu)$, or by definition of $\alpha_e(\mu)$ and $\gamma_e(\mu)$, the eigenvalues of the matrix $\mathbf{A}_N(\mu)$ are bounded by $\alpha_e(\mu)$ and $\gamma_e(\mu)$. then we have that $\lambda_{\min}(\mathbf{A_N}(\mu)) \geq \alpha_e(\mu)$ and $\lambda_{\max}(\mathbf{A_N}(\mu)) \leq \gamma_e(\mu)$. Finally, we have that

$$\kappa(\mathbf{A_N}(\mu)) = \frac{\lambda_{\max}(\mathbf{A_N}(\mu))}{\lambda_{\min}(\mathbf{A_N}(\mu))} \leq \frac{\gamma_e(\mu)}{\alpha_e(\mu)}$$

5.6 f)

Take into account the parameters L and t as parameters: the geometric transformation must be taken into account in the affine decomposition procedure.
To do this, we understand that the geometric transformation is a function of the parameters L and t, for example we know that the surface of ecah subdomain is a function of L and t, then we can write that

$$|\Omega_i| = L \times t$$

Then, we show that the bilinear form $a(w,v;\mu)$ can be decomposed as

$$a(w,v;\mu) = \sum_{q=0}^{Q} \theta_q(\mu,L,t) a_q(w,v), \quad \forall w,v \in X, \forall \mu \in D, \tag{55}$$

Where $\theta_q(\mu,L,t)$ is a function depending also on the parameters L and t. Or, we have that the bilinear form $a(w,w;\mu)$ given by

$$a(w,w;\mu) = \sum_{i=0}^{4} k_i \int_{\Omega_i} \nabla w \cdot \nabla w \, dA + \mathrm{Bi} \int_{\Gamma \backslash \Gamma_{\text{root}}} w^2 \, dS$$

Now, let's start to apply an change of variable between our domain Ω_i and the reference domain $\hat{\Omega}$, let $(x,y) \in \Omega_i$ and $(\hat{x},\hat{y}) \in \hat{\Omega}$, we have that $x = t\hat{x}$ and $y = L\hat{y}$, then we have that $\hat{x} = \frac{x}{t}$ and $\hat{y} = \frac{y}{L}$. So, jacbian of the transformation is given by

$$J = \begin{pmatrix} \frac{\partial x}{\partial \hat{x}} & \frac{\partial x}{\partial \hat{y}} \\ \frac{\partial y}{\partial \hat{x}} & \frac{\partial y}{\partial \hat{y}} \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & L \end{pmatrix} = tL$$

For an integrations on the domain Ω_i, we have that

$$\int_{\Omega_i} \nabla w \cdot \nabla v \, dA = \int_{\Omega_i} \nabla w(x,y) \cdot \nabla v(x,y) \, dA = \int_{\hat{\Omega}} \nabla w(\hat{x},\hat{y}) \cdot \nabla v(\hat{x},\hat{y}) \, |J| d\hat{A} = tL \int_{\hat{\Omega}} \nabla w(\hat{x},\hat{y}) \cdot \nabla v(\hat{x},\hat{y}) \, d\hat{A}$$

By the same way, we have that

$$|\Gamma \setminus \Gamma_{\text{root}}| = 2L + t$$

and

$$\int_{\Gamma\backslash\Gamma_{\text{root}}} wv\,dS = 2L + t\int_{\hat{\Gamma}\backslash\hat{\Gamma}_{\text{root}}} wv\,d\hat{S}$$

Then, we have that the bilinear form $a(w, v; \mu)$ in the references domain is given by

$$a(w, v; \mu) = \sum_{i=0}^{4} k_i tL \int_{\hat{\Omega}} \nabla w(\hat{x}, \hat{y}) \cdot \nabla v(\hat{x}, \hat{y})\,d\hat{A} + (2L + t)\text{Bi}\int_{\hat{\Gamma}\backslash\hat{\Gamma}_{\text{root}}} wv\,d\hat{S}$$

Then, we have that the bilinear form $a(w, w; \mu)$ can be decomposed as

$$a(w, v; \mu) = \sum_{q=0}^{Q} \theta_q(\mu, L, t)a_q(w, v), \quad \forall w, v \in X, \forall \mu \in D,$$

where

$$\theta^1(\mu, L, t) = k^0, \qquad a_1(w, v) = \int_{\hat{\Omega}} \nabla w \cdot \nabla v\,dA$$

$$\theta^2(\mu, L, t) = k^1 tL, \qquad a_2(w, v) = \int_{\hat{\Omega}} \nabla w \cdot \nabla v\,dA$$

$$\theta^3(\mu, L, t) = k^2 tL, \qquad a_3(w, v) = \int_{\hat{\Omega}} \nabla w \cdot \nabla v\,dA$$

$$\theta^4(\mu, L, t) = k^3 tL, \qquad a_4(w, v) = \int_{\hat{\Omega}} \nabla w \cdot \nabla v\,dA$$

$$\theta^5(\mu, L, t) = k^4 tL, \qquad a_5(w, v) = \int_{\hat{\Omega}} \nabla w \cdot \nabla v\,dA$$

$$\theta^6(\mu, L, t) = \text{Bi}(2L + t), \qquad a_6(w, v) = \int_{\hat{\Gamma}\backslash\hat{\Gamma}_{\text{root}}} wv\,dS$$

For Ω_0, i take it as reference domain.