Fair Division with Minimal Sharing

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Abstract

A set of goods has to be divided fairly among agents with different tastes, modeled by additive value-functions. If the goods cannot be shared, so that each good must be entirely allocated to a single agent, then fair division may not exist. What is the smallest number of goods that have to be shared between two or more agents in order to attain a fair division?

We focus on attaining a Pareto-optimal, envy-free and proportional division. We show that, for a typical instance of the problem (all instances except of a zero-measure set of degenerate problems), a *fair and Pareto-optimal division with the smallest possible number of shared goods can be found in polynomial time*, assuming the number of agents is fixed. However, the problem becomes hard for degenerate instances, where the agents' valuations are aligned for many goods.

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1 Introduction

What is a fair way to allocate discrete items without monetary transfers?

When the items are indivisible, it may be impossible to allocate them fairly — consider a single item and two people. A common approach to this problem is to look for an *approximately-fair* allocation. There are several definitions of approximate fairness, the most common of which are *envy-freeness except one item (EF1)* and *maximin share (MMS)*. An alternative solution is to "make items divisible" by allowing randomization and to ensure that the division is fair ex-ante, i.e, in expectation.

While approximate or ex-ante fairness are reasonable when dividing low-value items, such as seats in a course or in a school, it is not so reasonable when dividing high-value items, such as houses or precious jewels. Consider a divorcing couple deciding how to split children, or two siblings who have to divide three houses among them; it is unlikely that one of them will agree to receive a bundle that is envy-free except one child/house, or a lottery that gives him/her either one or two with equal probability.

In practical cases, when monetary transfers are undesired, the common solution is to find some way to share some of the items. For example, a shop can be jointly owned by several partners, sharing the costs and revenues of operation. A house can be jointly owned by several people, who live in the house alternately in different times. While such sharing may be inevitable if exact fairness is desired, it may be quite inconvenient due to the overhead in managing the shared property. Therefore it is desirable to minimize the number of items that have to be shared.

Our contribution.

Our paper advocates a new approach to the problem of fair division:

minimize the number of goods shared, given fairness and economic efficiency as constraints.

This approach gives a compelling alternative to approximate-fairness when items to divide are highly valuable and sharing is technically possible (as in all examples above and many real-life situations) but unwanted.

We assume that agents have additive utilities[1] and focus on classical fairness notions of *proportionality* (each agent gets a bundle worth at least $1/n$ of the total value, where n is the number of agents) or *envy-freeness* (each agent weakly prefers his/her bundle to the bundle of any other agent). Economic efficiency is captured by *fractional*-Pareto optimality: no other allocation, even without any restriction on sharing goods, improves the well-being of some agent without harming some others.

We focus on the case in which n is a fixed small number, as is common in inheritance division, so the problem size is determined by the number of goods, which we denote by m.

[11] showed that there is always a fair fractionally-Pareto-optimal allocation with at most $n-1$ goods shared. The bound is tight: when there are $n-1$ identical goods, all of them must be shared. In **Subsection 2.3** we show that such an allocation can be constructed in time $O(\text{poly}(m,n))$.

In **Section 3** we investigate algorithmic properties of fair and fractionally-Pareto-optimal allocations that, for a given valuation profile, minimize the number of shared goods. We demonstrate the following dichotomy:

- The problem of minimizing the number of shared goods is *algorithmically tractable if the valuation profile is non-degenerate.* We present an algorithm with a run-time *polynomial in the number of goods* m (for fixed n) for such profiles.

  A valuation profile is degenerate if it satisfies certain algebraic identities. Thus *a random valuation-profile is degenerate with probability 0, so our algorithm runs in polynomial time for almost all instances.* Formally, a valuation profile $v = (v_{i,g})_{i\in[n],g\in[m]}$ is non-degenerate if for any pair of agents i, j the ratios $v_{i,g}/v_{j,g}$ are different across all goods g.

  An extreme example of degenerate profile is when all agents have identical preferences.
- The problem of minimizing the number of shared goods is *NP-hard for degenerate problems*, for any fixed $n \geq 2$.

Our results confirm the common sense that *computationally-hard instances of resource-allocation problems are rare* and, in practice, resource-allocation problems can be often be solved efficiently.

However, the fact that *computationally-hard instances are those in which agents have identical valuations* is quite surprising. In fact, in many previous papers on fair item allocation (e.g. [38, 40]), computational hardness results are presented with a qualifier saying that the problem is hard "*even* when the valuations are identical"; our results show that the word "even" is unwarranted.

Another observation that may seem surprising is that *finding fair and fractionally-Pareto optimal allocations is easier than just fair (without Pareto optimality).* The underlying reason is that, for non-degenerate problems, fractional-Pareto-optimality is a strong condition that

[1] While additive utilities do not allow to express complementarities between items (e.g., a garage becomes more valuable together with a car), this class proved to be convenient in practice because of simplicity of formulating and reporting such preferences [30].

Allocation	k	**Valuations** Identical	General additive	Non-degenerate
Fair	$n-1$	$O(m)$ [footnote 3]		
Fair	$n-2$	$O(\mathrm{poly}(m))$ [T3.5]	OPEN	OPEN
Fair	$1,\ldots,n-3$	$O(\mathrm{poly}(m))$ [T3.5]	NP-hard [T3.11] (for envy-freeness)	
Fair	0	NP-hard [T3.2]	NP-hard [T3.4]	
Fair+fPO	$n-1$	$O(\mathrm{poly}(m))$ [C2.5]		
Fair+fPO	$n-2$	$O(\mathrm{poly}(m))$ [T3.5]	OPEN	$O(\mathrm{poly}(m))$ [T3.9]
Fair+fPO	$1,\ldots,n-3$	$O(\mathrm{poly}(m))$ [T3.5]	NP-hard [T3.12]	$O(\mathrm{poly}(m))$ [T3.9]
Fair+fPO	0	NP-hard [T3.2]		$O(\mathrm{poly}(m))$ [T3.9]

Table 1 Run-time complexity of dividing m goods among n agents (where n is fixed), with at most k shared goods. For $k = n-1$ a fair (or fair and fractionally-PO) allocation always exists, and the cell contains the runtime complexity of finding it. For $k < n-1$, the cell contains the runtime complexity of deciding if such a division exists, and finding it if exists. fPO = fractionally-Pareto-Optimal.

shrinks the search space to a polynomial number of structures (see Lemma 3.8 for a formal statement).

Note the important *contrast between fractional and discrete Pareto-optimality* (an allocation is discrete-Pareto optimal if it is not dominated by any allocations with zero sharings). For discrete-Pareto optimality, even basic questions are computationally hard: deciding whether a given allocation is discrete-PO is co-NP hard, and deciding whether there exists an envy-free and discrete-PO allocation is Σ_2^p-complete [23]. In contrast, deciding whether a given allocation is fractional-PO is polynomial in m and n (see Lemma 2.2), and deciding whether there exists an envy-free and fractionally-PO allocation is, for almost all instances, polynomial in m for fixed n. These observations suggest that *fractional-Pareto-optimality is a compelling concept of economic efficiency even for indivisible goods*; recent results by [8, 7] confirm this observation.

Our results (and some remaining gaps) are summarized in **Table 1**. Notation and preliminary results can be found in **Section 2** (we consider these results "preliminary" since they are similar to results proved in previous work). Our main results are proved in **Section 3**. Related work is surveyed in **Section 4**. In **Section 5** we consider some possible extensions of the model: truthful mechanisms, non-linear utilities, and chores instead of goods. Some open questions are mentioned along the way.

2 Preliminaries

2.1 Agents, goods, and allocations

There is a set of n agents, denoted $[n] = \{1,\ldots,n\}$. There is a set of m goods, denoted $[m] = \{1,\ldots,m\}$. Each agent $i \in [n]$ has an *additive* utility function. The value for i of good $g \in [m]$ is denoted by $v_{i,g}$. The value for agent i of receiving a fraction $z_{i,g} \in [0,1]$ of good g is $z_{i,g} \cdot v_{i,g}$. We assume that all valuations are weakly-positive, i.e., $v_{i,g} \geq 0$ for all i, g. Some results assume that the valuations are strictly-positive, i.e., $v_{i,g} > 0$ for all i, g.

The quantity of each good is normalized to 1. An *allocation* is specified by a matrix $\mathbf{z} := (z_{i,g})_{i\in[n],g\in[m]}$ such that, for each good g, $\sum_{i\in[n]} z_{i,g} = 1$, i.e, all goods are allocated — there is no disposal.

The bundle of agent i in the allocation is denoted by $\mathbf{z}_i := (z_{i,g})_{g\in[m]}$. The utility of agent i from bundle $\mathbf{z}_i$ is:

$$u_i(\mathbf{z}_i) = \sum_{g\in[m]} z_{i,g} v_{i,g}$$

The *utility-profile* of an allocation $\mathbf{z}$ is the vector $\mathbf{u}(\mathbf{z}) := (u_i(\mathbf{z}_i))_{i\in[n]}$. A vector $\mathbf{U} := (U_1,\ldots,U_n)$ is called a *feasible utility-profile* if there exists an allocation $\mathbf{z}$ with $\mathbf{U} = \mathbf{u}(\mathbf{z})$.

Fairness concepts and Pareto-optimality.

An allocation $\mathbf{z}$ is called *envy-free (EF)* if for every $i, j \in [n]$: $u_i(\mathbf{z}_i) \geq u_i(\mathbf{z}_j)$.

For each agent i, define $V_i := \sum_{g=1}^{m} v_{i,g}$ = the total utility of all goods. An allocation $\mathbf{z}$ is called *proportional* if for every agent i: $u_i(\mathbf{z}_i) \geq V_i/n$, i.e, each agent receives at least his/her *fair share* of the total utility.

An allocation $\mathbf{z}$ is Pareto-dominated by $\mathbf{y}$ if $\mathbf{y}$ gives at least the same utility for all agents and strictly more utility to at least one of them.

An allocation $\mathbf{z}$ is called *fractionally-Pareto-optimal (fractionally-PO)*[2] if no feasible $\mathbf{y}$ dominates it.

Measures of sharing

If for some $i \in [n]$, $z_{i,g} = 1$, then good g is not shared — it is fully allocated to agent i. Otherwise, good g is shared between two or more agents.

Throughout the paper, we consider two measures quantifying the amount of sharing in a given allocation $\mathbf{z}$.

The simplest one is *the number of shared goods*

$$|\{g \in [m] \,:\, z_{i,g} \in (0,1) \text{ for some } i \in [n]\}|\,.$$

Alternatively, one can count the number of times each good is shared. This multiplicity is captured by *the number of sharings*

$$\sum_{g\in[m]} \left(|\{i \in [n] :\, z_{i,g} > 0\}| - 1\right).$$

For an "indivisible" allocations both measures are zero, but they differ, for example, if only one good g is shared but each agent consumes a bit of g: the number of shared goods in this case is 1 while the number of sharings is $n-1$. Clearly, the number of *shared goods* in $\mathbf{z}$ is always at most the number of *sharings* in $\mathbf{z}$.

2.2 Agent-good graphs and a characterization of fractional-Pareto-optimality

Our algorithms will use several kinds of *agent-good graphs* — directed bipartite graphs in which the nodes one one side are the agents and the nodes on the other side are the goods:

- In the *undirected consumption-graph* of a given allocation $\mathbf{z}$, denoted $\mathcal{CG}_{\mathbf{z}}$, there is an edge between agent $i \in [n]$ and a good $g \in [m]$ iff $z_{i,g} > 0$. Note that the number of sharings in $\mathbf{z}$ equals the number of edges in $\mathcal{CG}_{\mathbf{z}}$ minus m.

[2] The literature on indivisible items, considers a weaker notion of economic efficiency: $\mathbf{z}$ is called *discrete-Pareto-optimal* if no feasible *indivisible* $\mathbf{y}$ dominates it (indivisibility means $y_{i,g} \in \{0,1\}$). While fractional-Pareto-optimality has good algorithmic properties, its discrete version does not (see the discussion below Table 1).

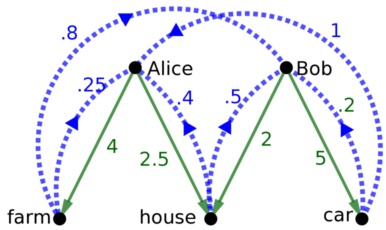

Figure 1 An example of a weighted directed consumption graph. The farm, house and car are valued by Alice at 4, 2.5 and 1, and by Bob at 1.25, 2 and 5. Alice gets the farm, Bob gets the car, and they share the house.

- The *weighted directed consumption-graph* of $\mathbf{z}$, denoted $\overrightarrow{\mathcal{CG}_{\mathbf{z}}}$, contains all edges of $\mathcal{CG}_{\mathbf{z}}$ oriented from an agent to a good and, additionally, an edge from every good $g \in [m]$ to every agent i is traced regardless of the allocation.
  Weights are assigned to the edges in the following way: the weight of an edge directed from an agent $i \in [n]$ to a good $g \in [m]$ is $v_{i,g}$, and the weight of an edge in the opposite direction (from the good to the agent) is $1/v_{i,g}$. An example is shown in Figure 1.

The *product* of a directed path P in a directed agent-good graph, denoted $\pi(P)$, is the product of weights of edges along the path. In particular, the product of a directed cycle $C = (i_1 \to g_1 \to i_2 \to g_2 \to \ldots \to i_L \to g_L \to i_{L+1} = i_1)$ is:

$$\pi(C) = \prod_{k=1}^{L} \left(v_{i_k,g_k} / v_{i_{k+1},g_k} \right) = \prod_{k=2}^{L+1} \left(v_{i_k,g_k} / v_{i_k,g_{k-1}} \right)$$

The importance of this product comes from the following lemma (proved in Appendix A):

▶ **Lemma 2.1.** *WIth strictly-positive valuations, An allocation $\mathbf{z}$ is fractionally-Pareto-optimal if-and-only-if the directed graph $\overrightarrow{\mathcal{CG}_{\mathbf{z}}}$ contains no directed cycle C with $\pi(C) < 1$.*

Lemma 2.1 has a useful computational implication, which is also proved in Appendix A.

▶ **Lemma 2.2.** *It is possible to decide in time $O(nm(n+m))$ whether a given allocation $\mathbf{z}$ is fractionally-Pareto-optimal.*

The following well-known lemma provides a useful necessary condition for fractional-Pareto-optimality of an allocation (necessary and sufficient for $n = 2$, see [11]).

▶ **Lemma 2.3.** *Let g, h be two goods such that $v_{i,h}/v_{j,h} > v_{i,g}/v_{j,g}$. If, in a fractionally-PO allocation, agent i gets a positive amount of g, then agent j gets none of h.*

Proof. Suppose that, in some allocation $\mathbf{z}$, agent i gets a positive amount of g while agent j gets a positive amount of h. Then, in $\overrightarrow{\mathcal{CG}_{\mathbf{z}}}$, we have the directed cycle $C = (i \to g \to j \to h \to i)$. The product $\pi(C) = v_{i,g}/v_{j,g} \cdot v_{j,h}/v_{i,h} < 1$ and hence by Lemma 2.1, $\mathbf{z}$ is not fractionally-PO. ◀

2.3 Worst-case bounds on sharing

When there are n agents and $n-1$ identical goods, a fair allocation must give each agent a fraction $(n-1)/n$ of a good, for any reasonable definition of fairness. This requires sharing all $n-1$ goods, so $n-1$ is a lower bound on the worst-case sharing. As we will see this lower bound can always be achieved.

[11] showed that for any fractionally-Pareto-optimal allocation $\mathbf{z}$, there exists an equivalent one $\mathbf{z}^*$ (all agents are indifferent between the two) with at most $n-1$ goods shared. A similar proof can be found in [7] (see Claim 2.2) for the so-called competitive equilibrium allocations.

The following lemma shows that $\mathbf{z}^*$ can be constructed efficiently. It also covers arbitrary allocations $\mathbf{z}$, not necessary fractionally-Pareto-optimal. It is proved in Appendix B.

▶ **Lemma 2.4.** *For any allocation $\mathbf{z}$, there exists a fractionally-Pareto-optimal allocation $\mathbf{z}^*$ such that:*

- *(a) $\mathbf{z}^*$ weakly Pareto dominates $\mathbf{z}$, i.e, for any agent i, $u_i(\mathbf{z_i^*}) \geq u_i(\mathbf{z_i})$.*
- *(b) the non-directed consumption graph $\mathcal{CG}_{\mathbf{z}^*}$ is acyclic.*
- *(c) $\mathbf{z}^*$ has at most $n-1$ sharings (hence at most $n-1$ shared goods).*

Such allocation $\mathbf{z}^*$ can be constructed in time $O(poly(m,n))$.

From this lemma we can get an upper bound on the number of sharings:

▶ **Corollary 2.5.** *In any instance with n agents, there exists a fractionally-Pareto-optimal, envy-free and proportional division with at most $n-1$ sharings.*[3] *Such an allocation can be found in time $O(poly((m,n)))$.*

Proof. For proportionality, consider an equal-split allocation $\mathbf{z}$ ($z_{i,g} = \frac{1}{n}$ for all i,g) and construct a fractionally-Pareto-optimal dominating allocation $\mathbf{z}^*$ by Lemma 2.4. Pareto-improvements preserve proportionality, and thus $\mathbf{z}^*$ is proportional, fractionally-Pareto-optimal and has at most $n-1$ sharings.

It is known that every allocation maximizing the *Nash product* $\prod_{i\in n} u_i(\mathbf{z_i})$ is both fractionally-Pareto-optimal and envy-free. Indeed, Eisenberg and Gale proved that this allocation can be alternatively viewed as an equilibrium allocation of the so-called Fisher market associated with the fair division problem, see [51]. Envy-freeness and fractional-Pareto-optimality of market equilibria follow from classic results, [50].

The exact market equilibrium can be computed in time $O((n+m)^4 \log(n+m))$, see [39]. After finding an equilibrium allocation $\mathbf{z}$ we apply Lemma 2.4 which gives an allocation $\mathbf{z}^*$ with $\mathbf{u}(\mathbf{z}) = \mathbf{u}(\mathbf{z}^*)$ and at most $n-1$ sharings. Since this allocation maximizes the Nash product too, it is envy-free. ◀

▶ **Question 1.** The requirement of a market equilibrium is much stronger than the requirements of envy-freeness and fractional-Pareto-optimality. Can we find an EF+fPO allocation using simpler and/or faster algorithms?

3 Pareto Optimal Fair Division: Minimizing the Sharing

As we saw in Subsection 2.3, in the worst case, it might be required to share $n-1$ goods. However, in some cases it may be possible to find a fair allocation with less sharing. This raises the following computational problem:

[3] Without Pareto-optimality, an analogue of Corollary 2.5 can be deduced from results on fair cake-cutting. Place goods arbitrary on a line (goods are represented by consecutive intervals with piece-wise constant utility functions, see [42] for the details of reduction) and find a fair partition with connected pieces. A connected partition makes $n-1$ cuts and hence at most $n-1$ goods are shared. For proportionality, such partition can be found constructively using $O(mn\log(n))$ operations by the Even-Paz algorithm, [27] (note that no assumption of fixed n is needed!); for envy-freeness, existence follows from [47, 49] but for general cakes and $n \geq 3$ such a partition cannot be found using a finite number of queries [48].

Given a specific instance of a fair allocation problem, find a solution that minimizes the number of shared goods and/or the number of sharings.

We will contrast between the two extreme cases: agents with *identical valuations* and agents with *non-degenerate valuations*.

▶ **Definition 3.1.** *An instance with n agents is called* degenerate[4] *if there exist two agents i, j and two goods g, h such that $v_{i,g}/v_{j,g} = v_{i,h}/v_{j,h}$. Otherwise it is called* non-degenerate.

Note that, if the agents' valuations are selected according to any continuous probability distribution, the resulting instance is non-degenerate with probability 1.[5]

Intuitively, it may seem that the case of identical valuations is computationally easier than the case of different valuations, since the input is smaller. However, as we will see the situation is the opposite: degenerate case of identical valuations constitutes computationally-hard instances of the problem.

3.1 Two agents

For a warm-up, we start with the case of $n = 2$ agents.

The upper bound on the number of sharings of Section 2.3 is 1, so the only remaining question is whether there exists a fair allocation with no sharing at all. The following theorem is well-known (e.g. [34]); we present it only to set the background for the theorem after it.

▶ **Theorem 3.2.** *When there are $n = 2$ agents with identical valuations, it is NP-hard to decide whether there exists an allocation with no sharings that is proportional / proportional and fractionally-PO (or discrete-PO) / EF / EF and fractionally-PO (or discrete-PO).*

The following theorem shows that identical valuations are, in a sense, a rare "worst case".

▶ **Theorem 3.3.** *When there are $n = 2$ agents with non-degenerate and strictly-positive valuations, it is possible to find in time $O(m \log(m))$ a division that is fractionally-PO and proportional/EF, and subject to these requirements, minimizes the number of sharings.*

Proof. Order the goods in descending order of the ratio $v_{1,g}/v_{2,g}$, for $g \in \{1, \ldots, m\}$ (this takes $O(m \log(m))$ operations). By the assumption of non-degeneracy, no two ratios coincide.

By Lemma 2.3, any fractionally-PO allocation $\mathbf{z}$ takes one of two forms:

- "0 sharings": there is a good g such that $\mathbf{z}$ gives all the prefix goods $1, ..g$ to agent 1, and all postfix goods $g + 1, ..m$ to agent 2.
- "1 sharing": there is a good g which is split between the two agents, while all goods $1, ..g - 1$ are consumed by agent 1 and all remaining goods $g + 1, ..m$ by agent 2.

Therefore we have $m + 1$ allocation with 0 sharings and each of them can be tested for fairness.

If there are no fair allocations among them, then we look for a fair allocation among those with one sharing. For any fixed g, this leads to solving a system of two linear inequalities with just one variable (the amount of g consumed by agent 1). ◀

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[4] [11] and [16] use a stronger definition of degeneracy: the complete agent-good graph has no cycles C with $\pi(C) = 1$. Their condition implies that $\mathcal{CG}_{\mathbf{z}}$ is acyclic for any fractionally-Pareto-optimal allocation $\mathbf{z}$ and that there is a bijection between Pareto-optimal utility profiles and fractionally-Pareto-optimal allocations. Our definition addresses only cycles of length 4 and thus can be easily checked in $O(n^2 \cdot m^2)$ operations. For 2 agents the definitions coincide.

[5] An interesting problem, that we leave to future work, is to calculate the probability that an instance is degenerate when the valuations are selected according to a discrete distribution, for example, from finite set of integers.

The non-degeneracy requirement of Theorem 3.3 can be relaxed as follows. Define *degree of degeneracy* as the smallest d such that for any ratio $r > 0$ there are at most d goods $g \in [m]$ such that $v_{1,g}/v_{2,g} = r$.

By Lemma 2.3, any efficient allocation has the following form: there is a ratio $r > 0$ such that all g with $\frac{v_{1,g}}{v_{2,g}} > r$ are allocated to agent 1, those where the sign is opposite are allocated to agent 2, while g with $\frac{v_{1,g}}{v_{2,g}} = r$ are allocated arbitrarily among the two agents.

Therefore we have at most $2^d \cdot m$ allocations with no sharings and at most $d \cdot 2^d \cdot m$ allocations with one sharing. If $d \in O(\log(m))$, exhaustive search gives a polynomial-time algorithm.

If $d = m$, then the valuations are identical (up to a multiplicative constant), so we fall into Theorem 3.2. Thus, as d it increases, we move gradually from the easiness of Theorem 3.3 to the hardness of Theorem 3.2.

Indeed, fix $c \in (0,1)$ and $\alpha \in (0,1]$, then for valuations with degree of degeneracy $d \geq c \cdot m^\alpha$, deciding whether an allocation with 0 sharings exists, is NP-hard. Starting from an instance $a_1, a_2, ...a_d$ of PARTITION pick m such that $c \cdot m^\alpha \leq d \leq m$ and define a fair division instance with m goods:

- for each $a_g,\ g \in [d]$ there is a good equally valued by agents $v_{1g} = v_{2g} = a_g$
- there are $Q = \frac{m-d}{2}$ pairs (w.l.o.g., $m - d$ is even) of goods $(q_k, \bar{q}_k)_{k=1,..Q}$ such that $v_{1q_k} = v_{2\bar{q}_k} = \frac{k+1}{3mk}$, $v_{1\bar{q}_k} = v_{2q_k} = \frac{1}{3mk}$

Since all the goods q and $\bar{q}$ are small ($\sum_k v_{1q_k} = \sum_k v_{2\bar{q}_k} < 1$) while $a_g,\ g \in [d]$ are positive integers, in any fair fractionally-PO allocation, agent 1 consumes all q and agent 2 all $\bar{q}$ and the allocation has 0 sharings if and only if $a_1, .., a_d$ can be partitioned into two subsets of equal sum.

Thus we reduced PARTITION to the division problem. The condition $c \cdot m^\alpha \leq d$ ensures that the length of binary representation of this fair-division problem is bounded by a polynomial of the length of $a_1, .., a_d$.

Theorem 3.3 considers fairness with fractional-PO. The following theorem shows that its analogue for fairness *without* fractional-PO is false.

▶ **Theorem 3.4.** *When there are $n = 2$ agents, even with non-degenerate and strictly-positive valuations, it is NP-hard to decide whether there exists an allocation with no sharings that is proportional / EF.*

Proof. We reduce from PARTITION. Consider a PARTITION instance with p positive integers, $a_1, \ldots, a_p$. We create a fair division problem with $m = p$ goods and two agents with the following valuations:

- Alice values each good g as a_g;
- Bob values each good g as $a_g + b_g$, where $\forall_{g \in [m]} : b_g > 0$ and $\sum_{g \in [m]} b_g < 1/2$, and b_g are selected such that the valuations are non-degenerate (there are infinitely many ways to select these, for example, $b_g = \frac{g}{3p(g+1)}$).

If there exists an equal-sum partition of the numbers $a_1, \ldots, a_p$ into two subsets (X_1, X_2), then there exists a fair allocation of the goods: since $u_A(X_1) = u_A(X_2)$, we can give Bob either X_1 or X_2 for which his utility is higher, and give Alice the other subset.

Conversely, if there exists a fair allocation (X_A, X_B) of the goods, then Bob does not envy, so $u_B(X_B) \geq u_B(X_A)$. The sum of all the b_g is less than $1/2$, so $u_B(X_B) - u_A(X_B) < 1/2$, so $u_A(X_B) > u_B(X_B) - 1/2$. All the b_g are positive, so $u_B(X_A) > u_A(X_A)$. Combining the latter two inequalities gives $u_A(X_B) > u_A(X_A) - 1/2$. But Alice does not envy too, so $u_A(X_A) \geq u_A(X_B)$. Since the values of all items for Alice are integers, this implies that $u_A(X_A) = u_A(X_B)$, so (X_A, X_B) is an equal-sum partition of the numbers $a_1, \ldots, a_p$. ◀

Allocation	Sharings	Valuations: Identical	Valuations: Non-degenerate
Fair	1	$O(m)$ [footnote 3]	
Fair	0	NP-hard [T3.2]	NP-hard [T3.4]
Fair+fractionally-PO	1	$O(\text{poly}(m))$ [$O(m^3)$, C2.5]	
Fair+fractionally-PO	0	NP-hard [T3.2]	$O(\text{poly}(m))$ [$O(m \log m)$, T3.3]

Table 2 Run-time complexity of dividing m goods between 2 agents. A fair or fair+fractionally-PO division with 1 sharing always exists, and the cell contains the runtime complexity of finding it. With 0 sharings, the cell contains the runtime complexity of deciding if such a division exists, and finding it if it exists.

Table 2 summarizes our results for the computational problem of finding an allocation between two agents, from this and the previous section.

3.2 Three or more agents

3.2.1 Identical valuations

Suppose there are $n \geq 3$ agents. Theorem 3.2 obviously holds here too: when there are n agents with identical valuations, it is NP-hard to decide whether there exists an allocation with no sharings that is: (1) proportional, (2) proportional and fractionally-PO, (3) EF, (4) EF and fractionally-PO.

If we allow at least one shared good, then the problem can be solved in polynomial time for identical valuations:

▶ **Theorem 3.5.** *For every integers $n \geq 2$ and $k \geq 1$, for every instance with n agents with identical valuations, it is possible to decide in time $O(poly(m))$ whether there exists an allocation with at most k shared goods that is: (1) proportional, (2) proportional and fractionally-PO, (3) EF, (4) EF and fractionally-PO. If such allocation exists, it can be found in time $O(poly(m))$, assuming n is fixed.*

Proof. Partition the set of goods $[m]$ into two subsets: M_k = the k high-value goods, and M_l = the $m-k$ low-value goods (breaking ties arbitrarily). Let $V_k = \sum_{g \in M_k} v_g$ = the sum of the k highest values, and $V_m = \sum_{g \in [m]} v_g$ = the sum of all m values. With identical valuations, all four variants are equivalent to the following problem:

> (*) Partition the goods in $[m]$ into n subsets, with at most k shared goods, such that the sum of values in each subset is exactly V_m/n.

The problem (*) can be reduced into the following problem [43]:

> (**) Partition the goods in M_l into n subsets, with *no* shared goods, such that the sum of values in each subset is *at most* V_m/n.

Indeed, if there is a solution to (**), then it is possible to use the goods in M_k to round up the value in each subset to exactly V_m/n, which is a solution to (*) with at most k shared goods.

Conversely, suppose there is a solution to (*).

Let M'_k be the set of k shared goods. If $M'_k = M_k$, then remove the shared goods from the allocation. This leaves n subsets of M_l, and the sum of each subset is at most V_m/n, which is a solution to (**).

If $M'_k \neq M_k$, then pick a good $g \in M_k \setminus M'_k$ and another good $h \in M'_k \setminus M_k$. By assumption, $v(g) \geq v(h)$, and g is not shared — it is given entirely to some agent i. Create a new allocation as follows: take g from agent i; give agent i all of h, plus some fraction of g whose value is $v(g) - v(h)$; divide the remainder of g (whose remaining value is now $v(h)$) among the other agents in the same fraction as the good h that was taken from them. The resulting allocation is still fair. In a similar way, we can replace each shared good in M'_k with a good in M_k. After at most k such steps, we get at an allocation in which only the goods in M_k are shared. Then, we can remove the shared goods from the allocation and get a solution to (**).

The problem (**) can be solved in time $O(\text{poly}(m))$ (for fixed n and k) using a known PTAS for the knapsack problem [43].

◀

We take a detour to discuss the k-knapsack problem. In the k-knapsack problem, we are given capacity c, and n items such that the ith item has size a_i. We are interested in finding k disjoint subsets $S_1, \dots, S_k$ of $\{1, \dots, n\}$, such that $\sum_{i \in S_j} a_i \leq c$ for all j, and $\sum_{j=1}^{k} \sum_{i \in S_j} a_i$ is maximized.

▶ **Theorem 3.6.** *One can find a $(1-\epsilon)$-approximation of the k-knapsack problem in $O(\frac{n^{2k+1}}{\epsilon^k})$.*

Proof. Given object sizes $a_1, \dots, a_n$. Define $D[b_1, \dots, b_k, i]$ as the maximum value of $\sum_{j=1}^{k} \sum_{i \in S_j} a_i$, where S_j are disjoint subsets of $\{1, \dots, n\}$ and $\sum_{i \in S_j} a_i \leq b_j$ for all $1 \leq j \leq k$. The desired solution is $D[c, \dots, c, n]$. The recurrence relation is $D[b_1, \dots, b_k, i] = \max(\max_j(D[b_1, \dots, b_j - a_i, b_k, i-1]) + a_i, D[b_1, \dots, b_k, i-1])$. The base case are simple, and the running time for solving this dynamic program is $O(c^k kn)$.

We now proceed into a FPTAS for the $(1-\epsilon)$-approximation of the k-knapsack problem.

Consider an instance of k-knapsack problem with object sizes $a_1, \dots, a_n$ and capacity c. Call this instance A. Let $a = \max\{a_1, \dots, a_n\}$ and assume $a \leq c$, since otherwise we can generate a k-knapsack problem instance where all objects are no larger than c. We define $B = \frac{\epsilon a}{n}$, and $a'_i = \lfloor a_i / B \rfloor$. Let $c' = \lfloor \min(c, na)/B \rfloor$. Consider the instance of k-knapsack with capacity c' and items sizes $a'_1, \dots, a'_n$, call it A'. By the above dynamic programming algorithm, it outputs a solution with running time $O(c'^k n)$. Since $c' \leq na/B = \frac{n^2}{\epsilon}$, the running time is $O(n^{2k+1}/\epsilon^k)$. We want to show the solution is a $(1-\epsilon)$-approximation for A.

Let OPT be optimum of A and OPT' be the optimum of A'. Note that the solution with value OPT' for A gives us a solution with value at least $B \cdot OPT'$. We also have that $B \cdot OPT' + nB \geq OPT$. Therefore $B \cdot OPT' \geq OPT - nB$. Note in particular $nB = a\epsilon \leq OPT \cdot \epsilon$. Hence we have $B \cdot OPT' \geq (1-\epsilon)OPT$. ◀

3.2.2 Non-degenerate valuations

As in the case of two agents, when the problem is non-degenerate, an allocation with minimal number of shared goods can be found in polynomial time.

The main ingredient of the algorithms is Lemma 3.8 which allows to enumerate all consumption graphs $\overrightarrow{\mathcal{CG}}_{\mathbf{z}}$ of fractionally-Pareto-optimal allocations z by dynamic programming. It uses the following lemma.

▶ **Lemma 3.7.** *Let $\mathbf{z}$ be a fractionally-PO allocation, and suppose we get an additional amount of some good g. It is possible to give the additional amount to some agent $i \in [n]$, such that the new allocation $\mathbf{z}'$ is fractionally-PO too.*

Proof. Denote by $\mathbf{z}'(i)$ the allocation resulting from giving the additional g to agent i. Assume by contradiction that for any $i \in [n]$, $\mathbf{z}'(i)$ is not fractionally-PO. Then by Lemma 2.1, the consumption graph $\overrightarrow{\mathcal{CG}}_{\mathbf{z'(i)}}$ has a simple cycle $C(i)$ with $\pi(C(i)) < 1$. Because such cycle did not exist in $\overrightarrow{\mathcal{CG}}_{\mathbf{z}}$, the cycle $C(i)$ necessarily contains an edge $i \to g$ and an edge from g to some $j = j(i)$. Consider an auxiliary directed graph on $[n]$ with edges $i \to j(i)$, $i \in [n]$ are traced. Since every vertex has out-degree 1, this graph has a simple cycle $i_1 \to i_2 \to .. \to i_L \to i_{L+1} = i_1$. Cycles $C(i_k)$ and $C(i_{k+1})$ share the node $j(i_k)$ and thus we can construct a (non-simple) cycle $\mathbf{C}$ obtained by gluing all $C(i_k)$, $k \in [L]$. We have $\pi(\mathbf{C}) = \prod_{k\in[L]} \pi(C(i_k)) < 1$. With each edge $i_k \to g$, the cycle $\mathbf{C}$ contains the inverse edge $g \to i_k$. Thus by eliminating all the edges $i \to g$ and $g \to i$, $i \in [n]$ from $\mathbf{C}$ we get a new cycle $\mathbf{C}'$ in $\overrightarrow{\mathcal{CG}}_{\mathbf{z}}$ with $\pi(\mathbf{C}') = \pi(\mathbf{C}) < 1$. By Lemma 2.1, we get a contradiction with fractional-Pareto-optimality of $\mathbf{z}$. ◀

The next Lemma is a key technical result: for non-degenerate problems it provides a dynamic programming approach for enumerating fractionally-PO allocations.[6]

▶ **Lemma 3.8.** *For every fixed integer $n \geq 2$, for every instance with n agents with non-degenerate and strictly-positive valuations, the number of consumption-graphs of fractionally-PO allocations is $m^{O(n^2)}$, and it is possible to enumerate all of them in time $m^{O(n^2)}$.*

Proof. We construct the *tree of consumption-graphs*. In this tree, each node represents a possible consumption-graph of a fractionally-PO allocation. For each $i \in [n]$, the nodes in depth i represent consumption-graphs of fractionally-PO allocations among only the agents $1, \ldots, i$.

The root, which we define as having a depth of 1, represents the fractionally-PO division in which all goods are given to agent 1 (the consumption-graph in which there are edges from agent 1 to all m goods, and no edges from another agent to any good).

The nodes in depth 2 represent all and only fractionally-PO divisions between agents 1 and 2. By Lemma 2.3, with non-degenerate valuations, there are $2m + 1$ such graphs. One graph is identical to the parent graph — no goods are given to agent 2. To generate the other $2m$ graphs, order the m goods in decreasing order of the ratio v_{2g}/v_{1g}. For each $g \in \{1, \ldots, m\}$, create one graph by adding an edge from agent 2 to g (representing a sharing of g between 1 and 2), and another graph by removing the edge from agent 1 to g (representing the good g entirely given to 2).

Each node in depth 3 is generated from a node in depth 2 in the following way. Consider a node in depth 2 in which there are m_1 edges from agent 1 and m_2 edges from agent 2 (where $m_1 + m_2 = m$ if the node represents 0 sharings and $m_1 + m_2 = m + 1$ if the node represents 1 sharing). By Lemma 2.3, with non-degenerate valuations, there are $(2m_1 + 1)$ ways to reallocate the former m_1 edges between agents 1 and 3, and $(2m_2 + 1)$ ways to reallocate the latter m_2 edges between agents 2 and 3. All in all, there are $(2m_1 + 1)(2m_2 + 1)$ combinations of reallocations, each of which generates a consumption-graph, so there are less than $(2m+1)^2$ new consumption-graphs per node in depth 2. The total number of consumption-graphs

[6] There are several alternative approaches. [16] recover all n-agent graphs by their 2-agent projections. [24] use a complicated technique of cell-enumeration from computational algebraic geometry. A dynamic-programming approach similar to ours was suggested by [26]. The difference is that instead of repeatedly adding new agents, this algorithm adds new goods and runs in $O(2^n m \cdot L)$, where L is the total number of different consumption graphs of fractionally-PO allocations. Lemma 3.8 provides an a-priori polynomial upper bound on L, which shows that the algorithm of [26] is also polynomial for non-degenerate problems.

generated in depth 3 is less than $(2m+1)^3$. Some of these are duplicates and can be removed; some of these may contain cycles with product less than 1 and can also be removed (by Lemma 2.1); so we remain with *only* consumption-graphs of fractionally-PO allocations. We now have to prove that *all* consumption-graphs of fractionally-PO allocations among agents 1,2,3 are generated by this process. Indeed, let $\mathbf{z}$ be an arbitrary fractionally-PO allocation among $1, 2, 3$. By Lemma 3.7, we can reallocate the goods allocated to agent 3, one by one, between agents 1 and 2, such that the new allocation remains fractionally-PO. Hence we get a new fractionally-PO allocation $\mathbf{z}'$, and its consumption-graph $\overrightarrow{\mathcal{CG}}_{\mathbf{z}'}$ is one of the nodes in depth 2. By construction, $\overrightarrow{\mathcal{CG}}_{\mathbf{z}}$ is generated as a child of this $\overrightarrow{\mathcal{CG}}_{\mathbf{z}'}$. All in all, the at most $(2m+1)^3$ nodes in depth 3 represent all and only the fractionally-PO allocations among agents 1,2,3.

We proceed in the same way down the tree. For each $i \in [n]$, for each node in depth $i-1$, we generate at most $(2m+1)^{i-1}$ nodes in depth i by reallocating some edges to agent i, and then remove duplicates and graphs with cycles of product less than 1. All in all, the nodes in depth i represent all and only the consumption-graphs of fractionally-PO allocations among the agents $1, \ldots, i$. The number of nodes in depth i is at most $(2m+1)^{1+2+\ldots+(i-1)} = (2m+1)^{i(i-1)/2}$. The leaves (in depth n) represent all and only the fractionally-PO allocations among all n agents. Their number is at most $(2m+1)^{n(n-1)/2} \in (m^{O(n^2)})$. The time required to construct the tree is proportional to the total number of nodes, which is roughly bounded by the depth times the total number of leafs: $n \cdot (2m+1)^{n(n-1)/2} = m^{O(n^2)}$. ◀

The following theorem extends Theorem 3.3:

▶ **Theorem 3.9.** *Fix an integer $n \geq 2$. For every n-agent instance with non-degenerate and strictly-positive valuations, it is possible to find in time $m^{O(n^2)}$ an allocation that minimizes the number of shared goods subject to (1) proportionality and fractional-PO, or (2) EF and fractional-PO.*

Proof. Apply Lemma 3.8 to enumerate all consumption graphs of fractionally-PO allocations. For each consumption graph G do:

- Check the number of shared goods in G (the number of goods with two or more incoming edges). If it is more than $n-1$ — skip G.
- For each shared good g, and each agent i with whom it is shared, create a variable $x_{i,g}$ representing the fraction of g allocated to i. The total number of variables is at most $2(n-1)$.
- Represent the required fairness condition (EF / proportional) as a set of linear inequalities in these variables. Solve the resulting LP. Note that the size of this LP depends only on n (at most $2(n-1)$ variables and at most $3(n-1)$ constraints), and thus it can be solved in constant time (assuming n is fixed).

Among those graphs G where the LP has a solution, select the one with the smallest number of shared goods and return the corresponding allocation. Since all graphs of fractionally-PO allocations are checked, a fractionally-PO+fair allocation with the smallest number of shared goods will be found. ◀

▶ Remark 3.10. Instead of minimizing the number of shared goods, it is possible to minimize the number of sharings using a similar algorithm.

If we remove the fractional-PO requirement from Theorem 3.9, we get at the hardness result extending Theorem 3.4 to n agents.

▶ **Theorem 3.11.** *For fixed integers $n \geq 3$ and $k \leq n-3$, it is NP-hard to decide whether a given non-degenerate instance with strictly-positive valuations admits an EF allocation with at most k shared goods.*

Proof. We will construct a reduction from PARTITION. It is enough to show the result *for degenerate valuations allowing zeros* and then to add small perturbations as it was done in Theorem 3.4 and thus guarantee strict positivity and non-degeneracy.

Given an instance $a_1, ..a_p$ of PARTITION, we build a fair division instance with $m = p + k + (n - k - 3)$ goods — p "number-goods" and k "special-goods" and $n - k - 3$ "personal-goods".

- Agents 1 and 2 assign a positive value only to the number-goods, with $v_{1,g} = v_{2,g} = a_g$ for all $g \in [p]$.
- If $k \geq 1$, there is a group $A = \{3, 4, \ldots, k+3\}$ of $k+1$ agents, who assign a positive value only to the special-goods — the values of all these agents to all special-goods are equal.
- All remaining agents (if any) are single-minded: each agent $i \in \{k+4, \ldots, n\}$ comes in pair with his favorite personal-good, while for all other goods, i has zero value.

For $k \geq 1$, in an envy-free allocation, each special-good must be shared: otherwise agents from the second group envy each other. Hence the number of shared goods in an envy-free allocation is at least k. Therefore an envy-free allocation with at most k shared items exists if and only if goods from $[p]$ can be divided in envy-free manner between 1 and 2 *without any sharing*, i.e., if $a_1, ..a_p$ can be partitioned in two subsets of equal sum. ◀

▶ **Question 2.** Theorem 3.11 leaves open the case of proportionality (instead of EF) and the case $k = n - 2$. Are they NP-hard too?

Going back to fractionally-PO+fair allocations: Theorems 3.5 and 3.9 imply that finding a fractionally-PO+fair allocation with one sharing is computationally easy both when the valuations are non-degenerate, and when the valuations are maximally degenerate (i.e., identical). Thus one would guess that this problem is computationally easy for general additive valuations. The following theorem shows that this guess is incorrect: in this "intermediate" case the problem is NP-hard!

▶ **Theorem 3.12.** *For every two integers $n \geq 3$ and $k \leq \max(1, n-3)$, considering the collection of all n-agent instances in which some valuations are identical, it is NP-hard to decide whether there exists an allocation with at most k shared goods that is: (1) proportional and fractionally-PO, (2) EF and fractionally-PO.*

In particular, with $n \geq 3$ agents, it is NP-hard to decide whether there exists a fractionally-PO+fair allocation with at most 1 sharing, and a fractionally-PO+fair allocation with at most $n - 3$ shared goods. This leaves open the case $n \geq 4$ with $k = n - 2$ shared goods.

Proof of Theorem 3.12. We reduce from PARTITION. Consider a PARTITION instance with p positive integers, $a_1, \ldots, a_p$. Denote the sum of these p integers by $2S$. Let $b_1, \ldots, b_p$ be numbers such that $\forall_{g \in [p]} : b_g > 0$ and $\sum_{g \in [p]} b_g = 1/4$, and the b_g are selected such that the valuations below are non-degenerate (similarly to Theorem 3.4, there are infinitely many ways to select the b_g).

We create a fair division problem with $m = p+k$ goods: p "number goods" and k "special goods". There are n agents: agents $1, 2$ have the same valuation and agents $3, \ldots, n$ have the same valuation, but the valuations of agents from different groups are different and non-degenerate. The values are shown below:

	p number-goods		k special-goods	
	value of good g	total	value of good	total
Agents $1,2$	$k \cdot a_g$	$2kS$	$(n-2)S + 1/(2k)$	$(n-2)kS + 1/2$
Agents $3,\ldots,n$	$k \cdot a_g - b_g$	$2kS - 1/4$	$(n-2)S + 3/(4k)$	$(n-2)kS + 3/4$

The important properties of this valuation profile are:

- For all agents, the sum of values of all goods is $nkS + 1/2$, so the proportional share of each agent is $kS + 1/(2n)$.
- In any proportional or EF division, agent 1 and/or agent 2 must get some of the special-goods (since their combined proportional share is $2kS + 1/n$, which is more than the sum of values of the number-goods).
- If g is a number-good and h is a special-good, then for every $i \in \{1,2\}$ and for every $j \in \{3,\ldots,n\}$, $v_{i,g}/v_{j,g} > 1 > v_{i,h}/v_{j,h}$. So by Lemma 2.3, in any fractionally-PO allocation, if agent i gets a positive amount of h, then agent j gets no g.

Hence, in any division that is fractionally-PO+proportional or fractionally-PO+EF:

1. Agents 1 and 2 get all number-goods and some special goods, while agents $3,\ldots,n$ get only special goods.
2. Agent 1 and agent 2 together must get a value of less than $3/4$ from the special-goods (since otherwise the remaining value of the special-goods for agents $3\ldots,n$ is less than $(n-2)kS$, while the combined proportional share of agents $3\ldots,n$ is $(n-2)kS + (n-2)/(2n) \geq (n-2)kS + 1/6$).
3. Each of the agents $3,\ldots,n$ must get a value of more than kS and less than $kS + 3/4$ from the special goods (otherwise there will not be enough value for all of them).

First, assume that $k = 1$.

If there exists a fractionally-PO+EF or fractionally-PO+proportional division with at most 1 shared good, then this single shared good must be the single special good (by property 1). So the number-goods are divided among agents 1 and 2 without sharing. The values of the number-goods are integers, and by property 2, the value given to agents 1 and 2 together from the special goods is at most $3/4$. Hence, each of them must get a value of exactly $kS = S$ from the number goods. This implies an equal-sum partition of the numbers $a_1,\ldots,a_p$.

Conversely, if there exists an equal-sum partition of the numbers, then a fractionally-PO+EF and proportional allocation with at most 1 shared good can be created as follows. Give each of agents $1,2$ a value of $kS = S$ from the number-goods. Give each of agents $3,\ldots,n$ a value of $kS = S$ from the single special-good. So far the division is envy-free: the share of agents $1,2$ is worth less than S for agents $3,\ldots,n$ and vice-versa. Divide the remainder of the special good equally among all n agents. The division remains EF. Hence it is proportional too.

Second, assume that $k \leq n-3$.

If there exists a fractionally-PO+EF or fractionally-PO+proportional division with at most k shared goods, then by property 3, the k shared goods must be the k special goods, since the value each agent is allowed to get from a special good is at most $kS + 3/4$ which is less than $(n-2)S + 3/(4k)$, so no agent may get a whole special good. So the number-goods are divided among agents 1 and 2 without sharing, and we get an equal-sum partition of the numbers as in the case $k = 1$.

Conversely, if there exists an equal-sum partition of the numbers, then a fractionally-PO+EF and proportional allocation with at most k shared goods can be created by giving agents $1,2$ a value of kS from the number-goods, giving agents $3,\ldots,n$ a value of kS from the special-goods, and dividing the remaining special goods equally among all agents. ◀

▶ **Question 3.** Can we extend Theorems 3.11 and 3.12 to $k = n - 2$ (in particular $n = 4$ and $k = 2$)?

4 Related Work

4.1 Fair division with bounded sharing

The idea of finding fair allocations with a bounded number of shared goods was first studied by [13, 14] who developed the *Adjusted Winner (AW)* procedure for allocating goods among two agents with additive utility functions. This procedure finds an allocation that is Pareto-optimal, envy-free, equitable, and shares at most a single good. Since in some cases it may be necessary to share a single good, this algorithm is optimal (in the worst case) in terms of the number of shared goods. The Adjusted Winner procedure has been applied (at least theoretically) to division problems in divorce cases and international disputes [15, 35]. It was also studied empirically [44, 22].

The Adjusted Winner procedure is designed for two agents. [13] present an example with three agents in which no allocation is simultaneously Pareto-optimal, envy-free and equitable. However, this example does not rule out the option of satisfying each two of these three properties.

In an unpublished manuscript, [53][section 3] proved that the lower bound of $n - 1$ on the number of sharings can be attained in two other kinds of fair allocations: *max-equitable* (an allocation in which each agent receives exactly the same subjective utility, and this same value is maximized over all allocations) and *max-min* (an allocation which maximizes the smallest utility of an agent). He also proves that there exists a *max-equitable-envy-free* allocation with at most $n^2 - 1$ sharings. Recently, [11][Lemma 1] proved that any (fractionally)-Pareto-optimal utility profile can be represented by an allocation in which only $n - 1$ goods are shared. We prove an algorithmic version of this result in Subsection 2.3.

4.2 Other fairness approximations

Envy-freeness except one good (EF1) was introduced by [34] and studied extensively by [17, 2, 8, 38] and others.

Maximin share guarantee (MMS) was introduced by [17] and studied extensively by [41, 29] and others. It has been applied for allocating seats in courses [18].

Indivisible goods can be made "divisible" by means of lotteries. However, in this approach fairness holds in a weaker sense: in expectation, before the lottery is implemented. This approach has been studied extensively for random assignment problem ([9], [1], [32]), its multi-unit and constrained modifications [19], as a device for tie-braking in matching markets [33], and in the context of online-fair division [10].

A new approach to fairness with discrete goods is using a *monetary subsidy* by a third party [31]. In contrast to other results on fair allocation of indivisible goods with monetary transfers (e.g., rent-division problem of [28]), [31] focus on minimization of monetary transfers and provide universal guarantees that hold for any valuation profile.

4.3 Existence of fair allocations

Fair allocation of discrete goods might not exist in all cases, but it may exist in some cases. A natural question is how to decide whether it exists in a specific instance. [34, 23, 12] study this problem for various fairness and efficiency notions and show that, in general, it is computationally hard (with some exceptions).

[25] study the existence of fair allocations on random instances; they show a phase transition — a fair allocation usually does not exist in problems with few goods, but does exist in problems with many goods.

4.4 Cake-cutting with few cuts

The goal of minimizing the number of "cuts" has also been studied in the context of *fair cake-cutting* — dividing a heterogeneous continuous resource [52, 46, 5, 6, 3, 45]. Since the resource is continuous, the techniques and results are quite different.

5 Future Work and Open Questions

5.1 Truthful Fair Division

A division algorithm is *truthful* if for every agent i, it is a weakly-dominant strategy to report the true $v_{i,g}$ values. [37] and [20] present a simple truthful fair division algorithm using a *consensus division*.

▶ **Definition 5.1.** *A division* $\mathbf{z}$ *is called a* consensus division *if for every two agents* i, j*:* $u_i(\mathbf{z}_j) = V_i/n$

Once we have a consensus division, we can allocate one part to each agent uniformly at random. The expected utility of any agent i, whether truthful or not, is V_i/n, so the agent cannot gain by false reporting. Moreover, a truthful agent gets a utility of exactly V_i/n with certainty, so for a truthful agent, the allocation is proportional and envy-free with certainty. A non-truthful agent might get more or less than V_i/n. Therefore, for an agent who is slightly risk-averse, truthfulness is a strictly dominant strategy.

▶ **Theorem 5.2.** *In any instance with* n *agents, there exists a consensus division with at most* $n(n-1)$ *sharings (hence at most* $n(n-1)$ *shared goods). Such allocation can be found in time* $O(poly(n, m))$*.*

Proof. The proof is based on an idea of [53]. The set of consensus allocations is defined by $n(n-1) + m$ linear constraints:

$$\sum_{i=1}^{n} z_{i,g} = 1 \qquad \text{for } g \in [m] \tag{5.1}$$

$$\sum_{g=1}^{m} v_{i,g} z_{j,g} = V_i/n \qquad \text{for } i \in [n], j \in [n-1]$$

The first m equalities assert that all goods are fully allocated. The next $n(n-1)$ equalities assert that each agent $i \in [n]$ believes that the share of each agent $j \in [n-1]$ is worth exactly V_i/n; this implies that the same is true when $j = n$, so we do not need an equality for that.

The LP is feasible — the constraints are satisfied, for example, by letting $z_{i,g} = 1/n$ for every i, g. Hence, it has a *basic feasible solution* — a solution with at most $n(n-1) + m$ nonzero elements [36]. Such a solution corresponds to an allocation with at most $n(n-1)$ sharings. ◀

Combining the algorithm of [37] and [20] with Theorem 5.1 gives:

▶ **Corollary 5.3.** *There exists a randomized truthful algorithm that always returns a proportional allocation with at most* $n(n-1)$ *sharings.*

▶ **Question 4.** (a) Can the upper bound for a consensus division be improved? Can we get a consensus division with $n(n-1)-1$ sharings? With $O(n)$ sharings?

(b) Can we find in polynomial time a consensus allocation with a minimal number of shared goods?

5.2 Non-linear utilities

So far we assumed that the utility of an agent from a good is a linear function of the fraction of the good he owns.

In reality, due to the overhead in managing shared ownership, the utility-decrease might be non-linear. In general, the utility of agent i from having a fraction $z_{i,g}$ of good g may be $f(z_{i,g}) \cdot v_{i,g}$, where f is an increasing function with $f(0) = 0$ and $f(1) = 1$. For example, if $f(x) = x^2$, then having 1/2 of good g gives an agent only 1/4 the utility of having the entire good. We call f the *overhead function*.

We still assume additivity between different goods, so for example having $z_{i,g}$ of g and $z_{i,h}$ of h gives agent i a utility of $f(z_{i,g}) \cdot v_{i,g} + f(z_{i,h}) \cdot v_{i,h}$.

5.2.1 Discontinuous overhead functions

Often, the very fact that a good must be shared, even if only a small fraction of it is shared, incurs a relatively high cost. Then, the overhead function might be discontinuous. We consider a particular family of overhead functions, parametrized by a fraction $r \in (0,1)$:

$$f_r(z_{i,g}) = \begin{cases} 1 & z_{i,g} = 1 \\ r \cdot z_{i,g} & z_{i,g} < 1 \end{cases}$$

For example, $r = 0.7$ means that the value of a shared good drops by 30%.

Unfortunately, with such a discontinuous overhead function, there might be no fractionally-PO+EF allocation at all.

▶ **Theorem 5.4.** *For any $r < 1$, when the overhead function is the discontinuous function f_r, there is an instance with two agents and identical valuations in which no division is both EF and fractionally-PO.*

Proof. Suppose there are 2 goods and 2 agents with identical valuations (for some $\epsilon > 0$):

$$v(g_1) = 1 + \epsilon \qquad\qquad v(g_2) = 1$$

If no goods are shared, then obviously the division is not EF.

If one good is shared, then this good must be g_1. So g_2 must be given entirely to one agent, say Alice. But then the value of Bob is at most $r \cdot (1 + \epsilon)$. When $\epsilon < 1/r - 1$, Bob's value is smaller than 1, so the division is not EF.

If both goods are shared, then the sum of values is $r \cdot (2 + \epsilon)$, so in an EF division each agent must get a value of exactly $r \cdot (1 + \epsilon/2)$. When $\epsilon/2 < 1/r - 1$, each agent's value is smaller than 1, so the division is Pareto-dominated by the division in which each agent gets a whole good. ◀

5.2.2 Continuous and convex overhead functions

Suppose now that the overhead function f is continuous and convex. A reasonable choice is $f(z_{i,g}) = (z_{i,g})^p$, for some constant $p > 1$. The first question of interest is whether there is an upper bound on the number of sharings.

At first glance it seems that, for convex f, an analogue of Lemma 2.4 should hold, guaranteeing an upper bound of $n-1$ sharings (by the same cycle-trading argument). The difficulty is that under such f the Pareto-frontier becomes non-convex, and proportional or envy-Free allocations may fail to exist.

We can also try to attain an upper bound using the "dual" approach (an analogue to the LP approach mentioned in Remark B.1). We would have to solve the following non-linear optimization problem:

$$\begin{aligned}
\text{maximize} \quad & \sum_{i=1}^{n}\sum_{g=1}^{m} v_{i,g}\cdot f(z_{i,g}) && (5.2)\\
\text{subject to} \quad & \sum_{i=1}^{n} z_{i,g} = 1 && \text{for } g\in[m]\\
& \sum_{g=1}^{m} v_{i,g}\cdot f(z_{i,g}) = U_i + s_i && \text{for } i\in[n]\\
& z_{i,g}\geq 0, s_i \geq 0 && \text{for } i\in[n], g\in[m]
\end{aligned}$$

▶ **Question 5.** When f is non-linear and convex, does the above problem have "basic feasible solutions" with a small number of nonzero variables? Can it be solved efficiently?

If the general answer is "no", does it change to "yes" for the special case $f(z_{i,g}) = (z_{i,g})^p$, for some $p \geq 1$?

5.3 Chores

Our focus is this paper was on goods (items with positive valuations), but some of the results hold for chores (items with negative valuations) too.

Lemma 2.4 holds as-is whether the valuations are positive, negative or mixed.

Corollary 2.5 holds for chores too. It holds for mixed valuations when the fairness criterion is proportionality. When the criterion is envy-freeness, it would hold if we knew how to calculate market equilibria.

What about the other results?

APPENDIX

A Characterization of fractional Pareto-optimality

In this section we prove Lemma 2.1: *With strictly-positive valuations, an allocation* $\mathbf{z}$ *is fractionally-Pareto-optimal if-and-only-if its consumption-graph* $\overrightarrow{\mathcal{CG}}_{\mathbf{z}}$ *contains no directed cycle* C *with* $\pi(C) < 1$.

The lemma was proved in [16] for the case of chores (items with negative utilities) and in [4], Section 8, for the problem of *cake-cutting*. For completeness, we present a stand-alone proof below.

Proof that fractional-PO $\implies$ no C. We assume by contradiction that $C = (i_1 \to g_1 \to i_2 \to g_2 \to \ldots \to i_L \to g_L \to i_{L+1} = i_1)$ is a directed cycle in $\overrightarrow{\mathcal{CG}}_{\mathbf{z}}$ with $\pi(C) < 1$. We show how to construct an exchange of goods among the agents in C such that the utility of all agents in C strictly increases without affecting the other agents. This will imply that $\mathbf{z}$ is Pareto-dominated.

Define $R := \pi(C)^{1/L}$; by assumption, $R < 1$.

For each $k \in [L]$, there is an edge from agent i_k to good g_k. Hence, by definition of $\overrightarrow{\mathcal{CG}}_{\mathbf{z}}$, i_k consumes a positive amount of g_k. Suppose each i_k gives some small positive amount ϵ_k of g_k to i_{k+1} (where $\epsilon_k \in (0, z_{i_k,g_k})$). Then, agent i_k loses a utility of $\epsilon_k \cdot v_{i_k,g_k}$ to the next agent, but gains a utility of $\epsilon_{k-1} \cdot v_{i_k,g_{k-1}}$ from the previous agent, so the net change in the utility of i_k is $\epsilon_{k-1} v_{i_k,g_{k-1}} - \epsilon_k v_{i_k,g_k}$ (where the arithmetic on the indices k is done modulo L such that the index is always in $\{1, \ldots, L\}$). To guarantee that all agents in C strictly gain from the exchange, it is sufficient to choose $\epsilon_1, \ldots, \epsilon_k$ such that the following L inequalities hold:

$$
\begin{aligned}
\forall k \in [L] : & \epsilon_{k-1} v_{i_k,g_{k-1}} > \epsilon_k v_{i_k,g_k} \\
\iff & \epsilon_{k-1}/\epsilon_k > v_{i_k,g_k}/v_{i_k,g_{k-1}} \\
\iff & \epsilon_k/\epsilon_{k-1} < v_{i_k,g_{k-1}}/v_{i_k,g_k}
\end{aligned}
\tag{A.1}
$$

For any $\epsilon_1 > 0$, define the ϵ_k as follows:

$$\epsilon_k := \epsilon_{k-1} \cdot R \cdot (v_{i_k,g_{k-1}}/v_{i_k,g_k}) \qquad \text{for } k \in \{2, \ldots, L\}$$

Since $R < 1$, the inequality (A.1) is satisfied for each $k \in \{2, \ldots, L\}$. It remains to show that it is satisfied for $k = 1$ too (note that in this case we define $k - 1 = L$). Indeed:

$$
\begin{aligned}
& \epsilon_L = \epsilon_1 \cdot R^{L-1} \cdot \prod_{k=2}^{L} (v_{i_k,g_{k-1}}/v_{i_k,g_k}) \\
\iff \quad & \epsilon_1/\epsilon_L = R^{1-L} \prod_{k=2}^{L} (v_{i_k,g_k}/v_{i_k,g_{k-1}}) \\
& = (R/R^L) \cdot \pi(C) \cdot (v_{i_1,g_L}/v_{i_1,g_1}) \\
& = R \cdot (v_{i_1,g_L}/v_{i_1,g_1}) \\
& < v_{i_1,g_L}/v_{i_1,g_1}.
\end{aligned}
$$

By choosing ϵ_1 sufficiently small, it is easy to guarantee that, for each $k \in [L]$, $\epsilon_k < z_{i_k,g_k}$, so this trade is possible. ◀

Proof that no $C \implies$ fractional-PO. We assume that $\overrightarrow{\mathcal{CG}}_{\mathbf{z}}$ contains no directed cycles C with $\pi(C) < 1$. We prove that $\mathbf{z}$ maximizes a monotonically-increasing function (a weighted sum) of the agents' utilities. This will imply that $\mathbf{z}$ is fractionally-PO.

Assume w.l.o.g. that agent 1 has a non-empty bundle, $\mathbf{z}_1 \neq \emptyset$. Then, in $\overrightarrow{\mathcal{CG}}_{\mathbf{z}}$, there is an edge from 1 to some good g, so there is a path $1 \to g \to j$ from 1 to every other agent j. For every other agent j, let $P_{1,j}$ be a directed path from 1 to j in $\overrightarrow{\mathcal{CG}}_{\mathbf{z}}$, for which the product $\pi(P_{1,j})$ is a minimum. The minimum is well-defined and attained on an acyclic path, since by assumption there are no cycles with product smaller than 1, so adding cycles to a path cannot make its product smaller.

Set the *weight* of each agent j as $w_j := \pi(P_{1,j})$ (in particular $w_1 = 1$). We prove that $\mathbf{z}$ maximizes the sum $\sum_{j=1}^{n} w_j \cdot u_j(\mathbf{z})$. It is sufficient to prove that $\mathbf{z}$ allocates each good g to an agent i for which the product $w_i \cdot v_{i,g}$ is maximized. Indeed, suppose some good g is allocated to an agent i ($z_{i,g} > 0$). Let j be any other agent. Consider the minimum-product path $P_{1,i}$. Since $z_{i,g} > 0$, there is an edge from i to g. There is always an edge from g to j. Consider the concatenated path $Q_{1,j} := P_{1,i} \cdot g \cdot j$. By definition of minimum-product paths:

$$\begin{aligned}
&\pi(Q_{1,j}) \geq \pi(P_{1,j}) \\
\implies &\pi(P_{1,i}) \cdot v_{i,g}/v_{j,g} \geq \pi(P_{1,j}) \\
\implies &w_i \cdot v_{i,g}/v_{j,g} \geq w_j \\
\implies &w_i \cdot v_{i,g} \geq w_j \cdot v_{j,g}
\end{aligned}$$

So indeed each good is allocated to an agent for whom the weight-value product is maximized. Now, in any Pareto-improvement allocation $\mathbf{z}'$, any monotonically-increasing function (in particular, a weigheted-sum with positive weights) attains a strictly-higher value. Since $\mathbf{z}$ maximizes such a function, it must be fPO. ◀

Lemma 2.1 has a useful computational implication, which we denoted by Lemma 2.2: *It is possible to decide in time $O(nm(n+m))$ whether a given allocation $\mathbf{z}$ is fractionally-Pareto-optimal.*

Proof. The basic idea is the following: construct the graph $\overrightarrow{\mathcal{CG}}_{\mathbf{z}}$, replace each weight with its logarithm, and look for a negative cycle using one of many existing algorithms [21] (e.g. Bellman-Ford). If there is a cycle C in which the sum of log-weights is negative, then $\pi(C) < 1$, so by Lemma 2.1, $\mathbf{z}$ is not fractionally-PO. Otherwise, $\mathbf{z}$ is fractionally-PO. A negative cycle can be found in time $O(|V| \cdot |E|)$. Here $|V| = m + n$ and $|E| \leq mn$.

Because of irrationality, logarithms can be computed only approximately and thus, to ensure the correctness of the algorithm, one has to adjust the quality of approximation depending on the input. However, these difficulties are easy to avoid by using a multiplicative version of any of the algorithms in [21]: multiplication replaces addition, division is used instead of subtraction, and one instead of zero.[7] This allows to avoid logarithms and keep the same estimate of $O(|V| \cdot |E|)$ on runtime. ◀

B Worst-case bound on sharing

In this section we prove Lemma 2.4: *For any allocation $\mathbf{z}$, there exists a fractionally-Pareto-optimal allocation $\mathbf{z}^*$ such that:*

[7] A similar idea is mentioned in slides by Robert Sedgewick and Kevin Wayn as an algorithm for *arbitrage detection*, https://algs4.cs.princeton.edu/lectures/44ShortestPaths-2x2.pdf

- *(a)* $\mathbf{z}^*$ *weakly Pareto dominates* $\mathbf{z}$*, i.e, for any agent* i*,* $u_i(\mathbf{z_i^*}) \geq u_i(\mathbf{z_i})$*.*
- *(b) the non-directed consumption graph* $\mathcal{CG}_{\mathbf{z}^*}$ *is acyclic.*
- *(c)* $\mathbf{z}^*$ *has at most* $n-1$ *sharings (hence at most* $n-1$ *shared goods).*

Such allocation $\mathbf{z}^*$ *can be constructed in time* $O(poly(m,n))$*.*

Proof. Let's call a cycle $C = (i_1 \to g_1 \to i_2 \to g_2 \to \ldots \to i_L \to g_L \to i_{L+1} = i_1)$ in the directed graph $\overrightarrow{\mathcal{CG}}_{\mathbf{z}}$ *simple* if each node is visited at most once and for any $i \in [n]$ and $g \in [m]$ only one of the edges $i \to g$ or $g \to i$ is contained in the cycle.

If there is a simple cycle C with $\pi(C) \leq 1$, then this cycle can be eliminated by the cyclic trade making all the agents weakly better off (similarly to the necessity proof of Lemma 2.1 in Appendix A). Transfer $0 < \varepsilon_k < z_{i_k,g_k}$ amount of g_k from i_k to i_{k+1}, where $\varepsilon_k v_{i_k,g_k} = \varepsilon_{k+1} v_{i_k,g_{k+1}}$ for $k = 1, ..., L-1$. Therefore, each agent i_k, $k = 2, ..L$, remains indifferent between old and new allocations while agent i_1 is weakly better off because of the condition $\pi(C) \leq 1$. By selecting epsilons as big as possible, we eliminate one of the edges $i_k \to g_k$ in the resulting allocation $\mathbf{z}'$.

Repeat this procedure again and again until there are no simple cycles with $\pi(C) \leq 1$. Note that we need at most $(n-1)m$ repetitions since each time at least one edge is deleted in the undirected graph $\mathcal{CG}_{\mathbf{z}}$ and the total number of edges is at most nm. Denote the resulting allocation by $\mathbf{z}^*$.

(a) By construction, $\mathbf{z}^*$ weakly improves the utility of each agent, has no cycles with $\pi(C) < 1$, and thus $\mathbf{z}^*$ is fractionally-Pareto-optimal by Lemma 2.1.

(b) The undirected consumption graph of $\mathbf{z}^*$ is acyclic: assume by contradiction that there is a cycle C in $\mathcal{CG}_{\mathbf{z}}$. Then in the directed graph $\overrightarrow{\mathcal{CG}}_{\mathbf{z}}$ there are two cycles: C passed in one direction and in the opposite. Denote them by $\overleftarrow{C}$ and $\overrightarrow{C}$. Since $\pi(\overrightarrow{C}) = \frac{1}{\pi(\overleftarrow{C})}$, by fractional-Pareto-optimality we get $\pi(\overrightarrow{C}) = \pi(\overleftarrow{C}) = 1$; however all such cycles were eliminated at the previous stages of the algorithm.

(c) Since any acyclic graph on $m+n$ vertices has at most $m+n-1$ edges, the number of sharings at $\mathbf{z}^*$ is at most $n-1$.

It remains to estimate the complexity of the algorithm. The run-time is determined by the number of operations needed to find a cycle C with $\pi(C) \leq 1$. To find cycles with $\pi(C) < 1$ we can use a multiplicative modification of the Bellman-Ford algorithm, as in Lemma 2.1.

When all cycles C with $\pi(C) < 1$ have been eliminated, it remains to delete all the cycles in the *undirected graph* $\mathcal{CG}_{\mathbf{z}}$ if any (note that all such cycles have $\pi(C) = 1$). Such cycles can be found using depth-first search which needs $O(|V| + |E|) = O(m)$ operations per cycle.

Since at most nm cycles are to be eliminated, the overall run-time of the algorithm is polynomial in m and n. ◀

▶ Remark B.1. An alternative "dual" approach to existence of $\mathbf{z}^*$ with $n-1$ sharings is based on linear programming (LP), see [53]: $\mathbf{z}^*$ can be constructed as a basic feasible solution (a vertex of the set of solutions) to a linear program where the objective is to maximize the utilitarian social welfare $\sum_{i\in[n]} u_i(\mathbf{z}_i^*)$ given the feasibility and domination constraints. Feasibility and domination can be expressed by $m+n-1$ constraints. Hence the LP has a "basic feasible solution" — a solution with at most $m+n-1$ nonzeros [36]. It corresponds to an allocation with at most $n-1$ sharings. We illustrate this approach in Subsection 5.1, where truthful fair allocation rules are discussed. Such LP-based approach leads to a weakly-polynomial algorithm.

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