"DIVIDE THE LAND EQUALLY" (Ezekiel 47:14)

Redividing the Cake

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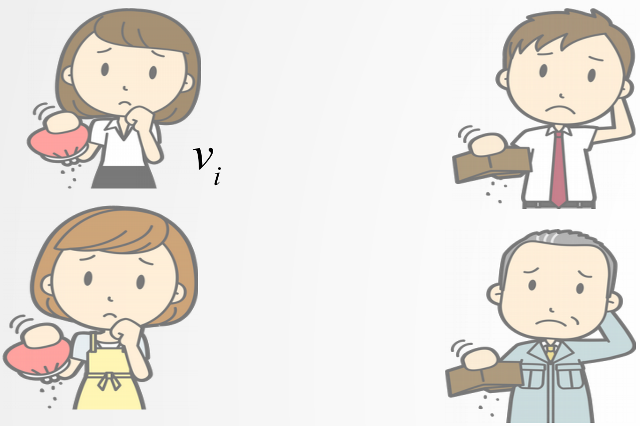
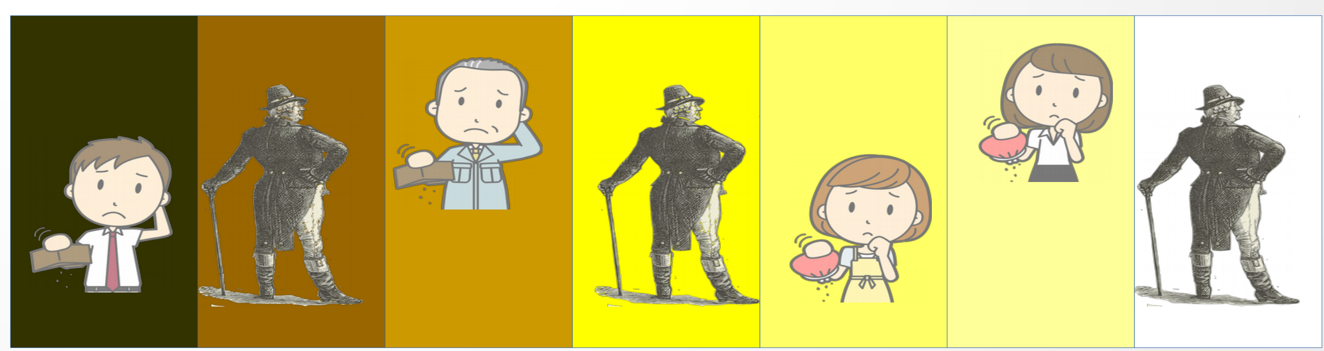
General Question

How to get from an existing, unfair division:



to a new, fair division:



Detailed Question

INPUT: Cake C; n agents;

- Each agent j has a value-measure V_j on C.
- Existing division: each j has a piece $\boldsymbol{Z_j}$ in C.

OUTPUT: allocate to each j a piece $\boldsymbol{X_j}$ in C.

DESIRED PROPERTIES (r, w are in [0,1]):

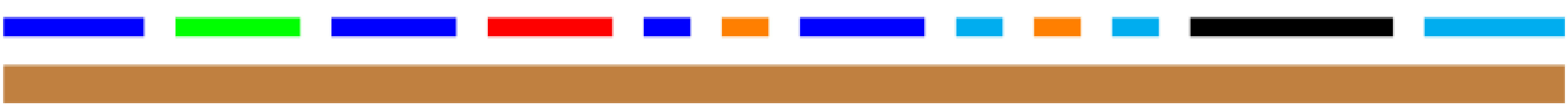
- Fairness: For each j, $V_j(X_j) \geq V_j(C)/n$.
- Ownership: For each j, $V_j(X_j) \geq V_j(Z_j)$.
- ***r*-fairness**: For each j, $V_j(X_j) \geq r * V_j(C)/n$.
- ***w*-ownership**: For each j, $V_j(X_j) \geq w * V_j(Z_j)$.

Q: What combinations of *r,w* are attainable?

Unrestricted pieces

Answer 1. r-fairness and w-ownership can be attained together if-and-only-if $r+w \le 1$.

- *"If" Proof idea*: Create a convex combination of the existing division with an arbitrary fair division. $O(n^2)$ queries.
- ***Caveat**. pieces might be disconnected:*



1-dimensional cake, connected pieces

Answer 2. If:

- The cake C is *a 1-dimensional interval*.
- Each piece must be *a connected interval*.

then, r-fairness and w-ownership cannot be attained together for *any* constant $r>0$, $w>0$.

Why? for every $r>0$ and integer k in $\{1,\dots,n\}$, there might be k **agents** who, in any r-fair division, get **at most ~k/n** of their old value.

Democratic Ownership

Definition. For every integer k in $\{1,\ldots,n\}$, there are **at least** $\boldsymbol{n-k}$ agents who get **at least** $\boldsymbol{\sim k/n}$ of their old value.

- *By previous slide, this is the best we can hope for if we want r-fairness for $r>0$.*

Revised question: what fairness can be attained together with democratic ownership?

1-dimensional cake, connected pieces

Answer 3. If:

- The cake C is *a 1-dimensional interval*.
- Each piece must be *a connected interval*.

then, democratic-ownership can be attained together with **1/3-fairness**.

Proof idea:

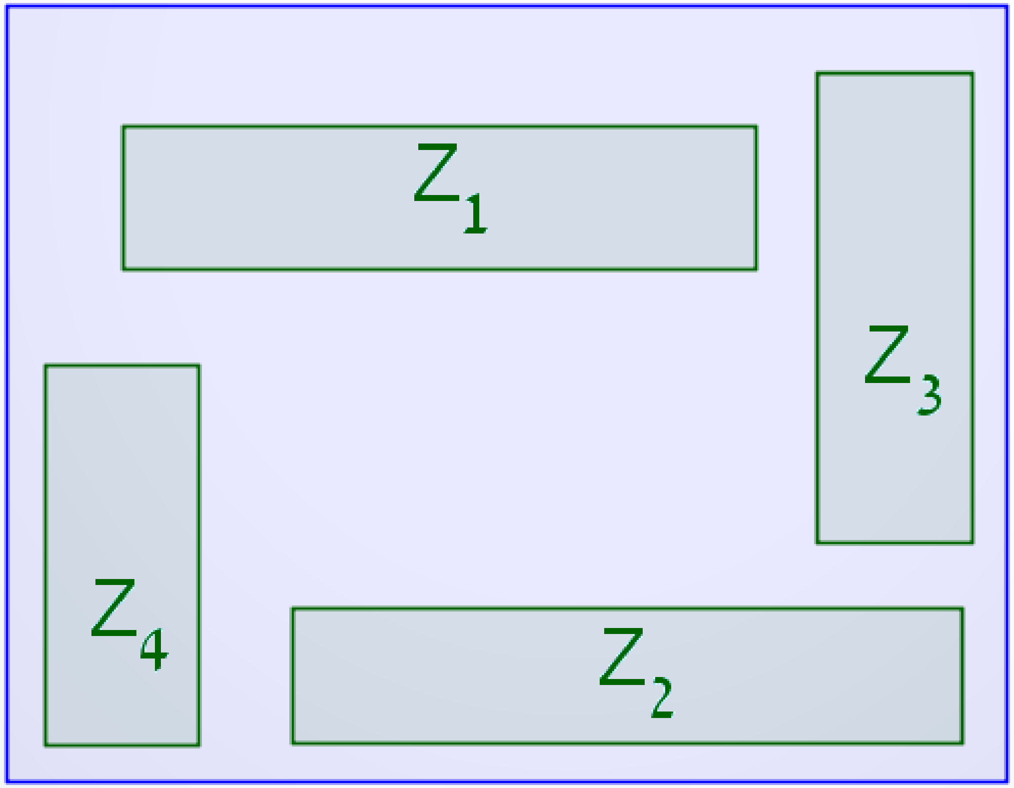
- Divide each Z_j among agents who want it most.
- *Pigeon principle*: many Z_j are thinly-populated.
- Value of agent j $\geq V_j(C) / (2n + \text{#pieces} - 1)$.

$> V_j(C) / 3n$

2-dimensional cakes

New challenge: division might have holes!

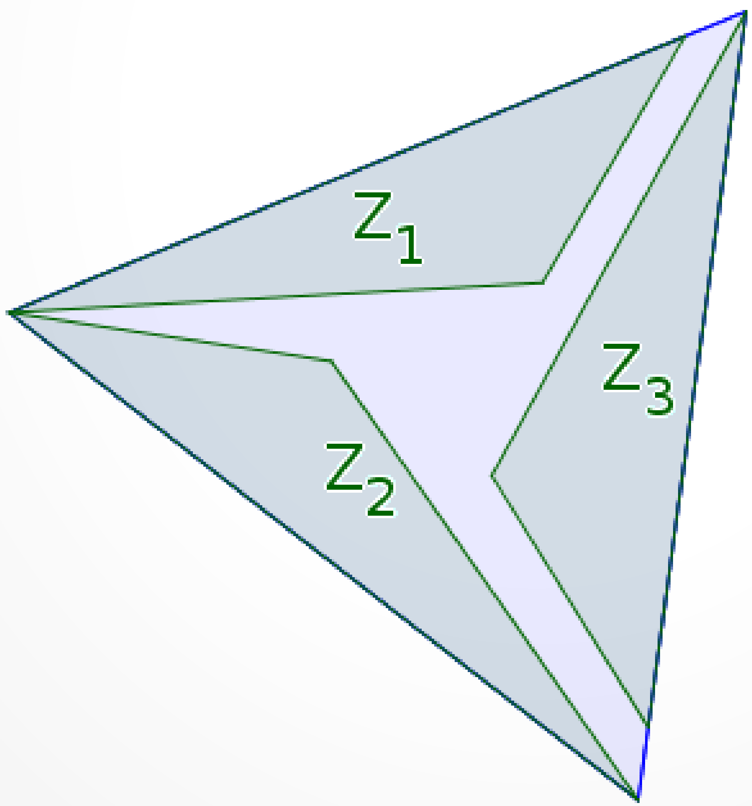
Rectangle cake and rectangle pieces:



2-dimensional cakes

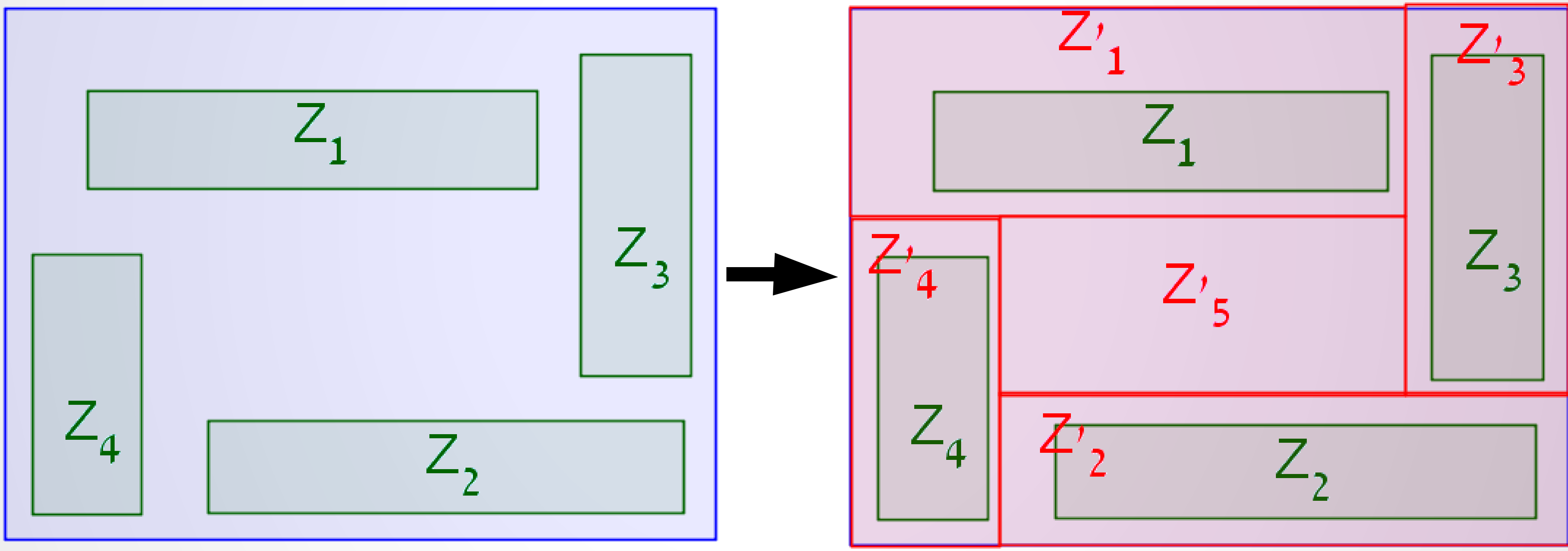
New challenge: division might have holes!

Convex cake and convex pieces:



Allocation-completion step

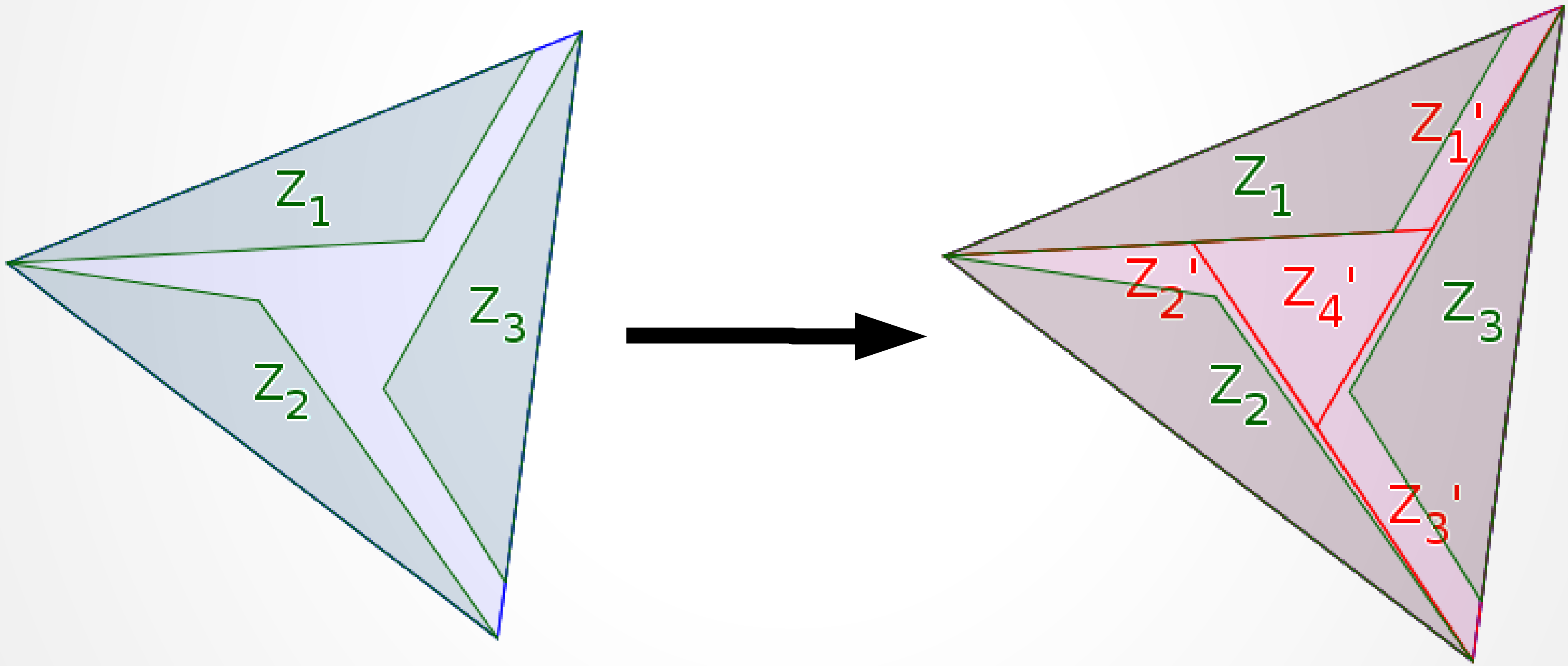
Rectangle cake and rectangle pieces:



#holes < n.

Allocation-completion step

Convex cake and convex pieces:



#holes $< 2\,n$.

Fairness levels attainable together with democratic ownership

The value of agent j is at least:

$$V_j(C) / (2n + \#\text{pieces} - 1).$$

Interval cake and pieces:

- #holes = 0, #pieces = n $\rightarrow$ 1/3-fairness.

Rectangle cake and pieces:

- #holes < n, #pieces < $2n$ $\rightarrow$ 1/4-fairness.

Convex cake and pieces:

- #holes < $2n$, #pieces < $3n$ $\rightarrow$ 1/5-fairness.
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Conclusion

To make fairness practical, we must handle the:

1) existing allocation;
2) geometric requirements.

Thank you very much!

Open Questions

What is the largest r such that r-fairness is compatible with democratic ownership:

- With 1-D connected intervals? ($\geq$ 1/3).
- With rectangles? ($\geq$ 1/4).
- With 2-D convex figures? ($\geq$ 1/5).
- With 2-D connected figures?
- With squares?
- With 3-D convex figures?
- With two 1-D intervals per agent?