Submission Deadline: October 18, 2024, 22:00 hrs

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1. (a) Use the following Monte Carlo estimator to approximate $I = E[e^U]$, where $U \sim U(0,1)$:

$$I_M = \frac{1}{M}\sum_{i=1}^{M} Y_i, \text{ where } Y_i = \exp(U_i) \text{ with } U_i \sim U(0,1).$$

Take the values of M to be $10, 10^2, 10^3, 10^4$ and 10^5. For each value of M, determine the sampling variance of the estimate. [You restart the random number generator from the same point (value) for each value of M.]

(b) Repeat the above exercise using antithetic variates and call the estimator $\widehat{I}_M$ (with comparable values of M).

(c) Repeat the above exercise using control variates taking U itself as the control variable, and call the estimator $\widetilde{I}_M$.

(d) Present the results that you have obtained in Parts (a), (b) and (c) in a tabular form. Your table must consist of the values of I_M, $\widehat{I}_M$ and $\widetilde{I}_M$, their sampling variances, and the percentage reduction in the variances of the improved estimates over the crude Monte carlo estimate. How do the values of the estimates compare with the actual value of I?

2. Suppose that the random variable Y is exponentially distributed with mean 1. Suppose further that, conditional on $Y = y$, the random variable X is Gaussian with mean y and variance 4. Taking the values of M to be $10, 10^2, 10^3, 10^4$ and 10^5, estimate by basic Monte Carlo the value of $P(X > 1)$. What would be the estimate if you use the variance reduction by conditioning idea, and how superior it is to the naive estimate? Can you improve it further by using either (a) antithetic variable technique, or (b) control variable technique? How much better these two new estimates are? Compare the estimates with sampling variances, and percentage reduction in variances.

3. Compound Poisson models are commonly used for rainfall and, here, we will look at stratifying such a model. In our simplified setting of such a model, we consider that the number of rainfall events (storms) in the coming month is given by the random variable N with $P\{N = 1\} = 0.19, P\{N = 2\} = 0.26, P\{N = 3\} = 0.24, P\{N = 4\} = 0.17, P\{N = 5\} = 0.14$. The depth of rainfall (in centimeters) in storm i is $D_i \sim Weib(k, \sigma)$ with shape $k = 0.8$ and scale $\sigma = 3$ (centimeters) and the storms are independent. The probability density function of $Weib(k, \sigma)$ distribution is given by

$$f(x) = \frac{k}{\sigma}\left(\frac{k}{\sigma}\right)^{k-1} e^{-(x/\sigma)^k} \quad \text{for } x > 0.$$

If the total rainfall is below 5 centimeters then an emergency water allocation will be imposed. Our goal is to approximate the probability of imposing the emergency water allocation in the coming month. Note that total rainfall is given by $\sum_{i=1}^{N} D_i$. Use simple Monte Carlo and stratification methods to approximate the probability based on $n = 10^2$ and 10^4. Also, provide the 99% confidence interval for the probability using both the methods.
