A PATH IN FOREST EXCHANGE PROPERTY OF SERIES-PARALLEL GRAPHS

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1. An Exchange Propery

It is well-known that a series-parallel subgraph, defined as a graph with no subgraph homeomorphic to K_4, is equivalent to a subgraph of an st-series-parallel network. Here an st-series-parallel network has two distinguished vertices s, t constructable from single edge st-series-parallel networks using the series and parallel connection operations. All graphs can have multiple edges unless otherwise indicated.

Theorem 1. *Let G be a series-parallel subgraph with two designated vertices s and t.*

(1) *Whenever F_0 and F_1 are forest subgraphs in G for which F_1 contains an st path P, there is a unique subset $A \subseteq P \setminus F_0$ for which $F_0 \cup A$ is a forest that contains an st path P' for which $A \subseteq P'$.*

(2) *In this situation, let $F_1' = F_0 \cup A$, which contains path P', and $F_0' = F_1 \setminus A$. A is also the unique subset $A \subseteq P' \setminus F_0'$ for which $F_0' \cup A$ is a forest that contains an st path P for which $A \subseteq P$.*

Remark: When $F_0 = \emptyset$, the property is satisfied with $A = P$. When F_0 contains an st path, the property is satisfied with $A = \emptyset$.

Proof. The theorem claims that there is a unique subset of zero or more edges $A \subseteq P \subseteq F_1$ from st path P which when added to F_0 comprises or completes an st path P' within a forest; and when the same set is removed from F_1 it is the unique subset of a path in $F_0 \cup A$ that comprises or completes an st path when added to $F_1 \setminus A$. Let us call A an "exchange set."

If F_1 doesn't contain an st path, there is nothing to prove. So, assume F_1 contains st path P. Note that if a forest contains an st-path then that st path is unique.

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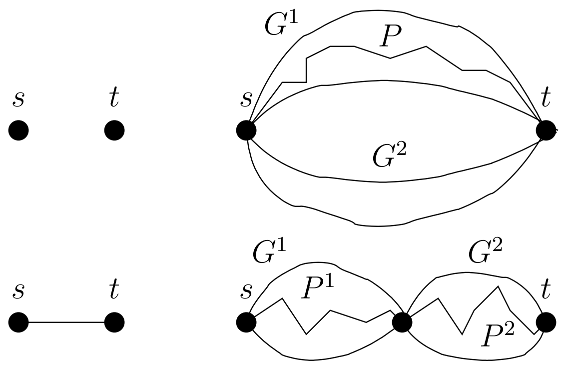
If F_0 already contains an st path, then $A = \emptyset$ is the only exchange set because additional paths cannot be produced by adding edges to F_0 without producing circuits. $F_1 \setminus A = F_1$ contains an st path, so $\emptyset$ is the only subset of a path that can be added to F_1 without producing circuits.

So, let us assume that F_0 does not contain an st path.

The following structural characterization of an st series-parallel subgraph facilitates the proof:

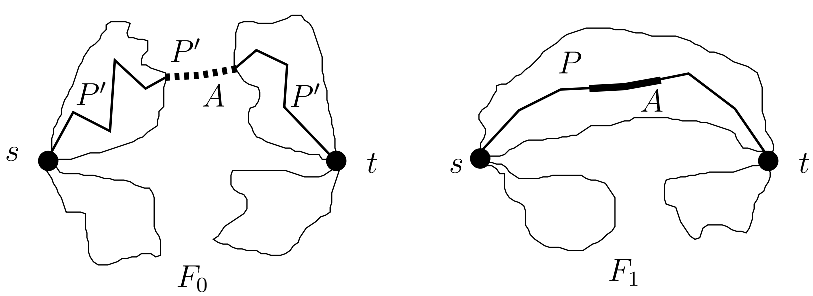
An st series-parallel subgraph is either the edgeless graph with vertices s, t, a one-edge graph with vertices s, t, or is graph composed from already constructed st series-parallel subgraphs by series or by parallel connection along the st terminals. See the figure.



We proceed by induction on the number of the above series or parallel construction steps for G.

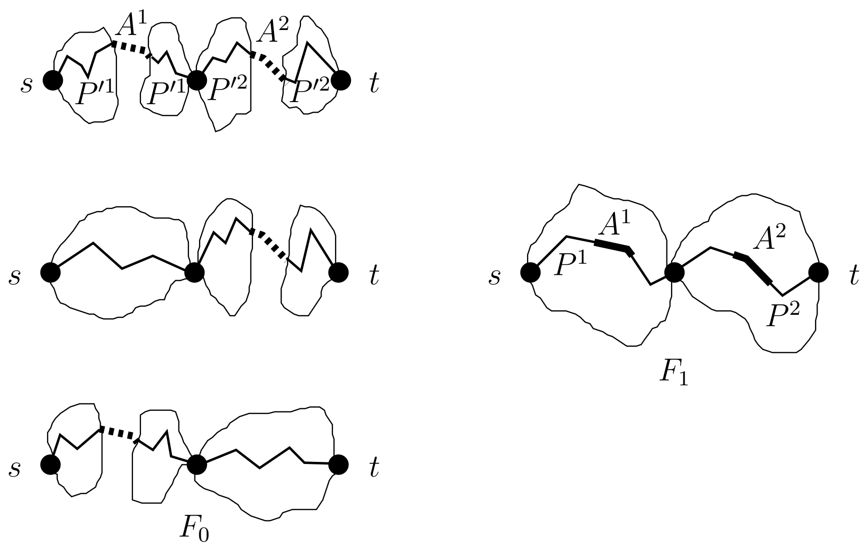
The theorem has been verified for the graph G with vertices s, t and no edges. For the one-edge graph $e = st$, we need check only $F_0 = \emptyset$ and $F_1 = \{e\}$. In this case, it is immediate that only $A = \{e\}$ satisfies the conclusion.

So, we assume G is the parallel connection along s, t of two graphs constructed with fewer steps, or is the series connection along s, t of such graphs. Let the subgraphs be G^1 and G^2. Let $F_i^j = F_i \cap G^j$ for $i = 0, 1$ and $j = 1, 2$. Recall that we can assume that F_0 does not contain an st path.



Consider the case of parallel connection. Assume the st path P in F_1 is contained in $F_1^1 \subseteq G^1$, otherwise reverse the roles of G^1, G^2. Let A be the exchange set for forests F_0^1, F_1^1 in G^1 that results from induction. $F_0 \cup A$ has as st path (since $F_0 \cup A \supset F_0^1 \cup A$) but no circuit, since otherwise, F_0^2 would have an st path, contradicting our assumption. Also, by induction, A is unique.

The path $P' \subseteq F_0^1 \subseteq G^1$, so none of its edges can comprise or complete an st path in $F_0^2 \setminus A = F_0^2 \subseteq G^2$. Therefore, A is the unique exchange set for F_0, F_1.



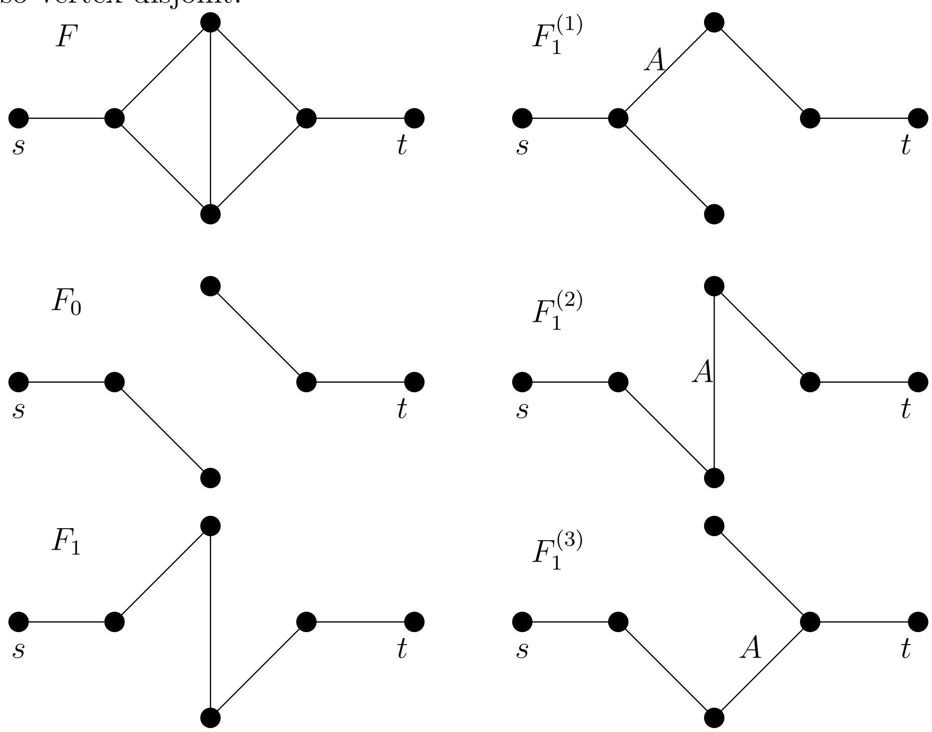
Now for series connection. The st path P in F_1 is uniquely partitioned into two interterminal paths P^1 and P^2 in F_1^1 and F_1^2 respectively. By induction, there are unique A^1, A^2 and P'^1, P'^2 satisfying the theorem for F_0^1, F_1^1 and F_0^2, F_1^2 respectively. $A^1 \cap A^2 = P'^1 \cap P'^2 = \emptyset$. Therefore $A = A^1 \cup A^2 \subset P'$ is the unique set for which $P' = P'^1 \cup P'^2$ is an st path in G with $A \subseteq P'$.

P'^1 and any subset of it must belong to G^1 and P'^2 and any subset of it must belong to G^2. Therefore, the properties and uniqueness of A^1, A^2 and P^1, P^2 following by induction from part (2) of the theorem imply the same properties for $A = A^1 \cup A^2$ and $P = P^1 = P^2$. $\square$

2. Converses

Clearly, if any subgraph F_1 contains an st path P, then zero or more edges $A \subset P$ can be added to any forest F_0 to comprise or complete an st path within forest F_0. The conditions that F_1 is a forest and $F_1 \cup F_0$ is a series-parallel subgraph are sufficient to guarantee the uniqueness of A and the property (2) in our theorem.

The only obstacle to the uniqueness seems to be two st paths with at least one common edge e where e is traversed in opposite directions when traversing the two st paths. Let G^+ be G with a new edge st added. In this situation, it is known that there is in $G+$ a subgraph homeomorphic to K_4. The K_4 homoeomorph is comprised of 6 edge-disjoint paths, and e and st are in two of the pairs of paths that are also vertex disjoint.



3. Perhaps Excessive Arguments

If F_0 also contains an st path, we claim the theorem is satisfied by $A = \emptyset$. $F_0 \cup \emptyset$ is a forest with st path P', and clearly $A = \emptyset \subseteq P'$. To prove $A = \emptyset$ is unique, suppose $A \neq \emptyset$ also satisfies the theorem, so $F_0 \cup A$ is a forest. Since $F_0 \cup A$ is a forest, P' is the unique st path in $F_0 \cup A$. Therefore, $A \subseteq P' \setminus F_0$. But $P' \subseteq F_0$, so $A \neq \emptyset$ is contradicted.

We therefore verified that when F_0 contains an st path, $A = \emptyset$ uniquely satisfies (1) of the theorem. In this situation, $F_0' = F_1 \setminus A = F_1$ contains the unique path P, so $\emptyset = A$ is the only set for which $F_0' \cup A$ is a forest that contains an st path P for which $A \subseteq P$.

——- **Parallel case:** There cannot be an st path in F_1^2, because that path together with P would be a circuit, which contradicts F^1 being a forest.

By induction, A is the unique subset of P for which $F_0^1 \cup A$ is a forest in G^1 that contains st path P' with $A \subseteq P'$. Since F_0^2 is a forest without an st path, $F_0 \cup A = F_0^1 \cup A \cup F_0^2$ is a forest in G that contains st path P'. Therefore, (1) is confirmed.

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