Restricted or Ported Tutte Decomposion and Analogs of All-Minors Laplacian Expansions

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What is a parametrized strong Tutte function?

Tutte equations are satisfied in a very general setup:

1. Elements $\{e\}$ each with parameters g_e, r_e.
2. A category $\mathcal{N}$ of objects $\mathbf{N}$ each with ground set $S = S(\mathbf{N})$ of elements.
3. For some *decomposible* $\mathbf{N}$, for one or more *separators* $e \in S(\mathbf{N})$, the *contraction* and *deletion* operations are defined with results $\mathbf{N}/e$ and $\mathbf{N}\backslash e$ in $\mathcal{N}$, with ground sets $S(\mathbf{N})\backslash\{e\}$
4. Some $\mathbf{N} = \mathbf{N}_1 \oplus \mathbf{N_2}$ are direct sums, where $S(\mathbf{N}_1) \cap S(\mathbf{N}_2) = \emptyset$.
5. For each indecomposible $\mathbf{N}$ with no separators there is an additional parameter $i_\mathbf{N}$ called the *initial value*.

Tutte equations, functions and Good Questions

1. For all $\mathbf{N}$ with separator $e \in S(\mathbf{N})$,

$$F(\mathbf{N}) = g_e F(\mathbf{N}/e) + r_e(\mathbf{N}\backslash e)$$

2. When $\mathbf{N} = \mathbf{N_1} \oplus \mathbf{N_2}$,

$$F(\mathbf{N}) = F(\mathbf{N_1})F(\mathbf{N_2})$$

3. When $\mathbf{N}$ is indecomposible,

$$F(\mathbf{N}) = i_{\mathbf{N}}$$

F is Tutte function when all the Tutte equations are satisfied. This MEANS $F(\mathbf{N})$ is what is computed by applying Tutte equations *in any order they are applicable*.
Good Questions: When does $\mathcal{N}$ and parameters ACTUALLY HAVE a Tutte function? If so, what is a *universal* Tutte function?

Some answers–for Graphs and Matroids

Only loops and coloops need initial values

The only $\mathbf{N}$ with no separators and no $\mathbf{N} = \mathbf{N}_1 \oplus \mathbf{N_2}$ for $\mathbf{N}_i \neq \emptyset$ are $\mathbf{loop}(e)$ and $\mathbf{coloop}(e)$.

The famous Tutte Polynomial

Adding all $g_e = r_e = 1$, the Tutte polynomial $F(\mathbf{N})(x, y)$ obtained from $i_{\mathbf{loop}(e)} = x$, $i_{\mathbf{coloop}(e)} = y$ and $i_\emptyset = 1.$ is a universal Tutte function.

Normal Tutte Functions

(Zaslavsky) With arbitrary g_e, r_e, the *normal* Tutte functions are obtained with $i_{\mathbf{coloop(e)}} = g_e y + x$, $i_{\mathbf{loop(e)}} = r_e x + y$ and $i_\emptyset = 1$.
(Zas. result abt. normal Tutte fun here.)

Our setup

- Matrices N_α, $N_\beta^\perp$; full row rank, columns indexed by $P \coprod E$.
  $\mathrm{rank}(N_\alpha) + \mathrm{rank}(N_\beta^\perp) = |E| + |P|$.
  $P_\alpha, P_\beta \leftrightarrow P$, $P_\alpha \cap P_\beta = \emptyset$.
- Weight (parameter) matrices
  $G = \mathrm{diag}\{g_e\}_{e\in E}$, $R = \mathrm{diag}\{r_e\}_{e\in E}$.
- Matrix with columns $P_\alpha \coprod P_2 \coprod E$

$$L\begin{pmatrix} N_\alpha \\ N_\beta^\perp \end{pmatrix} = \left[\begin{array}{c|c|c} N_\alpha(P) & 0 & N_\alpha(E)G \\ \hline 0 & N_\beta^\perp(P) & N_\beta^\perp(E)R \end{array}\right]$$

Define

$$F(L) = ((\binom{2p}{p})) - \text{tuple of determinants } L[Q_1 Q_2 E])$$

indexed by sequences $Q_\alpha Q_\beta \subseteq P_\alpha P_\beta$ where $Q_\alpha \subseteq P_\alpha$, $Q_\beta \subseteq P_\beta, |Q_\alpha Q_\beta| = p = |P|$.

Column e of L when $e \notin P$ is

$$\begin{bmatrix} N_{\alpha,1,e}g_e \\ N_{\alpha,2,e}g_e \\ \dots \\ N_{\alpha,r_1,e}g_e \\ N^{\perp}_{\beta,1,e}r_e \\ N^{\perp}_{\beta,2,e}r_e \\ \dots \\ N^{\perp}_{\beta,r_2,e}r_e \end{bmatrix} = \begin{bmatrix} N_{\alpha,1,e} \\ N_{\alpha,2,e} \\ \dots \\ N_{\alpha,r_1,e} \\ 0 \\ 0 \\ \dots \\ 0 \end{bmatrix} g_e + \begin{bmatrix} 0 \\ 0 \\ \dots \\ 0 \\ N^{\perp}_{\beta,1,e} \\ N^{\perp}_{\beta,2,e} \\ \dots \\ N^{\perp}_{\beta,r_2,e} \end{bmatrix} r_e$$

So,

$$F(L)_{Q_\alpha Q_\beta} = L[Q_\alpha Q_\beta E] =$$

$$g_e L \begin{pmatrix} N_\alpha / e \\ N^{\perp}_{\beta} \backslash e \end{pmatrix} [Q_\alpha Q_\beta E] + r_e L \begin{pmatrix} N_\alpha \backslash e \\ N^{\perp}_{\beta} / e \end{pmatrix} [Q_\alpha Q_\beta E].$$

Each determinant $L[Q_\alpha Q_\beta E]$ is one of $\binom{2p}{p}$ components, so

$$F(L) = g_e FL \left(\begin{array}{c} N_\alpha / e \\ N_\beta^\perp \backslash e \end{array} \right) + r_e FL \left(\begin{array}{c} N_\alpha \backslash e \\ N_\beta^\perp / e \end{array} \right)$$

where

N/e means remove the g_e or r_e but otherwise keep column e

$N \backslash e$ means replace column e by 0.

$$FL(M) = g_e FL\begin{pmatrix} N_\alpha/e \\ N_\beta^\perp\backslash e \end{pmatrix} + r_e FL\begin{pmatrix} N_\alpha\backslash e \\ N_\beta^\perp/e \end{pmatrix}$$

Real deletion/contraction removes e from the ground set of the matroid or other object, but $N/e, N\backslash e$ still have column e. But (*) holds for all $e \in E$, so Laplace's expansion is a basis expansion:

$$L[Q_\alpha Q_\beta E] = \sum_{A \subseteq E} g_A r_{\overline{A}} N_\alpha[Q_\alpha A] N_\beta^\perp[Q_\beta \overline{A}]\epsilon(Q_\alpha A, Q_\beta \overline{A})$$

The A term is $\neq 0$ iff $Q_\alpha A$ is a column basis for N_α and $Q_\beta \overline{A}$ is a column basis for $N_\beta^\perp$. So, for each $Q_\alpha Q_\beta$

$$L[Q_\alpha Q_\beta E] = \pm \sum_{A \subseteq E} g_A r_{\overline{A}} N_\alpha[Q_\alpha A] N_\beta^\perp[Q_\beta \overline{A}]\epsilon(A, \overline{A})$$

(The non-zero terms all have $|A| = \text{rank}(N_\alpha) - |Q_\alpha|$.)

Quick and dirty fix

1. Drag column e to the far right.
   Changes sign of $F(L)$ by $\epsilon(E'e)$.
2. Left multiply by a determinant 1 matrix that sends the last column to $(0, ..., 1g_e, 0, ..., 1r_e)^{\mathbf{t}}$ (if the top or bottom submatrix has just 1 row, do the hack: $\mathbf{N}/e$ is number $\mathbf{N}_{1,e}$ that acts like a matrix with columns E' and no rows.)
3. Drag the row with the $1g_e$ to the bottom.
   Changes sign of $F(L)$ by $(-1)^{\mathbf{rN}_\beta^\perp}$
4. With e deleted/contracted from the $\mathbf{N}$s defining L, define F by $FL_{Q_\alpha Q_\beta} = L[Q_\alpha Q_\beta E']$

Result

$$FL\begin{pmatrix} N_\alpha \\ N_\beta^\perp \end{pmatrix} = \epsilon(E'e)\left(g_e(-1)^{\mathrm{r}(N_\beta^\perp)}FL\begin{pmatrix} N_\alpha/e \\ N_\beta^\perp\backslash e \end{pmatrix} + r_e FL\begin{pmatrix} N_\alpha\backslash e \\ N_\beta^\perp/e \end{pmatrix}\right)$$

$$\begin{array}{ccc} (e_1) & (e_2) & (e_3) \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ \hline \multicolumn{3}{l}{\mathbf{N}[e_1 e_3] = (a_1 b_3 - a_3 b_1)} \end{array}$$

$$\begin{array}{rl} & (a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3) \\ \wedge & (b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3) \\ \hline \multicolumn{2}{l}{((a_1 b_3 - a_3 b_1)\mathbf{e}_1\mathbf{e}_3 + \cdots)} \end{array}$$