PORTED SEPARATOR-STRONG PARAMETRIZED TUTTE FUNCTIONS

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Abstract. We discuss Tutte functions of parametrized matroids in which certain "port" elements have been distinguished. The port elements are held back from deletion and contraction during Tutte decompositions. Conditions on the parameters and the values for matroids or oriented matroids on port elements necessary and sufficient for a ported Tutte function T to be well-defined are proved. Activities based expressions and the corresponding interval partitions of the boolean lattice of subsets without port elements are given, for all recursive computations of $T(M)$, not just those based on an element order. These are then specialized to a corank-nullity and a geometric lattice expansion for the ported generalization of normal Tutte functions.

1. Introduction

Two of the natural, known and useful generalizations of the Tutte equations and the resulting Tutte polynomials and decompositions occur when:

(1) A set P of distinguished elements, which we call "ports," is specified. The operations of deleting and contracting non-separator e, and the factoring out of a separator e so $T(G \oplus e) = XT(G)$ or $YT(G)$ depending on whether e is a coloop or loop, are then restricted only to $e \notin P$. See [12, 11] where the Tutte polynomials are called "set-pointed", [4], and Brylawsky's early work [3] on the $|P| = 1$ case.

(2) Four parameters x_e, y_e, X_e, Y_e are given for each element e. x_e and y_e are coefficients in the additive Tutte equation for contracting and deleting e when e is a non-separator. X_e and Y_e generalize the common X and Y above to different parameters for different e. When combined with (1), parameters are given only for $e \notin P$. See [15, 2, 7].

In this note, we give common generalizations for matroids of pertinent definitions and methods, and prove the results. Our synthesis began with [5], where the term "port" is motivated by an application; but this application requires only those Tutte functions that are both **strong** ($T(M^1 \oplus M^2) = T(M^1)T(M^2)$) and **normal** (have a corank-nullity generating function expression) [15]. Additional progress is reported in [6] where the functions are called "relative" Tutte polynomials; and they have a very different application. The Tutte functions we treat are those only required, in addition to the parametrized additive identity, to satisfy $T(M) = X_eT(M \setminus e)$ or $T(M) = Y_eT(M \setminus e)$ for separators $e \notin P$, rather than $T(M^1 \oplus M^2) = T(M^1)T(M^2)$. They are called **separator-strong**[8]. $E(M)$ denotes $S(M) \setminus P$, the elements of M not in P. We present two principal outcomes:

(1) The extension is straighforward but has the novel feature that the resulting Tutte functions might depend on the orientation of an oriented matroid M, not just on the underlying matroid. The Tutte function depends on $I(Q)$, the initial values on minors $Q = M/A\setminus(E\setminus A) = M/A|P$ which are called **P-quotients**, in addition to the

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parameters. The three known [15, 2, 7] families of polynomial conditions necessary and sufficient for the Tutte function to be well-defined are generalized to five; the two new ones involve pairs $\{e, f\} \subseteq E$ connected to P.

We were surprised that although each condition is polynomial in parameters for 2 or 3 elements of E, each polynomial is *a multiple of* $I(Q)$ for *only one* Q.

(2) When the conditions are satisfied, all recursive computations based on the ported Tutte equations give the same result. The Tutte computation tree formalism [10] enables us to define an activities-type expansion for every such computation, not just those based on a linear ordering of E. The known theory generalizes to the ported case, including for example, interval partitions of the Boolean lattice 2^E. There is one term and boolean interval for each independent subset $F \subseteq E$ for which $F \cup P$ is spanning; such F specialize to bases when $P = \emptyset$. The blocks of such interval partitions are then partitioned according to the common $Q = M/A|P$ for A in each block.

The conditions, similar to ours, on the parameters only given in [6] are necessary and sufficient for the Tutte function to be well-defined *for all* choices of initial values $I(Q)$. Our conditions are derived without this universal quantification. Hence a factor $I(Q)$ appears in each. They reveal exactly which polynomials involving any $I(Q)$ must vanish depending on the structure of the family on which the Tutte function is well-defined. Also, our proof technique is different; it generalizes that in [7].

We only address *separator-strong* Tutte functions of matroids. Ported generalizations of the formula $T(\emptyset)T(M^1 \oplus M^2) = T(M^1)T(M^2))$ and related results about strong Tutte functions[15], direct sums and graphs (based on [15, 2, 7]) are left for future publication.

2. Matroid Tutte Function Characterization

Definition 1. *Let P be a set of* ***ports*** *and $\mathcal{C}$ be a class of matroids or oriented matroids closed under taking minors or oriented minors by deleting or contracting only elements $e \notin P$. Such a $\mathcal{C}$ is called a* ***P-family***. *Suppose $\mathcal{C}$ is given with four parameters (x_e, y_e, X_e, Y_e) in a commutative ring R, for each $e \notin P$ that is an element in some $M \in \mathcal{C}$.*

A ***separator-strong parametrized P-ported Tutte function*** *T maps $\mathcal{C}$ to R or to an R-module and satisfies conditions* (TA) *and* (TSSM) *below for all $M \in \mathcal{C}$ and all e in $E(M)$.*

(TA)
$$T(M) = x_e T(M/e) + y_e T(M \setminus e)$$
if $e \notin P$ and e is a non-separator, i.e., neither a loop nor a coloop.

(TSSM)
If $e \notin P$ is a coloop in M then $T(M) = X_e T(M/e)$.
If $e \notin P$ is a loop in M then $T(M) = Y_e T(M \setminus e)$.

The ground set of $M \in \mathcal{C}$ is denoted by $S(M)$ and $E(M) = S(M) \setminus P$, the ground set elements of M that are not ports. $U_r^{ef\cdots}$ denotes the uniform rank r matroid on $\{e, f, \ldots\} \subseteq E$. The $Q \in \mathcal{C}$ for which neither (TA) nor (TSSM) apply, equivalently, $E(Q) = \emptyset$, are called the **indecomposibles** or **P-quotients** in $\mathcal{C}$.

Proposition 2. *Suppose e, f are in series, or are in parallel, in matroid or oriented matroid M. (1) The minors $M/e \setminus f = M/f \setminus e$ are equal as matroids. (2) If M is oriented, the oriented minors $M/e \setminus f = M/f \setminus e$ are equal as oriented matroids.*

Proof. If e, f are in series, note that $M/e \setminus f = M \setminus f/e$. e is a coloop in $M \setminus f$, so $M \setminus f/e$ $= M \setminus \{e, f\}$, which is clearly the same matroid or oriented matroid if e, f are interchanged. The relevant theory of minors of oriented matroids can be found in [1].

If e, f are in parallel, e, f are in series in the matroid or oriented matroid dual M^* of M. By the first case, $M^* \setminus e/f = M^* \setminus f/e$ as matroids or as oriented matroids. Thus $M/e \setminus f$ $= (M^* \setminus e/f)^* = (M^* \setminus f/e)^* = M/f \setminus e$ as matroids or as oriented matroids. □

Theorem 3 (Ported Matroid ZBR Theorem)**.** *The following two statements are equivalent.*

(1) *T from $\mathcal{C}$ to R or an R-module is a P-ported separator-strong parametrized Tutte function with R-parameters (x, y, X, Y) whose values $T(Q)$ on P-quotients $Q \in \mathcal{C}$ are the initial values $I(Q)$.*

(2) (a) *For every $M = U_1^{ef} \oplus Q \in \mathcal{C}$ with P-quotient Q (U_1^{ef} is a dyad),*

$$I(Q)(x_eY_f + y_eX_f) = I(Q)(x_fY_e + y_fX_e).$$

(b) *For every $M = U_2^{efg} \oplus Q \in \mathcal{C}$ with P-quotient Q (U_2^{efg} is a triangle),*

$$I(Q)X_g(x_ey_f + y_eX_f) = I(Q)X_g(x_fy_e + y_fX_e).$$

(c) *For every $M = U_1^{efg} \oplus Q \in \mathcal{C}$ with P-quotient Q (U_1^{efg} is a triad),*

$$I(Q)Y_g(x_eY_f + y_ex_f) = I(Q)Y_g(x_fY_e + y_fx_e).$$

(d) *If $\{e, f\} = E(M)$ is a parallel pair connected to P,*

$$I(Q)(x_eY_f + y_ex_f) = I(Q)(x_fY_e + y_fx_e)$$

where P-quotient $Q = M/e \setminus f = M/f \setminus e$.

(e) *If $\{e, f\} = E(M)$ is a series pair connected to P,*

$$I(Q)(x_ey_f + y_eX_f) = I(Q)(x_fy_e + y_fX_e)$$

where P-quotient $Q = M/e \setminus f = M/f \setminus e$.

When $P = \emptyset$, this reduces to the Zaslavsky-Bollobás-Riordan theorem for matroids[7]: Only $Q = \emptyset$ occurs and the last two conditions become vacuous.

Proof. Our proof is the immediate result of adding considerations of ports to the proof in [7], there described as “a straightforward adaption of the proof of Theorem 3.3 of [15].” We sketch the proof with a few details, pointing out differences from [7].

As in [7], the necessary relations are easy to deduce by applying (TA) and (TSSM) in two different orders to a general configuration in each of the five families.

Now on to the converse. As in [7], the strategy is to first verify that the conditions imply $T(M)$ is well-defined for $n = |E(M)| = 0, 1$ and 2. Note that for us, n does not count $|P \cap S(M)|$. Second, we rely on the hypothesis the $\mathcal{C}$ is closed under P-minors in order to verify that in a larger minimum n counterexample, the elements of $E(M)$ are either all in series or all in parallel, and then the conditions imply that all calculation orders give the same result. All the cases involve two different combinations of deleting and contracting of several elements in $E(M)$ where both combinations produce the same P-quotient matroids or oriented matroids.

Let $M \in \mathcal{C}$ be a counterexample with minimum $n = |E(M)|$. Therefore, whenever M' is a proper P-minor of M, $T(M')$ is well-defined. (TA) and (TSSM) have the property that given M and $e \in E(M)$, exactly one equation applies. Therefore, calculations that yield different values for $T(M)$ must start with reducing by different elements of $E(M)$. Since $T(M)$ is given unambiguously by the initial value $I(M)$ when $n = 0$, we can assume $n \geq 2$.

M cannot contain a separator $e \in E(M)$, because this e is a separator in every P-minor of M containing e. Therefore, as observed in [7], every computation has the same result $X_e T(M/e)$ or $Y_e T(M \setminus e)$ depending on whether e is a coloop or a loop.

Let e be one element in $E(M)$. Since no element in $E(M)$ is a separator, $V = x_e T(M/e) + y_e T(M \setminus e)$ is well-defined, and so is $x_{e'} T(M/e') + y_{e'} T(M \setminus e')$ for each other $e' \in E$. We follow [7] and define $D = \{e' \in E(M) \mid V = x_{e'} T(M/e') + y_{e'} T(M \setminus e')\}$. The induction hypothesis then tells us that there is at least one $f \in E(M) \setminus D$. (Recall $e, e', f \notin P$.)

Suppose that e is a separator in both $M \setminus f$ and M/f and f is a separator in both $M \setminus e$ and M/e. Then, T would be well-defined for all four of these P-minors and so we can write

$$T(M) = x_e x_f T(M/\{e,f\}) + x_e y_f T(M/e \setminus f) + y_e x_f T(M \setminus e/f) + y_e y_f T(M \setminus \{e,f\}).$$

Both computations give the same value because in this situation the reductions by e and f commute. So, for M to be a counterexample, there must be $e \in D$ and $f \notin D$ ($e, f \notin P$) to which one case of the following lemma applies:

Lemma 4. [15] *Let e, f be nonseparators in a matroid M. Within each column the statements are equivalent:*

(1) *e is a separator in $M \setminus f$.*	(1) *e is a separator in M/f.*
(2) *e is a coloop in $M \setminus f$.*	(2) *e is a loop in M/f.*
(3) *e and f are in series in M.*	(3) *e and f are in parallel in M.*
(4) *f is a separator in $M \setminus e$.*	(4) *f is a separator in M/e.*

We claim that one of the following five cases must be satisfied:

(1) $n = 2$ and $E(M) = \{e, f\}$ is a dyad.
(2) $n \geq 3$ and $E(M)$ is a circuit not connected to P.
(3) $n \geq 3$ and $E(M)$ is a cocircut not connected to P.
(4) $n \geq 2$ and for some $\emptyset \neq P' \subseteq P$, $P' \cup E(M)$ is a cocircuit.
(5) $n \geq 2$ and for some $\emptyset \neq P' \subseteq P$, $P' \cup E(M)$ is a circuit.

As in [7], we draw the conclusion that if $e \in D$ and $f \notin D$ then e, f are either series or parallel. It was further proven that a series pair and a parallel pair cannot have exactly one element in common. Therefore, the pairs e, f satisfying the conditions are either all series pairs or all parallel pairs. By minimality of n, $E(M)$ is either an n-element parallel class or an n-element series class. The last two cases are distinguished from the first three according to whether or not $E(M)$ is disconnected or not from elements of P in matroid M. We now use (TA) and (TSSM) to show that, in each case, the calculations that start with e and those that start with f have the same result, which contradicts $e \in D$ and $f \notin D$.

We give the details for case (d). By hypothesis, each of $T(M/e)$, $T(M \setminus e)$, $T(M/f)$, $T(M \setminus f)$, $T(M/f/e) = T(M/e/f)$, $T(M/e \setminus f)$ and $T(M/f \setminus e)$ is well-defined. (Remark: By Proposition 2, $M \setminus f/e = M \setminus e/f$.

Starting with e and with f, (TA) gives the two expressions:

$$V = x_e T(M/e) + y_e x_f T(M \setminus e/f) + y_e y_f T(M \setminus e \setminus f)$$

$$V \neq x_f T(M/f) + y_f x_e T(M \setminus f/e) + y_f y_e T(M \setminus f \setminus e)$$

Let M' be the P-minor obtained by deleting each element in $E(M)$ except for e and f ($M' = M$ if $n = 2$.) Since $E(M') = \{e, f\}$ and e, f are in parallel connected to P within

M', (d) of the hypotheses tells us that

$$I(Q)(x_eY_f - y_fx_e) = I(Q')(x_fY_e - y_ex_f),$$

where $Q = M'/e\backslash f$, $Q' = M'/f\backslash e$. Since e, f are in parallel within M', matroids or oriented matroids $Q = Q'$ by Proposition 2. Since $A = E(M) \setminus \{e, f\}$ is a set of loops ($\emptyset$ if $n = 2$) in $M/e \setminus f$ ($N/e \setminus f$) and in $M/f \setminus e$ ($N/f \setminus e$), we write $Y_A = \prod_{a \in A} Y_a$ (1 if $A = \emptyset$) and use (TSSM) to write

$$T(M/e \setminus f) = Y_A I(Q), \;\; T(M/f \setminus e) = Y_A I(Q'),$$
$$T(M/e) = Y_f Y_A I(Q), \quad \text{and} \quad T(M/f) = Y_e Y_A I(Q').$$

So, since $M \setminus f \setminus e = M \setminus e \setminus f$,

$$x_eT(M/e) + y_ex_fT(M \setminus e/f) = x_fT(M/f) + y_fx_eT(M \setminus f/e)$$

contradicts $V \neq x_fT(M/f) + y_fx_eT(M \setminus f/e) + y_fy_eT(M \setminus f \setminus e)$ The remaining cases can be completed analogously. It might be noted that our proof differs slightly from [7] in that the cases of $n = 3$ and $n \geq 4$ are not distinguished. □

2.1. **Universal Tutte Polynomial.** It is easy to follow [2, 7] to define a universal, i.e., most general P-ported parametrized Tutte function $T^{\mathcal{C}}$ for $\mathcal{C}$. We take indeterminates x_e, y_e, X_e, Y_e for each $e \in E(M), M \in \mathcal{C}$ and an indeterminate $[Q]$ for each P-quotient $Q \in \mathcal{C}$. Let $\mathbb{Z}[x, y, X, Y]$ denote the integer polynomial ring generated by the x_e, y_e, X_e, Y_e indeterminates, and define $\widetilde{\mathbb{Z}}$ to be the $\mathbb{Z}[x, y, X, Y]$-module generated by the $[Q]$. Let $I^{\mathcal{C}}$ denote the ideal of $\widetilde{\mathbb{Z}}$ generated by the identities of Theorem 3, comprising for example $[Q](x_eY_f + y_eX_f - x_fY_e - y_fX_e)$ for each instance of case (a), etc. The universal Tutte function has values in the quotient module $\widetilde{\mathbb{Z}}/I^{\mathcal{C}}$. Finally, observe that the range of Tutte function T can be considered to be the R-module generated by the values $I(Q)$ where ring R contains the x, y, X, Y parameters. If the $I(Q) \in R$, consider the ring R to be the R-module generated by R. We follow [7] to write the corresponding consequence of Theorem 3:

Corollary 5. *Let $\mathcal{C}$ be a P-minor closed class of matroids or oriented matroids. Then there is a $\widetilde{\mathbb{Z}}/I^{\mathcal{C}}$-valued function $T^{\mathcal{C}}$ on $\mathcal{C}$ with $T^{\mathcal{C}}(Q) = [Q]$ for each P-quotient $Q \in \mathcal{C}$ that is a P-ported parametrized Tutte function on $\mathcal{C}$ where the parameters are the $x, y, X, Y \in \widetilde{\mathbb{Z}}$. Moreover, if T is any R-parametrized Tutte function with parameters x'_e, y'_e, X'_e, Y'_e, then T is the composition of $T^{\mathcal{C}}$ with the homomorphism determined by $[Q] \to I(Q) = T(Q)$ for P-quotients Q and $x_e \to x'_e$, etc., for each $e \in E(\mathcal{C})$.*

3. Tutte Computation Trees and Activities

Zaslavsky[15] noted that the *basis* or *activities* expansion, introduced by Tutte [13, 14] for graphs, applies to all well-defined parametrized Tutte functions whereas the parametrized corank-nullity generating function expresses only a proper subset of them, which he called normal. We show how to give activities-type expansions comprised of one expansion for every recursive computation of a ported Tutte function value, not just those determined by a linear element ordering. We apply to ported matroids the expansions based on Tutte computation trees which were formalized by McMahon and Gordon [10] for the purpose of analyzing Tutte polynomials of greedoids. Unlike matroids, some greedoids do not have activities expansions for their Tutte polynomials that derive from element orderings.

Ellis-Monaghan and Traldi [7] remarked that the Tutte equation approach appears to give a shorter proof of the ZBR theorem than the activities expansion approach. The proofs by induction on $|E|$ demonstrate that every calculation of $T(M)$ from Tutte equations produces the same result when the conditions on the parameters and initial values are satisfied. We then know immediately that the polynomial expression resulting from a particular calculation equals the Tutte function value. We suggest a heuristic reason why the Tutte equation approach is more succinct: The induction assures that *every* computation with smaller $|E|$ gives the same result, not just those computations that are determined by linear orders on E. Proofs of activities expansions for matroids, and their generalizations for P-ported matroids, seem more informative and certainly no harder when the expansions are derived from a general Tutte computation tree, than when the expansions are only those that result from an element order. From the retrospective that the Tutte equations specify a non-deterministic recursive computation [9], it seems artificial to start with element-ordered computations and then prove first that all linear orders give the same result and second that it satisfies the Tutte equations, in order prove that all recursions give the same result. We therefore take advantage of the Tutte computation tree formalism and the more general expansions it enables.

3.1. Computation Tree Expansion.

Definition 6. *Given P-ported matroid or oriented matroid M, a P**-subbasis** F is an independent set with $F \subseteq E(M)$ (so $F \cap P = \emptyset$) for which $F \cup P$ is a spanning set for M (in other words, F spans M/P). $\mathcal{B}_P(M)$ denotes the set of P-subbases.*

Equivalent definitions were given in [12] and in [6]. The following is immediate and useful:

Proposition 7. *For every P-subbasis F there exists an independent set $Q \subseteq P$ that extends F to a basis $F \cup Q \in \mathcal{B}(M)$ of M. Conversely, if $B \in \mathcal{B}(M)$ then $F = B \cap E = B \setminus P$ is a P-subbasis.*

Definition 8 (Computation Tree, following [10])**.** *A P-ported (Tutte) computation tree for M is a binary tree whose root is labeled by M and which satisfies:*

1. *If M has non-separating elements not in P, then the root has two subtrees and there exists one such element e for which one subtree is a computation tree for M/e and the other subtree is a computation tree for $M \setminus e$.*
   The branch to M/e is labeled with "e contracted" and the other branch is labeled "e deleted".
2. *Otherwise (i.e., every element in $E(M)$ is separating) the root is a leaf.*
   Each leaf is labelled with a P-quotient Q plus a direct sum of loop or coloop matroids on elements respectively e or f in E, expressed as $[Q]\prod X_e \prod Y_f$.

Definition 9 (Activities with respect to a leaf)**.** *For a P-ported Tutte computation tree for M, a given leaf, and the path from the root to this leaf:*

- *Each $e \in E(M)$ labeled "contracted" along this path is called **internally passive**.*
- *Each coloop $e \in E(M)$ in the leaf's matroid is called **internally active**.*
- *Each $e \in E(M)$ labeled "deleted" along this path is called **externally passive**.*
- *Each loop $e \in E(M)$ in the leaf's matroid is called **externally active**.*

Proposition 10. *Given a leaf of a P-ported Tutte computation tree for M, the set of internally active or internally passive elements constitutes a P-subbasis of M which we say **belongs to the leaf**. Furthermore, every P-subbasis F of N belongs to a unique leaf.*

Proof. For the purpose of this proof, let us extend Definition 9 so that, given a computation tree with a given node i labeled by matroid M_i, $e \in E$ is called internally passive when e is labeled "contracted" along the path from root M to node i. Let IP_i denote the set of such internally passive elements.

It is easy to prove by induction on the length of the root to node i path that (1) $IP_i \cup S(M_i)$ spans M and (2) IP_i is an independent set in M. The proof of (1) uses the fact that elements labeled deleted are non-separators. The proof of (2) uses the fact that for each non-separator $f \in M/IP_i$, $f \cup IP_i$ is independent in M.

These properties applied to a leaf demonstrate the first conclusion, since each $e \in E$ in the leaf's matroid must be a separator by Definition 8.

Given a P-subbasis F, we can find the unique leaf with the algorithm below. Note that it also operates on arbitrary subsets of E.

Tree Search Algorithm: Beginning at the root, descend the tree according to the rule: At each branch node, descend along the edge labeled "e-contracted" if $e \in F$ and along the edge labeled "e-deleted" otherwise (when $e \notin F$). □

We leave the reader to check that the classsical element-order based activities expansion, as extended with ports explicitly in[6], is reproduced with the unique P-ported computation tree in which the greatest non-separator $e \in E$ is deleted and contracted in the matroid at each tree node, when the elements are ordered so $p \in P$ is before each $e \notin P$.

Definition 11. *Given a computation tree for P-ported (oriented) matroid M, each P-subbasis $F \subseteq E$ is associated with the following subsets of non-port elements defined according to Definition 9 from the unique leaf determined by the algorithm given above.*

- $IA(F) \subseteq F$ *denotes the set of internally active elements,*
- $IP(F) \subseteq F$ *denotes the set of internally passive elements,*
- $EA(F) \subseteq E \setminus F$ *denotes the set of externally active elements, and*
- $EP(F) \subseteq E \setminus F$ *denotes the set of externally passive elements.*
- $A(F) = IA(F) \uplus EA(F)$ *denotes the set of active elements.*

Proposition 12. *Given a P-ported Tutte computation tree for M, the boolean lattice of subsets of $E = E(M)$ is partitioned by the collection of intervals $[IP(F), F \uplus EA(F)]$ determined from the collection of P-subbases F, which correspond to the leaves. (Note $F \uplus EA(F) = IP(F) \uplus A(F)$.)*

Each $A \in [IP(F), F \uplus EA(F)]$ in one interval determines the same matroid or oriented matroid P-quotient by $M/A|P = M/IP(F)|P$.

The boolean lattice of subsets of $E = E(M)$ is also partitioned by the collection of intervals $[EP(F), (E \setminus F) \uplus IA(F)]$. (Note $(E \setminus F) \uplus IA(F) = EP(F) \uplus A(F)$.)

Each $A' \in [EP(F), (E \setminus F) \uplus IA(F)]$ of one of these intervals determines the same matroid or oriented matroid P-quotient by $M \setminus B'/(E \setminus B') = M/F|P$.

For a given $F \in \mathcal{B}_P(M)$, $A \subseteq E$ satisfies $A \in [IP(F), F \uplus EA(F)]$ if and only if $(E \setminus A) \in [EP(F), (E \setminus F) \uplus IA(F)]$.

Proof. Every subset $A \subseteq E = E(M) \setminus P$ belongs to the unique interval corresponding to the unique leaf found by the tree search algorithm given at the end of the previous proof.

P-quotient $M/A|P$ is independent of the order of the deletions and contractions. So let $IP(F) \subseteq A$ be contracted and $(E \setminus EA(F)) \subseteq (E \setminus A)$ be deleted first. The remaining elements of E are loops or coloops, so the P-quotient is independent of whether they are deleted or contracted.

The dual of that tree search algorithm, which descends along the edge labelled "e-deleted" if $e \in A'$, etc., will find the unique leaf whose interval $[EP(F), E \setminus F \cup IA(F)]$ contains A'.

When $A \in [IP(F), F \cup EA(F)]$, the dual algorithm applied to $A' = E \setminus A$ will find the same leaf. The P-quotient is determined by $IP(F)$. □

The following generalizes the activities expansion expression given in [15] to ported (oriented) matroids, as well as Theorem 8.1 of [12].

Proposition 13. *Given parameters x_e, y_e, X_e, Y_e, and P-ported matroid or oriented matroid M the Tutte polynomial expression determined by the sets in Definition 11 from a computation tree is given by*

$$\text{(PAE)} \qquad \sum_{F \in \mathcal{B}_P} [M/F|P]\ X_{IA(F)}\ x_{IP(F)}\ Y_{EA(F)}\ y_{EP(F)}.$$

Proof. (PAE) is an expression constructed by applying some of the Tutte equations. One monomial results from each leaf. In that leaf's matroid, each active element is a separator, and the active elements contribute $X_{IA(F)}Y_{EA(F)}$ to the monomial. The passive elements which contribute $x_{IP(F)}y_{EP(F)}$ are the tree edge labels in the path from the root to the leaf. Each $M/F|P$ denotes a P-quotient of M, so the expression is a polynomial in the parameters and in the initial values. Therefore, (PAE) expressions the result of the calculation when one substitutes $[M/F|P] = I(M/F|P)$. □

From Corollary 5 we conclude:

Theorem 14. *For every P-ported parametrized Tutte function T on $\mathcal{C}$ into ring R or an R-module, for every computation tree for $M \in \mathcal{C}$, (and so for every ordering of $E(M)$), the polynomial expression* (PAE) *equals $T^{\mathcal{C}}(M)$ of Corollary 5.*

3.2. **Expansions of Normal Tutte Functions.** After a notational translation, Zaslavsky's [15] definition of **normal** Tutte functions becomes those for which $T(\emptyset) = 1$, and for which there exist u, $v \in R$ so that for each $e \in E(M)$,

$$\text{(CNF)} \qquad X_e = x_e + uy_e \text{ and } Y_e = y_e + vx_e.$$

The equations of Theorem 3 are readily verified and so we naturally extend this definition, with the $T(\emptyset) = 1$ condition dropped, to ported separator-strong parametrized Tutte functions. All the expressions for normal Tutte functions are therefore in the ring freely generated by u, v, the x_e, y_e and the $[Q]$. We can now generalize some known expansions for the Tutte polynomial $T^{\mathcal{C}}$ after the (CNF) substitution. The rank function for M is denoted by r.

Corollary 15 (Boolean Interval Expansion)**.** *The following activities and boolean interval expansion formula is universal for normal Tutte functions:*

$$T^{\mathcal{C}}(M) = \sum_{F \in \mathcal{B}_P} [M/F|P] \Bigg(\sum_{\substack{IP(F) \subseteq K \subseteq F \\ EP(F) \subseteq L \subseteq E \setminus F}} x_{K \cup (E \setminus F \setminus L)}\ v^{|E \setminus F \setminus L|}\ y_{L \cup (F \setminus K)}\ u^{|F \setminus K|} \Bigg)$$

Proof. Substitute (CNF) into $T^{\mathcal{C}}(M)$ and use $IP(F) \,\dot\cup\, IA(F) = F$ and $EP(F) \,\dot\cup\, EA(F) = E \setminus F$ from Definition 11. □

Lemma 16. *Given $F \in \mathcal{B}_P$, $IP(F)$ spans $EA(F)$.*

The pairs (K, L) for which $IP(F) \subseteq K \subseteq F$ and $EP(F) \subseteq L \subseteq E \setminus F$ are in a one-to-one correspondance with the A satisfying $IP(F) \subseteq A \subseteq F \cup EA(F)$ given by $A = K \cup (E \setminus F) \setminus L$.

For every such A,

$$|F \setminus K| = r(M) - r(M/F|P) - r(A)$$

and

$$|E \setminus F \setminus L| = |A| - r(A).$$

Proof. (See Figure 1 in Appendix.) By our definition of activities, after all the elements of $IP(F)$ are contracted, all elements in $EA(F)$ are loops. (Note none of these elements are ports.)

Let $A = K \cup (E \setminus F \setminus L)$. By our definition of activities, $IP(F) \cup IA(F) = F$, so $IP(F) \subseteq A$. Similarly, $EP(F) \cup EA(F) = E \setminus F$, so $A \cap (E \setminus F) \subseteq EA(F)$. Hence $IP(F) \subseteq A \subseteq F \cup EA(F)$. Conversely, given such an A, take $K = A \cap F$ and $L = (E \setminus F) \setminus A$.

Since $IP(F)$ spans $EA(F)$ and $K \supseteq IP(F)$, K spans $EA(F)$. Since $A \subseteq K \cup EA(F)$, K spans A. $K \subseteq F$, F is a P-subbasis, so K and F are independent, hence $|K| = r(K) = r(A)$ and $|F| = r(F)$. Therefore, $|F \setminus K| = r(F) - r(A)$.

Since F is a P-subbasis, $r(F \cup P) = r(M)$. By definition of contraction, $r(M/F|P) = r(F \cup P) - r(F)$, so $r(M/F|P) = r(M) - r(F)$. We conclude $|F \setminus K| = r(M) - r(M/F|P) - r(A)$.

$E \setminus F \setminus L = A \setminus K$, so $|E \setminus F \setminus L| = |A| - |K|$. As above, $|K| = r(A)$, so the last equation follows. ☐

Corollary 17.

$$T^{\mathcal{C}}(M) = \sum_{F \in \mathcal{B}_P} [M/F|P] \Big(\sum_{IP(F) \subseteq A \subseteq (F \cup EA(F))} x_A y_{E \setminus A} u^{r(M) - r(M/F|P) - r(A)} v^{|A| - r(A)} \Big) \tag{1}$$

Proof. Apply Lemma 16 to the inner sum in Proposition 15. ☐

Theorem 18 (Corank-nullity expansion)**.**

$$T^{\mathcal{C}}(M) = \sum_{A \subseteq E(M)} [M/A \mid P] x_A y_{E \setminus A} u^{r(M) - r(M/A|P) - r(A)} v^{|A| - r(A)}. \tag{PGF}$$

Remark: This extends to separator-strong ported Tutte functions with parameters on possibly oriented matroids an expression from [11] reproduced in [4]. It can also be proved by corank-nullity generating function methods.

Proof. By Proposition 12, given any Tutte computation tree, the lattice of subsets of $E(M)$ is partitioned into intervals corresponding to P-subbases $\mathcal{B}_P$. In each interval, P-quotient $M/F|P$ is equal to $M/A|P$ (as a matroid or oriented matroid) for $A \in [IP(F), F \cup EA(F)]$. Hence we can interchange the summations in (1) and write (PGF). ☐

Proposition 19 (Geometric Lattice Flat Expansion)**.** *Let M be an oriented or unoriented. In the formula below, F and G range over the geometric lattice of flats $\mathcal{L} = \mathcal{L}(M|E)$ contained in M restricted to $E = E(M)$. (E is the top of $\mathcal{L}$ and $\leq$ is its partial order.)*

$$T^{\mathcal{C}}(M) = \sum_{Q} [Q] \sum_{\substack{F \leq E \\ [M/F|P] = [Q]}} u^{r(M) - r(Q) - r(F)} v^{-r(F)} \sum_{G \leq F} \mu(G, F) \prod_{e \in G} (y_e + x_e v)$$

Proof. This generalizes and follows the steps for theorem 8 in [4]. (See Appendix.) ☐

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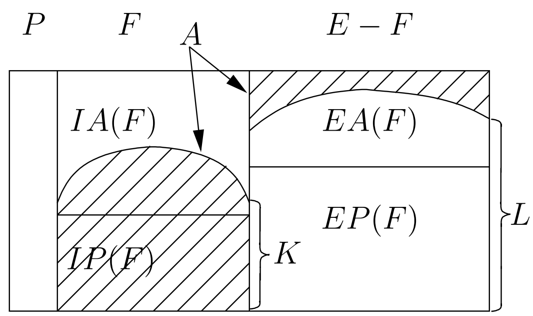

Figure 1. Illustration for proof of Lemma 16.

Appendix A. Proof of Proposition 19

For each subset $A \subseteq E$, $[Q] = [M/A|P] = [M/F|P]$ is determined by the unique flat F in $L(E)$ spanned by $A \subseteq E$. So, we write (PGF) by

$$T^{\mathfrak{C}}(M) = \sum_{Q}[Q] \sum_{\substack{F \in \mathfrak{L} \\ [M/F|P]=[Q]}} \sum_{\substack{A \subseteq F \\ A \text{ spans } F}} x_A y_{E\setminus A} u^{r(M)-r(M/A|P)-r(A)} v^{|A|-r(A)}.$$

Factoring, we get

$$T^{\mathfrak{C}}(M) = \sum_{Q}[Q] \sum_{\substack{F \in \mathfrak{L} \\ [M/F|P]=[Q]}} u^{r(M)-r(Q)-r(F)} v^{-r(F)} \sum_{\substack{A \subseteq F \\ A \text{ spans } F}} x_A y_{E\setminus A} v^{|A|}.$$

A is summed over the spanning sets of F. Let $Z(F)$ denote this last sum. Since every subset of F spans some flat in $\mathfrak{L}$,

$$\sum_{0 \le G \le F} Z(G) = \sum_{e \in F}(y_e + x_e v).$$

Möbius inversion gives

$$Z(F) = \sum_{0 \le G \le F} \mu(G, F)(y_e + x_e v).$$

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