What to Model ...

Seth Chaiken

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1 Of a circuit:

- KVL and KCL defined spaces and dependencies of edge voltages and currents, in a (1) graph-based network and more generally an orthogonal complemetary decomposition of a linear space.
- Equivalances to avoid circuits of voltage sources and capacitors and cocircuits of current sources and inductors, in order to better (how?) choose state variables (generalized coordinates?); existance of tree containing all the voltage sources and capacitors whose cotree contains all the current sources and inductors. (Starting point of [9]; find early references ??) Benefits of this concretely and for Lagrangian formulations. Generalizes of course.
- Basic Lagrange and Hamiltonian formulation: [9].
- Capacitor and inductor flux ($\phi = \int v(t)dt$) and charge ($q = \int i(t)dt$). General nonlinear laws: $L(i) = \phi$ and $C(v) = q$.
- Transforms of linear circuit functions.
- Resistance and other port quantities within the theory of **random walks** [16, 28]. Given a graph on states (as with Markov chains) the discrete Greens function is defined in [11] to be the inverse of $\Delta|_S$ where

  $$\Delta f(x) = \sum_y (f(x) - f(y)p_{xy}$$

  is the discrete Laplace equation when p_{xy} is the transition probability from x to y, and S is a connected subset with non-empty boundary $\{y \notin S | p_{xy} \neq 0 \text{ for some } x \in S\}$. [11] treats Dirichlet eigenvalues determining Greens functions, heat kernels, expected hitting times and Greens functions for paths, lattices and distance regular graphs.

  Spanning tree counting was done by these methods in [10].

  The normalized Laplacian is the symmetric matrix of weights.

- Electricity as statistics mechanincs of electron transfer. Relate to random walk models.
- Resistive port behavior; OM characterizes of forest term sign.
- Monotone non-linear well-posedness.
- OM of solution states.
- Predictions and constraints derived from OM properties:
    1. How orthogonality with an OM vector predicts one sign from others.
    2. How OM covector elimination predicts a zero value is possible.
    3. (Possible application of "fourientation") Suppose voltage and current source edges are specified with a source direction for each. A source directions is specified by an orientation, that may one direction to specify a non-zero source, or (perhaps) none to specify a zero source. The (1) resistor values and, in the case of two or more sources, (2) the source values determine a solution; the solution determines another orientation on all the edges, called the solution orientation. Rules for the solution orientation (SO):
        (a) SO is defined on all the edges.
        (b) SO on one edge may to in one direction for non-zero current and voltage in a resistor edge, or non-zero current or voltage in a voltage or current edge respectively, or none to indicate those solution quantities are zero.
        (c) SO on a source edge may be the same or different from the given source direction.

        Questions:

        (a) Characterize the solution orientations; maybe there is Tutte-like function whose evaluation is the number of them.
        (b) Can a graph be derived so that a fourientation determines the set of solution orientations?
        (c) How do these questions differ when the resistor values vary and the sources values are constant, versus constant resistances and variable sources. Are the problems of both varying any good?

        Geometry questions: What about the space of solutions with given sign pattern?
- OM of transients.
- Signs of transients that might not be OM modelled.
- Laplace transform polynomials.
    - Bode diagram approx.
      NEW THING: Tropical algebra approximates real algebra!

- Stability
- Phase margin.
- Discrete model for spanning tree count: Sandpile group order.

- Imaginary frequency $s = i\omega$.
- Anderson's complex OM (after learning it properly.)
- Cones of possible phases. Is that what Anderson did?
- Orders or levels of resistance, etc. magnitudes; as in symbolic simulation.
- Laplacian eigenvalues and eigenvectors (what are the good for in elec. circuit study?)

  Spectral analysis of a graph laplacian matrix has more applications today than spectral analysis of the adjacency matrix; see [12].

  Let L be the (non-normalized) Laplacian matrix. Its eigenvalues have the following electrical interpretation: Consider the graph to be a network of unit resistors with an added ground node 0 and a unit capacitor between each original node and the ground node. Let the function of time $v_k(t)$, $k \neq 0$ be the voltage of node k relative to the ground node. Then $d\mathbf{v}/\mathbf{dt} = -\mathbf{Lv}$, so the eigenvalue λ_i corresponds to the solution $\mathbf{v_i(t)} = \mathbf{v_i(0)e^{-\lambda_i t}}$.

  Discrete Greens function [11].
- Effect or information transfer directionality due to impedance differences, "half-resistors", **Bond graphs**.
- Hamiltonian generalized position and momentum; that Frankel stuff.
- Voltages and currents are orthogonal under a bilinear form that represents power. What's behind this?

2 Recent Inverse Problem Results

Short survey notice [26] cites work by Curtis, Ingerman and Morrow [14] and de Verdière, Gitler and Vertigan [13]. Starting with solutions to the electrical network inverse problems for planar networks whose response vertices are on their bounding circle, research moved into such for other surfaces.

See recent preprint by Lam [25].

Add more refs.

3 Ports

Alternative vocabulary: Distinguished elements, pointed mathematical objects, set pointed objects, restrictioned objects

In matroid theory literature, a "port" is a clutter of subsets; it is the discrete structure in a matroid associated with a single element extension. We call a port an element p; M on E is the single element extension of $M \setminus p$. From [6], $\mathcal{P} = \{C - e; C \in \mathcal{C}(\mathcal{M}), e \in C\}$. Chari attributes this to Lehman and cites Seymour, Brylawski and Huseby. Recently, it was seen to be the structure of secret sharing matroids.

4 Electric Analogies

1. inductance=mass, capacitance=spring stiffness, ...

2. Traffic flow: voltage=trip time, current=flow, resistance=route-time per unit flow, KVL=each driver adjusts his route to minimize his time.

   Braess paradox: Removing an edge reduces travel time. (Thanks, Michael Sattinger.) Simplest example [on Wikipedia] occurs with a *voltage source*, an edge with constant travel time. Is related to Nash equilibrium.

   `http://homepage.ruhr-uni-bochum.de/Dietrich.Braess/#paradox` is a bibliography including C.M. Penchina and L.J. Penchina, "The Braess paradox in mechanical, traffic, and other networks." [30] Amer. J. Phys. 71, 479 - 482 (2003)

   non-linear monotonic constituitive functions might be more applicable here, like when the flow saturates, the time becomes ∞ when the capacity is reached (or exceeded?). Maybe try to write a note on necessary conditions for a Braess paradox, which is in contrast to Raleigh's inequalities.

3. Spider web: voltage=position, current=force, resistance=spring constant.

5 Benefits of Formal Expression

Vectors: (1) time series (2) transform expansion. Good to formalize with generating functions IE? some general Fourier expansion. (How the same??)

Vectors: Multidimensional quantities: System state, multidim. signal, complex value: Good to formalize with the GROUP ALGEBRA or some of its generalization.

Connection: Generating functions are the SEMI-GROUP algebra (over the coefficient space) generated by $\{z^n\}$.

6 Engineering Intuitions

1. **A Study of paragraph-sized qualitative circuit operation descriptions**

   - In terms of causal tracing of increases or decreases.

- In terms of sinusoidal frequency response.
- In terms of exponential frequency response.

2. Impedance differences between parts of circuits; impedance in intuition.

3. **Approximation**

4. **Purposeful Design**

   - Need to account for **parasitic elements** (1) Sometimes have small effects. (2) Sometimes they must be compensated for.
   - Need for "robustness", researched with **Monte-Carlo methods** and **Sensitivity analysis**. [27]

5. **Effects and Strategies**

   - Effect of adding (or displacing) poles and zeros to a transfer function.
   - Compensation.
   - Feedback.
   - (Results of feedback): Miller effect, bootstrapping.

6. **Physical causes expressed in an electrical model**. Specifically, physics of devices (diode junctions, junction transtors, field-effect transistors, even non-ideal resistors, capacitor, inductor).

7. **Active vs. Passive: Bias distinguished from signal** "**Active device:** A device that can convert energy from a dc bias source to a signal at an RF frequency. Active devices are required for oscillators and amplifiers." Microwave Devices in *The Electrical Engineering Handbook*. [15, ch.37, Streer and Trew] (Find other definitions in the circuit theory literature?)

8. **Thermal Runaway** Old: BJT power transistor latchup. New: Hotspots in systems on a chip (SoC). There is a thermal network interacting with the electical network. Paper on it in VLSI [29].

9. (Non-thermal?) Latchup (parasitic transistors) CMOS [23]

7 Topology

Engineers use the term topology to mean "circuit shape", the network graph decorated with element types, which determines its instance of Kirchhoff law equations with a choice of constituitive laws. Topologists and mathematical physicists ([22] especially Appendix B, citing Eckmann[1] ([19]–perhaps [20], Bott [4] and a book by Bamberg and Sternberg [2]) recognize this first as a case for homology and cohomology theory of a 1-dimensional chain complex, and then as a source of analogies for more general spaces. Going beyond the formulated (generally) differential and transform space equations, the dynamical system solutions have been studied by topologists in their relation to the network topology [32].

Some authors pursued the topological view in terms of Lagrangian and Hamiltonian formulations [9].

7.1 Algebraic Topology

The Laplacian generalizes to chain complexes, matrix tree theorem generalizations. (among others, [18, 22]).

Weighted Laplacian (apps in persistent homology applications) [7].

More on weighted Laplacian, Hodge's theorem ker(Laplacian) $\cong H^i$ [24].

7.2 Back to Point Set Topology

Each network graph vertex models a conducting region of physical space, ignoring electromagnetics...

Idea: When we consider the system (circuit?) to be a physical object, the vertices are **topological contractions** of the conducting regions. **The homology group** of the conductor surface is trivial (since the conductor itself is simply connected.) The same level homology group for the whole system is the cycle space which is the subject for Kirchhoff's current law.

Magnetic phonomena make non-graphic topologies (recalled from one of Duffin's papers, where a cycle is assumed to have zero flux).

Recall conversation with Dan Silver and Susan Williams about topology, knots and Maxwell's equations. Work on knot invariants from Laplacian [31].

(Refer to homotopy methods for analyzing non-differential non-linear equations. Persistent homology is a new theme here. This may bridge statistical mechanical microscopic models to our macroscopic models.)

8 OTHER

The Boltzmann factor in interplay between combinatorics and statistical mechanics (after reading Feynman's Lectures on Physics account).

[1] From MR: "Let numerical values $f(x_i)$ be assigned on the vertices x_i of a subgraph R of a linear graph K. It is required to find f in $Q = KR$ so that $f(x_i)$ for vertices x_i in Q equals the mean of its values on the vertices joined to x_i by an edge. It is shown that f always exists, and is unique if and only if R has at least one vertex in each component of K. This is a generalization of a finite version of the Dirichlet problem. A similar theorem is given in n dimensions. The proof is based on the unique representation of a chain with real coefficients as a sum of a harmonic chain (which is both a cycle and a cocycle), a boundary and a coboundary. Reviewed by H. Whitney"

The Boltzmann factor is $p_\beta(x) = C\alpha^{-x} = Ce^{-\beta x}$ is the probability or probability density of the exponential distribution. Its origin in statistical mechanics: $E_\beta = x + (E_\beta - x)$ where the number of equally probable states of an environment (heat bath) is $Ce^{E_\beta - x}$ and so the probability of one state of a (system+environment) is proportional to $e^{-\beta x}$. It is the probability of one system state with energy x when the systema and environment are in **thermal equilibrium**. Then, β is defined as **temperature**.

Caticha: This is a special case of maximizing **Shannon entropy** subject to constraints on the expectations of one or more functions, such as $x(s)$. Then temperature like quantities emerge as Lagrange multipliers.

Partition and generating function:

$$Z = \sum_{s \in \text{states}} e^{-\beta x(s)} = \sum_x N(x) e^{-\beta x}$$

where $N(x)$ = number of states s with $x(s) = x$.

$$\frac{\partial \log Z}{\partial \beta} = \text{Expected Value}(x)$$

(Consequences of $\max \Omega(x)\Omega_{\text{bath}}(E_\beta - x)$ [5])

Caticha: The dynamical system on the microstates is called the "subject matter", which is an assumption part of what we must know or assume in order to do science. Particular equations or their form as Hamilton's equations are written down. The dynamical system of evolving an evolving microstate maps to a dynamical system of evolving probability distributions. Our knowledge is represented by a probability distribution on microstates.

Equal probabilities of all heat bath states with (heat bath) energy $E_\beta - x$ is equivalent to the system states' probability distribution has maximum entropy among those with expected (system) energy value x. But under the second interpretation, temperature emerges as the value of the Lagrange multiplier for particular energy value x. (Connection: The Max Ent distribution over all finite discrete probability distributions is the uniform distribution.)

Multiple interacting transport phenomena in transistors and other engineered systems (natural too!).

Objects categorized by computational effectiveness:

1. Statistical Mechanical Ensemble, through which a world traces merely a path of measure zero.
2. Countable set; RE set, recursive set.
3. A state of a world, existed physically, an approximation is impossible to store in computers.
4. A object that can possibly be stored using all the world's computers.
5. A object feasible to be a state in a computational simulation.
6. An object feasible to present to a human.

9 Laplace etc from other document

(a functional transformation, function $->$ function)

(a complex, more specifically, analytic function)

(a generating function, analog with $\int dt$ in place of $\sum$)

(approximation relations with dicrete generating functions, aka "z-transform". Digital signal processing implements or is approximately related to analog signal processing.)

(a formal ratio of an extensor's Plucker coordinates)

(eigenvalue of eigenvector e^{st} under a differential operator eg $(a\frac{d}{dx}^2 + b\frac{d}{dx} + c)(e^{st}) = (as^2 + bs + c)e^{st}$

(formal expression for impedances R, $1/Cs$, Ls)

(linear differential operator $s(f) = \frac{d}{dx}f$)

(deletion/contraction, "capacitor shorts out on high s, inductor opens on low s" [33])

(characterization of "responses" defined among port variables)

(?? Indicator of a step respone)

(asymptotic expansions as $s \to \infty$ and $s \to 0$)

More and more

(asymptotic behavior of a generating function's coefficients, which are the generated function's values, determined by analytic properties of the generated function. Relate EE, circuits and signals literature to statistical mechanics and combinatorics. [3, 8, 21])

(random generation: [17])

(geometry of phase shifts)

(risetime related to basic topology, graph theory, topology of lumped networks expressed by oriented matroids)

(FUTURE hopefully not too much: Complex oriented matroids of Anderson, et. al. [1])

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