# Tutte Polynomials and Electrical Networks

Seth Chaiken
Dept. of Computer Science
Univ. at Albany
schaiken@albany.edu

July 21, 2015

# Outline

1. Spanning trees and equivalent (linear) resistance.
2. An exterior algebra (extensor) Tutte function and a (linear) resistance network's behavior projected on distinguished coordinates.
3. Rayleigh's inequalities.
4. Tutte polynomials on pairs and (linear) amplifier networks.
5. Distinguished graph vertices and splitting formulas.
6. Electrical network analogies.

## Equiv. Resistance $:\equiv -(v_p/i_p)$ observed at a port $p$ by the environment (Port edge $p$ locates the *2 terminals*.)

**Theorem**

$$-\left(\frac{v_p}{i_p}\right) = \frac{WTS(G/p)}{WTS(G\backslash p)} = \frac{\textit{Matrix-Tree Det}(G/p)}{\textit{Matrix-Tree Det}(G\backslash p)}$$

This "Maxwell's rule" is proved via the Matrix Tree Thm. on 2 DIFFERENT GRAPHS $G/p$ and $G\backslash p$.

**Theorem**

*Weighted Tree Sum (WTS) is a colored Tutte function:*

$$WTS(G') = g_e WTS(G'/e) + r_e WTS(G'\backslash e) \text{ for all } e \notin P$$

$$WTS(coloop(e)) = g_e$$

$$WTS(loop(e)) = r_e$$

# Next steps

1. One (terminal-pair) port $\longrightarrow$ set of ports $P$.
2. 1-dim subspace of homogeneous coordinates of solutions $((v_p, i_p)) \longrightarrow$ p-dim subspace of $k^{2|P|}$.
3. p-dim subspace $\longrightarrow$ EXTENSOR (decomposible exterior algebra, i.e., anti-symmetric tensor) with $\binom{2p}{p}$ Plucker coordinates (determinants).

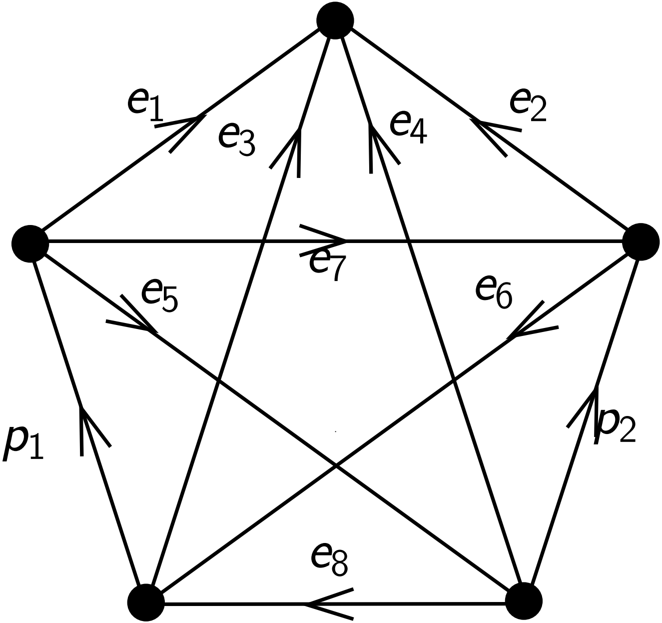
## Theorem

*(After careful definitions...) For fixed $P$,*

| | | |
|---|---|---|
| *each Plucker coordinate* | *and* | *this extensor in the exterior algebra* |

*satisfy weighed Tutte recursion, when $/e$ and $\backslash e$ are restricted to $e \notin P$.*

## Example



# Example

Here's $M$ of the electrical network equations $Mx = 0$. Kirchhoff's laws apply to all cycles and cocyles with $r_i x_{e_i}$ as voltage and and $g_i x_{e_i}$ as current of resistor (not port) edges. TWO SEPARATE voltage and current variables are used for each port edge.

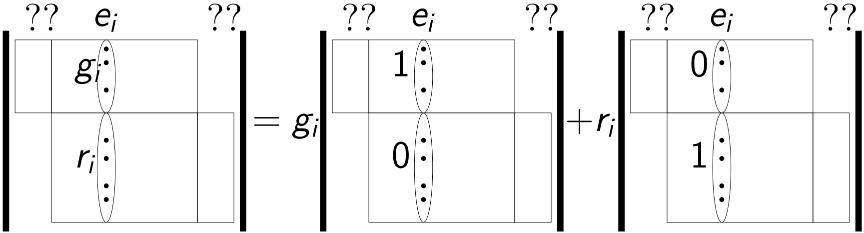
| $ip_1$ | $ip_2$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ | $e_8$ | $vp_1$ | $vp_2$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | $+g_3$ | 0 | 0 | $-g_6$ | 0 | $-g_8$ | | |
| $-1$ | 0 | $-g_1$ | 0 | 0 | 0 | $+g_5$ | 0 | $+g_7$ | 0 | | |
| 0 | $+1$ | 0 | 0 | 0 | $+g_4$ | $-g_5$ | 0 | 0 | $+g_8$ | | |
| 0 | $-1$ | 0 | $-g_2$ | 0 | 0 | 0 | $g_6$ | $+g_7$ | $g_8$ | | |
| | | $+r_1$ | 0 | $-r_3$ | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| | | 0 | $+r_2$ | 0 | $-r_4$ | 0 | 0 | 0 | 0 | 0 | 1 |
| | | $-r_1$ | 0 | 0 | $+r_4$ | $+r_5$ | 0 | 0 | 0 | 0 | 0 |
| | | 0 | $-r_2$ | $+r_3$ | 0 | 0 | $+r_6$ | 0 | 0 | 0 | 0 |
| | | $-r_1$ | $+r_2$ | 0 | 0 | 0 | 0 | $+r_7$ | 0 | 0 | 0 |
| | | 0 | 0 | $+r_3$ | $+r_4$ | 0 | 0 | 0 | $+r_8$ | 0 | 0 |

Top 4 rows: Basis for cocycle space. Represents graphic matroid.
Bot 6 rows: Basis for cycle space. Represents cographic matroid.

# The Tutte Decomposition

For all choices denoted by ?? of the $\binom{2|P|}{|P|}$ size $|P|$ subsets of the $2|P|$ columns $\{ip_k, vp_k\}$, the matrices in the equation below are square.

So the elementary multilinearity of determinants means Tutte decompostion holds for all $e_i \notin P$:




(Technical detail: Define the Tutte function on all graphs with distinguished or port subset $P$ so the det. signs are consistent with the decomposition.)

# Rayleigh Identity which $\Rightarrow$ inequality, "Neg. Spanning Tree Correlation"

$\Gamma_e(G)$ is equivalent conductance across $e$. Rayleigh: $0 \leq \frac{\partial \Gamma_p}{\partial g_f} = \frac{\partial \frac{T_G}{T_{G/e}}}{\partial g_f}$

is equivalent to

$$0 \leq \frac{\partial T_G}{\partial g_f} T_{G/e} - T_G \frac{\partial T_{G/e}}{\partial g_f} = T_{G/f} T_{G/e} - T_G T_{G/e/f}$$

## Theorem

$$T_{G/f} T_{G/e} - T_G T_{G/e/f} = \left( T^{+}_{G/e \ \& \ G/f} - T^{-}_{G/e \ \& \ G/f} \right)^2$$

$T^{\pm}_{G/e \ \& \ G/f}$ *enumerate the* $\pm$ *common spanning trees.*

Choe, Cibulka, Hladky, Lacroix and Wagner gave bijective proofs; we give det. based proofs and generalizations.

# Linear Alg./Oriented Matroid Proof of Rayleigh's Identity

Let $R$ be the transfer resistance matrix for 2 ports across $e$ and $f$. Our result implies that

$$\det R = \begin{vmatrix} R_{ee} & R_{ef} \\ R_{fe} & R_{ff} \end{vmatrix} = +\frac{T_{G/e/f}}{T_G}$$

It and better-known results tell us

$$R_{ee} = \frac{T_{G/e}}{T_G}; \quad R_{ff} = \frac{T_{G/f}}{T_G}; \quad R_{ef} = R_{fe} = \frac{T^+_{G/e\ \&\ G/f} - T^-_{G/e\ \&\ G/f}}{T_G}$$

$$T_{G/f}\,T_{G/e} - T_G\,T_{G/e/f} = \left(T^+_{G/e\ \&\ G/f} - T^-_{G/e\ \&\ G/f}\right)^2$$ is immediate after substituting these into

$$\det R = R_{ee}R_{ff} - (R_{ef})^2$$

The $+$ follows from physical grounds if the $g_e, r_e \geq 0$. Our characterization and proof are combinatorial.

# New Rayleigh's Identities!

The same method generates identities and inequalities from

$$\begin{vmatrix} R_{ee} & R_{ef} & R_{eg} \\ R_{fe} & R_{ff} & R_{fg} \\ R_{ge} & R_{gf} & R_{gg} \end{vmatrix} = +\frac{T_{G/e/f/g}}{T_G} \geq 0$$

when all $r_{..}, g_{..} \geq 0$, ETC...

(Applications???)

Might the same methods address a much harder problem: The same inequality for forests instead of spanning trees?

# Pairs: The Common Covector Model

The cycle space of $G_I$ GENERATES the covectors of an **oriented matroid** over $(E \cup P_I)$.

0

(signs indexed by $E$) (by $P_I$)

Non-linear monotone resistors CONSTRAIN SIGNS of voltage drops (from ↓) and flows (from ↑) TO BE EQUAL

(signs indexed by $E$) (by $P_V$)

The cocycle space of $G_V$ GENERATES the covectors of an oriented matroid over $E \cup P_V$.

$G_V$
SOMETIMES EQUALS
$G_I$

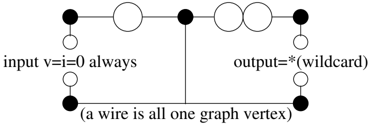
0

# Voltage and Current graphs $G_V$, $G_I$

“Voltage graph” $G_V$ (EE [8, 13], NOT Gross, ...) represents KVL $\mathbf{v} \in$ Cocycles W/ SOME $v_e \equiv 0$

“Current graph” $G_I$ represents KCL $\mathbf{i} \in$ Cycles WITH SOME FLOWS $\equiv 0$

- They are EQUAL GRAPHS for resistor networks.
- For networks with idealized amplifiers, they are not equal.




The output voltage and current are whatever makes the input voltage and current BOTH BE zero.

- (More) realistic amp. model = idealized amp. + resistors.

**open**

$G_v = G \backslash e$

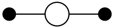
$G_I = G \backslash e$

**short**

$G_v = G / e$

$G_I = G / e$

**nullator**



$G_v = G / e$

$G_I = G \backslash e$

**norator**

$G_v = G \backslash e$

$G_I = G / e$

# Distinguished graph vertices and splitting formulas

Let $Q$ be a set of distinguished, labelled graph VERTICES, analogous to the distinguished port edges $P$

## Theorem

*Given graph $G(V \cupdot Q, E \cupdot P)$ let $T(G, P, Q)$ be the Tutte polynomial determined by restricting $/e$ and $\backslash e$ to $e \notin P$ AND carrying along the partition of $Q$ defined by the components of the contracted edges.*
*Construct $G^Q(V \cupdot Q, E \cupdot P \cupdot P_Q)$ by adding to $G$ a new vertex $Z$ and the $|Q|$ new port edges from $Z$ to each vertex in $Q$.*
*Then $T(G, P, Q)$ and $T(G^Q, P \cupdot P_Q)$ (the ported Tutte polynomial) determine each other by substitutions.*

So we can use ported Tutte polys to express splitting formulas for Tutte polynomials of graph, beginning with Crapo [5] and continuing with Andrzejak [1], Bonin and de Meir [4], and Narayanan [12, 14].

# etc

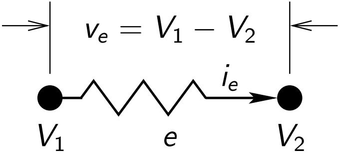
Extra slides...

# Why Electricity, EE?

- Scholarly topic suggested by G.-C. Rota $\approx$ 1980?.
- $\approx$ 100 yrs. geometry-like intuition of circuit configurations known by engineers, EE books: "Intuitive Analog Circuit Design (2013)" [15]; "Non-linear Circuits" [8] translates to our Oriented Matroid pair model.
- Geometry of linear spaces and oriented matroids; Tutte decomp. w/ techniques from Barnabi, Brini and Rota's Exterior Calculus [2])
- Real behavior $\approx$ ideal plus perturbations, ideal constraints predict intended real behavior,
- Interesting, accessible, intuitively understandable intentential designs, applicable, easy to both simulate and build physically, dimension $\approx$ 12 or 24, depending on formulation
- Analogs to chemical (and real algebraic geometry [11]), biological, elastic/tensegrity strs. etc., random walks ...
- Merely one scalar non-linearity can cause chaos.

# Kirchhoff (1847) [9] Maxwell (1891) [10] The equivalent resistance PROBLEM IS SOLVED by the Matrix Tree Theorem. (1) POSE! the VARIABLES or COORDINATES

$\boxed{v_e, i_e : e \in E}$

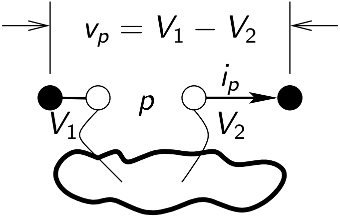



(Use **voltage drops along the flow**, not potentials $V_1$, $V_2$.)

**ordinary, resistor edge** $e \in E$

(IF linear Ohm's law use $|E|$ variables ($g_e v_e = r_e i_e$) ELSE use $2|E|$ variables.)

$\boxed{v_p, i_p : p \in P}$




**DISTINGUISHED, PORT edge** $p \in P$

The interface to an environment is modelled with $2|P|$ variables.

(math, not EE sign convention)

# (2) POSE: EQUATIONS. Preview the consequences.

- (KCL) $(i_e)_{e \in S}$ is a cycle (a flow).
- (KVL) $(v_e)_{e \in S}$ is a cocycle
- (constituitive Law) $i_e = g_e(v_e)$ non-linear, usually monotonic increasing $R \to R$.
  (Sometimes use Ohm's approximation $i_e = g_e v_e$)

## Combinatorics!

The signs $\{+, -, 0\}$ have a DUAL-PAIR ORIENTED MATROID structure (combinatorial, geometric, topological).

## Engineering with amplifiers!

There's good unique solvablility due to STRUCTURE, when the NON-DUAL PAIR (for voltages and currents) is ALMOST DUAL: No common covectors.

# Multiple Ports. (your stereo: 3=power plug & 2 speakers)

- One formula expresses $\binom{2|P|}{|P|}$ different Matrix Tree Theorems...
- ... long vertex-based proofs are shortened; Rayleigh inequalities too.
- Interesting **non-commutative ranges** of new ORIENTED MATROID Tutte invariants with pattern:

$$TF(N(P \cup E)) = F(N(P \cup E)/E)$$

(They distinguish DIFFERENT ORIENTATIONS of the SAME MATROID.)

- Ported/Relative OM Tutte Poly. terms embed SPECIFIC MINORS as variables, making proofs just with $\partial T/\partial x_e$ easier.
- Formalize composition of systems [12], Tutte poly. splitting formulas.
- Model practical devices (transistors, op amps); Label variables to observe.
- Align EE applications with knots [6] (Ported = "Relative") and combinatorial geometry [17] (Ported = "Set Pointed").

# Constraint/Generator Duals and 2 Results.

- (Part 1) Technique:

$$\text{Solution Space} = \bigcap \text{Constraint Subspaces}$$

- Result: An exterior algebraic algebraic Tutte function: Each of its $\binom{2|P|}{|P|}$ Plücker coordinates satisfies a Matrix Tree Theorem.
  This and det. formulas easily prove Rayleigh inequalities.

- (Part 2) Combine with:

$$\text{Solution Space} = \text{Closure(Set of Generators)}$$

- To apply: An oriented matroid's COVECTOR SET encodes ALL POSSIBLE $(+,-,0)$ coordinate behaviors or $\delta$s.
- Result: An oriented matroid pair model for some non-linear problem (AMPLIFIER!) well-posedness. (How? Sign contradictions $\Rightarrow$ a KERNEL$=\{(0)\}$.)

# Part 1) Use Matrix $M$ in CONSTRAINTS $MX = 0$ to get...

The Tutte-like function $\mathbf{M}_E()$ : Extensor $\mathbf{N} \to$ Extensor $\mathbf{M_E}(\mathbf{N})$.
(STUDENT NOTE: An EXTENSOR represents the row-space of an $r \times s$ $r-$rank matrix $M$ by the $\binom{s}{r}$-TUPLE of the DETERMINANTS of $M$'s $r \times r$ submatrices. Plücker coords.)
Given $N$ (matrix), construct $N^\perp$ with orthog. comp. row space.
Construct: $(G = \mathrm{diag}(g_e),\ R = \mathrm{diag}(r_e))$

$$M = \left[\begin{array}{c|c|c} N(P) & 0 & N(E)G \\ \hline 0 & N^\perp(P) & N^\perp(E)R \end{array}\right]$$

with columns labelled by $P_I \uplus P_V \uplus E$.
Extensor $\mathbf{M}$ over $k[g_e, r_e](P_V \uplus P_I \uplus E)$ is the $\wedge$-product of $M$'s **row vectors**. The contraction result $\mathbf{M}_E(\mathbf{N}) = \mathbf{M}/E$ appears:

$$\mathbf{M} = \mathbf{M}_E(\mathbf{N})\mathbf{e_1}\mathbf{e_2}\cdots\mathbf{e}_{|E|} + (\cdots)$$

$\mathbf{M}_E(\mathbf{N})$ is our Tutte function $\mathbf{N} \to$ Ext. Alg.

# Contracting means "Eliminate variables"

ELIMINATE the variables indexed by $E$, leaving $2|P|$ variables labelled by $P_I$ and $P_V$. ie, CONTRACT $E$. **Answer $\mathbf{M_E}$ IS:**

$$\mathbf{M}_E = \bigwedge_{\text{JOIN over rows}}^{\text{Exterior}} \left[\begin{array}{c|c} A_{I,I} & A_{I,V} \\ \hline A_{V_I} & A_{V,V} \end{array}\right] [\mathbf{p_{I_1}}, \cdots, \mathbf{p_{I_p}}; \mathbf{p_{V_1}}, \cdots, \mathbf{p_{V_p}}]^{\mathbf{t}}$$

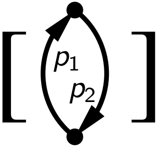
$$= \ldots + C_i\mathbf{XXX} + \ldots ; \text{Equiv. Resistance} = \text{certain } \; C_i/C_j$$

All the other $C_k$'s have similar interpretations.
$\binom{2|P|}{|P|}$ **Matr. Tree Theorems:** Each $C_k(N)$ (a PRINCIPAL MINOR of MATRIX $A$ ABOVE!) $= g_e C_k(N/e) + r_e C_k(N\backslash e)$ ($e \notin P$, $e$ not (co)loop).
Each $C_k$ is a signed weighted enumerator of forests satisfying **conditions ...**

# Conditions (what sets $F$ are enumerated by one det. $C_i$)

The **conditions ...** are on the rank, nullity of $F$ and, WHAT ORIENTED MINOR is $G/F \setminus (E \setminus F)$, the minor with ONLY PORT EDGES from contracting $F$ and deleting the other resistor edges, leaving the ports.
The conditions for a given $C_k$ *sometimes* make all the signs the same (eg: $C_i$ and $C_j$ in 1-port equivalent resistance $R = C_i/C_j$)
*Othertimes*, the oriented **P-minors** in the completed Tutte decomposition of $C_k$ determine some $+$ and some - signs.

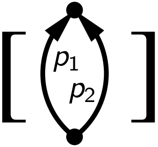
When $[G/F|P]$ is




the term is

$+g_F r_{E\setminus F}$

When $[G/F|P]$ is




the term is

$-g_F r_{E\setminus F}$

# Application: Rayleigh Identity, "Neg. Spanning Tree Correlation"

$\Gamma_e(G)$ is equivalent conductance across $e$. Rayleigh: $0 \leq \dfrac{\partial \Gamma_p}{\partial g_f} = \dfrac{\partial \frac{T_G}{T_{G/e}}}{\partial g_f}$

is equivalent to

$$0 \leq \frac{\partial T_G}{\partial g_f} T_{G/e} - T_G \frac{\partial T_{G/e}}{\partial g_f} = T_{G/f} T_{G/e} - T_G T_{G/e/f}$$

In fact,

$$T_{G/f} T_{G/e} - T_G T_{G/e/f} = \left( T^{+}_{G/e \;\&\; G/f} - T^{-}_{G/e \;\&\; G/f} \right)^2$$

$T^{\pm}_{G/e \;\&\; G/f}$ enumerate the $\pm$ common spanning trees.

# Known Partial and Full Combinatorial Proofs

$$T_{G/f}T_{G/e} - T_G T_{G/e/f} = \left(T^{+}_{G/e\ \&\ G/f} - T^{-}_{G/e\ \&\ G/f}\right)^2$$

$T^{\pm}_{G/e\ \&\ G/f}$ enumerate the $\pm$ common spanning trees.

Choe (2004) proved essentially this using the vertex-based all-minors matrix tree theorem, combinatorial cases and Jacobi's theorem relating the minors of a matrix to the minors of its inverse..

Cibulka, Hladky, Lacroix and Wagner (2008) gave a completely bijective proof that utilizes some natural 2:2 and 2:1 correspondances.

Difficulty: Some terms on the left cancel and some reduce to terms with coefficients $\pm 2$.

## “Colors” are parameters on every Tutte decomposition step

The Bollobos/Riordan/Zaslavsky [3, 18], Traldi-Ellis-Monaghan [16, 7], (sdc unpub) BRZ theory for well-definedness of “Relative Tutte Polynomials for Colored Graphs” ALL GOES THROUGH (Diao and Hetyei [6]): The 3 BRZ conditions on (colors,initial values) GENERALIZE TO 5; activity theory WORKS TOO, when based on linear orders on the non-port-elements.

### In a nutshell

The 5 conditions $\Longrightarrow$ activities define an unambiguous Tutte function from the deletion/contraction and initial value formulas. Additional conditions $\Longrightarrow$ the Tutte function has a rank-nullity expansion.

(The rank-nullity conditions are satisfied in our application.)

### To specify the activity/deletion-contraction linear order GLOBALLY is UNNECESSARY.

The Gordon/McMahon computation-tree-based activity theory also generalizes. (sdc).

# References I


Artur Andrzejak.
Splitting formulas for Tutte polynomials.
*Journal of Combinatorial Theory, Series B*, 70(2):346–366, 1997.

Marilena Barnabei, Andrea Brini, and Gian-Carlo Rota.
On the exterior calculus of invariant theory.
*Journal of Algebra*, 96:120–160, 1985.

B. Bollobas and O. Riordan.
A Tutte polynomial for colored graphs.
*Combin. Probab. Computat.*, 8(1–2):45–93, 1999.

J. Bonin and A. de Mier.

Henry H. Crapo.

# References II

Yuanan Diao and Gábor Hetyei.
Relative Tutte polynomials for colored graphs and virtual knot theory.
*Combin. Probab. Comput.*, 19(3):343–369, 2010.

J. A. Ellis-Monaghan and Lorenzo Traldi.
Parametrized Tutte polynomials of graphs and matroids.
*Combinatorics, Probability and Computing*, 15:835–854, 2006.

M. Hasler and J. Neirynck.
*Nonlinear Circuits*.
Artech House, Norwood, Mass., 1986.

# References III

G. Kirchhoff.
Über die auflösung der gleichungen, auf welshe man bei der untersuchung der linearen verteilung galvanischer ströme gefuhrt wird.
*Ann. Physik Chemie*, 72:497–508, 1847.
On the solution of the equations obtained from the investigation of the linear distribution of Galvanic currents, (J. B. O'Toole, tr.) *IRE Trans. Circuit Theory*, 5, 1958, pp. 238–249.

James Clerk Maxwell.
*A Treatise on Electricity and Magnetism*, volume 1, Part II, Appendix of Chapter VI, Mathematical Theory of the Distribution of Electric Currents, pages 409–410.
Claredon Press and reprinted by Dover, New York (1954), 3rd edition, 1891.

# References IV


Stefan Müller, Elisenda Feliu, Georg Regensburger, Carsten Conradi, Anne Shiu, and Alicia Dickenstein.
Sign conditions for injectivity of generalized polynomial maps with applications to chemical reaction networks and real algebraic geometry.
*Foundations of Computational Mathematics*, 2015.

H. Narayanan.
On the decomposition of vector spaces.
*Linear Algebra and its Applications*, 79:61–98, 1986.

C. Sanchez-Lopez, F. V. Fernandez, E. Tlelo-Cuautle, and S. X. Tan.
Pathological element-based active device models and their application to symbolic analysis.
*IEEE Transactions on Circuits and Systems-I-Regular Papers*, 58(6):1382–1395, 2011.

# References V


- Siva Theja and H. Narayanan.
  On the notion of generalized minor in topological network theory and matroids.
  *Linear Algebra and its Applications*, 458:1–46, 2014.

- Marc Thompson.
  *Intuitive Analog Circuit Design*.
  Newnes/Elsevier, 2nd edition, 2014.

- Lorenzo Traldi.
  A dichromatic polynomial for weighted graphs and link polynomials.
  *Proc. Amer. Math. Soc.*, 106:279–286, 1989.

- Michel Las Vergnas.
  The Tutte polynomial of a morphism of matroids I. set-pointed matroids and matroid perspectives.
  *Annales de l'Institut Fourier*, 49(3):973–1015, 1999.

# References VI

Thomas Zaslavsky.
Strong Tutte functions of matroids and graphs.
*Trans. Amer. Math. Soc.*, 334(1):317–347, 1992.