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An Exterior Algebra Valued Tutte Function on Linear Matroid Pairs

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Abstract. The matrix tree theorem expresses the basis enumeration Tutte function (spanning tree count of connected graphic matroids) as a determinant–a minor of the graph's Laplacian. We will generalize the choice of a root vertex for the trees by distinguishing a fixed set P of p matroid elements, which we will call ports, a word from engineering. We then define a function from k-linear matrix representations of matroid pairs (N_α, N_β) (often $N_\alpha = N_\beta$) whose ground sets include P, into pure (decomposible) exterior algebra elements which represent points in the Grassmannian of p dimensional k-linear subspaces of $k^{(2p)}$. We express this subspace value as a decomposible (i.e. product of vectors) element in the exterior algebra (of anti-symmetric tensors) over $K^{(2p)}$, in a standard way. The result is a restricted or set-pointed Tutte function $\mathbf{L}_E(N_\alpha, N_\beta)$ that satisfies sign-corrected forms of the two familiar identities for deletion/contraction of elements not in P, and for disjoint union. Note that these identities are in exterior algebra, whose multiplication is anti-commutative. This generalizes the all-minors directed graph matrix tree theorem (proved combinatorially by the author) because those minors are the coefficients in the expansion of T's value over the standard basis–they are the standardized Plücker coordinates for the dimension p subspace T's value represents. An unusual feature for this Tutte-like function is that when k is ordered so the matroids are oriented, T can distinguish different orientations of the same matroid.

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    refer to dim r in 3 equiv types of objects at the beginning of intro.
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1. Introduction

Let $\mathcal{A}(U)$ be the exterior algebra of the vector space $\mathcal{V}$ over K generated by finite basis U. We investigate parametrized Tutte functions on the following sets of objects, which are all bijectively equivalent (see, for example, [18]):

(1) All rank r pure (i.e., decomposible[18]) $\mathbf{N} \in \mathcal{A}$ equipped with a ground set $\subseteq U$.

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(2) All Grassmann representitives of r-dimensional subspaces of $\mathcal{V}$, i.e. particular representitives of their Plücker coordinates. Each coordinate of a representitive will be called a *component*. The component for a sequence of basis elements B will be denoted by $\mathbf{N}[B]$.
(3) All full-row-rank r matrix representations N of matroids with ground set $\subseteq U$, oriented if K is ordered, modulo left multiplication by SL_r.

We must consider each of our equivalent objects to be equipped with a ground set because the set of bases of a matroid with loops does not determine the ground set. Some elements of $e \in U$ will be equipped with commuting parameter coefficients g_e and r_e used in the additive Tutte identity 5. They are motivated by our electrical circuit linear equation application, add little overhead to the theory, and provide the subset generating function interpretation of the resulting functions.

This motivation led us to a Tutte function $\mathbf{L}(\mathbf{N}_\alpha; \mathbf{N}_\beta)$ on *pairs*, analogous to Welsh and Kayibi's linking polynomial[27] of matroid pairs. $\mathbf{N}_\alpha$, $\mathbf{N}_\beta$ are from the (1)-(3) class, with common ground sets S of the form $S = P' \coprod E$, where $P' \subseteq P$ and P is a *distinguished* set of basis elements. Our Tutte function of single $\mathbf{N}$ is the special case $\mathbf{L}_E(\mathbf{N}; \mathbf{N})$.

The Tutte function value is in the class (1)-(3) for which the ground set is $P'_\alpha \coprod P'_\beta$, where P'_α and P'_β are disjoint copies of P'. Interest in our application[8] leads us to prefer to call the elements of P *ports* and to call P-ported those objects equipped with a distinguished set P of ports, but other authors use terms for the same concept "a set-pointed matroid pointed by P[24, 25]", and the unnamed subset "$\mathcal{H}$" in the definition of relative Tutte polynomials[12]. The distinguishing property of $p \in P$ is that the additive Tutte identity is restricted so it applies only to $e \notin P$. Parameters g_p, r_p are not given when $p \in P$.

1.1. Origin of the Investigation. The results turn out apply to more general objects than (1)-(3) above. We begin with the original class.

As our construction generalizes the Laplacian matrix of a graph and the interpretation of its minors by the matrix tree theorem[6], we specify a subset P that will play the role of graph vertices. Along the same lines, to play the roles of the Laplacian's rows and columns, we postulate disjoint copies P_α and P_β respectively, with bijections $P \leftrightarrow P_\alpha \leftrightarrow P_\beta$. More generally, P_α and P_β will label certain variables in a system of linear equations, and the Tutte function will encode the linear relationships implied by the system on those labelled variables. When Δ is the Laplacian, that solution subspace is $\{(v_P, i_P) | \Delta v_P + i_P = 0\}$.

When $P = \varnothing$, the main theorem gives the well-known common basis enumerating interpretion of the Cauchy-Binet theorem applied to $|N_\alpha G N_\beta^t|$ where $G = \mathrm{diag}(\cdots, g_e, \cdots)$. When $P \neq \varnothing$, definition 1.3 gives what we prove is a function between elements in certain exterior algebras that satisfies Tutte functions' additive and multiplicative identities. Section 4 shows how every component has a Cauchy-Binet expansion that enumerates common bases in certain minors of the matroids represented by the function arguments.

1.2. Formalism, Duals and Minors. The order of an exterior algebra element $\mathbf{N}$, as a tensor, is denoted by $r\mathbf{N}$; it has been called "grade level", "rank", "degree", "valence", "number of indices" and occasionally[2] "step". Vectors, and other elements of the exterior algebra will be denoted by boldface characters, such as $\mathbf{e}_j$, $\mathbf{p}_{\alpha_k}$, $\mathbf{p}_{\beta_l} \in P_\alpha \coprod P_\beta \coprod E$, and $\mathbf{N}_\alpha$, $\mathbf{L}$, $\mathbf{L}_E$. Exterior products of vectors, or products of products, will be denoted by concatenation of boldface symbols, as in $\mathbf{AB} = \mathbf{a}_1\mathbf{a}_2 ... \mathbf{b}_1\mathbf{b}_2 ... = (\mathbf{a}_1\mathbf{a}_2 ...)(\mathbf{b}_1\mathbf{b}_2 ...) = \mathbf{AB} = \mathbf{A} \wedge \mathbf{B} = \mathbf{a}_1 \wedge \mathbf{a}_2 \wedge ... \wedge \mathbf{b}_1 \wedge \mathbf{b}_2 \wedge ....$ Sometimes the $\wedge$ will be included for clarity or for emphasis. A symbol for a set will also be used to denote its elements in a non-repeating sequence as in $A = a_1 a_2 \dots a_k$.

To investigate the Tutte functions, we must define operations corresponding to minors and the dual. If a volume form is specified for each $\mathcal{A}(U') \subseteq \mathcal{A}(U)$, we may use the resulting Hodge star operation to define an operation corresponding to matroid duality. We will specify this $\perp = \perp_\epsilon$ by one global alternating sign function ϵ on all sequences of elements from U. To be compatible with conventions established for oriented matroid chirotopes[3], we define the components of $\mathbf{N}^\perp$ in terms of the components of $\mathbf{N}$:

$$\mathbf{N}^\perp[B] = \mathbf{N}[\overline{B}]\epsilon(\overline{B}B) \tag{1}$$

where $\overline{B}$ is any sequence of the elements of $U' \setminus B$; the result is well-defined since both factors are alternating functions of $\overline{B}$. The $\epsilon \neq 0$ for distinct sets are arbitrary except that they must be compatible with the bijections $P \leftrightarrow P_\alpha \leftrightarrow P_\beta$.

For $A \subseteq U$, it is straightforward to define the operation $\mathbf{N} \setminus A$ corresponding to matroid deletion. Again by components, we define the contraction operation $\mathbf{N}/A$ so that the sign of $\pm(\backslash A) \circ ({}^\perp) = ({}^\perp) \circ (/A)$ is well-characterized by (3.3). (It can be defined in terms of the dual of basis U and interior product[18], except for our subsetting of the ground set.) So, for all $A \subseteq U$ and $B \subseteq U \setminus A$

Definition 1.1.

$$(\mathbf{N}\backslash A)[B] = \mathbf{N}[B], \tag{2}$$

$$(\mathbf{N}/A)[B] = \mathbf{N}[BA]\epsilon(BA). \tag{3}$$

Definition 1.2. ι_G and υ_R are exterior algebra homomorphisms depending on our commuting parameters $\{g_e, r_e\}$ generated by these actions on $\mathbf{e}_i \in E$ and $\mathbf{p}_j \in P$:

$$\iota_G(\mathbf{e}_i) = g_{e_i}\mathbf{e}_i, \;\; \upsilon_R(\mathbf{e}_i) = r_{e_i}\mathbf{e}_i, \;\; \iota_G(\mathbf{p}_j) = \mathbf{p}_{\alpha j}, \text{ and } \upsilon_R(\mathbf{p}_j) = \mathbf{p}_{\beta j}.$$

1.3. Results. We note that Tutte functions in exterior algebra, unlike a commutative algebra, can act between objects that represent linear subspaces. The following construction defines the objects of our study.

Definition 1.3. Let $\mathbf{N}_\alpha$ and $\mathbf{N}_\beta$ be P-ported exterior algebra elements with ground set $P \coprod E$ and $r\mathbf{N}_\alpha = r\mathbf{N}_\beta$.

$$\begin{aligned} \mathbf{L}\begin{pmatrix}\mathbf{N}_\alpha \\ \mathbf{N}_\beta^\perp\end{pmatrix} &= \iota_G(\mathbf{N}_\alpha) \wedge \upsilon_R(\mathbf{N}_\beta^\perp) \quad & \mathbf{L}_E\begin{pmatrix}\mathbf{N}_\alpha \\ \mathbf{N}_\beta^\perp\end{pmatrix} &= \mathbf{L}\begin{pmatrix}\mathbf{N}_\alpha \\ \mathbf{N}_\beta^\perp\end{pmatrix}/E \\ \mathbf{L}\left(\mathbf{N}_\alpha; \mathbf{N}_\beta\right) &= \mathbf{L}\begin{pmatrix}\mathbf{N}_\alpha \\ \mathbf{N}_\beta^\perp\end{pmatrix} & \mathbf{L}_E\left(\mathbf{N}_\alpha; \mathbf{N}_\beta\right) &= \mathbf{L}\left(\mathbf{N}_\alpha; \mathbf{N}_\beta\right)/E \end{aligned} \tag{4}$$

Remark: $\mathbf{L}\begin{pmatrix}\mathbf{N}_\alpha \\ \mathbf{N}_\beta^\perp\end{pmatrix}$ and $\mathbf{L}_E\begin{pmatrix}\mathbf{N}_\alpha \\ \mathbf{N}_\beta^\perp\end{pmatrix}$ as functions of $\mathbf{N}_\alpha$ and $\mathbf{N}_\beta^\perp$ do not depend on ϵ. However, $\mathbf{L}\left(\mathbf{N}_\alpha; \mathbf{N}_\beta\right)$ and $\mathbf{L}_E\left(\mathbf{N}_\alpha; \mathbf{N}_\beta\right)$ depend on ϵ through $\perp$'s dependence on ϵ.

Having started with exterior algebra elements, the values of the resulting Tutte functions will be in exterior algebra, not a commutative algebra. The definitions of minors and dual, our Tutte function, proofs of the Tutte identities, and a common basis expansion formula are all in terms of the *components* of exterior algebra elements $\mathbf{N}_\alpha$ and $\mathbf{N}_\beta$. These results apply to the wider class of the not just pure exterior algebra elements. These have expansions in terms of their components $\mathbf{N}_\alpha[B]$ or $\mathbf{N}_\beta[B]$, but do not generally have matrix representations and satisfy the Grassmann-Plücker identities. When the starting elements are pure, which means they represent linear subspaces, the values will also represent linear subspaces.

We can now state:

Theorem. *Given sequenced P and E, for all $e \in E$ and sequenced $E' = E \setminus e$,*

$$\epsilon(PE)\mathbf{L}_E\left(\mathbf{N}_\alpha;\mathbf{N}_\beta\right) = \epsilon(PE')\left(g_e\mathbf{L}_{E'}\left(\mathbf{N}_\alpha/e;\mathbf{N}_\beta/e\right) + r_e\mathbf{L}_{E'}\left(\mathbf{N}_\alpha\backslash e;\mathbf{N}_\beta\backslash e\right)\right) \tag{5}$$

Theorem. *Given, for $i = 1, 2$, $\mathbf{N}_\alpha^i$, $\mathbf{N}_\beta^i$, ${\mathbf{N}_\beta^i}^\perp$ and sequenced E, P^i, E^i, where $E = E^1 \coprod E^2$,*

$$\begin{gathered}\epsilon(P^1P^2E)\mathbf{L}_E\left((\mathbf{N}_\alpha^1 \wedge \mathbf{N}_\alpha^2);(\mathbf{N}_\beta^1 \wedge \mathbf{N}_\beta^2)\right) = \\ \epsilon(P^1E^1)\epsilon(P^2E^2)\left(\mathbf{L}_{E^1}\left(\mathbf{N}_\alpha^1;\mathbf{N}_\beta^1\right) \wedge \mathbf{L}_{E^2}\left(\mathbf{N}_\alpha^2;\mathbf{N}_\beta^2\right)\right)\end{gathered} \tag{6}$$

1.4. Matrix Illustration. To facilitate calculation, applications and a concrete explanation, we illustrate (1.3) as a linear algebra problem posed with matrices.

Let N_α and $N_\beta^\perp$ be two full row rank matrices (over field K_0) with columns indexed by $P \cup E$, $P \cap E = \varnothing$, for which $\operatorname{rank}(N_\alpha) + \operatorname{rank}(N_\beta^\perp) = |E| + |P|$.

Take column vector $\mathbf{z} = [\mathbf{p_1}, \mathbf{p_2}, \ldots, \mathbf{p}_p, \mathbf{e}_1, \ldots, \mathbf{e}_n]^t$. $\mathbf{N}_\alpha$ is the exterior product of the entries of the column of vectors $N_\alpha\mathbf{z}$ in order; briefly, "the rows of N_α", and similarly for $N_\beta^\perp$ and $\mathbf{N}_\beta^\perp$.

Let $G = \operatorname{diag}\{g_e\}_{e\in E}$ and $R = \operatorname{diag}\{r_e\}_{e\in E}$. $N(E)$ and $N(P)$ denote the submatrices of N with columns E and P respectively. With two disjoint copies of P and bijections $P \leftrightarrow P_\alpha \leftrightarrow P_\beta$, define the matrix L with columns indexed by $P_\alpha \coprod P_\beta \coprod E$, where each entry is in K_0 or is a K_0 multiple of a parameter:

$$L = L\begin{pmatrix} N_\alpha \\ N_\beta^\perp \end{pmatrix} = \left[\begin{array}{c|c|c} N_\alpha(P) & 0 & N_\alpha(E)G \\ \hline 0 & N_\beta^\perp(P) & N_\beta^\perp(E)R \end{array}\right]. \tag{7}$$

The exterior algebras

$$\bigoplus_{r=0}^{\left|P_\alpha \coprod P_\beta \coprod E\right|} \bigwedge^{(r)} KU,$$

with distinguished bases generated by $U = P_\alpha \coprod P_\beta \coprod E$ will contain our Tutte functions' values. The field $K = K_0(\cdots, g_e, \cdots, r_e, \cdots)$ is an extension containing all the commuting parameters g_e, r_e. The exterior algebras for our function domains will be over K_0.

The $\wedge$ and contraction $(/E)$ in (1.3) will simply take the exterior product of the rows of L, and then, for the Tutte function value, extract the terms that are divisible by $\wedge_{e\in E}\mathbf{e}$ when the product is expanded in terms of the exterior algebra basis generated by $P_\alpha \coprod P_\beta \coprod E$. (Some will recognize the latter step, up to sign, as interior product, when elements of E are dual to the column index elements. We choose notation so that the sign complies with oriented matroid practice[3] and because the generators of the codomain exterior algebra no longer include E.)

It is easy to explain this construction when it is weakened to apply to Grassmannians: Project the solution space $\{z\}$ of $Lz = 0$ onto the $P_\alpha \coprod P_\beta$ coordinates and then take the projection's orthogonal complement with respect to this coodinate basis. In other words, we eliminate all the variables indexed by E in the system $Lz = 0$ and take the exterior product of the rows from the result of the elimination; the result is a representitive (ambiguously defined) for a point in the Grassmannian. Such a result is nice for solving for some of the remaining z's in terms of the others, but a failure for the additive or multiplicative Tutte identities.

1.5. Outline. Section 2 gives familiar examples illustrated in matrix form, section 3 gives the proofs, and section 4 explains how common basis enumeration accounts for the partial connection (only for unimodular matroid representations) to the known generizations of the Tutte polynomials we mentioned above: P-ported Tutte functions, parametrized Tutte functions, and the Welsh and Kayibi's linking polynomial of a matroid pair. We provide more context, background and questions in 5.

2. K_4 Examples and Interpretation

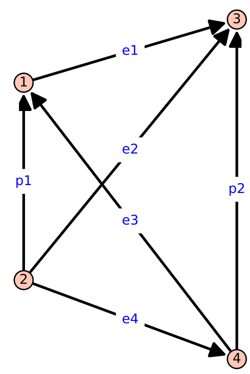
With the complete graph K_4, we will illustrate our construction, its evaluation and an interpretation in terms K_4's familiar linear spaces, oriented matroids, and electrical network problem model. $P = \{p_1, p_2\}$ is comprised of two non-adjacent edges and the rest, $K_{2,2}$, will comprise E.

We relate K_4 as a 1-dimensional abstract simplicial complex to its role as an electrical network model. To make cycles represent flows of positive current, coboundaries electrical potentials and resistance positive, our sign conventions for arbitrarilly oriented labeled (multi)graphs will be *opposite that of* the homology of 1-dimensional chain complexes: For $e : a \to b$, $\partial_1(e) = a - b$, $\partial^0(\hat{a}) = \hat{e} + \dots$, $\partial^0(\hat{b}) = -\hat{e} + \dots$, where e is edge with end vertices a and b oriented as indicated [(1)]. With $\partial^0(\phi(\hat{a})\hat{a} + \phi(\hat{b})\hat{b}) = (\phi(\hat{a}) - \phi(\hat{b}))\hat{e} + \cdots$, $\phi(\hat{a}) - \phi(\hat{b})$ is the voltage drop going from a to b. Edge sets E and P will respectively represent resistors and ports. When $e \in E$, Ohm's law says that $\phi(\hat{a}) - \phi(\hat{b})$ times the resistance of e equals the current through e, flowing from a to b. For us, the currents through and voltage drops along in all these edges are variables i_t and v_t for $t \in P \cup E$. Ohm's law is asserted in the homogeneous form[23, 10, 22]: For $e \in E$, the voltage-drop-to-current ratio $v_e : i_e = r_e : g_e$, the ratio of the resistivity parameters. Those parameters are not given for ports. Kirchhoff's voltage law asserts that $\sum_t v_t \hat{t}$ is the 1-coboundry $\partial^0(\sum \phi(\hat{a})\hat{a})$ for some $\phi : V \to K$; physically, ϕ is the electrical potential function. Kirchhoff's current law asserts that $\sum_t i_t t$ is a 1-cycle, i.e., in ker ∂_1. The problem of linear electrical network analysis is to characterize the linear relationships among port current and voltage drop variables implied by the network topology, Kirchhoffs' and Ohm's laws.

Let $P = \{p_1, p_2\}$ and $E = \{e_1, e_2, e_3, e_4\}$ together be the edges in the (oriented) graph below representing an electrical network, where E represents the resistors. N_α is a full-row-rank, all-column submatrix of the matrix form for ∂^1. It is the usual signed vertex-edge incidence matrix which represents our graph's graphic matroid, with one row deleted so N_α has full row rank. The rows of N_α hold the coefficients in Kirchhoff's current law, asserting that the edge currents constitute a 1-cycle, (i.e., a flow.) $N_\beta^\perp$ is a totally unimodular matrix whose rows are a basis for the 1-cycle space, conveniently obtainable by coding the incidences of edges with the oriented fundamental circuits each associated to an edge not in a fixed spanning tree. Its rows hold the coefficients in Kirchhoff's voltage law, asserting that the edge voltages drops constitute a 1-coboundary, differences of a potential along edges. This well-known construction gives two full row rank totally unimodular matrices whose row spaces are orthogonal complements, representing our graph G's graphic (with bases $\mathcal{B}(G)$) and cographic matroids respectively.

We begin with the familiar incidence matrix representation of K_4's matroid. We delete row 2 and take a representation for the dual that is consistent with our exterior algebra duality operator (1):

[(1)] On the other hand, for the sake of mathematical simplicity, we will use this sign convention the port edges (in P). In electrical engineering circuit theory writings the sign convention for ports is reversed so quantities characterizing typical system behavior observable at the ports are positive.



$$\left(\begin{array}{cc|cccc} 1 & 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 1 \end{array}\right) \quad N_\alpha = \left(\begin{array}{cc|cccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 1 \end{array}\right) \quad N_\beta^\perp = \left(\begin{array}{cc|cccc} 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right)$$

We form the following system of equations according to (7).

(8)
$$0 = L \begin{pmatrix} N_\alpha \\ N_\beta^\perp \end{pmatrix} z = \left[\begin{array}{c|c|c} N_\alpha(P) & 0 & N_\alpha(E)G \\ \hline 0 & N_\beta^\perp(P) & N_\beta^\perp(E)R \end{array}\right] \begin{bmatrix} i_{p_1} \\ i_{p_2} \\ v_{p_1} \\ v_{p_2} \\ x_{e_1} \\ x_{e_2} \\ x_{e_3} \\ x_{e_4} \end{bmatrix}$$
$$= \left(\begin{array}{cc|cc|cccc} 1 & 0 & 0 & 0 & -g_1 & 0 & g_3 & 0 \\ 0 & 1 & 0 & 0 & g_1 & g_2 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & -g_3 & g_4 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 & -r_3 & -r_4 \\ 0 & 0 & 0 & 1 & 0 & -r_2 & 0 & r_4 \\ 0 & 0 & 0 & 0 & r_1 & -r_2 & r_3 & r_4 \end{array}\right) z$$

The electrical network problem at hand is to determine linear constraints on the port variables i_p, v_p, $p \in P = \{p_1, p_2\}$ imposed by the system. In other words, we want a linear map M on the K-vector space with basis P_α, P_β whose kernel is the projection of this system's solution space. Below is a solution. We set $r_e = 1$ for $e \in E$, and $D = g_1g_2g_3 + g_1g_2g_4 + g_1g_3g_4 + g_2g_3g_4$.

(9)
$$M \begin{pmatrix} i_{p1} \\ i_{p2} \\ v_{p1} \\ v_{p2} \end{pmatrix} = \begin{pmatrix} (g_1+g_2)(g_3+g_4) & -g_2g_3 + g_1g_4 & D & 0 \\ -g_2g_3 + g_1g_4 & (g_1+g_3)(g_2+g_4) & 0 & D \end{pmatrix} \begin{pmatrix} i_{p1} \\ i_{p2} \\ v_{p1} \\ v_{p2} \end{pmatrix} = 0$$

To compute $\mathbf{L}_E$ from its definition, set $z = [\mathbf{p}_{\alpha 1}, \mathbf{p}_{\alpha 2}, \mathbf{p}_{\beta 1}, \mathbf{p}_{\beta 2}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4]^t$ in the right hand side of (8) and expand in terms of this basis the exterior product (in order) of the resulting column's entries. Finally, we select the terms that can be expressed as $\mathbf{T}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4$. The final value is the sum of such $\mathbf{T}$; it is a K-multiple of exterior row product of M, i.e., $M_1\mathbf{z} \wedge M_2\mathbf{z}$ where $\mathbf{z} = [\mathbf{p}_{\alpha 1}, \mathbf{p}_{\alpha 2}, \mathbf{p}_{\beta 1}, \mathbf{p}_{\beta 2}]^t$. Each $\mathbf{T}$ is the $\wedge$ of unique pair of distinct elements of $\{\mathbf{p}_{\alpha 1}, \mathbf{p}_{\alpha 2}, \mathbf{p}_{\beta 1}, \mathbf{p}_{\beta 2}\}$ with a coefficient that is a polynomial in the g_e, r_e. These coefficients are Plücker coordinates for the orthogonal complement of the electrical network's (itself parametrized by the g_e, r_e) solution space. Each polynomial term encodes a subset of $E = \{e_1, e_2, e_3, e_4\}$, for example, $\{e_1, e_2\}$ is encoded by $g_1g_2r_3r_4$. Note that each coefficient enumerates the common bases of a pair of not always distinct matroid minors. See 4.2.

(10)

basis element	coefficient	enumerates bases
$\mathbf{p}_{\alpha 1}\mathbf{p}_{\alpha 2}$	$g_1 + g_2 + g_3 + g_4$	$\mathcal{B}(G/\{p_1, p_2\})$
$\mathbf{p}_{\alpha 1}\mathbf{p}_{\beta 1}$	$g_2g_3 - g_1g_4$	$\mathcal{B}(G/p_1 \setminus p_2) \cap \mathcal{B}(G/p_2 \setminus p_1)$
$\mathbf{p}_{\alpha 1}\mathbf{p}_{\beta 2}$	$(g_1 + g_2)(g_3 + g_4)$	$\mathcal{B}(G/p_1 \setminus p_2)$
$\mathbf{p}_{\alpha 2}\mathbf{p}_{\beta 1}$	$-(g_1 + g_3)(g_2 + g_4)$	$\mathcal{B}(G/p_2 \setminus p_1)$
$\mathbf{p}_{\alpha 2}\mathbf{p}_{\beta 2}$	$-g_2g_3 + g_1g_4$	$\mathcal{B}(G/p_1 \setminus p_2) \cap \mathcal{B}(G/p_2 \setminus p_1)$
$\mathbf{p}_{\beta 1}\mathbf{p}_{\beta 2}$	$g_1g_2g_3 + g_1g_2g_4 + g_1g_3g_4 + g_2g_3g_4$	$\mathcal{B}(G \setminus \{p_1, p_2\})$

To verify that the coefficients of our $\mathbf{L}_E$ in (10) are Plücker coordinates, and indeed that $\mathbf{L}_E$ represents the space of linear constraints for the network solution problem, one can calculate that each 2×2 minor of M in (9) equals the D multiple of the corresponding coefficient in (10). The one non-trivial calculation is

$$(g_1 + g_2)(g_1 + g_3)(g_2 + g_4)(g_3 + g_4) - (-g_2g_3 + g_1g_4)^2 = D(g_1 + g_2 + g_3 + g_4)$$

This example also demonstrates that $\mathbf{L}_E$ might not have a matrix representation all of whose entries are K_0 polynomials in the r_e, g_e.

2.1. Example with $\mathbf{N}_\alpha \neq \mathbf{N}_\beta$. We take N_α from before, but form N_β from N_α by replacing -1s by 0. N_β now represents the partition matroid Π where the edges are partitioned according to which vertex in $\{1, 3, 4\}$ is its head.

$$N_\alpha = \left(\begin{array}{cc|cccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 1 \end{array}\right) \quad N_\beta = \left(\begin{array}{cc|cccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \quad N_\beta^\perp = \left(\begin{array}{cc|cccc} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{array}\right)$$

(11)

basis element	coefficient	enumerates bases
$\mathbf{p}_{\alpha 1}\mathbf{p}_{\alpha 2}$	g_4	$\mathcal{B}(G/\{p_1, p_2\}) \cap \mathcal{B}(\Pi/\{p_1, p_2\})$
$\mathbf{p}_{\alpha 1}\mathbf{p}_{\beta 1}$	0	$\mathcal{B}(G/p_1 \setminus p_2) \cap \mathcal{B}(\Pi/p_2 \setminus p_1)$
$\mathbf{p}_{\alpha 1}\mathbf{p}_{\beta 2}$	$(g_1 + g_2)g_4$	$\mathcal{B}(G/p_1 \setminus p_2) \cap \mathcal{B}(\Pi/p_1 \setminus p_2)$
$\mathbf{p}_{\alpha 2}\mathbf{p}_{\beta 1}$	$-g_3g_4$	$\mathcal{B}(G/p_2 \setminus p_1) \cap \mathcal{B}(\Pi/p_2 \setminus p_1)$
$\mathbf{p}_{\alpha 2}\mathbf{p}_{\beta 2}$	g_1g_4	$\mathcal{B}(G/p_1 \setminus p_2) \cap \mathcal{B}(\Pi/p_2 \setminus p_1)$
$\mathbf{p}_{\beta 1}\mathbf{p}_{\beta 2}$	$(g_1 + g_2)g_3g_4$	$\mathcal{B}(G \setminus \{p_1, p_2\}) \cap \mathcal{B}(\Pi \setminus \{p_1, p_2\})$

3. Proof and Corollaries

We begin with routine properties used in the proof.

Proposition 3.1. *Let S be the sequenced ground set for $\mathbf{N}$, A a sequenced subset of of the elements of S, and A_σ be A permuted by σ. Let C, D be other sequenced sets so A, C and D are pairwise disjoint.*

$$\mathbf{N}/A_\sigma = \epsilon(CAD)\epsilon(CA_\sigma D)\mathbf{N}/A = sign(\sigma)\mathbf{N}/A \tag{12}$$

Proof. Calculating componentwise from the definition,

$$(\mathbf{N}/A_\sigma)[B] = \mathbf{N}[BA_\sigma] = \epsilon(CAD)\epsilon(CA_\sigma D)\mathbf{N}[BA] = \epsilon(CAD)\epsilon(CA_\sigma D)(\mathbf{N}/A)[B]$$

since $\epsilon(CAD)\epsilon(CA_\sigma D) = \text{sign}(\sigma)$. $\square$

Proposition 3.2. *For $i = 1,2$, let S^i be sequenced ground sets for $\mathbf{N}^i$, pairwise disjoint. Let E^i be sequenced sets whose elements satisfy $E^i \subseteq S^i$.*

$$(\mathbf{N}^1 \wedge \mathbf{N}^2)/E^1E^2 = (-1)^{(r\mathbf{N}^2 - |E^2|)|E^1|}(\mathbf{N}^1/E^1 \wedge \mathbf{N}^2/E^2) \tag{13}$$

Proof. Proceeding componentwise, with sequenced $B^i \subseteq S^i \setminus E^i$

$$\begin{aligned}(\mathbf{N}^1 \wedge \mathbf{N}^2)/E^1E^2)[B^1B^2] &= (\mathbf{N}^1 \wedge \mathbf{N}^2))[B^1B^2E^1E^2] \\ &= (-1)^{(|B^2||E^1|)}(\mathbf{N}^1 \wedge \mathbf{N}^2))[B^1E^1B^2E^2] \\ &= (-1)^{(|B^2||E^1|)}(\mathbf{N}^1[B^1E^1])(\mathbf{N}^2[B^2E^2]) \\ &= (-1)^{(|B^2||E^1|)}(\mathbf{N}^1/E^1[B^1])(\mathbf{N}^2/E^2[B^2]) \\ &= (-1)^{(|B^2||E^1|)}(\mathbf{N}^1/E^1 \wedge \mathbf{N}^2/E^2)[B^1B^2]\end{aligned}$$

Since this calculation relevant only for $|B^2| = r\mathbf{N}^2 - |E^2|$, (13) follows. □

Proposition 3.3. *Let S be the ground set for $\mathbf{N}$. Let X be a sequenced subset of S. Let S' be any sequencing of $S\backslash X$.*

$$(\mathbf{N}\backslash X)^\perp = \epsilon(S')\epsilon(S'X)(\mathbf{N}^\perp/X) \tag{14}$$

$$(\mathbf{N}/X)^\perp = \epsilon(S')\epsilon(S'X)(-1)^{|X|r\mathbf{N}^\perp}(\mathbf{N}^\perp\backslash X) \tag{15}$$

Proof. Apply the definitions. These dualization formulas are well-defined since they are unchanged over different sequencings of S'. □

Proposition 3.4. *For $i = 1,2$, let S^i be sequenced ground sets for $\mathbf{N}^i$, pairwise disjoint.*

$$(\mathbf{N}^1 \wedge \mathbf{N}^2)^\perp = \epsilon(S^1)\epsilon(S^2)\epsilon(S^1S^2)(-1)^{(r{\mathbf{N}^1}^\perp)(r\mathbf{N}^2)}{\mathbf{N}^1}^\perp \wedge {\mathbf{N}^2}^\perp \tag{16}$$

Proof. Calculate componentwise, for any orderings $B_i \subseteq S_i$ and $\overline{B_i} \subseteq S_i \setminus B_i$:

$$\begin{aligned}(\mathbf{N}^1 \wedge \mathbf{N}^2)^\perp[B_1B_2] =&(\mathbf{N}^1 \wedge \mathbf{N}^2)[\overline{B_1}\ \overline{B_2}]\epsilon(\overline{B_1}\ \overline{B_2}B_1B_2) \\ =&\mathbf{N}^1[\overline{B_1}]\mathbf{N}^2[\overline{B_2}]\epsilon(\overline{B_1}\ \overline{B_2}B_1B_2) \\ =&\{{\mathbf{N}^1}^\perp[B_1]\epsilon(\overline{B_1}B_1)\}\{{\mathbf{N}^2}^\perp[B_2]\epsilon(\overline{B_2}B_2)\}\epsilon(\overline{B_1}\ \overline{B_2}B_1B_2) \\ =&({\mathbf{N}^1}^\perp \wedge {\mathbf{N}^2}^\perp)[B_1B_2]\epsilon(\overline{B_1}B_1)\epsilon(\overline{B_2}B_2)\epsilon(\overline{B_1}\ \overline{B_2}B_1B_2)\end{aligned}$$

Since $|\overline{B_2}| = r\mathbf{N}^2$ and $|B_1| = r{\mathbf{N}^1}^\perp$, $\epsilon(\overline{B_1}\ \overline{B_2}B_1B_2) = (-1)^{r{\mathbf{N}^1}^\perp r\mathbf{N}^2}\epsilon(\overline{B_1}B_1\ \overline{B_2}B_2)$. (16) follows since, for $i = 1,2$, simutaneously permuting the two appearances of $\overline{B_i}B_i$ into S^i in the combined sign expresssion does not change its sign. □

Remark: The sign $\epsilon(S^1)\epsilon(S^2)\epsilon(S^1S^2)$ may seem strange. The three ϵ's are functions of sequences from three different sets. They are used in defining $\perp$ of $\mathbf{N}^1$, $\mathbf{N}^2$ and $\mathbf{N}^1 \wedge \mathbf{N}^2$ respectively. The formula must connect three different positively oriented unit volumes used to define Hodge star in three different exterior algebras.

3.1. Tutte Identity Theorems.

Theorem 3.5. *Given sequenced P and E, for all $e \in E$ and sequenced $E' = E \setminus e$,*

$$\epsilon(PE)\mathbf{L}_E\left(\mathbf{N}_\alpha; \mathbf{N}_\beta\right) = \epsilon(PE')\left(g_e\mathbf{L}_{E'}\left(\mathbf{N}_\alpha/e; \mathbf{N}_\beta/e\right) + r_e\mathbf{L}_{E'}\left(\mathbf{N}_\alpha\backslash e; \mathbf{N}_\beta\backslash e\right)\right) \tag{17}$$

Proof. Within each of the two factors in

$$\mathbf{L} := \mathbf{L}\begin{pmatrix}\mathbf{N}_\alpha\\ \mathbf{N}_\beta^\perp\end{pmatrix} = \iota_G(\mathbf{N}_\alpha)\wedge \upsilon_R(\mathbf{N}_\beta^\perp)$$

let us group the terms according to those that don't and those that do contain $\mathbf{e}$ as a factor. Since for the given e, $\upsilon_R(\mathbf{e}) = r_e\mathbf{e}$, $\iota_G(\mathbf{e}) = g_e\mathbf{e}$,

$$\mathbf{L} = \{\iota_G(\mathbf{N}_\alpha\backslash e) + (g_e\iota_G(\mathbf{N}_\alpha/e))\mathbf{e}\} \ \bigwedge\ \left\{\upsilon_R(\mathbf{N}_\beta^\perp\backslash e) + (r_e\upsilon_R(\mathbf{N}_\beta^\perp/e))\mathbf{e}\right\}$$

Each factor has the form $\{\mathbf{X}+\mathbf{Y}\mathbf{e}\}$ where $\mathbf{X}$ (and $\mathbf{Y}$) are not multiples of $\mathbf{e}$, so

$$\mathbf{L} = r_e\left\{(\iota_G(\mathbf{N}_\alpha\backslash e))\wedge(\upsilon_R(\mathbf{N}_\beta^\perp/e))\mathbf{e}\right\} + g_e\left\{(\iota_G(\mathbf{N}_\alpha/e))\mathbf{e})\wedge(\upsilon_R(\mathbf{N}_\beta^\perp\backslash e)\right\} + \mathbf{J}$$

where $\mathbf{J}/E = 0$. The sign change from commuting $\mathbf{e}$ with $(\upsilon_R(\mathbf{N}_\beta^\perp\backslash e)$ gives

$$\begin{aligned}\mathbf{L} = \Big(r_e \qquad\qquad &\iota(\mathbf{N}_\alpha\backslash e)\wedge(\upsilon(\mathbf{N}_\beta^\perp/e))\wedge\mathbf{e}\\ +g_e(-1)^{r(\mathbf{N}_\beta^\perp)}&(\iota(\mathbf{N}_\alpha/e))\wedge(\upsilon(\mathbf{N}_\beta^\perp\backslash e))\wedge\mathbf{e}\Big) + \mathbf{J}.\end{aligned}$$

With sets E (and $E' = E\setminus e$ used below) given by sequences which are part of the theorem's data, we contract E to get

$$\mathbf{L}_E = \mathbf{L}/E = r_e\left(\mathbf{L}\begin{pmatrix}\mathbf{N}_\alpha\backslash e\\ \mathbf{N}_\beta^\perp/e\end{pmatrix}\wedge\mathbf{e}/E\right) + g_e(-1)^{r(\mathbf{N}_\beta^\perp)}\left(\mathbf{L}\begin{pmatrix}\mathbf{N}_\alpha/e\\ \mathbf{N}_\beta^\perp\backslash e\end{pmatrix}\wedge\mathbf{e}/E\right).$$

On dualizing with (14) and (15)

$$\mathbf{N}_\beta{}^\perp/e = \epsilon(S')\epsilon(S'e)(\mathbf{N}_\beta\backslash e)^\perp$$

$$\mathbf{N}_\beta{}^\perp\backslash e = \epsilon(S')\epsilon(S'e)(-1)^{|\{e\}|r\mathbf{N}\beta^\perp}(\mathbf{N}_\beta/e)^\perp,$$

we get

$$\mathbf{L}_E = \epsilon(S')\epsilon(S'e)\left[\mathbf{L}'\wedge e/E\right], \tag{18}$$

where the abbreviation

$$\mathbf{L}' = \left[r_e\mathbf{L}\begin{pmatrix}\mathbf{N}_\alpha\backslash e\\ (\mathbf{N}_\beta\backslash e)^\perp\end{pmatrix} + g_e\mathbf{L}\begin{pmatrix}\mathbf{N}_\alpha/e\\ (\mathbf{N}_\beta/e)^\perp\end{pmatrix}\right].$$

S' is an arbitrary ordering of $P\coprod(E\backslash e)$ introduced when we dualized $\mathbf{N}_\beta\backslash e$ and $\mathbf{N}_\beta/e$; the results are independent of this ordering. We use the same ordering for both dualizations. Our concluding step will show that the sign corrections in (17) depend only on the orderings of P, E and E' from the theorem's data, that is we must show (18) $\mathbf{L}_E = \epsilon(S')\epsilon(S'e)(\mathbf{L}'\wedge\mathbf{e})/E$ implies (17).

From (12), permuting E to $E'e$ changes the sign by $\epsilon(E)\epsilon(E'e)$, so

$$\mathbf{L}_E = \epsilon(S')\epsilon(S'e)\epsilon(E)\epsilon(E'e)(\mathbf{L}'\wedge\mathbf{e})/(E'e).$$

Since the ordering S' is arbitrary, take $S' = PE'$, to get

$$\mathbf{L}_E = \epsilon(PE')\epsilon(PE'e)\epsilon(E)\epsilon(E'e)(\mathbf{L}'\wedge\mathbf{e})/(E'e).$$

We now simulaneously permute the two appearances of $E'e$ to E and verify ((17):

$$\mathbf{L}_E = \epsilon(PE')\epsilon(PE)\epsilon(E)\epsilon(E)(\mathbf{L}'\wedge\mathbf{e})/(E'e) = \epsilon(PE')\epsilon(PE)(\mathbf{L}'\wedge\mathbf{e})/(E'e).$$

□

THEOREM 3.6. *Given, for $i = 1,2$, $\mathbf{N}_\alpha^i$, $\mathbf{N}_\beta^i$, ${\mathbf{N}_\beta^i}^\perp$ and sequenced $E = E^1\coprod E^2$, P^i, E^i,*

$$\begin{gathered}\epsilon(P^1P^2E)\mathbf{L}_E\left((\mathbf{N}_\alpha^1\wedge\mathbf{N}_\alpha^2);(\mathbf{N}_\beta^1\wedge\mathbf{N}_\beta^2)\right) =\\ \epsilon(P^1E^1)\epsilon(P^2E^2)\left(\mathbf{L}_{E^1}\left(\mathbf{N}_\alpha^1;\mathbf{N}_\beta^1\right)\wedge\mathbf{L}_{E^2}\left(\mathbf{N}_\alpha^2;\mathbf{N}_\beta^2\right)\right)\end{gathered} \tag{19}$$

Proof. Using (16),

$$\begin{aligned}\mathbf{L} :=& \mathbf{L}\begin{pmatrix}\mathbf{N}^1_\alpha \wedge \mathbf{N}^2_\alpha \\ (\mathbf{N}^1_\beta \wedge \mathbf{N}^2_\beta)^\perp\end{pmatrix} = \iota_G(\mathbf{N}^1_\alpha \wedge \mathbf{N}^2_\alpha) \wedge \left(\upsilon_R({\mathbf{N}_\beta}^1 \wedge {\mathbf{N}_\beta}^2)\right)^\perp \\ =& \iota_G(\mathbf{N}^1_\alpha) \wedge \iota_G(\mathbf{N}^2_\alpha) \wedge \upsilon_R({\mathbf{N}^1_\beta}^\perp) \wedge \upsilon_R({\mathbf{N}^2_\beta}^\perp)\epsilon(S^1)\epsilon(S^2)\epsilon(S^1S^2)(-1)^{(r{\mathbf{N}^1_\beta}^\perp)(r{\mathbf{N}_\beta}^2)} \\ =& \iota_G(\mathbf{N}^1_\alpha) \wedge \upsilon_R({\mathbf{N}^1_\beta}^\perp) \wedge \iota_G(\mathbf{N}^2_\alpha) \wedge \upsilon_R({\mathbf{N}^2_\beta}^\perp)\epsilon(S^1)\epsilon(S^2)\epsilon(S^1S^2)\end{aligned}$$

since $(-1)^{(r{\mathbf{N}^1_\beta}^\perp)(r{\mathbf{N}_\beta}^2)}\mathbf{N}^2_\alpha {\mathbf{N}^1_\beta}^\perp = {\mathbf{N}^1_\beta}^\perp \mathbf{N}^2_\alpha$. By (12) $\mathbf{L}/E = \epsilon(E)\epsilon(E^1E^2)\mathbf{L}/E^1E^2$, so

$$\mathbf{L}/E = \epsilon(E)\epsilon(E^1E^2)\epsilon(S^1)\epsilon(S^2)\epsilon(S^1S^2)\left(\iota_G(\mathbf{N}^1_\alpha) \wedge \upsilon_R({\mathbf{N}^1_\beta}^\perp) \wedge \iota_G(\mathbf{N}^2_\alpha) \wedge \upsilon_R({\mathbf{N}^2_\beta}^\perp)\right)/E^1E^2$$

$$= \epsilon(E)\epsilon(E^1E^2)\epsilon(S^1)\epsilon(S^2)\epsilon(S^1S^2)(-1)^{(r\mathbf{N}^2-|E^2|)(|E^1|)}\left(\mathbf{L}_{E^1}\left(\mathbf{N}^1_\alpha;\mathbf{N}^1_\beta\right) \wedge \mathbf{L}_{E^2}\left(\mathbf{N}^2_\alpha;\mathbf{N}^2_\beta\right)\right)$$

by (13) so we are done except to verify the sign correction. With $S^i = P^iE^i$ the sign is

$$\begin{aligned}&\epsilon(E)\epsilon(E^1E^2)\epsilon(P^1E^1)\epsilon(P^2E^2)\epsilon(P^1E^1P^2E^2)(-1)^{(r\mathbf{N}^2-|E^2|)(|E^1|)} \\ &\quad = \epsilon(E)\epsilon(E^1E^2)\epsilon(P^1E^1)\epsilon(P^2E^2)\epsilon(P^1P^2E^1E^2).\end{aligned}$$

We now obtain correction of (19) by simultaneously permuting the two occurrances of E^1E^2 to E:

$$\epsilon(E)^2\epsilon(P^1E^1)\epsilon(P^2E^2)\epsilon(P^1P^2E)$$

□

This corollary is very helpful for calculations.

COROLLARY 3.7. *Suppose all elements that could be included in sets we denote by P or E are linearly ordered, elements that can be in P are each before elements that can be in E, and each subset of such elements is considered to be sequenced in increasing order. With E, P, $\mathbf{N}_\alpha$, $\mathbf{N}_\beta$ and $e \in E \setminus P$ as above*

$$\mathbf{L}_E\left(\mathbf{N}_\alpha;\mathbf{N}_\beta\right) = g_e\mathbf{L}_{E\setminus e}\left(\mathbf{N}_\alpha/e;\mathbf{N}_\beta/e\right) + r_e\mathbf{L}_{E\setminus e}\left(\mathbf{N}_\alpha\backslash e;\mathbf{N}_\beta\backslash e\right) \tag{20}$$

Proof. Use for ϵ the orientation for which every increasing sequence is positive. □

4. Towards a Tutte Polynomial

An easy consequence of Theorem 3.5 is that $\mathbf{L}_E(\mathbf{N}_\alpha;\mathbf{N}_\beta)$ has an expansion over the common bases of the matroids represented by $\mathbf{N}_\alpha$ and $\mathbf{N}_\beta$.

DEFINITION 4.1. For $A \subseteq E$, the abbreviation $\mathbf{N}/A|P = \mathbf{N}/A\backslash\overline{A}$,
where $\overline{A} = (S(\mathbf{N}) \setminus P) \setminus A$. The analgous abbreviation for matroids is
$\mathcal{N}/A|P = \mathcal{N}/A\backslash\overline{A}$.

Notice that in the expansion below, the only non-zero terms in the sums are those for which A is a common basis in the matroids represented by $\mathbf{N}_\alpha/Q_\alpha$ and $\mathbf{N}_\beta/Q_\beta$. Because of our definition of contraction, if Q_α, Q_β are not independent sets in the matroids of $\mathbf{N}_\alpha, \mathbf{N}_\beta$ respectively, the term will be zero.

COROLLARY 4.2. *Given equal length sequences $Q_\alpha \subseteq P$, $Q_\beta \subseteq P$, let $\overline{Q_\beta} = P \setminus Q_\beta$.*

(21)

$$\begin{aligned}\epsilon(Q_\beta\overline{Q_\beta})\epsilon(PE)\mathbf{L}_E\left(\mathbf{N}_\alpha;\mathbf{N}_\beta\right)[Q_\alpha\overline{Q_\beta}] =& \epsilon(P)\sum_{A\subseteq E}(\mathbf{N}_\alpha/A|P)[Q_\alpha](\mathbf{N}_\beta/A|P)[Q_\beta]g_Ar_{\overline{A}} \\ =& \epsilon(P)\sum_{A\subseteq E}\mathbf{N}_\alpha[Q_\alpha A]\mathbf{N}_\beta[Q_\beta A]g_Ar_{\overline{A}}.\end{aligned}$$

Proof. By induction on (19),

$$\mathbf{L}_E\left(\mathbf{N}_\alpha;\mathbf{N}_\beta\right)[Q_\alpha\overline{Q_\beta}] = \epsilon(P)\epsilon(PE)\sum_{A\subseteq E}\mathbf{L}_\varnothing\left(\mathbf{N}_\alpha/A|P;\mathbf{N}_\beta/A|P\right)[Q_\alpha\overline{Q_\beta}]g_A r_{\overline{A}}.$$

By definitions 1.3 and 1.2,

$$\mathbf{L}_\varnothing\left(\mathbf{N}_\alpha/A|P;\mathbf{N}_\beta/A|P\right)[Q_\alpha\overline{Q_\beta}] = (\mathbf{N}_\alpha/A|P)[Q_\alpha](\mathbf{N}_\beta/A|P)^\perp[\overline{Q_\beta}].$$

Dualization $(\mathbf{N}_\beta/A|P)^\perp[\overline{Q_\beta}] = (\mathbf{N}_\beta/A|P)[Q_\beta]\epsilon(Q_\beta\overline{Q_\beta})$ gives the first equation. Our definition of contraction gives the second. ☐

Here we show: $\mathbf{L}_E$ is an evaluation of a common generalization of parametrized Tutte polynomials, relative Tutte polynomials and linking (matroid pair Tutte) polynomials. Based on these origins, it is straightforward to demonstrate the Tutte identities this generalization satisfies (Theorem 4.4.) This Tutte polynomial has matroid variables, and the ones that appear in the polynomial for a given pair of matroids are minors of those matroids. However, when the matroid minors in the generalization are replaced by exterior algebra minors, the Tutte equations fail to be satisfied. The only situation where those Tutte equations are satisfied is when the corank and nullity variables are set to zero and the matroids are unimodular (i.e. regular).

Let $\mathcal{N}$ be a matroid. In the following, $\boxed{\mathcal{N}}$ will denote the product of the connected components of $\mathcal{N}$, each regarded as a variable.

We will use the term "P-ported" for what has been called "set-pointed" and "relative" Tutte functions and polynomial. The following combines known definitions:

1. Las Vergnas' "big" Tutte polynomial of a P-ported matroid. Our references[24, 25, 8, 10, 12] document that all the fundamental features (expansions, universality, etc.) of Tutte function theory extend to porting/set-pointing/relativization.
2. "Normal" parametrized Tutte functions which are those that have rank-nullity generating function interpretations, according to classifications given by Zaslavsky[28], and Bollobas and Riordan[4], and unified by Monaghan and Traldi[14].
3. Welsh and Kayibi's linking polynomial of a matroid pair[27].

Definition 4.3.

$$\begin{aligned}
&L_E(\mathcal{N}_\alpha,\mathcal{N}_\beta;\dots \text{ matroid variables },\dots;v_\alpha,w_\alpha,v_\beta,w_\beta)\\
&\quad=\sum_{A\subseteq E}\boxed{\mathcal{N}_\alpha/A|P}\ \boxed{\mathcal{N}_\beta/A|P}\ g_A r_{\overline{A}}\\
&\qquad\qquad v_\alpha^{r_\alpha(\mathcal{N}_\alpha)-r_\alpha(\mathcal{N}_\alpha/A|P)-r_\alpha(A)}w_\alpha^{|A|-r_\alpha(A)}\\
&\qquad\qquad v_\beta^{r_\beta(\mathcal{N}_\beta)-r_\beta(\mathcal{N}_\beta/A|P)-r_\beta(A)}w_\beta^{|A|-r_\beta(A)}\\
&\quad=\sum_{A\subseteq E}\boxed{\mathcal{N}_\alpha/A|P}\ \boxed{\mathcal{N}_\beta/A|P}\ g_A r_{\overline{A}}\\
&\qquad\qquad v_\alpha^{r_\alpha(\mathcal{N}_\alpha)-r_\alpha(P\cup A)}w_\alpha^{|A|-r_\alpha(A)}\\
&\qquad\qquad v_\beta^{r_\beta(\mathcal{N}_\beta)-r_\beta(P\cup A)}w_\beta^{|A|-r_\beta(A)}.
\end{aligned}\tag{22}$$

Welsh and Kayibi's linking polynomial is obtained by taking $P=\varnothing$ and all $r_e = g_e = 1$ in the above. (Also, they use $x-1=v_\alpha$, $y-1=w_\alpha$, $u-1=v_\beta$ and $v-1=w_\beta$.)

The feature of normal Tutte polynomials manifest in this definition is that our variables v_α, v_β, w_α, and w_β are independent of e. The cited references prove that the coloop and loop values for $e\notin P$ must be evaluations of our definitions' expressions with these four variables in order for the polynomial to be a Tutte function (i.e., satisfy the parametrized Tutte equations over a class of matroids and parameters) *and* have a corank-nullity expansion.

THEOREM 4.4.

e *in* N_α	e *in* N_β	$L_E(\mathcal{N}_\alpha, \mathcal{N}_\beta) =$
non-separator	*non-separator*	$g_e L_{E'}(\mathcal{N}_\alpha/e, \mathcal{N}_\beta/e) + r_e L_{E'}(\mathcal{N}_\alpha\backslash e, \mathcal{N}_\beta\backslash e)$
non-separator	*coloop*	$g_e L_{E'}(\mathcal{N}_\alpha/e, \mathcal{N}_\beta/e) + r_e v_\beta L_{E'}(\mathcal{N}_\alpha\backslash e, \mathcal{N}_\beta\backslash^\dagger e)$
non-separator	*loop*	$g_e w_\beta L_{E'}(\mathcal{N}_\alpha/e, \mathcal{N}_\beta/^\dagger e) + r_e L_{E'}(\mathcal{N}_\alpha\backslash e, \mathcal{N}_\beta\backslash e)$
coloop	*non-separator*	$g_e L_{E'}(\mathcal{N}_\alpha/e, \mathcal{N}_\beta/e) + r_e v_\alpha L_{E'}(\mathcal{N}_\alpha\backslash^\dagger e, \mathcal{N}_\beta\backslash e)$
loop	*non-separator*	$g_e w_\alpha L_{E'}(\mathcal{N}_\alpha/^\dagger e, \mathcal{N}_\beta/e) + r_e L_{E'}(\mathcal{N}_\alpha\backslash e, \mathcal{N}_\beta\backslash e)$
coloop	*coloop*	$g_e L_{E'}(\mathcal{N}_\alpha/e, \mathcal{N}_\beta/e) + r_e v_\alpha v_\beta L_{E'}(\mathcal{N}_\alpha\backslash^\dagger e, \mathcal{N}_\beta\backslash^\dagger e)$ $= (r_e v_\alpha v_\beta + g_e) L_{E'}(\mathcal{N}_\alpha - e, \mathcal{N}_\beta - e)$
coloop	*loop*	$g_e w_\beta L_{E'}(\mathcal{N}_\alpha/e, \mathcal{N}_\beta/^\dagger e) + r_e v_\alpha L_{E'}(\mathcal{N}_\alpha\backslash^\dagger e, \mathcal{N}_\beta\backslash e)$ $= (r_e v_\alpha + g_e w_\beta) L_{E'}(\mathcal{N}_\alpha - e, \mathcal{N}_\beta - e)$
loop	*coloop*	$g_e w_\alpha L_{E'}(\mathcal{N}_\alpha/^\dagger e, \mathcal{N}_\beta/e) + r_e v_\beta L_{E'}(\mathcal{N}_\alpha\backslash e, \mathcal{N}_\beta\backslash^\dagger e)$ $= (r_e v_\beta + g_e w_\alpha) L_{E'}(\mathcal{N}_\alpha - e, \mathcal{N}_\beta - e)$
loop	*loop*	$g_e w_\alpha w_\beta L_{E'}(\mathcal{N}_\alpha/^\dagger e, \mathcal{N}_\beta/^\dagger e) + r_e L_{E'}(\mathcal{N}_\alpha\backslash e, \mathcal{N}_\beta\backslash e)$ $= (r_e + g_e w_\alpha w_\beta) L_{E'}(\mathcal{N}_\alpha - e, \mathcal{N}_\beta - e)$

For $e \in E$, $\mathcal{N}\backslash^\dagger e$ *and* $\mathcal{N}/^\dagger e$ *denote that* e *is a separator in* $\mathcal{N}$. *In that case,* $\mathcal{N}\backslash e = \mathcal{N}/e$ *is also denoted by* $\mathcal{N} - e$. *Furthermore,* $\mathcal{N}/(A\backslash e)|P = \mathcal{N}/(A \cup e)|P = \mathcal{N}/A|P$.

Proof. Apply straightforward adaptions of proofs given by the aforementioned authors, which use the following elementary properties of separators e. Let $e \notin A$. When e is a coloop,

$$r(\mathcal{N}) - r_{\mathcal{N}}(A \cup e \cup P) = r(\mathcal{N} - e) - r_{\mathcal{N}-e}(A \cup P)$$
$$r(\mathcal{N}) - r_{\mathcal{N}}(A \cup P) = 1 + r(\mathcal{N} - e) - r_{\mathcal{N}-e}(A \cup P)$$
$$|A \cup e| - r_{\mathcal{N}}(A \cup e \cup P) = |A| - r_{\mathcal{N}}(A \cup P) = |A| - r_{\mathcal{N}-e}(A \cup P).$$

When e is a loop,

$$r(\mathcal{N}) - r_{\mathcal{N}}(A \cup e \cup P) = r(\mathcal{N}) - r_{\mathcal{N}}(A \cup P) = r(\mathcal{N} - e) - r_{\mathcal{N}-e}(A \cup P)$$

and

$$|A \cup e| - r_{\mathcal{N}}(A \cup e \cup P) = 1 + |A| - r_{\mathcal{N}-e}(A \cup P)$$
$$|A| - r_{\mathcal{N}}(A \cup P) = |A| - r_{\mathcal{N}-e}(A \cup P).$$

□

4.1. COUNTEREXAMPLE. It is tempting to investigate whether the expansion identities of Theorem 4.4 are satisfied, when the matroid minor operations are replaced by our exterior algebra minor operations, by some function of exterior algebra pairs $(\mathbf{N}_\alpha, \mathbf{N}_\beta)$. After all, Theorem 3.5 tells us the identities are satisfied for $\mathbf{L}_E(\mathbf{N}_\alpha; \mathbf{N}_\beta)$ when we set $v_\alpha = v_\beta = w_\alpha = w_\beta = 0$. The simplest example, where $P = \varnothing$, reveals the answer is no.

Let $\mathbf{N}_\alpha = a\mathbf{e}_1 + b\mathbf{e}_2$ and $\mathbf{N}_\beta = c\mathbf{e}_1 + d\mathbf{e}_2$. Expansion in the order of e_1, e_2 gives the result

$$g_1 g_2 w_\alpha w_\beta ac + g_1 r_2 ac + r_1 g_2 bd + r_1 r_2 v_\alpha v_\beta bd$$

but in the order e_2, e_1 the result is

$$g_1 g_2 w_\alpha w_\beta bd + g_1 r_2 ac + r_1 g_2 bd + r_1 r_2 v_\alpha v_\beta ac$$

As expected, the results are equal when the w's and v's are set to zero. Equality holds iff $ac = bd$, so it fails when $\mathbf{N}\alpha = \mathbf{e_1} + \mathbf{e}_2$ and $\mathbf{N}\alpha = \mathbf{e_1} - \mathbf{e}_2$, which are unimodular matroid representations. It also fails for $\mathbf{N}_\alpha = \mathbf{N}_\beta = \mathbf{e}_1 + 2\mathbf{e}_2$.

4.2. When $\mathbf{L}_E$ is a Tutte polynomial evaluation. When the exterior algebra elements $\mathbf{N}_\alpha$ and $\mathbf{N}_\beta$ are unimodular representations of (oriented) matroids matroids, we can take the matroid variables to encode oriented matroid representations by chirotopes χ in which one of chirotopes $\pm\chi$ is distinguished. Then, we can identify these oriented matroid representations with the above $\mathbf{N}_\alpha$ and $\mathbf{N}_\beta$. In this situation, $\mathbf{L}_E(\mathbf{N}_\alpha; \mathbf{N}_\beta)$ is the result of substituting $\mathbf{N}_\alpha/A|P$ and $\mathbf{N}_\beta/A|P$ for the matroid variables in 22, together with setting the v's and w's to zero.

5. Discussion and Summary

Among the many currently active research subjects are algebraic theories for variations of Tutte decomposition and functions, and convolutions for Tutte functions of various compositions of objects. For example, Krajewski, Moffatt and Tanasa give a common Hopf algebra framework for the deletion and contraction operations for many of these variations[17]. This approach was generalized and further systemized by Dupont, Fink and Moci[13]. See also Crapo and Schmitt's work on the Hopf algebra of matroids and call for attention to "naturally occurring algebraic structures" in matroid theory[11].

Our motivating linear algebra construction, applied to electrical netwoks, and its the matroid abstraction doesn't appear to be addressed in the above cited literature. Current research along these lines is in Laplacians and their determinants of simplicial complexes of dimensions greater than 1.

We see our contribution is to Tutte functions of objects like these when they have ports.

One of the first applications of combinatorial enumeration appeared in a little known 1847 paper by Kirchhoff[16] in which a solution was given in terms of tree enumerations, and proven by a graphic matroid basis exchange argument, rather than by the matrix tree theorem. Our linear algebra construction and its application to electrical networks was studied early on by Bott and Duffin's[5] study of the "constrained inverse". When the g_e, r_e parameters are algebraically independent, $\mathbf{L}\begin{pmatrix}\mathbf{N}_\alpha\\ \mathbf{N}_\beta^\perp\end{pmatrix}$1.3 represents a minor of a matroid union– a discusson that relates this to the constrained inverse appears in [7]. A product formula for the P-ported Tutte polynomial of a matroid union appears in [8].

The target for us is exterior algebras, which occur naturally when problems of partial solution of systems of linear equations[**?**], i.e., Schur complements, are studied from a matroid theory, sometimes oriented, point of view. In exterior algebra, unlike commutative algebras, linear subspaces (as matroid representations and solutions to systems of homogeneous linear equations) are represented by exterior products of vectors; also called *decomposible* elements, *decomposibles*, or "pure" antisymmetric tensors $\mathbf{N}$. When $\mathbf{N}$ is expanded in terms of the basis generated by the elements corresponding to a matroid ground set, the result is a basis enumeration.

Our motivating example of such subspaces occur in problems posed from graphs that generalize inversion of graph Laplacians. These are defined in terms of linear resistive linear electrical networks, under Kirchhoffs' and Ohm's laws.

An examination shows that our notion of $\mathbf{N}$ and definitions $\mathbf{N}^{\perp}$ and $\mathbf{L}_E$ make sense even when $\mathbf{N}$ is a arbitrary element of the exterior algebra, not necessarilly a pure element. Furthermore, the properties and Tutte identity theorems hold in this wider context. Of course, non-pure elements do not represent subspaces, so our matroid representation and subspace interpretions no longer apply. On the other hand, a chirotope of any oriented matroid can be encoded by the sum over the bases of an ordered product of the basis elements.

The oriented matroid abstraction of objects like $\mathbf{N}$ and values of $\mathbf{L}_E$ (when K_0 is ordered) is an oriented matroid with one of its two chirotopes $\pm\chi$ distinguished; that distinguished chirotope is $\chi(B) = \operatorname{sign}(\mathbf{N}[B])$.

Consider definition (1.3) and Theorems (3.5,3.6) restricted to totally unimodular representations of unimodular (i.e., regular) oriented matroids with a distinguished chirotope. In this situation, $\chi(B) = \mathbf{N}[B] \in \{0, \pm 1\}$. So restricted, these definitions and theorems constitute a special case of a common generalization of three Tutte polynomial variations: set-pointed/relative/ported Tutte polynomial Tutte polynomials, Bollobos and Riordan's parameterized or colored Tutte polynomials[4], and Welsh and Kayibi's linking polynomial of a matroid pair[27]. The idea of set-pointed Tutte polynomials was likely first recoginzed by Las Vergnas[24] and was subsequently rediscovered and developed[8, 25, 12]. The common feature is an enumeration of the common bases in a pair of oriented matroids. Definition (1.3) is a function between exterior algebras, but the three variations give polynomial functions of matroids. In our special case, (1.3) is a evaluation of this common generalization polynomial.

For general $\mathbf{N}$ and the matroids they represent, given two indepentent sets A and A', $\mathbf{N}/A|P$ and $\mathbf{N}/A'|P$ might represent the same matroid but may be different decomposibles and even represent different elements of Grassmannians of K^P. Then there is no map from matroids to decomposibles whose evaluation on the ported Tutte polynomial gives the value of $\mathbf{L}_E(\mathbf{N};\mathbf{N})$. The basis expansion of $\mathbf{L}_E$ for general $\mathbf{N}_\alpha$, $\mathbf{N}_\beta$ we do have is essentially the Cauchy-Binet theorem:

Las Vergnas defined the big Tutte polynomial[] of a set-pointed matroid as follows, in our notation[] where we called the matroid "ported" because of our application to electrical networks. Each connected matroid is considered to be a variable and the symbol $\boxed{\mathcal{N}}$ denotes the product of the connected components of matroid $\mathcal{N}$.

$$r_P(\mathcal{N}) = \sum_{A \subseteq E} \boxed{\mathcal{N}/A|P}\, V^{r(\mathcal{N})-r(\mathcal{N}/A|P)-r(X)} W^{|A|-r(A)} \tag{23}$$

Welsh and Kayibi[27] defined (our notation):

Definition 5.1. Let $\mathcal{N}_\alpha, \mathcal{N}_\beta$ be matroids on S with rank functions r_α and r_β respectively. Define the 4-variable polynomial Q by

$$\begin{aligned} Q(\mathcal{N}_\alpha, \mathcal{N}_\beta; x, y, u, v) = \sum_{A \subseteq S} & (x-1)^{r_\alpha(S)-r_\alpha(A)}(y-1)^{|A|-r_\alpha(A)} \\ & (u-1)^{r_\beta(S)-r_\beta(A)}(v-1)^{|A|-r_\beta(A)} \end{aligned} \tag{24}$$

Let us combine these constructions and obtain the polynomial below whose variables include the parameters g_e, r_e for $e \in E$:

Tutte functions are a well-known, long-standing contstruction for which new applications, variations and generalizations continue to be active research subjects. For example, Krajewski, Moffatt and Tanasa give a common Hopf algebra framework for the deletion and contraction operations for many of these variations[17]. This approach was generalized and further systemized by Dupont, Fink and Moci[13]. See also Crapo and Schmitt's work on the Hopf algebra of matroids and call for attention to "naturally occurring algebraic structures" in matroid theory[11]. The target for us is exterior algebras, which occur naturally when problems of partial solution of systems of linear equations[**?**], i.e., Schur complements, are studied from a matroid theory, sometimes oriented, point of view. In exterior algebra, unlike commutative algebras, linear subspaces (as matroid representations and solutions to systems of homogeneous linear equations) are represented by exterior products of vectors; also called *decomposible* elements, *decomposibles*, or "pure" antisymmetric tensors $\mathbf{N}$. When $\mathbf{N}$ is expanded in terms of the basis generated by the elements corresponding to a matroid ground set, the result is a basis enumeration. Our motivating example of such subspaces occur in problems posed from graphs that generalize inversion of graph Laplacians. These are defined in terms of linear resistive linear electrical networks, under Kirchhoffs' and Ohm's laws. One of the first applications of combinatorial enumeration appeared in a little known 1847 paper by Kirchhoff[16] in which a solution was given in terms of tree enumerations, and proven by a graphic matroid basis exchange argument, rather than by the matrix tree theorem.

To connect this with Tutte decomposition, we make minor notational variations to the machinery of exterior algebra, also called Grassmann algebra, as presented by Marcus[18]. First, we specify the particular basis S so that a decomposible in the exterior algebra generated by S can represent a matroid with ground set S. Second, since we study the solution space projected into a subspace of distinguished coordinates, we specify a distinguished subset $P \subseteq S$ and our Tutte function will be valued in exterior algebras generated by sets related to P. Third, as we will see, since dualization by Hodge star is needed, an orientation must be specified for each space. Our identities are on pseudo-forms or pseudo-vectors[15]. So corrections depending on by those orientations are required in our Tutte identities. In all, the above projection plus parametrization of electrical networks by resistance r_e and/or conductance g_e values leads us to parametrized Tutte functions with a restricted Tutte decomposition identity:

(25) $F(N) = g_e F(N/e) + r_e F(N \backslash e)$ if e can be deleted and contracted, and $e \notin P$.

Our results for single $\mathbf{N}$ were originally announced in 2006[10], but their extension to pairs is new. Beside from their being an exterior algebra special case of Welsh and Kayibi's Tutte-like "linking" polynomial of a pair of matroids[27], we summarize the electrical engineering circuit theory motivation. When taking an arbitrarilly oriented labelled multigraph for a matroid theory style model of an electrical network, one uses coordinates for voltage drops across edges and current flows through edges. Kirchhoff's laws postulate that such currents and voltages belong to (KCL) the graph's cycle space and (KVL) its orthogonal complement, respectively. However, in circuit theory applicable to electronics[20], some graph models are used of which the linear spaces for voltages and currents are not orthogonal. For example, to model an ideal (so-called operational) amplifier, one can postulate that in a graph to which Kirchhoff's laws apply, both the current and voltage of the amplifier's input edge are zero and both those quantities of the output edge are not constrained. Such models are equivalent to using two graphs (the same when there are no amplifiers) with common edge labels: The "voltage graph" and the "current graph".[(2)] The equations are comprised of KCL from the current graph, KVL from the voltage graph and Ohm's law to relate current to voltage in each resistor. We have related oriented matroid properties of such graph pairs to questions of well-posedness of electrical circuit theory problems[9]. This is the origin of our investigations of (25) where the object $\mathbf{N}$ is generalized to a *pair* $\mathbf{N}_\alpha$, $\mathbf{N}_\beta$, corresponding to the "current graph" and "voltage graph" respectively. One result is that the solution problem is well-posed for generic resistance parameters iff the two graphs have a common spanning tree. A much more familiar instance is the matrix tree theorem for directed graphs. Its proof by the Cauchy-Binet theorem reveals that

(2) These two graphs are minors of a common graph, so the pair can be considered to be a matroid perspective as studied by Las Vergnas[25].

the enumerated rooted arborescences are the common bases in the graphic matroid and the partition matroid on edges that encodes the edge head and vertex incidences. Thus (25) becomes

$$F(\mathbf{N}_\alpha \ \ \mathbf{N}_\beta) = g_e F(\mathbf{N}_\alpha/e \ \ \mathbf{N}_\beta/e) + r_e F(\mathbf{N}_\alpha \backslash e \ \ \mathbf{N}_\beta \backslash e) \tag{26}$$

if e can be deleted and contracted, and $e \notin P$.

One of the earliest Tutte functions is the basis enumerator for matroids. It is a determinant in the case of graphic, and more generally, regular matroids; this is a consequence of the Cauchy-Binet theorem. The well-known Matrix Tree theorem tells us this determinant is (up to sign) any full-rank minor of the graph's Laplacian matrix. More generally, every minor is an enumerator for forests (signed, in some non-principal minor cases) satisfying the following condition: Each vertex indexing a deleted row is in exactly one of the trees, and similary for each vertex indexing a deleted column[6]. (Each forest's sign depends on the sign of the matching the forest determines between the deleted row vertices and deleted column vertices; of course when the minor is principal, the only such matching is the identity. We will see that this variation of sign is due to which oriented matroids are the result of contracting the forest edges.) We give a key observation: *Each* such minor, when e is a given edge, equals the sum of the corresponding minor when e is deleted plus the determinant expansion terms (table 10 gives examples) for forests that contain e. The latter can be also be expressed as a determinant. In the following, we will develop a formulation in which *all these minors together*, when they are constituted as an element in an exterior algebra, satisfy the well-known Tutte identities. The formulation applies to pairs of row-spaces of matrices, with rank conditions for non-triviality, and our Tutte decomposition operates on matrix columns as does (possibly oriented) matroid deletion and contraction.

In order for a tuple of determinants to be a Tutte function value, one must label them in a combinatorial way. For this purpose, we use the *set-pointed*[26, 26], also called the *relative*[12] or *ported*[8, 10] variation of Tutte functions. Let P be a set. A function F is a P-ported when the identity for deletion and contraction of e is asserted, and rules for reducing loops or coloops apply, only when $e \notin P$. P will play the role of vertices in our all-minors of the Laplacian matrix tree theorem specialization. We prefer the engineering term "port" because it eludes to both an object through which systems can be interconnected, and can be manipulated and observed by their environments[21, 19]. Engineers would call our graph theory example (**??**) a "p-port", with $p = 2$.

Let P_α and P_β be disjoint copies of P. We will see that our Tutte function's value will be an exterior product of $|P|$ vectors in the $2|P|$ dimensional vector space over K with basis $P_\alpha \cup P_\beta$. I.e., our value will have order $|P|$. As such, (when non-zero) it is a *Grassmann representitive*[18][p 18] for some dimension P subspace, a point in the Grassmannian $Gr(|P|, 2|P|)$. However, in order for Tutte's additive identity to make sense, the value cannot be a point in the $\binom{2|P|}{|P|} - 1$ dimensional projective space containing the variety $Gr(|P|, 2|P|)$. It must be in a module, which for our construction is a linear space.

A possibly novel feature is that unlike the usual commutative ring values for Tutte functions, our functions' values, being in an exterior algebra, are graded and have *anticommutative* multiplication. Hence a sign correction must be included in Tutte's multiplicative identity. Our Tutte function merely reduces to the parametrized basis enumerator obtained from setting, $u = v = 0$ in Tutte decomposition (25) with coloop e valued at $g_e + r_e u$ and loop e valued at $r_e + g_e v$. "Porting" is necessary for our approach to yield anything interesting. To find a Tutte function, like the

Tutte polynomial's Whitney rank-nullity generating function form in variables u, v, that reduces to our exterior algebra version when $u = v = 0$ is an interesting question. Our references[24, 25, 8, 10, 12] document that all the fundamental features (activities and flat expansions, universality, etc.) of Tutte function theory extend to porting/set-pointing/relativization. Throughout, we restrict our attention to the "normal" parametrized Tutte functions which, according to classifications given by Zaslavsky[28], and Bollobas and Riordan[4], and unified by Monaghan and Traldi[14]. have rank-nullity generating function interpretations. (They prove this requires the coloop and loop values be what is above, with u, v independent of e.)

Another possibly interesting feature is that, in a rather trivial way, our Tutte function can distinguish different orientations of an oriented matroid. When decomposition defined by our Tutte identities is applied to a represented oriented matroid, the irreducibles are representations of some of its minors for which their ground set only contains elements in P. Hence different orientations of the same matroid minor are distinguished. Recently, a new Tutte function variation that distinguishes properties of orientations more finely was found[1].

Indeed, our chief motivation is to give an algebraic combinatorial description of solutions to electrical network problems, where the parameters will represent the coefficient in Ohm's law. It is convenient (and elegant) to write that law in homogeneous form: For each resistor e there are two parameters r_e, g_e ("proresistance", "proconductance" apparently originated in [23] and are used in [10, 22]) such that the current and voltage drop combination of a resistor $(i_e, v_e) = x(g_e, r_e)$ for some x. In other words, when they are non-zero and not infinite, the conventional resistance is r_e/g_e and conductance is g_e/r_e(3). Shorts (zero resistance) and opens (zero conductance) are thus accomodated. To make this long story short (see section (**??**)), our Tutte function's value will encode the set of voltage drops across and current flows through each of the network edges in P that are compatible with Kirchhoff's and Ohm's laws applied to $E \cup P$. This set is the linear solution space of an underdetermined system projected into the dimension $2|P|$ space with a voltage (v_p) and a current (i_p) coordinate for each $p \in P$.

Thus, our function's value can represent a set of linear subspaces parametrized by the g_e, r_e. This opens questions for research into the topological and geometric properties of the set of those subspaces as the parameters range say over $\mathbb{R}$ or $\mathbb{C}$, say for a given graph or pair of matrices. This would be of interest to engineers because it parametrizes the behaviors of systems with the same structure as values of its parts or settings vary.

The coefficients of our Tutte function's value when it is expanded in the exterior algebra basis generated by the $2|P|$ elements of two copies of P are polynomials affine in the g_e, r_e and homogeneous in the g_e and also in the r_e. These polynomials are coefficients in the linear relations among the v_p and i_p. A more interesting connection to matroid orientation is how it affects the term signs of each coefficient polynomial, especially the relative sign of separate terms. See the example of table (10).

6. Motivation and Definition

In applications, columns of our full-row-rank matrices will be labelled by elements representing or related to matroid ground set elements, and to generators of our exterior algebras. Thus the row space is encoded by the exterior product of the row

(3) Electrical engineers sometimes call these reciprocal quantities "impedance" and "admittance" especially when they are in $\mathbb{C}$ or $\mathbb{C}(s)$.

vectors. Matroid, and, in the case of graphic matroids, graph operations will involve these elements.

For clarity and simplicity, we demonstrate this with matrices and and give the example of solving the electrical network equations for a graph.

The Tutte function is defined on pairs of decomposibles. The explicit vectors or matrices do not play a role except in applications.

It is important to recognize that the two given decomposibles, and the Tutte function's value, are decomposible exterior algebra elements, not the point in the Grassmannian whose Plücker coordinates are equivalence classes of non-zero multiples. We *do* distinguish different such multiples. Another way to put it is that our data and constructed objects are matrices modulo left multiplication by SL_r, not by GL_r the latter which gives the Grassmannian. Of course all such matrix classes surject onto the Grassmannian.

6.1. Definition of a Tutte Function of P-Ported Pairs. We need some careful formalism so we can algebraically relate the resulting Tutte function value to its values for our to-be-defined minors and direct product factors. Those will have a proper subset for E and possibly so for P.

Let $p = |P|$, $n = |E|$ and L_i be row i of L for $i \in \{1, \ldots, n+p\}$. We form $\mathbf{L}$ by exterior multiplication (in order) of the vectors each given by the rows of $(n+p) \times (n+2p)$ matrix L for the coefficients when written in terms of basis $P_\alpha \coprod P_\beta \coprod E$:

$$\mathbf{L} = \wedge_{i=1}^{n+p} L_i [\mathbf{p}_{\alpha 1} \ \cdots \ \mathbf{p}_{\alpha p} \ \mathbf{p}_{\beta 1} \ \cdots \ \mathbf{p}_{\beta_p} \ \mathbf{e}_1 \ \cdots \ \mathbf{e}_n]^t. \tag{27}$$

We can now define what will turn out to be our Tutte function. One cannot say what it means for it to be a Tutte function until we get to defining the deletion and contraction operations!

Let us fully expand $\mathbf{L}$ into an exterior polynomial in the vectors $P_\alpha \coprod P_\beta \coprod E$. We will abbreviate expressions like $(\mathbf{e_1} \wedge \mathbf{e_2} \cdots \wedge \mathbf{e_n})$ by $(\mathbf{e_1 e_2} \cdots \mathbf{e_n})$ or $\mathbf{E}$ where *set E is specified as a sequence of its elements*[(4)]. We define $\mathbf{L}_E$ by

$$\mathbf{L} = (\mathbf{L}_E) \wedge (\mathbf{e_1} \wedge \mathbf{e_2} \cdots \wedge \mathbf{e_n}) + \mathbf{J}.$$

where $\mathbf{J}$ is the sum of terms none of which is a multiple of $\mathbf{E}$. In other words, $\mathbf{L}_E = \Gamma_E(\mathbf{L})$ with homomorphism $\Gamma_E : \wedge^{(2p+n)} U \to \wedge^{(2p)} U$ defined on the basis of products of vectors corresponding to $P_\alpha \coprod P_\beta \coprod E$ to that of $P_\alpha \coprod P_\beta$ as follows:

$$\begin{aligned} \mathbf{ZE} &\longmapsto \mathbf{Z} \\ \mathbf{ZE}' &\longmapsto 0 \end{aligned}$$

where $Z \subseteq P_\alpha \coprod P_\beta$ and $E' \subsetneq E$.

Proposition 6.1. *$\mathbf{L}_E(N_\alpha, N_\beta^\perp)$ is invariant under left actions of SL_r on N_α or $N_\beta^\perp$.*

Proof. All the coefficients are products of an all row minor of N_α, an all-row minor of $N_\beta^\perp$, a sign, and parameters. By definition, the actions of SL_r will multiply those determinants by 1. ☐

We can therefore use the similarly formed exterior products (respecting order) of the rows of N_α and $N_\beta^\perp$ for our data, denote them by $\mathbf{N}_\alpha$ and $\mathbf{N}_\beta^\perp$ and justify

(4) Sometimes explicit $\wedge$ symbols are shown for notational clarity or emphasis.

Definition 6.2. Given E, P and $(\mathbf{N}_\alpha, \mathbf{N}_\beta^\perp)$ as above, $\mathbf{L}_E \begin{pmatrix} \mathbf{N}_\alpha \\ \mathbf{N}_\beta^\perp \end{pmatrix}$ is the unique exterior algebra element satisfying

$$\mathbf{L} \begin{pmatrix} \mathbf{N}_\alpha \\ \mathbf{N}_\beta^\perp \end{pmatrix} = (\mathbf{L}_E) \wedge (\mathbf{e_1} \wedge \mathbf{e_2} \cdots \wedge \mathbf{e_n}) + \mathbf{J} = (\mathbf{L}_E \begin{pmatrix} \mathbf{N}_\alpha \\ \mathbf{N}_\beta^\perp \end{pmatrix})\mathbf{E} + \mathbf{J},$$

where $\mathbf{J}$ is the sum of terms none of which is a multiple of $\mathbf{E} = \mathbf{e_1} \wedge \mathbf{e_2} \cdots \wedge \mathbf{e_n}$, when $\mathbf{L}$ is expanded using the basis generated by $P_\alpha \coprod P_\beta \coprod E$. In other words,

$$\mathbf{L}_E \begin{pmatrix} \mathbf{N}_\alpha \\ \mathbf{N}_\beta^\perp \end{pmatrix} = \Gamma_E(\mathbf{L} \begin{pmatrix} \mathbf{N}_\alpha \\ \mathbf{N}_\beta^\perp \end{pmatrix}).$$

7. Discussion

[[[Wikipedia articles on Pseudovector and Pseudotensor referred via article on Volume form may help, they give references not Frankel. Frankel explains it best!]]]

A pseudo-form[15, p 85-87] is an orientation dependent form ϕ_ϵ (that is, a map from orientations ϵ into forms) such that for all orientations ϵ, for all tuples (permutations) B of vectors from the set $B_{\text{set}} = \{b_1, b_2, \ldots, b_r\}$, $\epsilon(B)\phi_\epsilon(B)$ is constant. An example is the volume form Δ_ϵ. For any orientation ϵ we chose, if B is positively oriented and linearly independent, i.e., $\epsilon(B) = +$ then $\Delta_\epsilon(B) > 0$. Since both values change sign under an odd permutation of B, we have the (positive) geometric volume of the parallelopiped spanned by B given by the formula $\epsilon(B)\Delta_\epsilon(B)$. Frankel explains that many quantities in physics, like magnetic field $\mathbf{B}$, "don't make sense" without an orientation, like the right-hand rule, being specified.

Another way to account for the sign corrections in our Tutte identities is due to the fact each value can be considered to be a form (anti-symmetric multilinear function) on the dual space. But the value should only depend on the arguments $\mathbf{N}_\alpha$ and $\mathbf{N}_\beta$. This alone indicates that our function's value should be a *pseudo-form*, that is, be a form parametrized by orientations ϵ. Thus we recognize that our identities are sums and products of the $\epsilon(EP)\phi_\epsilon$ terms that are independent of ϵ and have the property that $\epsilon(P'_\alpha P'_\beta)\phi_\epsilon(P'_\alpha P'_\beta)$ depend only on the *sets* given by sequences P'_α and P'_β. It might be the case that Tutte functions are impossible in exterior algebra; they must be in an exterior algebra with pseudo-forms and pseudo-vectors!

7.1. Minors in Exterior Algebra.

Our definitions of minors in exterior algebra differ from the corresponding definitions in matroids in an important way that is relevant to our study of Tutte functions in exterior algebras. For simplicity, consider the minors obtained by deleting and contracting a single element. Deleting coloop e in a matroid is the same as contracting e. However in exterior algebra, the result of deleting e is 0. The value that represents the matroid result is the exterior contraction of e.

The distinction between deleting and contraction of a coloop in a matroid Tutte decomposition is encoded by multiplying by the U variable at the step where the coloop is deleted. When we try to replicate Tutte decomposion in exterior algebra, we replace deletion by contraction and multiply by the U variable. Hence $\mathbf{N}$ decomposes into $(1 + U)\mathbf{N}/e$.

Let $\mathbf{N}_\alpha$ and $\mathbf{N}_\beta$ be elements of order r in exterior algebra ... . Since a decomposible element represents a linear subspace, it is a representation of a sometimes oriented matroid. In the following, we define particular exterior algebra operations that represent matroid deletion, contraction and dualization. Dualization is defined using Hodge star, so it requires that orientations be given for each grade of each exterior algebra that the operations will be applied to. We then define the function $\mathbf{L}_E(\mathbf{N}_\alpha;\mathbf{N}_\beta)$ and prove:

$$\epsilon(PE)\mathbf{L}_E\left(\mathbf{N}_\alpha;\mathbf{N}_\beta\right)=\epsilon(PE')\left(g_e\mathbf{L}_{E'}\left(\mathbf{N}_\alpha/e;\mathbf{N}_\beta/e\right)+r_e\mathbf{L}_{E'}\left(\mathbf{N}_\alpha\backslash e;\mathbf{N}_\beta\backslash e\right)\right) \tag{28}$$

$$\begin{gathered}\epsilon(P^1P^2E)\mathbf{L}_E\left((\mathbf{N}_\alpha^1\wedge\mathbf{N}_\alpha^2);(\mathbf{N}_\beta^1\wedge\mathbf{N}_\beta^2)\right)=\\ \epsilon(P^1E^1)\epsilon(P^2E^2)\left(\mathbf{L}_{E^1}\left(\mathbf{N}_\alpha^1;\mathbf{N}_\beta^1\right)\wedge\mathbf{L}_{E^2}\left(\mathbf{N}_\alpha^2;\mathbf{N}_\beta^2\right)\right)\end{gathered} \tag{29}$$

With this connection between basic elements of represented matroids and exterior algebras in place, present the following theorem:

We naturally take Hodge star for the operation on decomposibles that represents dualization of their matroids. However, the minors and (nontrivial) direct product factors of matroid M have ground sets that are proper subsets of M's ground set. We just postulate an orientation for every finite set—an alternating function from sequences of elements to $\{+1,-1\}$. For simplicity, the same symbol ϵ is used throughout. We only require that ϵ commute with identity extensions of the bijections $P\leftrightarrow P_\alpha\leftrightarrow P_\beta$ introduced above. Exterior algebra operations representing matroid deletion and contraction will therefore be consistent with the operation for duality by using our global orientations ϵ for our Hodge star.

Throughout, boldface letters like $\mathbf{N}$, $\mathbf{N}^i$, etc. will denote exterior products of vectors for which a distinguished ordered basis, denoted by S, S^i, PE, etc, for the underlying vector space has been specified, which we will call the *ground set*. In the following and elsewhere, A, B, $\overline{B}$, P, E, etc. are sets expressed as sequences. (Many formulas will also hold when $\mathbf{N}$ is a non-necessarilly decomposible exterior algebra element. Essentially, the formulas apply to the *components* $\mathbf{N}[B]$ of $\mathbf{N}$. These are the coefficients when $\mathbf{N}$ is expanded into an exterior polynomial in the basis vectors S. If $\mathbf{N}$ is considered to be an n-form on $\wedge U(S)$, then $\mathbf{N}[B]$ would be $\mathbf{N}(\mathbf{b}_1,\cdots,\mathbf{b}_n)$.) When $U=\mathbb{R}S$, we recognize $B\mapsto\operatorname{sign}(\mathbf{N}[B])$ is the *chirotope* function χ for the oriented matroid represented by the linear subspace represented by $\mathbf{N}$.

We do not put any other conditions between our orientations of of different sets other than commutation of ϵ with the $P\leftrightarrow P_\alpha\leftrightarrow P_\beta$ bijections. Our formulas that involve two sequenced sets S_1 and S_2 will therefore include the factor $\epsilon(S_1)\epsilon(S_2)$.

Remark: With $\mathbf{A}$ the exterior product of the elements of A in order, $\mathbf{N}/A$ can be defined by interior product[18] $\mathrm{i}_{\mathbf{A}}(\mathbf{N})(-1)^{(r\mathbf{N}-1)|A|}$.

In the definition of duality, B is sequenced, but $\overline{B}$ is an arbitrary sequencing of the complement of B with respect to the ground set of $\mathbf{N}$. The result is unchanged for all reorderings of $\overline{B}$, so dualization is well-defined. It represents linear matroid duality since Hodge star maps a subspace V's representitive to a representitive of the orthogonal complement of V when the basis defining it is declared to be orthonormal, see, e.g.[18].

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