REPORT: (February 10, 2015) AN ORIENTED MATROID PAIR MODEL FOR ELECTRICAL AND MECHANICAL NETWORKS

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ABSTRACT

Both resistive electrical networks and elastic mechanical systems such as trusses have a topological or geometric structure together with constitutive laws for the elements prior to their interconnection. Oriented matroids provide a common discrete mathematical model for such structure in which relationships on the signs of element quantities can be expressed. Pairing of oriented matroids enables non-linear monotone constitutive laws to be fit into the abstraction in a way that accomodates port and nullor insertions as well.

The resulting mathematical model clarifies some mechanical analogies for these circuit theory concepts, relates apparently dissimilar published theories for existance and uniqueness and shows how to handle elastic mechanical systems with small displacements. It also enables constraints on the signs of system quantities to be predicted from the structure when this is possible. Finally, it derives topological solution formulas for linearized mechanical systems in which the analog of a tree-sum is a sum over minimally rigid trusses.

1. INTRODUCTION

Our work [1] shows that the model and results of [2, 3] generalize from a graph model to a linear subspace pair model; the pair of linear subspaces defines a pair of oriented matroids; this oriented matroid pair is a discrete structure that generalizes a graph with designated resistor, source, nullator and norator edges and enables topological conditions for the existance or uniqueness of solutions to be expressed. Two real matrices, which can be easily generated from the system design, represent these oriented matroids so that it is practical to work with them. (Some oriented matroids, indeed most, are not representable by real matrices and would require much more space to store, but they do not occur in our application.)

Part of this reseach was done during a Sabbatical from the University at Albany in 2001.

In the electrical circuit theory literature, the circuit "topology" means the network graph (to which Kirchhoff's laws apply) together with particular kinds of "device elements" such as resistors, capacitors, voltage sources (batteries), current sources, etc., associated with single graph edges, and possibly "multiport" elements associated with multiple graph edges. Problems with multiport elements are reduced to those with only single edge elements, in order to apply the theory of [2], through the use of "nullator" and "norator" elements. Detailed exposition of the problems, reductions, theory and applications is given in [3].

In many cases, interesting conclusions can be reached by "calculations" done in a very simple "algebra of signs", a form of qualitative reasoning, starting with the sign patterns of the matrices. In cases when the qualitative calculations show the outcome depends on numeric values, the numeric information can then be used; or inequalities among system parameters for each outcome can be derived. Our work applies to models whose non-linearities are monotone. However, in order to understand results like those of [2, 3] matroid theoretically, we believe a pair of oriented matroids (not just one) on a common ground set is necessary. Then, the non-linear monotonicity is modeled by the sign of an element in one oriented matroid covector being equal to the sign of the same element in the other. We further motivate pairs by drawing the analogy between the electrical and mechanical elastic (whose subspace pairs are not graphic cycle and cocycle spaces) applications by way of the common model.

The present report will clarify how our generalizations of Fosséprez, Hasler and Neirynck's conditions for unique solvability are equivalent to the characterization of $\mathcal{W}_0$ matrix pairs shown by Sandberg and Willson [4, 5]: They are equivalent combinatorial conditions for the unique solvability of a *common* problem with monotone non-linearities. A third known variant due to Nishi and Chua [6, 7] is based on deletion/contraction operations applicable to various kinds of primative 1 and 2-port elements to produce "cactus graph" networks, which are then analyzed with the signs of determinants of their matrices.

Theory valid for all oriented matroid pairs, not just those represented by a pair of linear subspaces, was presented in [1]. For lack of space, we omit determinant sign conditions shown equivalent in [1] to the cases of common covector conditions that we cover. Although oriented matroids can also be axiomatized with "chirotopes" which abstract determinant signs, the oriented matroid common covector approach has the intuitive advantage of relating the covector directly to qualitative properties of a system state or differences.

The introduction continues with single oriented matroids and then succinct example of our approach that reproduces a case of a known result.

1.1. Single Oriented Matroids

We think of the oriented matroid $\mathcal{M}(M)$ *represented* by matrix M as its finite set of *covectors* $\mathcal{L}(M)$, where each covector is the tuple of the *signs* $\{+,-,0\}$ or *signature* $X = \sigma(l)$ of the real coordinates of a member l of the linear subspace $L(M)$ in $\mathbf{R}^U$ spanned by the rows of M. (We use italics to denote a *term being defined*.) Hence $\mathcal{L}(M)$ has at most $3^{|U|}$ covectors. For example, when M is the signed incidence matrix of a network graph, each covector represents a combination of branch voltage drop signs feasible under Kirchhoff's voltage law; the finite *ground set* U labels the branches. One can call X a *signed set*, in which elements of subset X^+ occur with $+$ sign and those in X^- have $-$ sign. The *support* supp X is the subset of $e \in U$ for which $X_e \neq 0$, i.e., supp $X = X^+ \cup X^-$. Bachem and Kern's book [8] motivates oriented matroids from linear subspaces this way. For the sake of brevity, we take for the defining axioms of oriented matroids some properties of the "sign algebra" operations that we will use.

Given two sign tuples X^1, X^2, their *composition* $Z = X^1 \circ X^2$ has supp $Z =$ supp $X^1 \cup$ supp X^2 and for $e \in$ supp Z, $X_e = X^i_e$ where i is the smallest index for which $X^i_e \neq 0$. Note that if $X^i = \sigma(l^i)$ for $l^i \in \mathbf{R}^U$, then $X^1 \circ X^2 = \sigma(l^1 + \epsilon l^2)$ for some sufficiently small $\epsilon > 0$. Hence $\mathcal{L}(M)$ is closed under the $\circ$ operation.

Definition 1 *The collection $\mathcal{L}$ of sign tuples over U is the set of covectors of an oriented matroid if it satisfies:*
(L0) $0 \in \mathcal{L}$.
(L1) If $X \in \mathcal{L}$ then $-X \in \mathcal{L}$.
(L2) For all X, $Y \in \mathcal{L}$, $X \circ Y \in \mathcal{L}$.
(L3) For all X, $Y \in \mathcal{L}$, $e \in X^+ \cap Y^-$ there is $Z \in \mathcal{L}$ such that $Z^+ \subset (X^+ \cup Y^+) - e$, $Z^- \subset (X^- \cup Y^-) - e$, and $(\text{supp } X - \text{supp } Y) \cup (\text{supp } Y - \text{supp } X) \cup (X^+ \cup Y^+) \cup (X^- \cup Y^-) \subset \text{supp } Z$.

Note that property (L3) says $Z_e = 0$ and it predicts Z_g for all $g \neq e$ except when $X_g Y_g = -$; i.e., g has opposite signs in X and Y. The logical equivalence of this definition to various apparently weaker axiomatizations is due to work of Edmonds, Fukada and Mandel cited and surveyed in [9].

Other oriented matroid notions such as orthogonality and independence can be expressed by properties of covector sets that are motivated by linear algebra. The covectors $\mathcal{L}(L^\perp)$ of the orthogonal complement of linear subspace $L \subset \mathbf{R}^U$ form another oriented matroid. We say $X \perp Y$ for signed sets X, Y when either supp $X \cap$ supp $Y = \emptyset$ or there are $e, f \in U$ with $X_f Y_f = -X_e Y_e \neq 0$. This abstracts a necessary condition for two real vectors to be orthogonal. In fact, for every oriented matroid $\mathcal{M}$ with covectors $\mathcal{L}(\mathcal{M})$ the collection $\mathcal{V} = \mathcal{L}^\perp$ defined by $\{Y | Y \perp X \text{ for all } X \in \mathcal{L}\}$ satisfies the covector axioms ([9], Prop. 3.7.12); it is called the set of *vectors* of $\mathcal{M}$ and is the set of covectors of the *dual* or *orthogonal* oriented matroid $\mathcal{M}^\perp$. The vectors code all combinations of coefficient sign that occur among all linear dependencies of the columns of M, when $\mathcal{M} = \mathcal{M}(M)$. More directly, an independent set $I \subset U$ is characterized by: for all $3^{|I|}$ "input" assignments i of $e \in I$ to $\{+,-,0\}$, there exists a covector $l \in \mathcal{L}(\mathcal{M})$ for which $l_e = i_e$ for all $e \in I$. Abstractly, an *independent set* $I \subset U$ satisfies supp $V \not\subset I$ for all non-zero vectors $V(M)$.

KVL, KCL and analogous mechanical structural or geometric laws are each formulated by a constraint of the form $v \in L = \text{row space}(M)$ where $L \subset \mathbf{R}^U$. The problem of reformulating such a law by a system of linear equations is solved as follows: A maximal subset $B \subset U$ corresponding to a linearly independent set of columns of M is found. Such a B is a maximal independent set, called a *basis in the matroid* $\mathcal{M}(M)$. Row operations and possibly deletion of zero rows can transform M to $(I\ ;\ M^{\overline{B}})$ (after column permutation) where I is the $r \times r$ identity matrix, where r $=$ rank$(M) = \dim(L) =$ rank$(L) =$ rank$(\mathcal{M}(L))$. It is now clear that $v \in L$ is characterized by $v_{\overline{B}} = v_B M^{\overline{B}}$. For each independently chosen $v_B \in \mathbf{R}^B$, $v = (v_B; v_{\overline{B}})$ is unique tuple for which the B coordinates equal v_B.

The *cocircuits* (resp. *circuits*) of an oriented matroid are the non-zero covectors (resp. vectors) whose support is minimal. Minty's painting property, most popularly known as a theorem about directed graphs[10], is generally true about the cocircuit $\mathcal{C}^*$ and circuit $\mathcal{C}$ collections of an oriented matroid $\mathcal{M}$. (Note $\mathcal{C}$ is the cocircuits of the orthogonal oriented matroid $\mathcal{M}$.) In fact, when the simple nontriviality, symmetry, and minimal support properties are assumed, the painting property characterizes when $\mathcal{C}^*$ and $\mathcal{C}$ are the cocircuit/circuit collections of an oriented matroid.

Theorem 1 *([9], Th. 3.4.4(4P); [8], Prop. 5.12) For every partition $U = R \cup G \cup B \cup W$ and for every $e \in R \cup G$,* ***either*** *(a) There exists $X \in \mathcal{C}^*$ so $e \in$ Supp$X$, $X_R \geq 0$, $X_G \leq 0$, X_B unrestriced and $X_W = 0$* ***or***
(b) There exists $Y \in \mathcal{C}$ so $e \in$ Supp$Y$, $Y_R \geq 0$, $Y_G \leq 0$, $Y_B = 0$ and Y_W unrestricted ***but not both.***

This led us to generalize in [1] Hasler and Neirynck's notion of a "pair of conjugate trees" to a "complementary pair of bases"; and of a "non-trivial uniform partial orientation of the resistors" to a "common (non-zero) covector".

1.2. Example

We illustrate the oriented matroid approach by reproducing the result of [11] that a particular configuration of a "feedback structure" with two Ebers-Moll transistors and one port cannot exhibit negative differential resistance by itself, and it can exhibit NDR if one resistor is added and $\alpha_1 + \alpha_2 - 1 > 0$. Under KVL, the voltages across the port, resistor and two linearized Ebers-Moll diodes are given by the row space member of M_V when the three rows are multiplied by the 3 independent voltages V_1, V_2 and the port voltage V. The space of current values feasible in the same 4 elements under KCL and the two Ebers-Moll current controlled current source laws is only one dimensional; it is spanned by the one row of M_I. Beginning with the signatures of the rows of the matrices, we can apply the covector axioms to explore what common covectors are possible under several variations.

2. BASIC

A *subspace pair* (L_V, L_I) is a pair of linear subspaces of $\mathbf{R}^U$, where the elements of finite set U index the coordinates. The scalar product $v \cdot w = \sum_{e \in U} v_e w_e$ is used to define that $v, w \in \mathbf{R}^U$ are *orthogonal* when $v \cdot w = 0$. An *orthogonal subspace pair* satisfies $v \cdot w = 0$ for all $v \in L_V$ and $w \in L_I$. A subspace pair has *full rank* when $\mathrm{rank}(L_V) + \mathrm{rank}(L_I) = |U|$. Hence an orthogonal full rank subspace pair is a linear subspace paired with its orthogonal complement.

The structure of an electrical network is defined beginning with the *network graph* $\mathcal{N}$ with *nodes* N and *arcs* U. (The generality obtainable by port, nullator/norator or nullor, and device characteristic insertions will be treated later.) Each arc has a fixed but arbitrary direction to define the sign of its voltage drop and current flow. The *incidence matrix* M_V, with rows indexed by N and columns indexed by U is defined so $M_V(n, e) = +1$ when the tail of e is n, -1 if the head of e is n, and 0 if n and e are not incident.

When L_V is the row space of M_V and L_I is the orthogonal complement of L_V, the members of L_V are voltage drop tuples in $\mathcal{N}$ feasible under Kirchhoff's voltage law and the members of L_I are the current flow tuples feasible under Kirchhoff's current law. These facts restate Kirchhoff's laws and Tellegen's theorem. Note that we can determine (L_V, L_I) from one of these subspaces given and Tellegen's theorem: The role of nodes here is not strictly necessary.

Kirchhoff's voltage law can be expressed by the statement: The feasible voltage drops are the image of the map $\mathbf{R}^N \to \mathbf{R}^U$ given by $\phi \to \phi M_V$. Kirchhoff's current law says the feasible current flows are the kernel of the map $\mathbf{R}^U \to \mathbf{R}^N$ given by $u \to u M_I^t$. Tellegen's theorem is the observation that M_V and M_I^t are adjoints. See [12].

The definition of mechanical network structure begins with the (undirected) *framework graph* $\mathcal{F}$ with *vertices* N and *edges* U. A *framework* $\mathcal{F}(\mathbf{p})$ in d *dimentions* is a framework graph $\mathcal{F}$ and an *embedding* $\mathbf{p} : N \to \mathbf{R}^d$. The embedding assigns each vertex to a point in d-dimensional space. The *rigidity matrix* M_V has $d|N|$ rows, each indexed by one coordinate in $\mathbf{R}^d$ of the point that embeds one vertex. For edge $e = (i, j) \in U$, column $M_V(e)$ of the rigidity matrix is defined (when vertices are numbered 0 through $|N| - 1$):

$$(0, \ldots, 0, \underset{\text{positions } di \ldots di+d-1}{\mathbf{p}(n_i) - \mathbf{p}(n_j)},$$

$$0, \ldots, 0, \underset{\text{positions } dj \ldots dj+d-1}{\mathbf{p}(n_j) - \mathbf{p}(n_i)}, 0, \ldots, 0)^T$$

This definition is echoed from the literature [13] on rigidity theory, except we interchange rows and columns. Just as we deemphasized nodes of electrical networks, we will use merely the row space of M_V for most of what follows.

The rigidity matrix as a function of the embedding $\mathbf{p}$ is denoted $M_V(\mathbf{p})$. The row vector $\mathbf{p}$ left multiplied with $M_V(\mathbf{p})$ is the row tuple denoted $\mathbf{L} = \mathbf{p} M_V(\mathbf{p})$ Then, $\mathbf{L}_e = (\mathbf{p} M_V(\mathbf{p}))_e = |\mathbf{p}_i - \mathbf{p}_j|^2$ for each edge $e = (i, j)$. Now if each $\mathbf{p}_i$ is a differentiable function of t, $d\mathbf{L}/dt = 2\mathbf{p}' M_V(\mathbf{p})$. Framework $\mathcal{F}(\mathbf{p})$ is *first-order rigid* when $d\mathbf{L}/dt = 0$ for all $\mathbf{p}'$ implies $|\mathbf{p}_i - \mathbf{p}_j|^2$ is constant for all pairs i, j, not just endpoints of edges.

Let $\mathbf{p}(i) - \mathbf{p}(j)$ for edge $e = \{i, j\}$ be called the *vector from j to i*. It is known that the row space L_V of M_V consists of tuples in $\mathbf{R}^U$ such that component $v(e) \in \mathbf{R}$ for $e \in U$ is the projection of the relative velocity of vertex i with respect to vertex j projected onto the vector from j to i, for some combination of vertex velocities $\mathbf{v} : N \to \mathbf{R}^d$: $v(e) = (\mathbf{v}(i) - \mathbf{v}(j)) \cdot (\mathbf{p}(i) - \mathbf{p}(j))$.

It is also known that the L_I, the orthogonal complement of L_V, is comprised of the tuples $\sigma : U \to \mathbf{R}$ of scalars for which the framework is in static equilibrium when each edge e exerts force $\sigma(e)(\mathbf{p}(j) - \mathbf{p}(i))$ on vertex i. By this convention, $\sigma(e) > 0$ means e is under tension and $\sigma(e) < 0$ means e is under compression. Each tuple $\sigma \in L_I$ is called a *self-stress*.

Under this analogy, (1) KVL corresponds to geometric consistancy of first order edge length changes under changes in the embedding, (2) KCL corresponds to Newton's laws of static equilibrium, and (3) Tellegen's theorem corresponds to a virtual work principle, that static equilibrium is characterized by the internal forces of every virtual embedding change doing zero virtual work.

2.1. Elastic Analog of the Nodal Admittance Matrix

Reduced nodal admittance matrix. Nodal resistance matrix. Interaction with a physical framework with it's environment. A framework is first order rigid iff it "resolves all applications of static equilibrium forces". However, every physical bar has some elasticity: An ideal rigid bar is analogous to an ideal voltage source. Hence, given an elastic framework, for every application of static equilibrium forces on the vertices, the vertex positions will change as the bars stretch or shrink under the forces they now carry to resolve the applied force. These first order vertex position changes are given by $Z\mathbf{f}$.

The environment might interact by "forcing" some vertices to change position relative to one another. Intuitively, the framework will "push back". The other vertices are free to move as adjacent vertices move and incident bars change length in response to the forces developed in them to resolve the forces required to hold the framework in its new position. The position changes of the free vertices V can be calculated by solving for the unknown position changes in the system of equations $(Y\mathbf{v}_V)(V) = 0$.

For our purposes, we insert port elements in order to make interactions with the environment explicit. This enables a coordinate of an environmental interaction quantity to correspond to an oriented matroid element, so that its sign can be read off from the corresponding entry in a covector.

It's yet to be done to handle simutaneous application of force to more than 2 vertices....

3. THE SUBSPACE PAIR MODEL AND PORTS

Questions of existance and uniquenss of solution for various combinations of kinds of sources can be formulated after modeling devices. Each port element provides separate output and input variables for an electrical current or voltage *kind of* source, or its mechanical analog. Unlike device variables, the two variables of each port are not directly related by a constitutive law which is part of the system model.

Ports also facilitate formal operations to compose larger systems from smaller ones. We believe ports are important for investigations of rigidity because they model how a framework interacts with its environment, for example, what a mechanical model "feels like" when you squeeze it. We have also found that electrical port characteristics of unit resistance ported electrical networks are ratios of coefficients in certain partial evaluations of a generalization of the Tutte polynomial[14].

Familiar topological conditions on dependencies among source values pertain to the *matroids* of the subspaces L_V and L_I. Questions about existance and uniqueness of solution will be answered in terms of supplementary subspace pair models which are obtained by the familiar operations of opening and shorting ports. Finally, operations on subspace pairs that model nullor insertion are defined, so that such ideal elements can be modeled combinatorially or geometrically.

The supplemental subspace pair derived after nullor insertion will typically not be orthogonal. One might also choose to model linearized CCCSs or VCVSs within one of the subspaces. Each port insertion generally increases $\mathrm{rank}(L_V)+\mathrm{rank}(L_I)-|U|$; system behavior for linearized constitutive laws will be shown to be represented by the intersection of two linear spaces. Hence we do not assume any rank or orthogonality conditions on subspace pairs in the definitions below.

Given a subspace pair (L_V, L_I) and element $p \in U$ not already a port, we define the operation of *inserting a port at* p as follows: A new subspace pair (L'_V, L'_I) is defined with $U' = U - \{p\} \cup \{p_V, p_I\}$, $L'_V = L_V \oplus \mathbf{R}$ (direct sum) with p replaced by p_V; and the coordinate of the added $\mathbf{R}$ indexed by p_I, together with $L'_I = L_I \oplus \mathbf{R}$ with p now replaced by p_I and the added subspace indexed by p_V. Note that (going to (L'_V, L'_I)) the ranks of L_V and L_I each increase by 1, and $|U'| = |U| + 1$. After p port insertions, we denote the final U by $E \cup P_V \cup P_I$ with pairwise disjoint E, P_V and P_I, $|P_V| = |P_I| = p$, $P_V \cup P_I$ being the replacement elements. (((??? Let P denote $P_V \cup P_I$.)))

The *subspace pair model* $\mathbf{M} = (E, \Gamma, P, (L_V, L_I))$ consists of finite set E of *device elements*, *constitutive law relations* $\Gamma = \{\Gamma_e \subset \mathbf{R} \times \mathbf{R} | e \in E\}$, a finite set $P = P_V \cup P_I$ that result from inserting ports as defined above, and a subspace pair (L_V, L_I) over $\mathbf{R}^U$ with $U = E \cup P$.

The *variables* of $\mathbf{M}$ are $\{u_{Ve}, u_{Ie} | e \in E\} \cup \{u_{Vp}, u_{Ip} | p_I, p_V \in P\}$. (For brevity, subscript "Vp" means port element $p_V \in P_V$, etc.) A *subspace pair model with sources* S is a subspace pair model $(E, \Gamma, P, (L_V, L_I), S)$ together with a subset S of exactly $|P|$ of the $2|P|$ elements in P. A V*-driven port* is a port $p \in P$ for which $p_V \in S$ and $p_I \notin S$, then u_{Vp} is called an *input variable*. Reverse V and I to define an I*-driven port* and its input variable.

A *solution* of $\mathbf{M}$ with sources is a real valued extension to *all* variables of $\mathbf{M}$ of a given *input* assignment to the input variables that satisfies $(u_{VP}, u_{IP}, u_V) \in L_V$, $(u_{VP}, u_{IP}, u_I) \in L_I$ and $(u_{Ve}, u_{Ie}) \in \Gamma_e$ for all $e \in E$. Note that in this model, the constraint $(u_{VP}, u_{IP}, u_V) \in L_V$ does not (by itself) imply any constraint on a "I" type port variable u_{Ip}, similarly, u_{Vp} is not constrained by $(u_{VP}, u_{IP}, u_I) \in L_V$. Port variables are not constrained by the constitutive laws Γ (by themselves) either.

In the language of matroid theory, we can call the element p_I an *isthmus* of the matroid $\mathcal{M}(L_V)$; similarly, p_V is an isthmus of $\mathcal{M}(L_I)$. In general, the matroid represented by a matrix is characterized by the collection $\mathcal{I}$ of *independent sets* of matrix columns, where a set of columns is called independent when it is linearly independent. (Matroid theory studies what can be deduced by the following three axioms satisfied by $\mathcal{I}$: (1) $\mathcal{I} \neq \emptyset$. (2) If $A \subset B \in \mathcal{I}$ then $A \in \mathcal{I}$. (3) If A, $B \in \mathcal{I}$ and $|A| < |B|$, then there exists

$e \in B - A$ for which $A \cup \{e\} \in \mathcal{I}$. For example, an isthmus e is characterized by $A \cup \{e\} \in \mathcal{I}$ for all $A \in \mathcal{I}$. The *rank* of a subset $C \in U$ is the size of the largest independent subset of C.

We say a subspace pair problem with sources S is *well-posed* when for all input assignments there is a unique solution.

The condition that there is no cycle of voltage source branches in the "voltage" graph nor a cutset of current source branches in the "current graph" is well-known to be necessary for an electrical network to have a unique solution for all choices of source values. This generalizes to:

Theorem 2 *(1) If all "V" source port values are feasible under the L_V constraint then $\{p_V | p$ is V-driven$\}$ is an independent set in the matroid $\mathcal{M}(L_V)$. (2) If every solution is unique then $\{p_I | p$ is V-driven$\}$ must be co-independent in $\mathcal{M}(L_I)$. (3) If all "I" source port values are feasible under the L_I constraint then $\{p_I | p$ is I-driven$\}$ is an independent set in the matroid $\mathcal{M}(L_I)$. (4) If every solution is unique then the set of elements $\{p_V | p$ is I-driven$\}$ must be co-independent in $\mathcal{M}(L_V)$.*

Proof of (1) and (3): If set S of input variables is dependent, then there is some combination of input values that is not feasible. Proof of (2) and (4): A non-co-independent set S must contain a cocircuit, so there is a non-zero covector supported by S. Hence there is a feasible variable assignment that is non-zero on the some port output variables only.

When the constitutive laws are linear, the solutions of **M** are found from the *intersection* of two linear subspaces: Let G be the diagonal matrix with "conductances" g_e in its positions indexed by $e \in E$ (so $\Gamma_e = \{(v, g_e v) | v \in \mathbf{R}\}$ and 1 in its other diagonal positions. The solution set projected onto the w_I variables is $L_V G \cap L_I$.

4. DELETION AND CONTRACTION

Given a subspace $L \subset \mathbf{R}^U$ and $e \in U$, the subspace $L - e$ "*L with e deleted*" is the subspace $L - e \subset \mathbf{R}^{U - \{e\}}$ defined by $L - e = \{l(U-e) | l(U) \in L\}$, where $l(U-e)$ denotes the tuple $l(U) \in \mathbf{R}^U$ with component labeled by e dropped. Thus, $L - e$ is the *projection* of L into $\mathbf{R}^{U-\{e\}}$. If L is the row space of matrix M, then $L - e$ is the row space of $M(U - e)$, which is M with column e deleted.

The subspace L/e "*L with e contracted*" is the subspace of $\mathbf{R}^{\{U-e\}}$ defined by $L/e = \{l(U-e) | l(U) \in L$ and $l(e) = 0\}$. In other words, L/e is the intersection of L with the (hyperplane) subspace of $\mathbf{R}^U$ with $l(e) = 0$ projected into $\mathbf{R}^{\{U-e\}}$.

We now define deletion and contraction on subspace pairs: $(L_V, L_I) - e = (L_V - e, L_I/e)$ and $(L_V, L_I)/e = (L_V/e, L_I - e)$. One can recognize that deleting element $e \in S$ from a subspace pair modeling an electrical network corresponds to *opening* the corresponding branch. Dually, contraction corresponds to *shorting* the branch. Mechanically, deletion of an edge corresponds to "breaking" the corresponding bar: ignore any distance change between its ends and transmit no force. Contraction corresponds to declaring the bar to be rigid, which rules out all (first order) distance changes between the endpoints and allows the bar to transmit arbitrary force of tension or compression.

4.1. Nullators and Norators

A *nullator* element $e \in E$ expresses the ideal constitutive law $u_{Ve} = 0$ and $u_{Ie} = 0$ which approximates conditions at the input to an amplifier when a system is stabilized by feedback. Hence a nullator is declared by *contracting e in both L_V and L_I*. Ordinarilly, this reduces both their ranks by 1.

A *norator* element $e \in E$ indicates that the constitutive law put no direct constraint on u_{Ve} and u_{Ie}; the amplifiers approximately adjust the output state so the feedback produces zero input. Hence a norator is declared by *deleting e in both L_V and L_I*. Ordinarilly, their ranks don't change.

Thus, for each ordinary nullator/norator pair, the rank balance conditions are preserved.

The conditions in Theorem 2 apply to the subspace pair obtained from all declarations of nullators, norators, opens and shorts.

There is a subtle difference between declaring a V-source with 0 input value and contracting the same element. If a set S of k such elements is not independent in $\mathcal{M}(L_V)$, then the rank of $\mathcal{M}(L_V/S)$ will be more than rank$(\mathcal{M}(L_V)) - k$ but the given combination of input values will still be feasible. If they are not co-independent in $\mathcal{M}(L_I)$ (which will certainly be true when there are no nullors). then the rank of $\mathcal{M}(L_I - S)$ will be less than rank$(\mathcal{M}(L_I))$ but there will be a non-zero combination of output only varibles. (Physically, that corresponds to non-zero current circulating in a loop of ideal wires; or a non-zero self-stress in an overbraced subframework of rigid bars. The dual dual physical situation is that is possible for the disconnected parts of an electrical network to differ in electical potential when a cut-set of branches are removed; mechanically, more flexes of the framework can exist when some edges are removed. For this reason, we are careful to distinguish deletion/contraction from port insertion.)

4.2. Topological Formulas

They can come out of the subspace pair formulation three ways:

- Besides the "g" or "r" variables, do not insert ports but do use an extra variable "x" to relate the voltage to current of one port. Then the equation $\det(M_V G; M_I) = 0$ has the form $Ax + B = 0$, so $x = -B/A$.
- Use the Plücker coordinate formulation of the intersection subspace to identify a minor of a hybrid or other description matrix of as the determinant of the solution submatrix for

a system of equations; then use the Cramer's rule generalization to find the ratios of minors of the equation matrix to analyzed. This was done for my ISCAS 98 paper.

- Use Rota's Grassmann-Cayley algrabra meet formula to extract expansion directly from subspace pair matrices, together with the previous way to identify Plücker coordinate ratios with description matrix minors.

5. SUPPLEMENTAL SUBSPACE PAIR

The *supplemental subspace pair* of a subspace pair model with sources is constructed by *zeroing* all the sources. Specifically, (1) for each "V" source element p, p_V is contracted in both L_V and L_I, (2) for each "I" source element p, p_I is contracted in both L_V and L_I.

p $\qquad e_1 \qquad e_2$

(M_V matrix)

0	1	0	0
1	0	1	1
p_V	p_I	e_1	e_2
1	0	0	0
0	1	-1	0
0	0	1	-1

(M_I matrix)

1	0	0
p_I	e_1	e_2
0	0	0
1	-1	0
0	1	-1

ZIR analysis for voltage source input: p_V contracted. Zero response (unique solution) even if negative resistances $\neq 0$ are allowed.

0	0	0
1	1	1
p_V	e_1	e_2
1	0	0
0	1	-1

ZIR analysis for current source input: p_I deleted. Zero response (unique solution) *unless* $g_1 = -g_2$.

Fig. 1. Simple example that illustrates the solution is unique when the port is V-driven provided each $g_e \neq 0$, but the I-driven system has a unique solution provided each $g_e > 0$ because the resulting supplemental oriented matroid pair has complementary bases and no common covector.

6. NO-COMMON-COVECTOR PROPERTY AND $\mathcal{W}_0$ PAIRS

The following theorem demonstrates, by means of Sandberg and Willson's theory of $\mathcal{W}_0$ pairs, that any subspace pair model (with its separation of geometric/topological and constitutive constraints) can be analyzed for unique solvability from the oriented matroid pair it generates. Conversely, a matrix pair $(A, B) \in \mathcal{W}_0$ is characterized by a rank condition and a no-common-covector property.

Theorem 3 *The subspace pair model has a unique solution for all source values when the supplementary oriented matroid pair has a complementary base pair and no common covector.*

Proof: The following matrices must be square and order $|U|$ for solutions of (..) to be unique: $A = \begin{pmatrix} M_V \\ 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 \\ -M_I \end{pmatrix}$. The theorem follows with these matrices used in the equivalance of 2. and 5. in the theorem below.

Theorem 4 *For a pair of $n \times n$ matrices (A, B), the following conditions are equivalent.*

1. *$(A, B) \in \mathcal{W}_0$ in the sense of Sandberg and Willson [4, 5]; e.g., $|AD + B| \neq 0$ for all positive diagonal D, etc.*

2. $\operatorname{rank} \mathcal{M}[A\ B] = n$ *and* $\mathcal{L}[A\ B] \cap \mathcal{L}[I\ \ {-I}] = \{0\}$.

3. $\operatorname{rank} \mathcal{M}[A\ B] = n$ *and* $\mathcal{V}[A\ B] \cap \mathcal{V}[I\ \ {-I}] = \{0\}$.

4. *(Fundamental theorem of Sandberg and Willson [4, 5]) For all functions $F : \mathbf{R}^n \to \mathbf{R}^n$ of the form $F(x)_k = f_k(x_k)$ where each f_k is a strictly monotone increasing function from $\mathbf{R}$* onto *$\mathbf{R}$ and for all $c \in \mathbf{R}^n$, the equation*

$$AF(x) + Bx = c$$

has a unique solution x.

5. *For all functions $G : \mathbf{R}^n \to \mathbf{R}^n$ of the form $G(w)_k = g_k(w_k)$ where each g_k is a strictly monotone increasing function from $\mathbf{R}$* onto *$\mathbf{R}$ and for all $d', d'' \in \mathbf{R}^n$, the equations*

$$u^t = z^t A + d', \quad w^t = z^t B + d'', \quad u = -G(w) \quad (1)$$

have a unique solution (u, w, z).

Note: A direct inductive proof is obtainable by generalizing the proofs given in [3]. This approach has the advantage of revealing circuit theoretic concepts that occur. See also [15]. One of the steps is to prove that if the no-common-covector property is true for (M_V, M_I), then it is true for the matrix pair from the system obtained by replacing one of the non-linear elements by a source.

Theorem 5 *With constitutive laws given by monotone increasing functions from $\mathbf{R}$ onto $\mathbf{R}$, every subspace pair problem can be posed as a case of theorem 4, and every case of theorem 4 can be posed as a subspace pair problem.*

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