Ported Parametrized Tutte Functions: Old and New Applications

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Our Ported Parametrized separator-strong Tutte Equations

- $T(G) = x_e T(G/e) + y_e T(G \setminus e)$
  if e is a non-separator and $e \notin P$.
- $T(G) = X_e T(G/e)$ if e is a coloop (isthmus) and $e \notin P$.
- $T(G) = Y_e T(G \setminus e)$ if e is a loop and $e \notin P$.

Zaslavsky, Bollobas-Riordan, Ellis-Monaghan-Traldi; $P \neq \emptyset$ (sdc).

The Famous Tutte Polynomial

Take $P = \emptyset$, $x_e = y_e = 1$, $X_e = X$ and $Y_e = Y$ for all e,
define $T(\emptyset) = 1$:
$T(G)(X, Y)$ is then a well-defined polynomial in X, Y.

Theorem (Tutte, Brylawski)

$$T(X, Y) = \sum_{Bases\ B \subseteq E} X^{Internal\ Activity(B)} Y^{External\ Activity(B)}$$

independently of E's order used to define the activities.

Reminder about Activities

Given a linear order on E,

Given a basis B (spanning tree if G is connected):

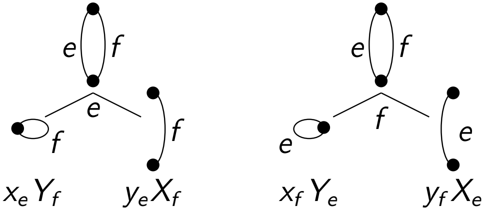
- $e \notin B$ is externally active if e is the smallest element of the (unique) circuit in $B \cup \{e\}$.
- $e \in B$ is internally active if e is the smallest element of the (unique) cocircuit in $E \setminus B \cup \{e\}$.
- Internal (External) Activity(B) is the number of internally (externally) active elements.

Huh?? We will get intuition for this and extend it with $P \neq \emptyset$ with a Tutte (Computation) Tree (Gordon-MacMahon) view.

H. Crapo also proved the well-definedness of the Tutte polynomial from its corank-nullity polynomial expression. But that doesn't fully generalize to parametrized Tutte functions (Zaslavsky).

$T(2-\text{circuit})$ is not defined unless...

$$x_e Y_f + y_e X_f = x_f Y_e + y_f X_e$$



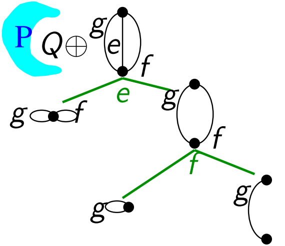
These are Tutte (computation) trees, defined formally and used by Gordon-MacMahon to study Tutte polynomials of greedoids, where sometimes, the same element order cannot be used under each branch.

A Detail

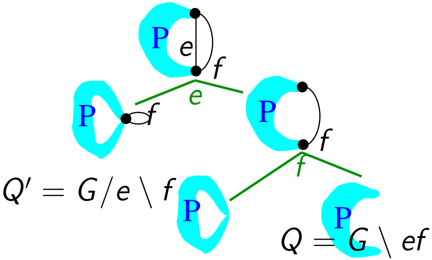
$T(\text{loop matroid on } e) = Y_e T(\emptyset(\text{empty matroid}))$, etc. so the real ZBR condition is

$$T(\emptyset)(x_e Y_f + y_e X_f) = T(\emptyset)(x_f Y_e + y_f X_e)$$

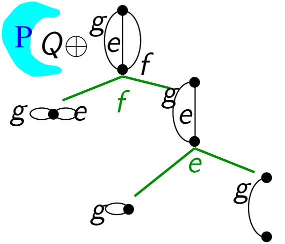
Problems: 2nd ZBR-type and 1st for $P \neq \emptyset$



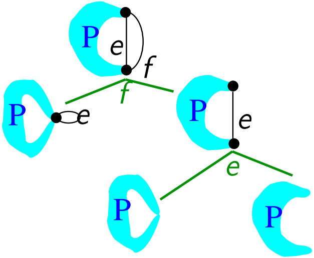
$I(Q)[Y_g(x_e Y_f + y_e x_f) + X_g y_e y_f]$



$I(Q')(x_e Y_f + y_e x_f) + I(Q) y_e y_f$

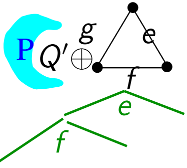


$I(Q)[Y_g(x_f Y_e + y_f x_e) + X_g y_f y_e]$

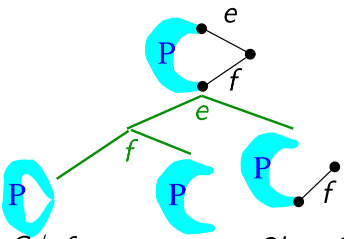


$I(Q')(x_f Y_e + y_f x_e) + I(Q) y_f y_e$

More Problems..., 2nd for $P \neq \emptyset$

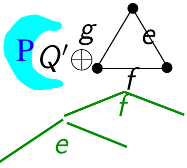


$I(Q')[Y_g x_e x_f + X_g(x_e y_f + y_e X_f)]$

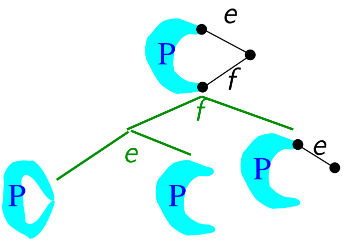


$Q = G/ef$ $\quad$ $Q' = G/e \setminus f$

$I(Q)x_e x_f + I(Q')(x_e y_f + y_e X_f)$



$I(Q')[Y_g x_f x_g + X_g(x_f y_e + y_f X_e)]$



$I(Q)x_f x_e + I(Q')(x_f y_e + y_f X_e)$

Solution—Setup

When do recursive equations have a solution?

"Have a solution" here means "Every calculation of $T(G)$ using the Tutte equations and initial values on members of $\mathcal{F}$ gives the same answer.

Definition (Sep. Strong Ported Parametrized Tutte Function)

Let P be a set and $\mathcal{F}$ be a family of graphs, oriented matroids or matroids that is closed under deletion and contraction of elements not in P. Deletion of loops and contraction of coloops is allowed.
Let ring R elements X_e, Y_e, x_e and y_e (for each $e \notin P$) and $R-$module elements $I(Q)$ for every $Q \in \mathcal{F}$ with Q over elements of P only also be given.
This structure has a Tutte function if and only if the Ported Parametrized Tutte Equations have (a necessarilly unique) solution over all of $\mathcal{F}$.

The X_e, Y_e, x_e, y_e and $I(Q)$ are called parameters and initial values.

Solution—Theorem

Theorem (After Zaslavsky, Bollobas-Riordan, Ellis-Monaghan-Traldi)

$\mathcal{F}$ and values as above has a Tutte function iff the following equations are satisfied whenever they arise from a member $G \in \mathcal{F}$:

1. *Suppose $G = Q \oplus G'$ where $S(Q) \subseteq P$.*
   1.1 *With G' a 2-circuit $\{e, f\}$ (and so 2-cocircuit too),*
   $I(Q)(x_e Y_f + y_e X_f) = I(Q)(x_f Y_e + y_f X_e)$.
   1.2 *With G' a 3-circuit $\{e, f, g\}$,*
   $I(Q)X_g(x_e y_f + y_e X_f) = I(Q)X_g(x_f y_e + y_f X_e)$.
   1.3 *With G' a 3-cocircuit $\{e, f, g\}$,*
   $I(Q)Y_g(x_e Y_f + y_e x_f) = I(Q)Y_g(x_f Y_e + y_f x_e)$.

   These generalize the 3 ZBR equations merely by replacing $I(\emptyset)$ with $I(Q)$.
2. *With $\{e, f\} = E$ in series and not isolated (from P),*
   $I(G/e \setminus f)(x_e y_f + y_e X_f) = I(G/e \setminus f)(x_f y_e + y_f X_e)$.
3. *With $\{e, f\} = E$ in parallel and not isolated,*
   $I(G/e \setminus f)(x_e Y_f + y_e x_f) = I(G/e \setminus f)(x_f Y_e + y_f x_e)$.

Proof Outline

Ported ZBR equations are necessary

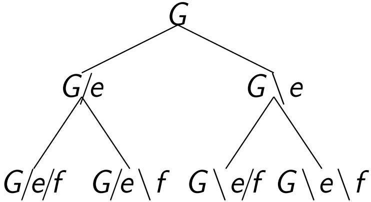
Consider the $1 + 4$ matroid/graph classes with $E(G) = \{e, f\}$ or $E(G) = \{e, f, g\}$, where $E(G) = S(G) \setminus P$, corresponding to the 5 ZBR conditions.
For each, show (as I illustrated before) that assuming certain pairs of computations of $T(G)$ give equal results implies the condition.

Ported ZBR equations are sufficient

Induction: Assume G is a minimum $|E(G)|$ counter example, where $E(G) = S(G) \setminus P$. So: $T(G/e)$ and $T(G \setminus e)$ are well-defined from the Tutte Equations for every $e \in E(G)$.
Lemma (Zaslavsky) shows all of $E(G)$ is a series class or a parallel class.
The relevent Tutte equations
(Is E isolated? Or is E connected to some of P?)
show there's a smaller E counterexample.

Some Details

- $|E| \geq 2$.
- No $e \in E$ is a separator in G.
- For no $e, f \in E(G)$ is this a Tutte tree:

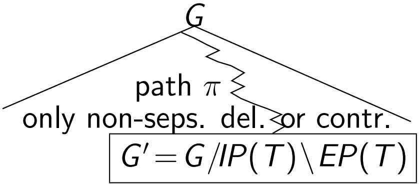


The Tutte Tree formalism here means e is a non-separator in G and f is a non-separator in both G/e and $G \setminus e$.

- Lemmas: Each $e \in E(G), f \in E(G)$, $e \neq f$, is series pair or a parallel pair.
  e, f parallel and f, g series is impossible.
  So all of E is a series class or is a parallel class.

Tutte (Computation) Trees and Internal/External Activities

A P-subbasis $T \subseteq E(G)$ ("contracting set" [Diao-Hetyei]) is an independent set (forest) for which $T \cup P$ is spanning.



Path π contributes $[G'|P]x^{IP(T)}y^{EP(T)}X^{IA(T)}Y^{EA(T)}$ to our Tutte Poly.

E is partitioned: $T = IP(T) \cup IA(T)$, $E \setminus T = EP(T) \cup EA(T)$.
$IP(T) = \{\text{elements contracted along } \pi\}$.
$EP(T) = \{\text{elements deleted along } \pi\}$.
In G', $IA(T)$ is all coloops, $EA(T)$ is all loops.
2^E is partitioned into intervals $\{[X_T, Y_T] | P - \text{subbasis}\, T\}$,
$X_T = IP(T) \subseteq (T = IP(T) \cup IA(T)) \subseteq (T \cup EA(T)) = Y_T$.

All is determined by the Tutte tree, NOT an element order!

Tutte Polynomials and Activities

1. When the conditions in our P-ported ZBR theorem are satisfied, all Tutte trees yield the same value in the R-module, called THE Tutte polynomial (because trees$\leftrightarrow$computations.) This value has multiple polynomial expressions.
2. The P-quotient $[G/IP(T)|P]$ in the term contributed by P-subbasis T is determined by the internally passive elements of T.

P-subbasis (Spanning Tree) Polynomial

$$\mathcal{T}_P(G) = \sum_{T:P-\text{subbasis}} [G/T|P] x^T y^{E\setminus T} = \sum_{T:P-\text{subbasis}} [G/IP(T)|P] x^T y^{E\setminus T}$$

$$\text{with notation: } z^S = \prod_{e\in S} z_e (z = x \text{ or } y).$$

Remark: If G is a tree, then E is the one P-subbasis, $IP(E) = \emptyset$ and $IA(E) = E$.

$$\mathcal{T}(G) = x_e \mathcal{T}(G/e) + y_e \mathcal{T}(G \setminus e) \text{ if } e \notin P \text{ is a non-separator.}$$

$$\mathcal{T}(G) = y_e \mathcal{T}(G \setminus e) \text{ if } e \notin P \text{ is a loop.}$$

$$\mathcal{T}(G) = x_e \mathcal{T}(G/e) \text{ if } e \text{ is an isthmus.}$$

To enumerate the spanning trees in E, set

$$[\text{all loops}] \leftarrow 1 \quad \text{and} \quad [\text{other } P\text{-quotients}] \leftarrow 0$$

P-ported Forest Polynomial

$$\mathcal{F}(G) = \sum_{F \subseteq E : F \text{ is a forest}} [G/F|P] x^F$$

The $\sum x^F$ coefficient of each $[Q]$ variable enumerates a class of forests. Examples: $[p\,q] \to F$ spans both p and q; $[p\,q] \to F$ spans neither p nor q.

$$\mathcal{F}(G) = \sum_{T : P-\text{subbasis}} [G/T|P] x^{IP(T)} (1+x)^{IA(T)}$$

$$= \sum_{T : P-\text{subbasis}} [G/IP(T)|P] x^{IP(T)} \sum_{F \subseteq IA(T)} x^F$$

$$\text{where } (1+x)^S = \prod_{e \in S} (1 + x_e)$$

$$\mathcal{F}(G) = x_e \mathcal{F}(G/e) + \mathcal{F}(G \setminus e) \;\text{ if } e \notin P \text{ is a non-separator.}$$

$$\mathcal{F}(G) = \mathcal{F}(G \setminus e) \;\text{ if } e \notin P \text{ is a loop.}$$

$$\mathcal{F}(G) = (1 + x_e) \mathcal{F}(G/e) \;\text{ if } e \notin P \text{ is an isthmus.}$$

Spanning Tree polys solve Equations of Kirchhoff and Ohm

Variables

For each $e \in E(G)$ or $p \in P$, v_e or v_p is the voltage drop across e or p; i_e or i_p is the current flow through e or p.

Equations

For some (unimodular) basis for the cocycle space ($k^j_{e \text{ or } p}$), $j = 1, \ldots, \mathrm{rank}(G)$, 0 net flow across cuts:

$$\sum_{e \in E} k^j_e i_e + \sum_{p \in P} k^j_p i_p = 0 \text{ for } j = 1, \ldots, \mathrm{rank}(G).$$

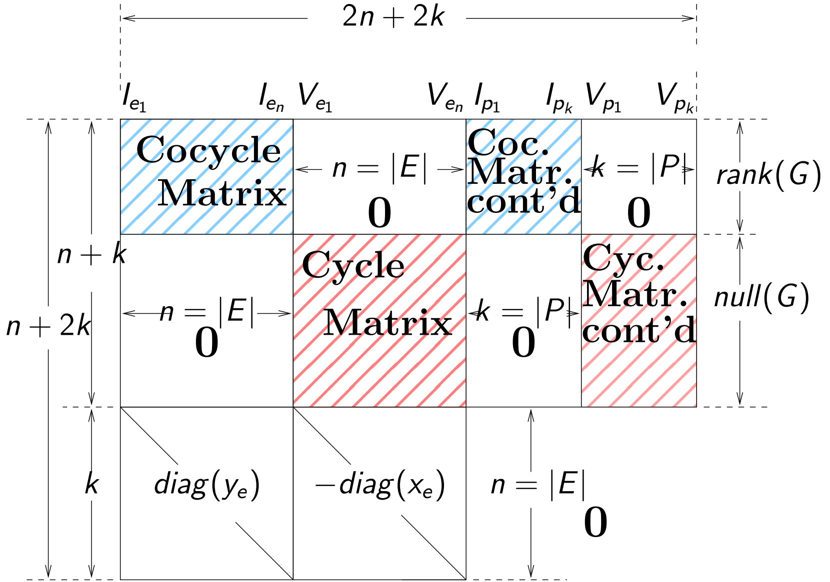
For some (unimodular) basis for the cycle space ($c^h_{e \text{ or } p}$), $h = 1, \ldots, \mathrm{nullity}(G)$, 0 sum of diffs. of potential around cycles:

$$\sum_{e \in E} c^h_e v_e + \sum_{p \in P} c^h_p v_p = 0 \text{ for } h = 1, \ldots, \mathrm{nullity}(G).$$

For each $e \in E$: $\quad x_e v_e - y_e i_e = 0$

Physical Conductance $= x_e : y_e$, Resistance $= y_e : x_e$

Matrix M:Flow, Potential Eqs. (Kirchhoff's) and Ohm's



Solution using Determinants

Let us apply current i_{p_β} through port edge p_β, leave the other ports "open" (means $i_{p_\gamma} = 0$, $\gamma \neq \beta$) and determine what is voltage drop v_{p_α} across port p_α.

$$v_{p_\alpha} = -\frac{M[I_{e_1}, \ldots, I_{e_n}, V_{e_1}, \ldots, V_{e_n}, V_{p_1} \ldots, I_{p_\beta}, \ldots, V_{p_k}]}{M[I_{e_1}, \ldots, I_{e_n}, V_{e_1}, \ldots, V_{e_n}, V_{p_1} \ldots, V_{p_\alpha}, \ldots, V_{p_k}]} \cdot i_{p_\beta}$$

In short: Coefficients in the linear relationships among port voltage and current variables are ratios of full-rowed minors of M, all with

- $k = |P|$ of the $2k$ column labels $I_{p_1}, \ldots, I_{p_k}, V_{p_1}, \ldots, V_{p_k}$.
- All $2n = 2|E|$ column labels $I_{e_1}, \ldots, I_{e_n}, V_{e_1}, \ldots, V_{e_n}$.

Theorem

Each of these minors (with carefully defined sign) satisfies the P-ported parametrized Tutte equations.

Remark: Ratios of these minors with numerator gotten by replacing more than one denominator label are higher order minors of a square matrix relating some k port variables with some k port variables.

Why the Tutte equations? Sketch

The last row is $(0 \cdots 0 y_{e_n} 0 \cdots - x_{e_n} 0 \cdots 0)$.
So, each minor can be written

$$-x_{e_n}(\text{a } (2n+k-1) \text{ by } (2n+k-1) \text{ minor})$$

$$+y_{e_n}(\text{a } (2n+2k-1) \text{ by } (2n+k-1) \text{ minor})$$

In the first minor, the column for e_n of the cycle matrix was deleted.
Do row operations so the column for e_n of the cocycle matrix becomes $(1, 0, \cdots, 0)^t$.
The resulting matrix corresponds to G/e_n. The other minor corresponds to $G \setminus e_n$.

Application: Rayleigh Identity, "Neg. Spanning Tree Correlation"

$\Gamma_e(G)$ is equivalent conductance across e. Rayleigh: $0 \leq \dfrac{\partial \Gamma_p}{\partial g_f} = \dfrac{\partial \frac{T_G}{T_{G/e}}}{\partial g_f}$

is equivalent to

$$0 \leq \frac{\partial T_G}{\partial g_f} T_{G/e} - T_G \frac{\partial T_{G/e}}{\partial g_f} = T_{G/f} T_{G/e} - T_G T_{G/e/f}$$

In fact,

$$T_{G/f} T_{G/e} - T_G T_{G/e/f} = \left(T^{+}_{G/e \text{ \& } G/f} - T^{-}_{G/e \text{ \& } G/f} \right)^2$$

$T^{\pm}_{G/e \text{ \& } G/f}$ enumerate the $\pm$ common spanning trees.

Known Partial and Full Combinatorial Proofs

$$T_{G/f}T_{G/e} - T_G T_{G/e/f} = \left(T^{+}_{G/e\ \&\ G/f} - T^{-}_{G/e\ \&\ G/f}\right)^2$$

$T^{\pm}_{G/e\ \&\ G/f}$ enumerate the $\pm$ common spanning trees.

Choe (2004) proved essentially this using the vertex-based all-minors matrix tree theorem, combinatorial cases and Jacobi's theorem relating the minors of a matrix to the minors of its inverse..

Cibulka, Hladky, Lacroix and Wagner (2008) gave a completely bijective proof that utilizes some natural 2:2 and 2:1 correspondances.

Difficulty: Some terms on the left cancel and some reduce to terms with coefficients ± 2.

Linear Alg./Oriented Matroid Proof of Rayleigh's Identity

Let R be the transfer resistance matrix for 2 ports across e and f. Our result implies that

$$\det R = \begin{vmatrix} R_{ee} & R_{ef} \\ R_{fe} & R_{ff} \end{vmatrix} = +\frac{T_{G/e/f}}{T_G}$$

It and better-known results tell us

$$R_{ee} = \frac{T_{G/e}}{T_G}; \quad R_{ff} = \frac{T_{G/f}}{T_G}; \quad R_{ef} = R_{fe} = \frac{T^{+}_{G/e\ \&\ G/f} - T^{-}_{G/e\ \&\ G/f}}{T_G}$$

$T_{G/f}\,T_{G/e} - T_G\,T_{G/e/f} = \left(T^{+}_{G/e\ \&\ G/f} - T^{-}_{G/e\ \&\ G/f}\right)^2$ is immediate after substituting these into

$$\det R = R_{ee}R_{ff} - (R_{ef})^2$$

The $+$ follows from physical grounds if the $g_e, r_e \geq 0$. Our characterization and proof are combinatorial.

New Rayleigh's Identities!

The same method generates identities from

$$\begin{vmatrix} R_{ee} & R_{ef} & R_{eg} \\ R_{fe} & R_{ff} & R_{fg} \\ R_{ge} & R_{gf} & R_{gg} \end{vmatrix} = +\frac{T_{G/e/f/g}}{T_G}$$

ETC...

(Applications???)

Might the same methods address a much harder problem: The same inequality for forests instead of spanning trees?

Negative Correlation in Forests Conjecture

Conjecture[Grimmett-Winker, Kahn and Pemantle]: For every pair of edges p, q in a graph, let F_p (F_q) enumerate all forests with p (q resp.), F_{pq} enumerate those with both p and q, and F enumerate all forests.

$$F_pF_q - FF_{pq} \geq 0 \text{ if } x_e, x_p, x_q \geq 0 \text{ for all } e \in E \text{ and } p, q.$$

Consider random forests, take for e, p, q: $x_e = Pr(e)/(1 - Pr(e))$.

$$\mathrm{Corr}(p \in RF, q \in RF) = \frac{-1}{\sigma^2}\left(\frac{F_p}{F}\frac{F_q}{F} - \frac{F_{pq}}{F}\right).$$

$$\text{By calculation: } W(G) = \mathcal{F}_{p|}\mathcal{F}_{q|} - \mathcal{F}_{\text{all}}\mathcal{F}_{p|q|} = \frac{F_pF_q - FF_{pq}}{x_px_q}$$

Example where the conjecture is true: If p and q are in series, then $\mathcal{F}_{p|} = \mathcal{F}_{q|} = \mathcal{F}_{\text{all}}$ and $\mathcal{F}_{p|q|} = \mathcal{F}_{\text{all}} - \mathcal{F}_{p()q}$, so

$$W(G) = \mathcal{F}_{\text{all}}\mathcal{F}_{\text{all}} - \mathcal{F}_{\text{all}}(\mathcal{F}_{\text{all}} - \mathcal{F}_{p()q}) = \mathcal{F}_{\text{all}}\mathcal{F}_{p()q} \geq 0$$

Wagner's Conjectured Formula

$$W(G) = \mathcal{F}_{p|}\mathcal{F}_{q|} - \mathcal{F}_{\mathrm{all}}\mathcal{F}_{p|q|} =? \sum_{A \subseteq E} x^A \left((\sum \pm x^L)^2 \right)$$

For each $A \subseteq E$, the sum is over (some?) forests L, $L \cap A = \emptyset$, for which there is some $B \subseteq L$, $A \cup B \cup \{p, q\}$ is a circuit.

We call $A \cup B \cup \{p, q\}$ a linking circuit. The signs are related to the relative orientations of p and q in the linking circuit.

$W^?(G)$ denotes Wagner's formula. Since $L \cap A = \emptyset$, any x_e^2 can only come from one or more $(\sum \cdots)^2$ expressions.

We sketch a P-ported Tutte decomposition approach to the conjecture. (It remains unproven.)

Towards Tutte Decompositions for p, q-in-Forest Correlation

$$W(G) = \mathcal{F}_{p|}\mathcal{F}_{q|} - \mathcal{F}_{\text{all}}\mathcal{F}_{p|q|}$$

Each of the four $\mathcal{F}_R$ satisfy separator-strong P-ported Tutte equations with $P = \{p, q\}$:

$$\mathcal{F}_R(G) = x_e\mathcal{F}_R(G/e) + \mathcal{F}_R(G \setminus e) \text{ for non-sep. } e \in E.$$

$$\mathcal{F}_R(G) = x_e\mathcal{F}_R(G/e) \text{ for isthmus } e \quad \mathcal{F}_R(G) = \mathcal{F}_R(G\setminus e) \text{ for loop } e.$$

Therefore;

$$W(G) = W(G \setminus e) + x_e^2 W(G/e) + x_e B(G/e, G \setminus e)$$

Where:

$$\begin{aligned} B(G_1, G_2) = \; & \mathcal{F}_{p|}(G_1)\mathcal{F}_{q|}(G_2) + \mathcal{F}_{q|}(G_1)\mathcal{F}_{p|}(G_2) \\ & -\mathcal{F}_{\text{all}}(G_1)\mathcal{F}_{p|q|}(G_2) - \mathcal{F}_{p|q|}(G_1)\mathcal{F}_{\text{all}}(G_2) \end{aligned}$$

Towards Tutte Decompositions...

$$W(G) = W(G \setminus e) + x_e^2 W(G/e) + x_e B(G/e, G \setminus e)$$

$$\text{Where } B(G_1, G_2) = \mathcal{F}_{p}(G_1)\mathcal{F}_{q}(G_2) + \mathcal{F}_{q}(G_1)\mathcal{F}_{p}(G_2) - \mathcal{F}_{\text{all}}(G_1)\mathcal{F}_{pq}(G_2) - \mathcal{F}_{pq}(G_1)\mathcal{F}_{\text{all}}(G_2)$$

$$B(G_1, G_2) = x_e B(G_1/e, G_2) + B(G_1 \setminus e, G_2) \text{ if } e \notin \{p, q\} \text{ is non-sep. in } G_1.$$

$$B(G_1, G_2) = (x_e + 1) B(G_1/e, G_2) \text{ if } e \notin \{p, q\} \text{ is an isthmus in } G_1.$$

$$B(G_1, G_2) = B(G_1 \setminus e, G_2) \text{ if } e \notin \{p, q\} \text{ is a loop in } G_1.$$

And similarly for G_2.

Values on indecomposibles: $W({}^{p}\,()\,{}^{q}) = 1$, $W($ 4 others$) = 0$.

Values of B on pairs of indecomposibles are expressed in a symmetric 5x5 table.

An Approach to an Inductive Proof

Verify $W(G) = W^?(G)$ for small cases. Use induction to verify $W(G) = W^?(G)$ for separable G.

$$W(G) = x_e^2 W(G/e) + W(G \setminus e) + x_e B(G/e, G \setminus e)$$

$$= \text{(by induction) } x_e^2 W^?(G/e) + W^?(G \setminus e) + x_e B(G/e, G \setminus e)$$

From a combinatorial definition (of $\sigma(A, L_1, L_2) = \sigma_G$)

$$W^?(G) = \sum_{A \subseteq E} x^A (\sum_L \pm x^L)^2 = \sum_{A \subseteq E} x^A (\sum_{L_1, L_2 \subseteq E \setminus A} (\sigma(A, L_1, L_2) x^{L_1} x^{L_2}))$$

extract a combinatorial description of the terms with degree 1 in x_e. There are 3 kinds: x_e in x^A, x_e in x^{L_1}, x_e in x^{L_2}. This will be completed to a proof if we verify (combinatorially) that all non-separable G:

$$B(G/e, G \setminus e) = \frac{\partial W^?(G)}{\partial x_e}\Big|_{x_e=0}$$

Bi-Tutte Decomposition for B

Remember, B is well defined and

$$B(G/e, G\backslash e) = \sum_{F_1,F_2\text{: forests in } E} x^{F_1}x^{F_2}B(G/e/F_1|P, G\backslash e/F_2|P).$$

Since $B(G/e/F_1|P, G\backslash e/F_2|P) = B(G/F_1/e|P, G/F_2\backslash e|P) = B((G/F_1|P \cup e)/e, (G/F_2|P \cup e)\backslash e)$, we can use ONE $\{p, q, e\}$-ported Tutte tree, and then at each leaf labelled by Q, append at most two branches, "Left" for Q/e and "Right" for $Q \setminus e$.

The monomials have the form $c_{A,B}x^{2A}x^{B}$. There are at most $2^{|B|}$ Left/Right pairs of new leaves $(Q_L/e,\ Q_R \setminus e)$; the pair contributes $B(Q_L/e, Q_R \setminus e)$ to $c_{A,B}$.

This at least helps us extract the coefficients of particular terms, and how various partitions of B contribute....