Restricted or Ported Tutte Decomposion and Analogs of All-Minors Laplacian Expansions

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What is a parametrized strong Tutte function?

Tutte equations are satisfied in a very general setup:

1. Elements $\{e\}$ each with parameters g_e, r_e.
2. A category $\mathcal{N}$ of objects $\mathbf{N}$ each with ground set $S = S(\mathbf{N})$ of elements.
3. For some *decomposible* $\mathbf{N}$, for one or more *separators* $e \in S(\mathbf{N})$, the *contraction* and *deletion* operations are defined with results $\mathbf{N}/e$ and $\mathbf{N}\backslash e$ in $\mathcal{N}$, with ground sets $S(\mathbf{N})\backslash\{e\}$
4. Some $\mathbf{N} = \mathbf{N}_1 \oplus \mathbf{N_2}$ are direct sums, where $S(\mathbf{N}_1) \cap S(\mathbf{N}_2) = \emptyset$.
5. For each indecomposible $\mathbf{N}$ with no separators there is an additional parameter $i_{\mathbf{N}}$ called the *initial value*.

Tutte equations, functions and Good Questions

1. For all $\mathbf{N}$ with separator $e \in S(\mathbf{N})$,

$$F(\mathbf{N}) = g_e F(\mathbf{N}/e) + r_e(\mathbf{N} \backslash e)$$

2. When $\mathbf{N} = \mathbf{N_1} \oplus \mathbf{N_2}$,

$$F(\mathbf{N}) = F(\mathbf{N_1})F(\mathbf{N_2})$$

3. When $\mathbf{N}$ is indecomposible,

$$F(\mathbf{N}) = i_{\mathbf{N}}$$

F is Tutte function when all the Tutte equations are satisfied. This MEANS $F(\mathbf{N})$ is what is computed by applying Tutte equations *in any order they are applicable*.
Good Questions: When does $\mathcal{N}$ and parameters ACTUALLY HAVE a Tutte function? If so, what is a *universal* Tutte function?

Some answers–for Graphs and Matroids

Only loops and coloops need initial values

The only $\mathbf{N}$ with no separators and no $\mathbf{N} = \mathbf{N}_1 \oplus \mathbf{N_2}$ for $\mathbf{N}_i \neq \emptyset$ are $\mathbf{loop}(e)$ and $\mathbf{coloop}(e)$.

The famous Tutte Polynomial

Adding all $g_e = r_e = 1$, the Tutte polynomial $F(\mathbf{N})(x, y)$ obtained from $i_{\mathbf{loop}(e)} = x$, $i_{\mathbf{coloop}(e)} = y$ and $i_\emptyset = 1.$ is a universal Tutte function.

Normal Tutte Functions for Matroids

(Zaslavsky, Bollobás/Riordan) With arbitrary g_e, r_e, and x,y, the *normal* Tutte functions for matroids are obtained with $i_{\mathbf{coloop(e)}} = g_e y + x$, $i_{\mathbf{loop(e)}} = r_e x + y$ and $i_\emptyset = 1.$ They are exactly the ones with a weighted rank-nullity generating function.
There's a big story about what relationships among the $g_e, r_e, i_{\mathbf{coloop(e)}}, i_{\mathbf{loop(e)}}, i_\emptyset$ give others.

Hopf Alg. from Minor Systems (Krajewski, Moffatt, Tanasa 2017)

Definition (Minor System)

- Finite combinatorial objects $\{N\}$ w/ ground sets $E(N)$, graded by $|E(N)|$; unique 1 with $E(1) = \emptyset$; $E(N)$ consists of objects at level $|E(N)|$.
- For distinct $e, f \in E(N)$, deletion & contraction ops so both ($\backslash\backslash e$ or $//e$) commute with both ($\backslash\backslash f$ or $//f$).

.

Tutte Functions using determinants: Our setup

- Matrices N_α, $N_\beta^\perp$; full row rank, columns indexed by $P \coprod E$.
  $\mathrm{rank}(N_\alpha) + \mathrm{rank}(N_\beta^\perp) = |E| + |P|$.
  $P_\alpha, P_\beta \leftrightarrow P$, $P_\alpha \cap P_\beta = \emptyset$.
- Weight (parameter) matrices
  $G = \mathrm{diag}\{g_e\}_{e\in E}$, $R = \mathrm{diag}\{r_e\}_{e\in E}$.
- Matrix with columns $P_\alpha \coprod P_2 \coprod E$

$$L = L\begin{pmatrix} N_\alpha \\ N_\beta^\perp \end{pmatrix} = \left[\begin{array}{c|c|c} N_\alpha(P) & 0 & N_\alpha(E)G \\ \hline 0 & N_\beta^\perp(P) & N_\beta^\perp(E)R \end{array}\right]$$

Define

$$F(L) = (({2p \choose p})) - \text{tuple of determinants } L[Q_\alpha \overline{Q_\beta} E(\text{all of } E)])$$

indexed by length $p = |P|$ sequences $Q_\alpha \overline{Q_\beta} \subseteq P_\alpha P_\beta$ where $Q_\alpha \subseteq P_\alpha$ and $\overline{Q_\beta} \subseteq P_\beta$.

Column e of L when $e \notin P$ is

$$\begin{bmatrix} N_{\alpha,1,e}g_e \\ N_{\alpha,2,e}g_e \\ \cdots \\ N_{\alpha,r_1,e}g_e \\ N^{\perp}_{\beta,1,e}r_e \\ N^{\perp}_{\beta,2,e}r_e \\ \cdots \\ N^{\perp}_{\beta,r_2,e}r_e \end{bmatrix} = \begin{bmatrix} N_{\alpha,1,e} \\ N_{\alpha,2,e} \\ \cdots \\ N_{\alpha,r_1,e} \\ 0 \\ 0 \\ \cdots \\ 0 \end{bmatrix} g_e + \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ N^{\perp}_{\beta,1,e} \\ N^{\perp}_{\beta,2,e} \\ \cdots \\ N^{\perp}_{\beta,r_2,e} \end{bmatrix} r_e$$

So, for all $e \in E$, that is $e \notin P$:

$$F(L)_{Q_\alpha \overline{Q_\beta}} = L[Q_\alpha \overline{Q_\beta} E] =$$

$$g_e L \begin{pmatrix} N_\alpha / e \\ N^{\perp}_\beta \backslash e \end{pmatrix} [Q_\alpha \overline{Q_\beta} E] + r_e L \begin{pmatrix} N_\alpha \backslash e \\ N^{\perp}_\beta / e \end{pmatrix} [Q_\alpha \overline{Q_\beta} E].$$

Since deletion and contraction are done only for $e \notin P$

we get a **Ported** (sdc) or **Set-pointed** (Las Vergnas) or **restricted** (Dao and Hetyei) Tutte Function.

$|Q_\alpha \overline{Q_\beta}| = p$, so $\binom{2p}{p}$ determinants $L[Q_\alpha \overline{Q_\beta} E]$ make the tuple:

$$F(L) = g_e FL \begin{pmatrix} N_\alpha / e \\ N_\beta^\perp \backslash e \end{pmatrix} + r_e FL \begin{pmatrix} N_\alpha \backslash e \\ N_\beta^\perp / e \end{pmatrix}$$

where

N/e means remove the g_e or r_e but otherwise keep column e

$N \backslash e$ means replace column e by 0.

Plücker coordinates

These determinants can be considered an *affine* version of the (projective) Plücker coordinates for the row space of L projected into $K^{P_\alpha \coprod P_\beta}$. We need affine so Tutte's $+$ identity makes sense.

$$FL\begin{pmatrix} N_\alpha \\ N_\beta^\perp \end{pmatrix} = g_e FL\begin{pmatrix} N_\alpha/e \\ N_\beta^\perp\backslash e \end{pmatrix} + r_e FL\begin{pmatrix} N_\alpha\backslash e \\ N_\beta^\perp/e \end{pmatrix} \qquad (*)$$

Real deletion/contraction removes e from the ground set of the matroid or other object, but $N/e, N\backslash e$ still have column e. But (*) holds for all $e \in E$, so Laplace's expansion is a basis expansion:

$$L[Q_\alpha \overline{Q_\beta} E] = \sum_{A \subseteq E} g_A r_{\overline{A}} N_\alpha[Q_\alpha A] N_\beta^\perp[\overline{Q_\beta A}] \epsilon(Q_\alpha A, \overline{Q_\beta A})$$

The A term is $\neq 0$ iff $Q_\alpha A$ is a column basis for N_α and $\overline{Q_\beta A}$ is a column basis for $N_\beta^\perp$. So, for each $Q_\alpha \overline{Q_\beta}$

$$L[Q_\alpha \overline{Q_\beta} E] = \pm \sum_{A \subseteq E} g_A r_{\overline{A}} N_\alpha[Q_\alpha A] N_\beta^\perp[\overline{Q_\beta A}] \epsilon(A, \overline{A})$$

(The non-zero terms all have $|A| = \mathrm{rank}(N_\alpha) - |Q_\alpha|$.)

Quick and dirty fix

1. Drag column e to the far right.
   Changes sign of $F(L)$ by $\epsilon(E'e)$.
2. Left multiply by a determinant 1 matrix that sends the last column to $(0,\dots,1g_e,0,\dots,1r_e)^{\mathbf{t}}$ (if the top or bottom submatrix has just 1 row, do the hack: $\mathbf{N}/e$ is number $\mathbf{N}_{1,e}$ that acts like a matrix with columns E' and no rows.)
3. Drag the row with the $1g_e$ to the bottom.
   Changes sign of $F(L)$ by $(-1)^{\mathrm{r}\mathbf{N}_\beta^\perp}$
4. With e deleted/contracted from the $\mathbf{N}$s defining L, define F by $FL_{Q_\alpha\overline{Q_\beta}} = L[Q_\alpha\overline{Q_\beta}E']$

Result

$$FL\begin{pmatrix} N_\alpha \\ N_\beta^\perp \end{pmatrix} = \epsilon(E'e)\left(g_e(-1)^{\mathrm{r}(N_\beta^\perp)}FL\begin{pmatrix} N_\alpha/e \\ N_\beta^\perp\backslash e \end{pmatrix} + r_e FL\begin{pmatrix} N_\alpha\backslash e \\ N_\beta^\perp/e \end{pmatrix}\right)$$

Simplify calculations /w minors via Exterior Algebra

Full r-row minors of matrix N with columns indexed by S:

$$\begin{array}{ccc} (e_1) & (e_2) & (e_3) \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ \hline \multicolumn{3}{c}{\mathbf{N}[e_1 e_3] = (a_1 b_3 - a_3 b_1)} \end{array}$$

Coefficients when the exterior product of N's row vectors $\mathbf{N}$ are expressed in basis $\{\mathbf{e}_{i_1} \wedge \mathbf{e}_{i_2} \cdots \mathbf{e}_{i_r} | i_1 < i_2 \cdots < i_r\}$:

$$\begin{array}{cc} & (a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3) \\ \wedge & (b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3) \\ \hline \multicolumn{2}{c}{((a_1 b_3 - a_3 b_1)\mathbf{e}_1\mathbf{e}_3 + \cdots)} \end{array}$$

We sometimes omit the $\wedge$ and we can always write:

$$(\text{Exterior product})\mathbf{N} = \sum_{A \subseteq S; |E|=r} \mathbf{N}[A]\mathbf{A}$$

Each subset A is ordered $a_1 a_2 \ldots a_r$ **arbitrarilly** but $\mathbf{A}$ denotes the exterior product of (row coordinate vectors) **in the same order**

$$\mathbf{A} = \mathbf{a_1 a_2 \ldots a_r}$$

Catalogs of Oriented Matroid operations on the OM of matrix N and on $\mathbf{N} = \wedge(\text{rows}(N))$

domain D of operations:	chirotopes	exterior products
D is which functions:	$\pm\chi : B \mapsto \text{sign}(N[B])$	$\mathbf{N} : B \mapsto \mathbf{N}[B]$
type of fun. value	sign $\in \{0, +, -\}$	**field value; number**
deletion $\bullet\backslash A$	$\pm\chi' : B \mapsto \chi(B)$	$\mathbf{N}\backslash A : B \mapsto \mathbf{N}[B]$
contraction $\bullet/A$	$\pm\chi' : B \mapsto \chi(BA)$	$\mathbf{N}/A : B \mapsto \mathbf{N}[BA]$
duality $\bullet^{\perp}$	$\pm\chi^{\perp} : B \mapsto \chi(\overline{B})\epsilon(\overline{B}B)$	$\mathbf{N}^{\perp} : B \mapsto \mathbf{N}[\overline{B}]\epsilon(\overline{B}B)$

We must choose some global orientation ϵ in order to define duality as an exterior alg. operation!
ϵ is an alternating sign function on all finite sequences of elements.

This implies commutations

$$(\mathbf{N}\backslash X)^{\perp} = \epsilon(S')\epsilon(S'X)(\mathbf{N}^{\perp}/X)$$
$$(\mathbf{N}/X)^{\perp} = \epsilon(S')\epsilon(S'X)(-1)^{|X|r\mathbf{N}^{\perp}}(\mathbf{N}^{\perp}\backslash X)$$

Our setup - again

- Matrices N_α, $N_\beta^\perp$; full row rank, columns indexed by $P \coprod E$.
  $\mathrm{rank}(N_\alpha) + \mathrm{rank}(N_\beta^\perp) = |E| + |P|$.
  $P_\alpha, P_\beta \leftrightarrow P$, $P_\alpha \cap P_\beta = \emptyset$.
- Weight (parameter) matrices
  $G = \mathrm{diag}\{g_e\}_{e\in E}$, $R = \mathrm{diag}\{r_e\}_{e\in E}$.
- Matrix with columns $P_\alpha \coprod P_2 \coprod E$

$$L\begin{pmatrix} N_\alpha \\ N_\beta^\perp \end{pmatrix} = \left[\begin{array}{c|c|c} N_\alpha(P) & 0 & N_\alpha(E)G \\ \hline 0 & N_\beta^\perp(P) & N_\beta^\perp(E)R \end{array}\right]$$

Define

$$F(L) = (({2p \choose p})) - \text{tuple of determinants } L[Q_\alpha\overline{Q_\beta}E])$$

indexed by sequences $Q_\alpha\overline{Q_\beta} \subseteq P_\alpha P_\beta$ where $Q_\alpha \subseteq P_\alpha$, $\overline{Q_\beta} \subseteq P_\beta, |Q_\alpha\overline{Q_\beta}| = p = |P|$.

$$L\begin{pmatrix} N_\alpha \\ N_\beta^\perp \end{pmatrix} = \left[\begin{array}{c|c|c} N_\alpha(P) & 0 & N_\alpha(E)G \\ \hline 0 & N_\beta^\perp(P) & N_\beta^\perp(E)R \end{array}\right] \qquad F(L) = \text{tuple } (L[Q_\alpha \overline{Q_\beta} E])$$

Translate into exterior algebra definitions:

$$\mathbf{L}\begin{pmatrix} \mathbf{N}_\alpha \\ \mathbf{N}_\beta^\perp \end{pmatrix} := (\iota(\mathbf{N}_\alpha)(P_\alpha) + \iota_G(\mathbf{N}_\alpha(E))) \wedge (\upsilon(\mathbf{N}_\beta^\perp)(P_\beta) + \upsilon_R(\mathbf{N}_\beta^\perp)(E))$$
$$= (\iota_G(\mathbf{N}_\alpha) \wedge \upsilon_R(\mathbf{N}_\beta^\perp))$$

$$\mathbf{F}_E(\mathbf{L}) := \mathbf{L}/E = \sum_{Q_\alpha, \overline{Q_\beta}} \mathbf{L}[Q_\alpha \overline{Q_\beta} E]\mathbf{Q}_\alpha \overline{\mathbf{Q}_\beta}$$
$$= ((\iota(\mathbf{N}_\alpha)\backslash e(\text{no } \mathbf{e}) + g_e(\iota(\mathbf{N}_\alpha)/e) \wedge \mathbf{e})$$
$$\wedge (\upsilon(\mathbf{N}_\beta^\perp)\backslash e(\text{no } \mathbf{e}) + r_e(\upsilon(\mathbf{N}_\beta^\perp)/e) \wedge \mathbf{e}))/E$$

$$\text{2 of 4 terms} = \Big(r_e \qquad \iota(\mathbf{N}_\alpha)\backslash e \wedge (\upsilon(\mathbf{N}_\beta^\perp)/e) \wedge \mathbf{e}$$
$$\text{vanish} \quad + g_e(-1)^{r(\mathbf{N}_\beta^\perp)}(\iota(\mathbf{N}_\alpha)/e) \wedge (\upsilon(\mathbf{N}_\beta^\perp)\backslash e) \wedge \mathbf{e}\Big)/E$$

$$L\begin{pmatrix} N_\alpha \\ N_\beta^\perp \end{pmatrix} = \left[\begin{array}{c|c|c} N_\alpha(P) & 0 & N_\alpha(E)G \\ \hline 0 & N_\beta^\perp(P) & N_\beta^\perp(E)R \end{array}\right] \qquad F(L) = \text{tuple } (L[Q_\alpha \overline{Q_\beta} E])$$

$$\begin{aligned}\mathbf{F}_E(\mathbf{L}) = \mathbf{L}/E = \Big(r_e \qquad & \iota(\mathbf{N}_\alpha\backslash e) \wedge (\upsilon(\mathbf{N}_\beta^\perp/e)) \wedge \mathbf{e} \\ & + g_e(-1)^{r(\mathbf{N}_\beta^\perp)}(\iota(\mathbf{N}_\alpha/e)) \wedge (\upsilon(\mathbf{N}_\beta^\perp\backslash e)) \wedge \mathbf{e}\Big)/E\end{aligned}$$

$$= r_e\left(\mathbf{L}\begin{pmatrix} \mathbf{N}_\alpha\backslash e \\ \mathbf{N}_\beta^\perp/e \end{pmatrix} \wedge \mathbf{e}/E\right) + g_e(-1)^{r(\mathbf{N}_\beta^\perp)}\left(\mathbf{L}\begin{pmatrix} \mathbf{N}_\alpha/e \\ \mathbf{N}_\beta^\perp\backslash e \end{pmatrix} \wedge \mathbf{e}/E\right)$$

$$(\mathbf{N}\backslash e)^\perp = \epsilon(S')\epsilon(S'e)(\mathbf{N}^\perp/e) \; ; \; (\mathbf{N}/e)^\perp = \epsilon(S')\epsilon(S'e)(-1)^{|\{e\}| r\mathbf{N}^\perp}(\mathbf{N}^\perp\backslash e)$$

Result

$$= \epsilon(S)\epsilon(S'e)(r_e\left(\mathbf{L}\begin{pmatrix} \mathbf{N}_\alpha\backslash e \\ (\mathbf{N}_\beta\backslash e)^\perp \end{pmatrix} \wedge \mathbf{e}/E\right) + g_e\left(\mathbf{L}\begin{pmatrix} \mathbf{N}_\alpha/e \\ (\mathbf{N}_\beta/e)^\perp \end{pmatrix} \wedge \mathbf{e}/E\right))$$

With $\mathbf{L}(\mathbf{N}_\alpha \ \ \mathbf{N}_\beta) = \mathbf{L}\begin{pmatrix} \mathbf{N}_\alpha \\ \mathbf{N}_\beta^\perp \end{pmatrix}$, and more sign calculations:

Definition

For E, P sets written as ordered sequences,

$$\mathbf{F}_E(\mathbf{N}_\alpha \ \ \mathbf{N}_\beta) = \mathbf{L}(\mathbf{N}_\alpha \ \ \mathbf{N}_\beta)/E$$

Theorem

$$\epsilon(PE)\mathbf{F}_E(\mathbf{N}_\alpha \ \ \mathbf{N}_\beta) = \\ \epsilon(PE')\left(g_e\mathbf{F}_{E'}(\mathbf{N}_\alpha/e \ \ \mathbf{N}_\beta/e) + r_e\mathbf{F}_{E'}(\mathbf{N}_\alpha\backslash e \ \ \mathbf{N}_\beta\backslash e)\right)$$

Corollary

1. $\mathbf{F} = \mathbf{F}_E(\mathbf{N}_\alpha \;\; \mathbf{N}_\beta) = \pm \sum_{H\subset E} g_H r_{\overline{H}} \mathbf{F}_\emptyset(\mathbf{N}_\alpha/H\backslash\overline{H} \;\; \mathbf{N}_\beta/H\backslash\overline{H})$
2. *Componentwise,* $\sum_{Q_\alpha,Q_\beta} \mathbf{F}_E[Q_\alpha\overline{Q}_\beta]\mathbf{Q}_\alpha\overline{\mathbf{Q}_\beta} =$

$$= \pm \sum_{Q_\alpha,Q_\beta} \sum_{H\in E} g_H r_{\overline{H}} \mathbf{N}_\alpha[Q_\alpha H]\mathbf{N}_\beta^\perp[\overline{Q_\beta H}]$$

$$= \pm \sum_{Q_\alpha,Q_\beta} \sum_{H\in E} g_H r_{\overline{H}} \mathbf{N}_\alpha[Q_\alpha H]\mathbf{N}_\beta[Q_\beta H]$$

3. *Two expr. for products of numbers* $\mathbf{N}_\alpha[Q_\alpha H]\mathbf{N}_\beta[Q_\beta H]$*:*

$$(\mathbf{N}_\alpha/Q_\alpha)[H]\cdot(\mathbf{N}_\beta/Q_\beta)[H] = (\mathbf{N}_\alpha/H)[Q_\alpha]\cdot(\mathbf{N}_\beta/H)[Q_\beta]$$

4. *It's non-zero iff H is a common basis (in the matroids of)* $\mathbf{N}_\alpha/Q_\alpha$ *and* $\mathbf{N}_\beta/Q_\beta$
   iff Q_α is a basis in $\mathbf{N}_\alpha/H$ *and Q_β is a basis in* $\mathbf{N}_\beta/H$

Weighted Laplacian-like matrices

Generalize a graph's incidence matrix: Make P label the rows, E the columns of any matrices A_α, A_β. Take all $r_e \neq 0$. Then, $N_\alpha = (I(P) \;\; A_\alpha(E))$ and $N_\beta = (I(P) \;\; A_\beta(E))$, and

$$L\begin{pmatrix} N_\alpha \\ N_\beta^\perp \end{pmatrix} = \left[\begin{array}{c|c|c} I & 0 & A_\alpha G \\ \hline 0 & -A_\beta^t & IR \end{array}\right] = L\left(N_\alpha \quad N_\beta\right). \text{ Do row ops:}$$

$$\begin{pmatrix} I & -A_\alpha G R^{-1} \\ 0 & R^{-1} \end{pmatrix} L = \begin{pmatrix} I & A_\alpha G R^{-1} A_\beta^t & 0 \\ 0 & -R^{-1} A_\beta^t & I \end{pmatrix}, \text{ and therefore}$$

$$\epsilon(Q_\alpha \overline{Q_\alpha}) \mathbf{F}_E(\mathbf{L})[Q_\alpha \overline{Q_\beta}] = \frac{1}{r_E} \sum_{B \in E} g_B r_{\overline{B}} A_\alpha[\overline{Q_\alpha} B] A_\beta[\overline{Q_\beta} B]$$

is the Cauchy-Binet expansion of any minor $(\overline{Q_\alpha}, \overline{Q_\beta})$ of the weighted graph Laplacian-like matrix $A_\alpha G R^{-1} A_\beta^t$.
(Note $\frac{1}{r_E} r_{\overline{B}} = (r^{-1})_B$.)

Examples

$N_\alpha = N_\beta = N; A =$ graph's incidence matrix w/ columns $(0,..0,1,0,..,-1,0,..,0)^t$ for each edge; reps. graphic matroid.

The all-minors Matrix Tree Theorem for weighted undirected graphs

A_α, N_α as above. $A_\beta =$ only the $+1$ entries of A for a directed graph, so $+1$ is for an edge head on a vertex.

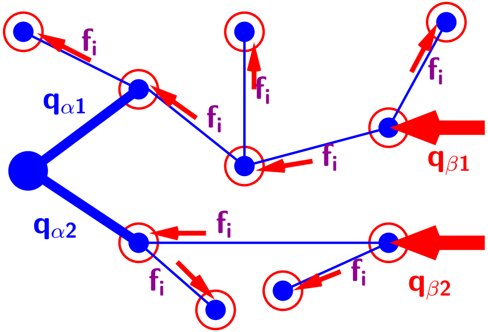
The all-minors Matrix Tree Theorem for weighted directed graphs

$N_\alpha = N_\beta = N; A =$ gain graph's incidence matrix w/ columns $(0,..,0,1,0,..,-\gamma_e,0,..,0)^t$ for e with gain $\gamma_e \in \mathbf{C}$.

All-minors expansions of the gain graph's Laplacian

NB: Edge Gains γ_e are DIFFERENT ATTRIBUTES from weights/parameters g_e, r_e

All-Minors Digraph Matrix Tree Theorem Example

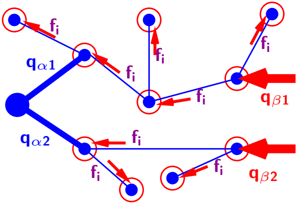


This contributes the term

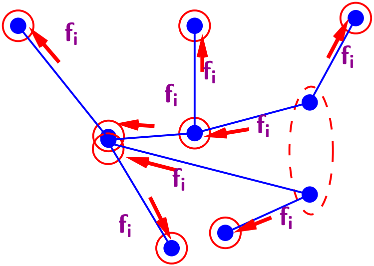
$g_F r_{\overline{F}} \mathbf{N}_\alpha[Q_\alpha F]\mathbf{N}_\beta[Q_\beta F]$.

The $\mathbf{q}_{\alpha 1}, \mathbf{q}_{\alpha 2}$ port edges $\cup$ the f_i elements as edges in the graphic matroid comprise a spanning tree.

The $\mathbf{q}_{\beta 1}, \mathbf{q}_{\beta 2}$ port arrows $\cup$ the f_i elements as arrows in a partition matroid comprise a basis. Each part (a red cirle) of the partition is the set of arrows incident to a vertex, except the star vertex.



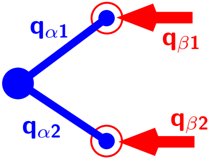
Contract the ports.



Count the bases in common.

$g_F \, r_{\overline{F}} \mathbf{N}_\alpha / Q_\alpha [F] \mathbf{N}_\beta / Q_\beta [F]$.

Contract the non-ports.



α and β ports are bases in the contracted N_α and N_β matroids.

$g_F \, r_{\overline{F}} \mathbf{N}_\alpha / F[Q_\alpha] \mathbf{N}_\beta / F[Q_\beta]$.

Chain Complexes View (Alg. Topology, Homological Alg.)

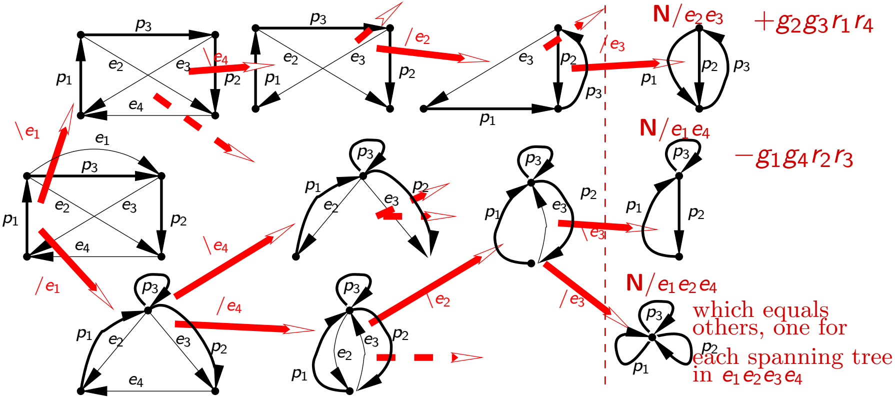
A graph is a k-dim simplicial complex X with $k = 1$.
In general, for us, the k-chains $C_k = Z[P \coprod E] = \{\sum_{x \in P \coprod E} c_e e\}$ are the free abelian group with basis $P \coprod E$.
The k-cochains $C^k = \mathrm{Hom}(C_k, \mathbb{R})$ is the $\mathbb{R}$-module of linear maps from C_k to a coefficient ring $\mathbb{R}$.
The k-complex $X = \coprod_{j=0}^{k} X_j$ (X_j is the set of j-simplices) determines, (or the chain complex might just be subspaces given with) **boundary maps** $\partial_j : C_j \to C_{j-1}$ for $j = 0, ..., k$ that satisfy $\partial_{j-1} \circ \partial_j = 0$ for each j.
The dual $\delta^j : C^{j-1} \to C^j$ is defined by $(\delta^j(u^*))(v) = u^*(\partial_j(v))$.

In the case $N_\alpha = N_\beta$, generlizing:

- $\mathbf{N}$ ($\wedge$ of the rows on N_α) represents the k-coboundary group $B^k = \mathrm{img}(\delta_k)$.
- The equation $N_\alpha \begin{pmatrix} I \\ G \end{pmatrix} (J_P \;\; X_E)^t = 0$ says $\begin{pmatrix} I \\ G \end{pmatrix} (J_P \;\; X_E) \in Z_1$, is a k-cycle. (Electrically, a flow of currents in edges.)
- $\mathbf{N}^\perp$ ($\wedge$ of the rows of $N^\perp$) represents the k-cycle group $Z_k = \ker(\partial_k)$.
- The equation $N^\perp \begin{pmatrix} I \\ R \end{pmatrix} (V_P \;\; X_E)^t = 0$ says $\begin{pmatrix} I \\ R \end{pmatrix} (V_P \;\; X_E) \in Z_1$, is a k-coboundary $\delta_k \psi$. (Electrically, $\delta_1\psi$ maps each edge (1-simplex) to the difference of electrical potential assigned to vertices (a 1-cochain) $\delta_1(\psi)(v_0 v_1) = \psi(v_1) - \psi(v_2)$.

Ported Tutte Decomposition (incomplete)

The decomposition ends with minors $\mathbf{N}/F\overline{F}$.
We show component $\mathbf{L}_E(\mathbf{N}\ \mathbf{N})[p_{\alpha 1}p_{\beta 1}p_{\alpha 3}]$:



Known to EEs: Linear electrical networks with IDEAL AMPLIFIERS

$N_\alpha i(P,E) = 0$ expresses Kirchhoff's current law on currents i_e in the network edges (along edge direction) and currents i_p into vertices from external connections.

$N_\beta^\perp v(P,E) = 0$ expresses Kirchoff's voltage law: The voltage rise along a network edge $v_e = v_h - v_t$ is the difference of the head and tail vertex potentials. (Sometimes the vertex potentials are imposed by external connections.)

$N_\alpha = N_\beta$ in ordinary resistor networks.

Different Graphs for N_α and N_β

W. K. Chen models networks with ideal amplifiers by N_α by one graph on (P,E) called the **Current Graph** and another graph also on (P,E) called the **Voltage Graph**.