REPORT (June 19, 2002: 874) THE ORIENTED MATROID PAIR MODEL FOR MONOTONE DC ELECTRICAL AND ELASTIC NETWORK UNIQUE SOLVABILITY

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Abstract. Resistive electrical networks and elastic mechanical systems such as trusses have a topological or geometric structure together with constitutive laws for the elements prior to their interconnection. Oriented matroids provide a common discrete mathematical model for such structure in which relationships on the signs of element quantities can be expressed. Pairing of oriented matroids enables non-linear monotone constitutive laws to be fit into the abstraction in a way that allows port and nullor insertions and provides discrete unique solvability conditions.

The resulting mathematical model clarifies some mechanical analogies for these circuit theory concepts, relates apparently dissimilar published theories for existence and uniqueness and shows how to handle elastic mechanical systems with small displacements. It also enables constraints on the signs of system quantities to be predicted from the structure when this is possible. Finally, it derives topological solution formulas for linearized mechanical systems in which the analog of a tree-sum is a sum over minimally rigid trusses.

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1. Introduction

Our topic in non-linear systems is DC equations, say for operating points/resistive circuits, whose only non-linearities are monotone increasing bijections $\mathbf{R} \to \mathbf{R}$. Special case conditions for existence and uniqueness of solutions for all such non-linear functions and additive constants were given by Duffin, Minty, and Rockafellar. The more general determinant based theory of $\mathcal{W}_0$ matrix pairs due to Sandberg and Willson [**SW69**] extended Fiedler and Pták's work [**FP66**]. Nielsen and Willson [**NW80**] used it to prove that disallowing the 2 transistor feedback structure in a transistor circuit is sufficient for uniqueness. Graph based theories that identified structures forbidden for general solvability and uniqueness were given by Hasler, Neirynck, and others [**HMOdW94, HN86, Fos92**] for nullor/resistor networks; and by Nishi and Chua [**NC84, NC86b**] for networks with all kinds of 2-port controlled sources, applied in [**NC86a**] to reproduce Nielsen and Willson's result.

Part of this reseach was done during a Sabbatical from the University at Albany in 2001. This report represents on-going research.

Hasler *et. als.* conclusion is that a "pair of conjugate spanning trees" and the absence of a "non-trivial uniform partial orientation of the resistor [edges]" are necessary and sufficient for existence and uniqueness of solutions for all suitable functions and source values. Nishi and Chua's structures are "cactus graph" networks with negative determinants obtained by deletion/contraction operations particular to each kind of controlled source.

Our paper shows oriented matroid (OM) theory [**BK92, BVS+99**] covers Hasler *et. als.* "conjugate spanning tree" and "orientation" concepts to provide a theory equivalent to Sandberg and Willson's theory of $\mathcal{W}_0$ pairs to solve the same problem. We also review the key oriented matroid concepts and demonstrate them on one feedback structure case of [**TW90**]. We believe distinguished port elements, pairings of oriented matroids with a common ground set, and common covectors (explained below) are crucial. Although oriented matroids can also be axiomatized with "chirotopes" which abstract determinant signs, (so suitable abstractions of graph orientation properties are mathematically equivalent to principles behind determinant signs) the oriented matroid pair common covector approach has an intuitive advantage for qualitative reasoning because the common covector displays precisely the signs of all state quantities or their differences.

Preliminary connections between our approach and the issue of DC operating point stability [**GW96**] were published [**Cha98**], but relating these determinant based results to common covectors is still under investigation.

The recognition of Minty's painting property[**VC80**] and other facts from OM theory used by Hasler *et. al.* [**HN86, Fos92, HMOdW94**] led us to generalize in [**Cha96**] their graph model notion of a "pair of conjugate trees" to a "complementary pair of bases" in a pair of matroids (which abstract the "voltage and current graphs" of [**Che76**] and others); and of a "non-trivial uniform partial orientation of the resistors" to a "common (non-zero) covector in an oriented matroid pair". We showed that the graph model generalizes to a linear subspace pair model; the pair of linear subspaces defines a pair of oriented matroids. This OM pair is the discrete structure that generalizes a graph with designated resistor, source, nullator and norator edges. Topological conditions for the existence or uniqueness of solutions are expressed in it. Two real matrices, which can be easily generated from the system design, represent the OMs so that it is practical to work with them. (Some OMs, indeed most, are not representable by real matrices and would require much more space to store, but they do not occur in our application.) Theory valid for all OM pairs, not just those represented by a pair of linear subspaces, was presented in [**Cha96**]. For lack of space, we omit determinant sign conditions shown equivalent in [**Cha96**] to the cases of common covector conditions that we cover.

In the electrical circuit theory literature, the circuit "topology" means the network graph (to which Kirchhoff's laws apply) together with particular kinds of "device elements" such as resistors, capacitors, voltage sources (batteries), current sources, etc., associated with single graph edges, and possibly "multiport" elements associated with multiple graph edges. Each graph edge is associated with one voltage and one current variable. Kirchhoff's laws and the graph determine homogeneous linear equations on these variables.

The electrical network model consists of the "topology" plus particular numeric or symbolic constitutive laws and or parameters the define the characteristics of each device. Idealizations of certain devices such as operational amplifiers enable the

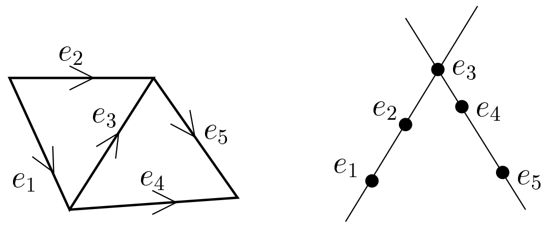


FIGURE 1. $\{e_1, e_2, e_3\}, \{e_3, e_4, e_5\}, \{e_1, e_2, e_4, e_5\}$ are supports of cycles and are minimal dependencies in the graphic matroid and in its equivalent affine dependency matroid of the 2-dimensional point arrangement.

modeller to use more a general "topology" in lieu of these devices. In particular, nullators and norators are two-terminal elements that obey Kirchhoff's laws but have constitutive relations $(v, i) = (0, 0)$ and $(v, i) \in \mathbf{R}^2$ (unrestricted) respectively. Such idealization requires that the actual system is stabilized by feedback as a dynamical system. Problems with multiport elements are reduced to those with only single edge elements, in order to apply the theory of [**HMOdW94**], through the use of these nullator and norator elements. Detailed exposition of the problems, reductions, theory and applications is given in [**HN86**].

1.1. Topology and Geometry. The term "topology" of an electrical network applies to the network graph as a 1-dimensional finite simplicial complex (together with edge labels to distinguish kinds of edges.) This graph determines the particular Kirchhoff law constraints on the voltage and current variables of the network model. The relevant information is coded by the graph's oriented matroid. Matroid theory lets us study certain combinatorial properties of graphs in geometric terms[1]. Figure 1.1 illustrates shows how the circuits in a graph, which are the minimal dependencies among voltage drops under KVL, are the affine dependencies among points in the 2-dimensional plane. These dependencies comprise the same matroid. A theme in our work is to extend analytical theories of graph based (electrical) networks to applications where a geometric object plays the role of the graph. In rigidity theory, this geometric object is the embedded framework (graph with nodes embedded in $\mathbf{R}^d$). The relevent oriented matroids do depend on the embedding as well as the graph.

Graphic oriented matroids belong to the special class of *regular* (oriented) matroids, also called totally unimodular matroids. These are the matroids that be represented with matrices all of whose minors are $+1$, -1 or 0. The rigidity oriented matroids (from frameworks) are generally not regular.

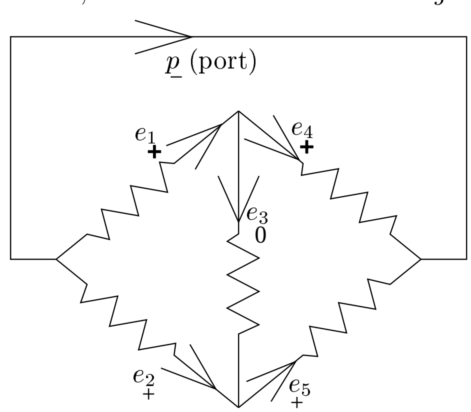
[1]Recall that combinatorial geometry has been used as alternative term for simple matroid.

Heuristic Comment ("Fuzz") 1. The table below indicates some of the prior results:

Minty[]	Dual pair of regular matroids, abstracting a graph.
Rockafellar[**Roc67**]	Orthogonal complementary subspaces
Hasler, Neirynck[**HN86, Fos92**]	Pair of graphic and cographic matroids, from different graphs.
First-order elastic frameworks	A case of a dual pair of realizable oriented matroids.
Sandberg, Willson[**SW69, SW72**]	We show their theory is for a pair of realizable oriented matroids with the complementary base/no-common-covector property.

1.2. Qualitative Reasoning. In many cases, interesting conclusions can be reached from the sign patterns of feasible subspace members by "calculations" in an "algebra of signs", a form of qualitative reasoning that uses easy, fundamental oriented matroid theoretic operations upon matrix sign patterns. In cases when the qualitative calculations show the outcome depends on numeric values, the numeric information can then be used, say to calculate a new matrix whose signs reveal better information; or else, inequalities on system parameters for each case of outcome can be derived. The pairing seems to be needed because the non-linear monotonicity constrains two quantities to only have a common sign. The list of those signs for one state or state difference is the common covector. (Our structural/constitutive law separation by OM pairing handles issues different from "imprecise constitutive law constants" of [**Mur00**] and others.)

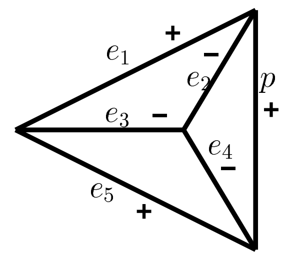
1.2.1. *Introductory examples.* Here are two examples of inferences from oriented matroid theory. First, consider the *Wheatstone bridge* network below.



Among the covectors in $\mathcal{L}_I$ are $X_{e_1}X_{e_2}X_{e_3}X_{e_4}X_{e_5}X_p = +0+0+-$ and $0+-+0-$. These are indeed cocircuits; they signify the signature of the current flows in the elementary cycles $\{e_1, e_3, e_5, p\}$ $\{e_2, e_3, e_4, p\}$. It's an immediate consequence of Definition 1 (**L3**) (see 1.3) that $++0++- \in \mathcal{L}_I$.

When the resistance values are positive or, more generally, the resistors have nonlinear characteristics satisfying $v_e i_e \geq 0$ and $v_e i_e = 0$ if and only if $v_e = 0$ and $i_e = 0$, covectors $X \in \mathcal{L}_V$ and $Y \in \mathcal{L}_I$ indicate signatures of feasible electrical solutions only if $X_{e_i} = Y_{e_i}$, $i = 1, \ldots, 5$. The duality of $\mathcal{M}_V$ and $\mathcal{M}_I$ then implies $X_p Y_p = -$, so neither can be 0.

Our second example is the 2-dimensional framework below.



The only non-zero covectors in $\mathcal{L}_I$ now is $\pm(+--+-+)$. This describes the signature of the only non-zero self-stress. The signatures of first-order bar length changes comprise the covectors of the oriented matroid dual $\mathcal{L}_V$. Oriented matroid orthogonality now rules out length change combinations of single bars only (the framework is rigid), certain changes in adjacent pairs of bars (such as $e_1p = \pm(+-)$) or triples (such as $e_1e_2p = +-+$).

1.2.2. *Monotone source dependency.* Hasler, Wang and Chauffoureaux [**CH90, HW92**] gave a condition that distinguished when a current or voltage in one resistor of a nonlinear resistor/nullator network depends monotonically on one source value and predicts the direction of the dependence. Their condition generalizes to the relative sign of the corresponding two elements in all common covectors. They also show that when the direction of dependence is unpredicatable then there exist non-linear monontonic increasing constitutive law functions for which the corresponding dependency is not monotonic.

Despite the intuitive appeal of common covectors, it is not resolved whether search for a common covector or an algorithm based on determinant signs is more efficient either in practice or theoretically. In [**Cha96**] is is shown how the existence of a common covector in the cases of supplemental subspace pairs with balanced rank and free union is as hard as the problem of telling if a digraph has an even directed circuit. The latter problem was recently shown to have a polynomial time solution in [**RST99**]

1.3. Single Oriented Matroids. See [**BVS$^+$99**] for full details about OMs; [**BK92**] is a good introduction to our point of view. Other ways to apply matroids to electrical and other systems are given in [**Rec89, Mur00, GSS93**] and EE literature on symbolic simulation[**?**]. A full survey is omitted.

We think of the oriented matroid $\mathcal{M}(M)$ *represented* by matrix M as its finite set of *covectors* $\mathcal{L}(M)$, where each covector is the tuple of the *signs* $\{+,-,0\}$ or *signature* $X = \sigma(l)$ of the real coordinates of a member l of the *linear subspace* $L(M)$ in $\mathbf{R}^U$ spanned by the rows of M. (Italics denote a *term or symbol being defined.*) Hence $\mathcal{L}(M) = \mathcal{L}(L(M))$ has at most $3^{|U|}$ covectors. For example, when M is the signed incidence matrix of a network graph, each covector represents a combination of branch voltage drop signs feasible under Kirchhoff's voltage law; the finite *ground set* U labels the branches. One can call X a *signed set*, (formally speaking, a function $X : U \to \{+,-,0\}$ denoted by $e \to X_e$) in which elements of subset X^+ occur with $+$ sign and those in X^- have $-$ sign. The *support* supp X is the subset of $e \in U$ for which $X_e \neq 0$, i.e., supp $X = X^+ \cup X^-$. Bland [**Bla77**] defined the oriented matroid represented by a linear subspace this way in order to abstract the combinatorics of linear programming. For brevity's sake, we define oriented matroids using some "sign algebra" operation properties we will then use.

Given two sign tuples X^1, X^2, their *composition* $Z = X^1 \circ X^2$ has supp $Z =$ supp $X^1 \cup$ supp X^2 and for $e \in$ supp Z, $X_e = X_e^i$ where i is the smallest index for which $X_e^i \neq 0$. Note that if $X^i = \sigma(l^i)$ for $l^i \in \mathbf{R}^U$, then $X^1 \circ X^2 = \sigma(l^1 + \epsilon l^2)$ for some sufficiently small $\epsilon > 0$. Hence $\mathcal{L}(M)$ is closed under $\circ$.

Definition 1. The collection $\mathcal{L}(\mathcal{M})$ of signed sets with ground set U is the set of *covectors* of an oriented matroid $\mathcal{M}$ if it satisfies:
(L0) $0 \in \mathcal{L}$. **(L1-2)** If $X, Y \in \mathcal{L}$ then $-X$ and $X \circ Y \in \mathcal{L}$.
(L3) For all X, $Y \in \mathcal{L}$ and $e \in X^+ \cap Y^-$ there is $Z \in \mathcal{L}$ such that $Z^+ \subset (X^+ \cup Y^+) \backslash \{e\}$, $Z^- \subset (X^- \cup Y^-) \backslash \{e\}$, and (supp $X \backslash$ supp $Y) \cup$(supp $Y \backslash$ supp $X) \cup (X^+ \cup Y^+) \cup (X^- \cup Y^-) \subset$ supp Z.

Property (L3) says $Z_e = 0$ and it predicts Z_g for all $g \neq e$ except those with $X_g Y_g = -$; i.e., g having opposite signs in X and Y. The logical equivalence of this definition to various apparently weaker axiomatizations is due to work of Edmonds, Fukada and Mandel cited and surveyed in [**BVS+99**].

Other oriented matroid notions such as duality (orthogonality) and independence can be expressed by properties of covector sets that are motivated by linear algebra. The covectors $\mathcal{L}(L^\perp)$ of the *orthogonal complement* $L^\perp$ of linear subspace $L \subset \mathbf{R}^U$ form another oriented matroid. We say $X \perp Y$ for signed sets X, Y when either supp $X \cap$ supp $Y = \emptyset$ or there are $e, f \in U$ with $X_f Y_f = -X_e Y_e \neq 0$. This abstracts a necessary condition for two real vectors to be orthogonal under the usual dot product. In fact, for every covector set $\mathcal{L}(\mathcal{M})$, the set $\mathcal{V} = \mathcal{L}^\perp$ defined by $\{Y | Y \perp X \text{ for all } X \in \mathcal{L}\}$ satisfies the covector axioms ([**BVS+99**], Prop. 3.7.12); $\mathcal{V}(\mathcal{M})$ is called the set of *vectors* of $\mathcal{M}$ and is the set of covectors of the *dual* oriented matroid $\mathcal{M}^\perp$. The OM vectors display all combinations of coefficient sign that occur among all linear dependencies of the columns of M, when $\mathcal{M} = \mathcal{M}(M)$. More directly, an independent set $I \subset U$ is characterized by: for all $3^{|I|}$ "input" assignments $i : I \to \{+, -, 0\}$, there exists a covector $X \in \mathcal{L}(\mathcal{M})$ for which $X_e = i_e$ for all $e \in I$. Abstractly, an *independent set* $I \subset U$ satisfies supp $V \not\subset I$ for all non-zero vectors $V \in \mathcal{V}(M)$.

We will use the terms *orthogonal* for linear subspaces (and signed sets for which $X \perp Y$) but reserve *dual* for dual oriented matroids although some authors use orthogonal for the latter.

KVL, KCL and analogous mechanical structural or geometric laws are each formulated by a constraint of the form $v \in L =$ row space(M) where $L \subset \mathbf{R}^U$. To reformulate this law by a system of linear equations, a maximal subset $B \subset U$ corresponding to a linearly independent set of columns of M is found. Such a B is a maximal independent set, called a *basis in the matroid* $\mathcal{M}(M)$. The collection of all bases in $\mathcal{M}$ is denoted by $\mathcal{B}(\mathcal{M})$. Row operations and possibly deletion of zero rows can transform M to $[I\ M^{\overline{B}}]$ (after column permutation) where I is the $r \times r$ identity matrix, where $r = \text{rank}(M) = \dim(L) = \text{rank}(L) = \text{rank}(\mathcal{M}(L))$. Thus $v \in L$ if and only if $v_{\overline{B}} = v_B M^{\overline{B}}$. For each independently chosen $v_B \in \mathbf{R}^B$, $v = (v_B; v_{\overline{B}}) \in L$ is unique with its B coordinates equal to v_B.

The *cocircuits* (resp. *circuits* $\mathcal{C}$) of an oriented matroid are the non-zero covectors (resp. vectors) whose support is minimal. Minty's painting property, most popularly known as a theorem about directed graphs[**VC80**], is generally true about the cocircuit $\mathcal{C}^*(\mathcal{M})$ and circuit $\mathcal{C}(\mathcal{M})$ collections. Note $\mathcal{C}$ is the cocircuits of the dual oriented matroid $\mathcal{M}^\perp$. In fact, when the simple non-triviality, symmetry, and minimal support properties are assumed, the painting property characterizes when $\mathcal{C}^*$ and $\mathcal{C}$ are the cocircuit/circuit collections of an oriented matroid.

THEOREM 1. *([**BVS**$^+$**99**], Th. 3.4.4(4P); [**BK92**], Prop. 5.12) For every partition $U = R \cup G \cup B \cup W$ and for every $e \in R \cup G$, **either** (a) There exists $X \in \mathcal{C}^*$ so $e \in SuppX$, $X_R \geq 0$, $X_G \leq 0$, X_B unrestricted and $X_W = 0$ **or***
(b) There exists $Y \in \mathcal{C}$ so $e \in SuppY$, $Y_R \geq 0$, $Y_G \leq 0$, $Y_B = 0$ and Y_W unrestricted ***but not both.***

1.4. Tableaux. We will use the tableau notation (defined and used for oriented matroid programs developed in Chapter 10 of [**BVS**$^+$**99**][2]) to express fundamental cocircuits associated with a given oriented matroid basis B so that, when the oriented matroid $\mathcal{M}$ is realizable, the tableau encodes the sign pattern of the matrix $M = [I \ M^{\overline{B}}]$, with B indexing the first $|B|$ columns. This M is called the fundamental cocycle matrix associated with basis B. M is illustrated by the tableau:

$$M: \begin{array}{c|c} B & \overline{B} \\ \hline \diagdown & M^{\overline{B}} \end{array}$$

1.5. Pseudosphere Arrangements. We first review the geometric interpretation of oriented matroid $\mathcal{M}(M)$ as an arrangement of hyperplanes with distinguished sides that is described in section 1.2 of [**BVS**$^+$**99**]. Next, the topological representation theory of Folkman, Lawrence, Edmonds and Mandel is described. Finally, a heuristic explanation is given for the significance of oriented matroid objects to systems defined and observed in terms of states and functions of states.

A *central hyperplane* H_e in $\mathbf{R}^r$ is the solution subset $\{y \in \mathbf{R}^r | ym_e = 0\}$ for the linear constraint $ym_e = 0$ where m_e is column e of say the $r \times U$ matrix M. The closed halfspace $H_e^+ = \{y \in \mathbf{R}^r | ym_e \geq 0\}$ is called the closed *positive side* of H_e. In this way, matrix M defines an *arrangement of halfspaces.* The oriented matroid $\mathcal{M}(M)$

> encodes a lot of geometric information about the hyperplane arrangement in a very simple and explicit way. For this, observe that every hyperplane arrangement decomposes its ambient space into a collection of pieces (that are in fact relatively open topological cells [homeomorphic to balls] of various dimensions). Each of these *cells* is exactly determined by the information whether for hyperplane H_e, the cell is on its positive side, its negative side, or on the hyperplane itself. This leads us to associate a sign vector with every cell.
>
> These sign vectors are exactly the covectors of the oriented matroid

$\mathcal{M}(M)$. This is simply another way at looking at how a realizable oriented matroid is defined by its covectors from the row space $L(M) \subset \mathbf{R}^U$ of matrix M. As y ranges over $\mathbf{R}^r$, the covector $\sigma(yM)$, which is the tuple of $(\sigma(ym_e))_{e \in U}$, is a code for the cell that contains y. There are two advantages to this point of view:

[2]Our notation differs slightly as we include the current basis elements in the column set of the tableau.

(1) When we think of y as an exact "state," the covector $\sigma(yM)$ indicates the combination of signs of coordinates and/or any other linear functions of y. (2) The Topological Representation Theorem of Folkman and Lawrence says that *every* oriented matroid (system defined by Definition 1) has a signed arrangement of suitable generalizations of hyperplanes whose cells are coded by its covector set $\mathcal{L}$.

Every signed arrangement of hyperplanes $(H_e | e \in U)$ is determined by its intersections $S_e^\sigma = H_e^\sigma \cap S^{(r-1)}$ within the sphere $S^{(r-1)} = \{y \in \mathbf{R}^r || y| = 1\}$. S_e^0 is a sphere of dimension $r-2$ in $S^{(r-1)}$ and S_e^+ is (topologically) a $(r-1)$ dimension ball in $S^{(r-1)}$. It is convenient to state the Folkman and Lawrence Theorem in terms of a a signed arrangement of generalized spheres (called pseudospheres) within $S^{(r-1)}$. The relevent definitions and statements are paraphrased from [**BVS$^+$99**] below.

A subset S of S^{r-1} is called a *pseudosphere* if $S = h(S^{r-2})$ for some homeomorphism $h : S^{(r-1)} \to S^{(r-1)}$ [continuous one-to-one correspondance with a continuous inverse]. A pseudosphere S has two sides (hemispheres) S^+ and S^-, and is, of course, topologically indistinguishable from a linear $(r-2)$-subsphere. We define an *arrangement of pseudospheres* $\mathcal{A} = (S_e)_{e \in U}$ to be a finite set of pseudospheres S_e in S^{r-1} such that

Definition 2. (Arrangement of Pseudospheres)

- A1 Every non-empty intersection $S_A = \cap_{e \in A} S_e$ is homeomorphic to a sphere of some dimension, for $A \subseteq U$.
- A2 For every non-empty intersection S_A and every $e \in U$ such that $S_A \not\subseteq S_e$, the intersection $S_A \cap S_e$ is a pseudosphere in S_A with sides $S_A \cap S_e^+$ and $S_A \cap S_e^-$.
- A3 The intersection of a arbitrary collection of closed sides is either a sphere or a ball. (This axiom is redundant. Edmonds and Mandel [] showed that A3 is implied by A1 and A2 but the proof is difficult.)

The stronger version, proved by Edmonds and Mandel, of the Topological Representation Theorem can now be stated:

Theorem 2. *Let $\mathcal{L} \subseteq \{+, -, 0\}^U$. Then the following three conditions are equivalent:*

- (i) *$\mathcal{L}$ is the set of covectors of a loop-free oriented matroid of rank r. (I.e., $\mathcal{L}$ satisfies Definition 1 and for no e is $X_e = 0$ for all $X \in \mathcal{L}$. For $\mathcal{L} = \mathcal{L}(M)$, this means M has no zero columns.)*
- (ii) *$\mathcal{L} = \mathcal{L}(\mathcal{A})$ for some signed arrangement $\mathcal{A} = (S_e)_{e \in U}$ of pseudospheres in S^{r+k}, such that $\dim(\cap_{e \in U} S_e) = k$. (Note $\dim(\emptyset) = -1$, so the arrangement is in S^{r-1} when $\cap_{e \in U} S_e = \emptyset$. Such an arrangement is called essential.)*
- (iii) *$\mathcal{L} = \mathcal{L}(\mathcal{A})$ for some signed arrangement $\mathcal{A}$ of pseudospheres in S^{r-1}, which is essential and centrally symmetric (and whose induced cell complex $\Delta(\mathcal{A})$ is shellable.)*

We can give a heuristic description of situations modeled by a pseudosphere arrangement $\mathcal{A}$. Each $e \in U$ corresponds to a scalar function m_e on a set of states. The open pseudohemisphere models the states y for which $m_e(y) > 0$. S_e models the set of states for which $m_e(y) = 0$. Hence S_A models the states that satisfy all the constraints $(m_e(y) = 0)_{e \in A}$. The set A is not independent if there is an element $e \in A$ for which $m_e(y) = 0$ is satisfied for every y that satisfies the constraints

$m_f(y) = 0$ for all $f \in A \backslash e$. The axiom (A1) means that all combinations of equality constraints have feasible sets homeomorphic to spheres. Axiom (A2) means that for every feasible combination of equality constraints, the remaining constraints together define a (lower dimensional) oriented matroid situation.

The rank of the oriented matroid is the minimum size of a set $I \subseteq E$ for which $m_e(y) = 0$ for all $e \in I$ implies $m_f(y) = 0$ for all $f \in E$. When the hyperplane arrangement is central, that is, $\cap_{e \in E} H_e^0 = 0$, this implies $y = 0$. The essential pseudosphere arrangement is the generalization of this. For this case, the rank corresponds heuristically to the number of degrees of freedom; one of freedom corresponds to a state vector's length. Hence the dimension of the ambient topological sphere is the rank of the oriented matroid minus 1[3].

We can give the following heuristic description of oriented matroid *circuits* and *cocircuit*. This description is precise when the constraints are actually central linear hyperplanes.

- A circuit is the signature of a minimal dependency among constraints. More specifically, suppose X is a circuit. The support $S = \text{supp } X$ is the minimal $S \subseteq U$ for which the following is true: Let $e \in S$. For all states y, if for all $f \in S \setminus e$, $\sigma(m_f(y)) = 0$ then $\sigma(m_e(y)) = 0$.

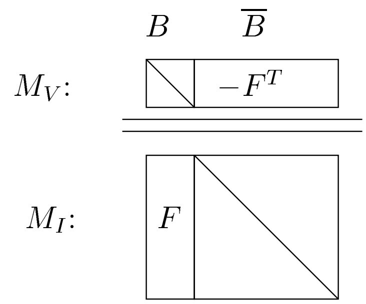
  Furthermore, if for all $f \in S \backslash e$, $\sigma(m_f(y)) = X_f$ or $\sigma(m_f(y)) = 0$ with at least one $f \in S \setminus e$ for which $\sigma(m_f(y)) = X_f$, then $\sigma(m_e(y)) = -X_e$.
- A cocircuit X is a signature of a feasible one-dimensional behavior. It is a setwise minimal non-zero intersection of equality constraints $m_f(y) = 0$. When these constraints $m_f(y) = 0$ are satisfied for all $f \in U \setminus \text{supp } X$, the signs of $m_e(y)$ are all predicted by $\sigma(m_e(y)) = X_e$ for all $e \in \text{supp } X$. In general, a covector is the signature of some feasible behavior.

2. Pairing of Subspaces and Oriented Matroids

A *subspace pair* (L_V, L_I) is a pair of linear subspaces of $\mathbf{R}^U$, where the elements of finite set U index the coordinates. The scalar product $v \cdot w = \sum_{e \in U} v_e w_e$ is used to define that $v, w \in \mathbf{R}^U$ are *orthogonal* when $v \cdot w = 0$. An *orthogonal subspace pair* satisfies $v \cdot w = 0$ for all $v \in L_V$ and $w \in L_I$. A subspace pair has *full rank* when $\text{rank}(L_V) + \text{rank}(L_I) = |U|$. Hence an orthogonal full rank subspace pair is a linear subspace L paired with its orthogonal complement $L^\perp$.

It is convenient to illustrate subspace and oriented matroid pairs with a cocycle matrix M_V or tableau for $\mathcal{M}(M_V) = \mathcal{M}_V$ printed above a cocycle matrix or tableau for $\mathcal{M}(M_I) = \mathcal{M}_I$. Thus an orthogonal pair of subspaces is illustrated by

[3] An *affine* oriented matroid is an oriented matroid with one distinguished element e_0 that is neither a loop nor an isthmus. We can take the homogeneous coordinates with $h_{e_0} = 1$ to be the state space. The number of degrees of freedom equals the oriented matroid rank r minus 1. Then, the state space is homeomorphic to a dimension $r - 1$ ball and to dimension $r - 1$ hemisphere H_e^+.



The columns B, $\overline{B}$ form an identity submatrices in M_V, M_I respectively.

An oriented matroid pair is a pair of oriented matroids with common ground set. Each subspace pair as defined above naturally represents the oriented matroid pair $\mathcal{M}(L_V, L_I) = (\mathcal{M}(L_V), \mathcal{M}(L_I)) = (\mathcal{M}_V, \mathcal{M}_I)$. The ground set is denoted $U = U(\mathcal{M}_V, \mathcal{M}_I) = U(\mathcal{M}_V) = U(\mathcal{M}_I)$.

When $\mathcal{M}_V, \mathcal{M}_I$ are duals as oriented matroids we say $(\mathcal{M}_V, \mathcal{M}_I)$ is a dual oriented matroid pair. It is almost immediate from the definition of oriented matroid dual that a dual oriented matroid pair has no common covector and has at least one complementary pair of bases. A key theme of this subject is how the latter property generalizes duality.

??? The *rank excess* of pair $(\mathcal{M}_V, \mathcal{M}_I)$ is $\text{rank}(\mathcal{M}_V) + \text{rank}(\mathcal{M}_I) - |U|$.

2.1. Subspace Pair Examples. Some applications begin with basic structural laws modeled by an orthogonal complementary pair of linear subspaces (L_V, L_I), i.e., $L_I = L_V^\perp$. This pair is then modified by port insertion, deletion, contraction, and nullator/norator insertion operations which can destroy the original orthogonality and/or $\dim(L_V) + \dim(L_I) = |U|$ properties. Other applications can begin with one or both of L_V, L_I expressing some constitutive laws too (see our example).

Electrical network structure is defined with the *network graph* $\mathcal{N}$ with *nodes* N and *arcs* U. The *incidence matrix* M_V has rows indexed by N, columns indexed by U, and $M_V(n, e) = +1$ when the tail of e is n, -1 if the head of e is n, and 0 if n and e are not incident. When $L_V = L(M_V)$ and $L_I = L_V^\perp$, each $u \in L_V$ is a combination of voltage drops in $\mathcal{N}$ feasible under KVL and each $w \in L_I$ is an arc current flow feasible under KCL. These facts restate Kirchhoff's laws and Tellegen's theorem.

We can determine (L_V, L_I) from one of these subspaces given and Tellegen's theorem: The role of nodes here is not strictly necessary. Kirchhoff's voltage law can be expressed by the statement: The feasible voltage drops are the image of the map $\mathbf{R}^N \to \mathbf{R}^U$ given by $y \to yM_V$. Kirchhoff's current law says the feasible current flows are the kernel of the map $\mathbf{R}^U \to \mathbf{R}^N$ given by $u \to uM_I^t$. Tellegen's theorem is the observation that M_V and M_I^t are adjoints. See [**WP84**].

A practical and familiar way to generate M_V and M_I matrices whose rows provide some covectors to use for qualitative analysis is to chose a spanning set of (directed) cuts and cycles respectively, say the fundamental cutsets and cycles of a spanning tree. A single spanning tree however for both M_V and M_I is not necessary. There is no advantage, at least for hand calculations, for M_V and M_I to have the minimum number of rows. Particular cutsets and cycles, or linear combinations of them, can be devised and used to provide covectors that represent approximations of particular "modes" of system operation or change from one operating point to another.

The definition of mechanical elastic structure begins with the (undirected) *framework graph* $\mathcal{F}$ with *vertices* N and *edges* U. *Framework* $\mathcal{F}(\mathbf{p})$ is $\mathcal{F}$ and an *embedding* $\mathbf{p} : N \to \mathbf{R}^d$. The *rigidity matrix* M_V has $d|N|$ rows in groups of d corresponding to the vertices. For $e = (i,j) \in U$, column $M_V(e)$ is defined [**GSS93**] $(0, \ldots, 0, \mathbf{p}(n_i) - \mathbf{p}(n_j), 0, \ldots, \mathbf{p}(n_j) - \mathbf{p}(n_i), 0, \ldots, 0)^T$ where $\mathbf{p}(n_i) - \mathbf{p}(n_j)$ occupies n_i's group of positions and $\mathbf{p}(n_j) - \mathbf{p}(n_i)$ occupies n_j's. This definition is echoed from the literature [**GSS93**] on rigidity theory, except we interchange rows and columns. Just as we deemphasized nodes of electrical networks, we will use merely the row space of M_V for most of what follows. The rigidity matrix as a function of the embedding $\mathbf{p}$ is denoted $M_V(\mathbf{p})$. The row vector $\mathbf{p}$ left multiplied with $M_V(\mathbf{p})$ is the row tuple denoted $\mathbf{L} = \mathbf{p}M_V(\mathbf{p})$ Then, $\mathbf{L}_e = (\mathbf{p}M_V(\mathbf{p}))_e = |\mathbf{p}_i - \mathbf{p}_j|^2$ for each edge $e = (i,j)$. Now if each $\mathbf{p}_i$ is a differentiable function of t, $d\mathbf{L}/dt = 2\mathbf{p}'M_V(\mathbf{p})$. Framework $\mathcal{F}(\mathbf{p})$ is *first-order rigid* when $d\mathbf{L}/dt = 0$ for all $\mathbf{p}'$ implies $|\mathbf{p}_i - \mathbf{p}_j|^2$ is constant for all pairs i, j, not just endpoints of edges. From [**GSS93**], $u \in L_V = L(M_V) \subset \mathbf{R}^U$ iff for some combination of *vertex velocities* $\mathbf{v} : N \to \mathbf{R}^d$, $u_e = (\mathbf{v}(i) - \mathbf{v}(j)) \cdot (\mathbf{p}(i) - \mathbf{p}(j))$ for each $e = (i,j) \in U$. Also, the *self-stress subspace* $L_I = L_V^{\perp}$is shown to be all $w \in \mathbf{R}^U$ for which the framework is in static equilibrium when each edge $e = (i,j)$ exerts force $w_e(\mathbf{p}(j) - \mathbf{p}(i))$ on vertex i. By this convention, $w_e > 0$ means e is under tension and $w_e < 0$ means e is under compression. (The rigidity matrix in [**GSS93**] is M_V^T, not M_V.)

Under this analogy, (1) KVL corresponds to geometric consistency of first order bar length changes under changes in the embedding, (2) KCL corresponds to Newton's laws of static equilibrium, and (3) Tellegen's theorem corresponds to a virtual work principle, that static equilibrium is characterized by the internal forces against every virtual embedding change doing zero virtual work.

2.1.1. *Display Elements.* We can also declare "display" quantities, say for the difference of two independent quantities, by appending the appropriate column to M_V or M_I. Any dependency of the display quantity on element quantities will appear as a dependency in the oriented matroid. The parallel extension operation (defined below) inserts an element whose quantity will duplicate the element quantity of the parallel element.

2.2. Reduction Operations. The matroid contraction and contraction operations are defined on oriented matroid pairs by $\mathcal{M}/e = (\mathcal{M}_V, \mathcal{M}_I)/e =$ (by definition of *contraction*) $(\mathcal{M}_V/e, \mathcal{M}_I \setminus e)$; and $\mathcal{M} \setminus e =$ (by definition of *deletion*) $(\mathcal{M}_V \setminus e, \mathcal{M}_I/e)$. In other words, the operation is applied to the V part and its dual is applied to the I part. Oriented matroid pair deletion and contraction preserve dual pairs.

THEOREM 3. *Suppose $(\mathcal{M}_V, \mathcal{M}_I)$ have complementary base pair $B_V \cup B_I = E$, $B_V \cap B_I = \emptyset$ (denoted $B_V \uplus B_I = E$) and no common covector. Let $e \in E$. If $e \in B_V$ then $(\mathcal{M}_V, \mathcal{M}_I)/e$ also has a complementary base pair $((B_V \setminus e) \uplus B_I)$ and no common covector. Dually, if $e \in B_I$ then $(\mathcal{M}_V, \mathcal{M}_I) \setminus e$ has a complementary base pair and no common covector.*

PROOF. The smaller oriented matroid pair has complementary base pairs because of the condition on which basis contains e and the definition of deletion and contraction. Suppose say $e \in B_V$ but $(\mathcal{M}_V, \mathcal{M}_I)/e = (\mathcal{M}_V/e, \mathcal{M}_I \setminus e)$ have a

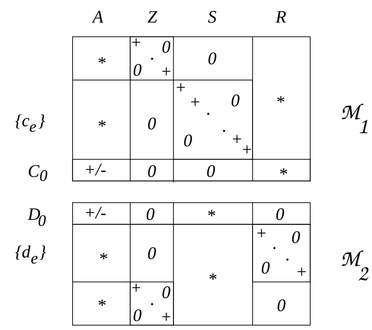


FIGURE 2. The tableau for $\mathcal{M}_1$ shows the cocircuits c_e for $e \in S$, other cocircuits for $e \in Z$, and the covector C_0. These cocircuits are fundamental with respect to some basis that extends $Z \cup S$. The algorithm constructs a covector C by starting with $C \leftarrow C_0$ and composing $C \leftarrow C \circ (\pm c_e)$ to make $C(e) = D(e)$ when necessary, while similar operations are applied to D using cocircuits from $\mathcal{M}_2$.

common covector X. Let $A = \operatorname{supp} X$, $S = (B_V \setminus A) \cup \{e\}$, $R = (B_I \setminus A)$ and $Z = \emptyset$. Let signed set C_0 be defined by $(C_0)_f = X_f$ for all $f \neq e$ and $(C_0)_e = 0$. C_0 is a covector in $\mathcal{M}_V$ by definition of the contraction $\mathcal{M}_V/e$. Let signed set D_0 be a covector of $\mathcal{M}_I$ $(D_0)_f = X_f$ for all $f \neq e$ which exists since X is a covector of $\mathcal{M}_I \setminus e$. If $(D_0)_e = 0$ then $X = C_0 = D_0$ is a common covector of $(\mathcal{M}_V, \mathcal{M}_I)$ and we have a contradiction. Otherwise, the following Theorem 4 applied to the above sets and covectors demonstrates a contradiction with C.

THEOREM 4. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be oriented matroids on the ground set $A \uplus Z \uplus S \uplus R$. Assume the covectors $C_0 \in \mathcal{L}(\mathcal{M}_1)$ and $D_0 \in \mathcal{L}(\mathcal{M}_2)$ satisfy the properties:*

(1) *$A \neq \emptyset$ and $C_0(a) = D_0(a) \neq 0$ for all $a \in A$,*
(2) *$S \uplus Z$ is independent in $\mathcal{M}_1$ and $C_0(S \uplus Z) = 0$, and*
(3) *$R \uplus Z$ is independent in $\mathcal{M}_2$ and $D_0(R \uplus Z) = 0$.*

Then $\mathcal{M}_1, \mathcal{M}_2$ have a common (non-zero) covector $C \in \mathcal{L}(\mathcal{M}_1) \cap \mathcal{L}(\mathcal{M}_2)$ that is compatible with both C_0 and D_0. In other words, $C_0 \preceq C$ and $D_0 \preceq C$.

The proof of Theorem 4 given in [**Cha96**] contains an algorithm to construct a common covector efficiently by composing covectors with cocircuits. Figure 2 illustrates the setup.

The dual case is similar. □

Given a subspace $L \subset \mathbf{R}^U$ and $e \in U$, the subspace "*L with e deleted*" is $L \setminus e = \{l(U \setminus e) | l \in L\} \subset \mathbf{R}^{U \setminus \{e\}}$, where $l(U \setminus e)$ denotes the $l \in \mathbf{R}^U$ with the component labeled by e dropped. Thus, $L \setminus e$ is the *projection* of L into $\mathbf{R}^{U \setminus \{e\}}$. If $L = L(M)$ then $L \setminus e = L(M(U \setminus \{e\}))$ is the row space of $M(U \setminus \{e\})$, which is M with column e deleted. The subspace "*L with e contracted*" $L/e = \{l(U \setminus e) | l(U) \in L \text{ and } l(e) = 0\}$. So, L/e is the intersection of L with the (hyperplane) subspace of $\mathbf{R}^U$ with $l(e) = 0$, projected into $\mathbf{R}^{U \setminus \{e\}}$.

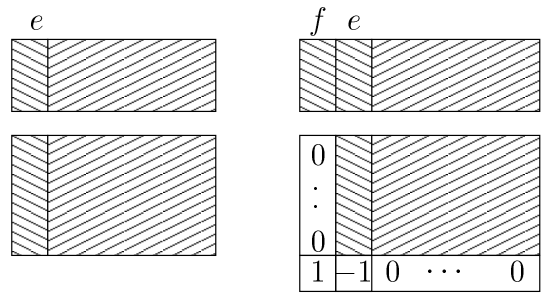


FIGURE 3. Tableaux for an oriented matroid pair and its parallel extension at element e.

We now define *deletion* and *contraction* on subspace pairs: $(L_V, L_I) \setminus e = (L_V \setminus e, L_I/e)$ and $(L_V, L_I)/e = (L_V/e, L_I \setminus e)$. Deleting e in the electrical application corresponds to *opening* the corresponding branch. Dually, contraction corresponds to *shorting*. Mechanically, deletion of an edge corresponds to "breaking" the corresponding bar: ignore any distance change between its ends and transmit no force. Contraction corresponds to declaring the bar to be inelastic, which rules out all (first order) distance changes between the endpoints and rules out any constitutive law referring to tension or compression in that bar.

2.2.1. *Nullor Elements.* A *nullator* element $e \in E$ expresses the ideal constitutive law $u_{Ve} = 0$ and $u_{Ie} = 0$ which approximates conditions at the input to a high-gain amplifier when a system is stabilized by feedback. Hence a nullator is declared by *contracting* e *in both* L_V *and* L_I. In this way, the effects of an ideal constitutive law are modeled by the subspace rather than the constitutive law part of the subspace pair model.

In non-degenerate situations, the variables associated with e are not already constrained to zero so e is not a loop in either $\mathcal{M}_V$ or $\mathcal{M}_I$. When this is true, nullator insertion reduces both ranks by 1. A *norator* element $e \in E$ indicates that the constitutive law puts no direct constraint on u_{Ve} or u_{Ie}; the amplifiers approximately adjust the output state so the feedback results in zero input. Hence a norator is declared by *deleting* e *in both* L_V *and* L_I. In non-degenerate situations, e is not an isthmus in either $\mathcal{M}_V$ or $\mathcal{M}_I$, so their ranks don't change.

2.3. Extension Operations.

2.3.1. *Series and Parallel Extension.* The parallel extension of a non-loop element e in oriented matroid $\mathcal{M}$ over E has ground set $E \cup e'$ and covectors $\mathcal{L}'$ defined by $X' \in \mathcal{L}'$ with $X'_e = X'_{e'} = X_e$ and $X'_f = X_f$ for all $f \in E \setminus e$ and $X \in \mathcal{L}$. [**BVS$^+$99**, p.??] proves it is an oriented matroid. Series extension is the dual operation: If e is not an isthmus, then the series extension of e is $((\mathcal{M}^*)')^*$. We define parallel extension of e in pair $(\mathcal{M}_V, \mathcal{M}_I)$ when e is not a loop in $\mathcal{M}_V$ and not an isthmus in $\mathcal{M}_I$ by parallel extending e in $\mathcal{M}_V$ and series extending e in $\mathcal{M}_I$. Series extension is defined dually. It is clear that series and parallel extension preserves dual oriented matroid pairs.

HEURISTIC COMMENT ("FUZZ") 2. Does series/parallel extension preserve the complementary base and no-common-covector property of a non-dual oriented matroid pair?

2.3.2. *Ports.* A single *port* in a single oriented matroid or subspace of $\mathbf{R}^U$ is a single distinguished element or coordinate $p \in U$. We will define a port in a pair of oriented matroids or subspaces to be a pair $\{p_V, p_I\} \subset U$. Each member of the port will be a matroid *isthmus* (element not in any circuit[4]) in one of OMs or subspaces because this notation will enable us to formulate some solution sets (when the constitutive laws are linear) as intersections of subspaces. We will construct oriented matroid or subspace pairs with one or more ports by applying a *port insertion operation* to such a pair.

Each port of a subspace pair provides two variables: When one of them is considered a parameter, it is called an *input* or a *source*. The non-input port variables are called *outputs*. Such a descriptions of variables can apply to the corresponding oriented matroid elements or coordinate names.

Environmental constraints in physical problems often occur so that exactly one variable of each port is an input and the other is an output[5]. Which of $\{p_V, p_I\}$ is for the input variable is described by what *kind of source* is *driving* the port: electrical voltage (battery) or current sources; or their mechanical analogs of displacement or force. Unlike device variables, the two variables of each port are not directly related by a constitutive law which is part of the system model. Ports are introduced so the response of an electrical network to current and/or voltage sources, and the mechanical analogs, can be formulated. Questions of existence and uniqueness of solution for various combinations of kinds of sources are formulated after modeling devices.

Ports facilitate formal operations to compose larger systems from smaller ones. We believe ports are important for investigations of rigidity because they model how a framework interacts with its environment, for example, what a mechanical model "feels like" when you squeeze it. We have also found that electrical port characteristics of unit resistance ported electrical networks are ratios of coefficients in certain partial evaluations of a generalization of the Tutte polynomial[**Cha89**].

Here are the formal definitions. Given (L_V, L_I) and $p \in U$ not already a port, the operation of *inserting a port at p* defines a new subspace pair (L'_V, L'_I) with $U' = U \setminus \{p\} \cup \{p_V, p_I\}$, $L'_V = L_V \oplus \mathbf{R}$ (direct sum) with p replaced by p_V; and the coordinate of the added $\mathbf{R}$ indexed by p_I, together with $L'_I = L_I \oplus \mathbf{R}$ with p now replaced by p_I and the added subspace indexed by p_V. Note that (going to (L'_V, L'_I)) the ranks of L_V and L_I each increase by 1, and $|U'| = |U| + 1$. After p port insertions, we denote the final U by $E \cup P_V \cup P_I$ with pairwise disjoint E, P_V and P_I, $|P_V| = |P_I| = p$, $P_V \cup P_I$ being the replacement elements.

Inserting ports into a subspace pair is easily abstracted to an oriented matroid pair. An oriented matroid pair has ports $P_V \uplus P_I$ when $U = E \uplus P_V \uplus P_I$ where $|P_V| = |P_I|$, P_V is a set of isthmuses in $\mathcal{M}_I$ P_I is a set of isthmuses in $\mathcal{M}_I$, and

[4]In matroid theory an element like p_I of $\mathcal{M}(L_V)$ that is independent of all others is called an *isthmus*. Similarly, p_V is an isthmus of $\mathcal{M}(L_I)$. In general, the matroid represented by a matrix is characterized by the collection $\mathcal{I}$ of *independent sets* of matrix columns, where a set of columns is called independent when it is linearly independent. (Matroid theory studies what can be deduced by the following three axioms satisfied by $\mathcal{I}$: (1) $\mathcal{I} \neq \emptyset$. (2) If $A \subset B \in \mathcal{I}$ then $A \in \mathcal{I}$. (3) If A, $B \in \mathcal{I}$ and $|A| < |B|$, then there exists $e \in B \setminus A$ for which $A \cup \{e\} \in \mathcal{I}$. For example, an isthmus e is characterized by $A \cup \{e\} \in \mathcal{I}$ for all $A \in \mathcal{I}$. The *rank* of a subset $C \in U$ is the size of the largest independent subset of C.

[5]Formulations in which both variables are inputs are used to analyze interconnected systems. The result of analyzing a linear network this way is called a transmission matrix. **We should try to understand this distinction better.**

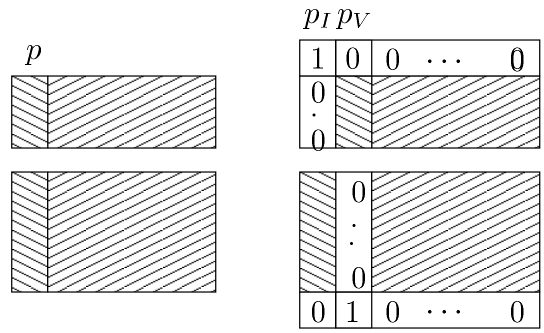


FIGURE 4. Tableaux for an oriented matroid pair and the result of inserting a port at element p

the elements of P_V and P_I are in one-to-one correspondance. A corresponding pair (p_V, p_I) with p_V an isthmus in $\mathcal{M}_I$ and p_I an isthmus in $\mathcal{M}_V$ is denoted briefly by $p \in P$.

Given $p \in E(\mathcal{M})$, not already a port, the oriented matroid pair $\mathcal{M}' = \text{make-port}_p(\mathcal{M})$ is defined as follows: $E(\mathcal{M}') = E(\mathcal{M}) \setminus p$, $P_W(\mathcal{M}') = P_W(\mathcal{M}) \uplus p_W$ for $W = V$ and I, $\mathcal{M}'_V$ is the extension of $\mathcal{M}_V$ with isthmus p_I and with element p renamed by p_V, and $\mathcal{M}'_I$ is the extension of $\mathcal{M}_I$ with isthmus p_V and with element p renamed by p_I. It is immediate that inserting a port causes the rank excess to increase by 1. The operation make-port$_p()$ may be iterated over each $p \in P$ given $P \subseteq E$ with the same resulting oriented matroid pair independent of the order of iteration.

2.3.3. *Zeroing of Ports.* Given port $p = (p_V, p_I)$ in $\mathcal{M}' = \text{make-port}_p(\mathcal{M})$ the operations V-zero$_p(\mathcal{M}')$ and I-zero$_p(\mathcal{M}')$ are defined as follows: V-zero$_p(\mathcal{M}') = (\mathcal{M}'/p_V/p_I, \mathcal{M}' \setminus p_V \setminus p_I) = \mathcal{M}/p$. The symbol p in the last expression of course denotes the element of $E(\mathcal{M})$ that was made into a port. Dually, I-zero$_p(\mathcal{M}') = (\mathcal{M}' \setminus p_V \setminus p_I, \mathcal{M}'/p_V/p_I) = \mathcal{M} \setminus p$.

PROPOSITION 1. *If p is in basis B of $\mathcal{M}_V$ with $U \setminus B$ a basis of $\mathcal{M}_I$, then $(\mathcal{M}_V, \mathcal{M}_I)/p =$ Vdrive-zero$_p$(make-port$_p(\mathcal{M}_V, \mathcal{M}_I)$).*

If p is in basis B of $\mathcal{M}_I$ with $U \setminus B$ a basis of $\mathcal{M}_v$, then $(\mathcal{M}_V, \mathcal{M}_I) \setminus p =$ Idrive-zero$_p$(make-port$_p(\mathcal{M}_V, \mathcal{M}_I)$).

2.4. Subspace and Matroid Combination Along Ports. Suppose we can perform row operations on M so it has the its first two rows have the form $(1, 0, r_1)$ and $(0, 1, r_2)$, and the other $r - 2$ rows have the form $(0, 0, r_i)$. Let p_1 and p_2 be the elements corresponding to the first two columns.

To combine L along p_1, p_2 *in parallel*, we replace the first two rows by the single row $(1, 1, r_1 + r_2)$, delete the second column and then identify elements p_1, p_2. The resulting subspace is obtained by intersecting $L = L(M)$ with the hyperplane defined by $u_{p_1} = u_{p_2}$ and then projecting into $R^{U \setminus p_1}$.

To combine L along p_1, p_2 *in series*, we replace the first two rows by the two rows $(1, 1, r_1)$ and $(1, 1, r_2)$, delete the second column and then identify elements p_1, p_2. The resulting subspace is the image of $L = L(M)$ under the linear map $\mathbf{R}^U \to \mathbf{R}^{U \setminus p_2}$ by the map that takes $(1, 0, 0, \ldots, 0)$ and $(0, 1, 0, \ldots, 0)$ to $(1, \ldots, 0)$.

3. Parameters, Constitutive Laws and Solutions

In this section, we will define the subspace pair model and its solutions, give matroid theoretic conditions necessary for solutions to exist or to be unique, and show how no-common-covector conditions are necessary for uniqueness. The section concludes with the main existance and uniqueness theorem, explained and proved by demonstrating its equivalance to Sandberg and Willson's result.

The *subspace pair model* $\mathbf{M} = (E, \Gamma, P, (L_V, L_I))$ consists of finite set E of *device elements*, *constitutive law relations* $\Gamma = \{\Gamma_e \subset \mathbf{R} \times \mathbf{R} | e \in E\}$, the finite set $P = P_V \cup P_I$ that results from inserting ports as defined above, and a subspace pair (L_V, L_I) over $\mathbf{R}^U$ with $U = E \cup P$. Note that P_V (resp. P_I) is a set of isthmuses in $\mathcal{M}_I$ (resp. P_V).

The *variables* of $\mathbf{M}$ are $\{u_{Ve}, u_{Ie} | e \in E\} \cup \{u_{pV}, u_{pI} | p_I, p_V \in P\}$. (For brevity, subscript "pV" means port element $p_V \in P_V$, etc.) A *subspace pair model with sources* S is a subspace pair model $(E, \Gamma, P, (L_V, L_I), S)$ together with a subset S of exactly $|P|$ of the $2|P|$ elements in P. A *V-driven port* is one for which $p_V \in S$ and $p_I \notin S$, then u_{pV} is called an *input variable.* Reverse V and I to define an *I-driven port* and its input variable.

A *solution* of $\mathbf{M}$ with sources is a real valued extension to *all* variables of $\mathbf{M}$ of a given *input assignment* to the input variables that satisfies $(u_{PV}, u_{PI}, u_V) \in L_V$, $(u_{PV}, u_{PI}, u_I) \in L_I$ and $(u_{Ve}, u_{Ie}) \in \Gamma_e$ for all $e \in E$. The constraint $(u_{PV}, u_{PI}, u_V) \in L_V$ does not (by itself) imply any constraint on a I-driven port variable u_{pI}, similarly, u_{pV} is not constrained by $(u_{PV}, u_{PI}, u_V) \in L_I$. Port variables are not constrained by the constitutive laws Γ (by themselves) either.

MUST FIX THIS::: Assume as usual no port is both I-driven and V-driven. The well-known necessary condition for an electrical network to have a unique solution for all choices of source values is condition of no cycle of voltage source branches in the "voltage" graph nor a cutset of current source branches in the "current graph" (see, e.g., [**Che76**]) generalizes to:

THEOREM 5. ***(1)*** *If all V input assignments are feasible under the L_V constraint then $\{p_V | p \text{ is V-driven}\}$ is an independent set in the matroid $\mathcal{M}(L_V)$.* ***(2)*** *If every solution is unique then $\{p_I | p \text{ is V-driven}\}$ must be coindependent in $\mathcal{M}(L_I)$.* ***(3)*** *If all I input assignments are feasible under the L_I constraint then $\{p_I | p \text{ is I-driven}\}$ is an independent set in the matroid $\mathcal{M}(L_I)$.* ***(4)*** *If every solution is unique then $\{p_V | p \text{ is I-driven}\}$ must be coindependent in $\mathcal{M}(L_V)$.*

PROOF. (1,3): If set S of input variables is dependent, then there is some combination of input values that is not feasible. (2,4): S is not a *coindependent* set iff S contains a cocircuit, so there is a non-zero covector supported by S. Hence there is a feasible variable assignment that is non-zero on the some port output variables only. □

Each port insertion increases the rank excess $\text{rank}(L_V) + \text{rank}(L_I) - |U|$ by 1. When the constitutive laws are linear, the solutions of $\mathbf{M}$ are found from the *intersection* of two linear subspaces: Let G be the diagonal matrix with "conductances" g_e in its positions indexed by $e \in E$ (so $\Gamma_e = \{(v, g_e v) | v \in \mathbf{R}\}$ and 1 in its other diagonal positions. The solution set projected onto the u_I variables is $L_V G \cap L_I$. (Here, $L_V G$ means $L(M_V G)$.)

$S = \{p_V \mid p \text{ is V-driven.}\} \in \mathcal{I}(\mathcal{M}_V)$	$\leftarrow$ All of $\mathbf{R}^{\lvert S \rvert}$ feasible as V-input assignments.
$S = \{p_V \mid p \text{ is V-driven.}\} \in \mathcal{I}(\mathcal{M}_V^*)$	$\leftrightarrow$ All V-input assignments $\mathbf{R}^{\lvert S \rvert}$ make some other V variables non-zero.
$S = \{p_I \mid p \text{ is V-driven.}\} \in \mathcal{I}(\mathcal{M}_I)$	$\leftrightarrow$ No KCL dependencies among output variables
$\{p_I \mid p \text{ is V-driven.}\} \in \mathcal{I}(\mathcal{M}_I^*)$	$\leftarrow$ I-output values unique.

FIGURE 5. Electrical network application interpretation of matroid theoretic independence conditions on the V-driven port subset. Corresponding dual statements apply to the I-driven port subset.

3.1. Supplemental Subspace Pair and Uniqueness Theory. The *supplemental subspace pair* is constructed by *zeroing* all the sources. For each V-driven (resp. I-driven) port p, p_V (resp. p_I) is contracted in both L_V and L_I, and p_I (resp. p_V) is deleted in both L_V and L_I. The resulting subspaces, etc. are denoted L_V^0, L_I^0, etc.

The following proposition expresses elementary properties of the subspace pair model formulation that have been observed by Willson and Sandberg [] and by Hasler, Nernyck *et. al.* [] in their own formulations.

PROPOSITION 2. *Let* $\mathbf{M} = (E, \Gamma, P, (L_V, L_I), S)$ *be subspace pair model with sources in which the contitutive relations are monotone increasing. (a) If* $\mathbf{M}$ *has a solution and the supplemental subspace pair* (L_V^0, L_I^0) *has no common covector, then the solution is unique. (b) If* (L_V^0, L_I^0) *has a common covector, then there exist linear monotone increasing constitutive relations* Γ *for which* $\mathbf{M}$ *has multiple solutions.*

We note here that when (L_V^0, L_I^0) have a common covector, the existence of multiple solutions results from either a "degeneracy in the problem formulation" (which we will see below as a failure of conditions **(1)** or **(2)** of Theorem 6, below and is called a "never-well-posed" problem th Hasler and Neiryck) or an exceptional choice of monotone constitutive laws. A problem that is well-posed (has a unique solution) for almost all linear constitutive laws, but which has non-unique solutions for particular postive constitutive law coefficients is called "sometimes-ill-posed" by Hasler and Neiryck.

The signs of each solution to a subspace pair problem with sources comprise a common covector of the subspace pair (L_V, L_I). Indeed, with linear constitutive laws, a solution will exist when dim L_V + dim $L_I > |U|$. The no-common-covector condition implies such a solution (a tuple of reals) is unique. The solution depends on the constitutive laws. For different constitutive laws, there will typically be different solutions. Please note that the (non-zero) common covectors of (L_V, L_I) are not necessarilly unique (more precisely, for common convectors $X_1, X_2 \neq 0$, $X_1 \neq X_2$ and $X_1 \neq -X_2$,) even when (L_V^0, L_I^0) have no common covector.

3.2. Main Theorem on No-Common-Covectors and $\mathcal{W}_0$ Pairs. Theorem 6 uses Sandberg and Willson's $\mathcal{W}_0$ pairs to show every subspace pair model (with its separation of geometric/topological and constitutive constraints) can be analyzed for unique solvability from the oriented matroid pair it generates. Theorem

7 shows how $(A, B) \in \mathcal{W}_0$ is characterized by a rank condition and a no-common-covector property. Theorem 8 is our unifying conclusion.

THEOREM 6. *The subspace pair model has a unique solution for all input assignment values and positive monotone constitutive laws iff* ***(1)****,* ***(2)*** *and* ***(3)*** *are satisfied:*
(1). *There are bases $B_V \in \mathcal{B}(L_V)$, $B_I \in \mathcal{B}(L_I)$ for which all V-driven ports p satisfy $p_V \in B_V$ (note p_I must be in B_V since it's an isthmus in $\mathcal{M}_V$) and $p_I \notin B_I$, and all I-driven ports p satisfy $p_I \in B_I$ and $p_V \notin B_V$.*
(2). *$B_V \cup B_I = U$, for the bases in* ***(1)****.*
(3). *The oriented matroid pair of supplemental pair (L_V^0, L_I^0) have no common (nonzero) covector. (i.e., $\mathcal{L}(L_V^0) \cap \mathcal{L}(L_I^0) = \{0\}$.)*

PROOF. (Sketch) If: When (L_V^0, L_I^0) is constructed given **(1-2)**, we find rank $(L_V^0)+$ rank $(L_I^0) \geq |E|$. By Theorem 1.3 of [**Cha96**] and $|E| \geq$ rank of the matroid union ([**Wel76**], 8.3) of $\mathcal{M}_V^0 \vee \mathcal{M}_I^0$, we must have equality else **(3)** would be contradicted. Form square matrices $A = \begin{bmatrix} M_V^0 \\ 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ -M_I^0 \end{bmatrix}$ and verify from **(2)** they satisfy the rank part of cond. **(2)** of Th. 7. $\mathcal{L}[A\ B] \cap \mathcal{L}[I\ -I] = \{0\}$. follows directly from hypothesis **(3)**. We can then formulate the monotonic subspace pair problem as a case of Theorem 7 part **(5)** to complete the proof.

Only if: Use Theorem 5 to verify **(1)** and **(2)**. If **(3)** were violated, positive linear constitutive laws could be constructed (as in [**HN86, SW69**]) for which a non-zero solution for zero input exists. □

THEOREM 7. *(From* [**Cha96**]*) For a pair of $n \times n$ matrices (A, B), the following conditions are equivalent.*
(1) *$(A, B) \in \mathcal{W}_0$ in the sense of Sandberg and Willson* [**SW69**]*; e.g., $|AD + B| \neq 0$ for all positive diagonal D, etc.*
(2) rank $\mathcal{M}[A\ B] = n$ *and* $\mathcal{L}[A\ B] \cap \mathcal{L}[I\ -I] = \{0\}$.
(3) rank $\mathcal{M}[A\ B] = n$ *and* $\mathcal{V}[A\ B] \cap \mathcal{V}[I\ -I] = \{0\}$.
(4) *(Fundamental theorem of Sandberg and Willson* [**SW69, Wil70**]*) For all functions $F : \mathbf{R}^n \to \mathbf{R}^n$ of the form $F(x)_k = f_k(x_k)$ where each f_k is a strictly monotone increasing function from $\mathbf{R}$* onto *$\mathbf{R}$ and for all $c \in \mathbf{R}^n$, the equation $AF(x)+Bx = c$ has a unique solution x.* [**Wil70**].
(5) *For all functions $G : \mathbf{R}^n \to \mathbf{R}^n$ of the form $G(w)_k = g_k(w_k)$ where each g_k is a strictly monotone increasing function from $\mathbf{R}$* onto *$\mathbf{R}$ and for all $d', d'' \in \mathbf{R}^n$, the eqns. $u^t = z^tA + d',\ \ w^t = z^tB + d'',\ \ u = -G(w)$ have a unique solution.*

REMARK 1. Our proof in [**Cha96**] of equivalence of **(2)** and **(3)** depends on general properties of relative orientations of complementary base pairs in realizable oriented matroid pairs (i.e., subspace pairs). The rank condition is equivalent to $|AD + B| \neq 0$ for some diagonal D. The no-common-vector condition of **(3)** means that whenever $Ax + By = 0$ and $\sigma(x) = \sigma(y)$, x and y must both be $0 \in \mathbf{R}^n$. The $\mathcal{W}_0$ property follows from[**SW72**] a stronger property that (A, B) is a passive pair: $Ax + By = 0$ implies $x \cdot y \leq 0$. An example of a non-passive $\mathcal{W}_0$ pair is $A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. This matrix pair comes from the elementary operational amplifier example in [**Cha96**].

REMARK 2. The direct inductive proof given in section 5 which generalizes [**HN86, Fos92**]'s has the advantage of revealing circuit theoretic concepts that occur. One step uses Theorem 3 to prove that the no-common-covector hypothesis is preserved by replacing one of the nonlinear elements by a source.

THEOREM 8. *With constitutive laws given by monotone increasing functions from* $\mathbf{R}$ *onto* $\mathbf{R}$*, every subspace pair problem can be posed as a case of Theorem 7 (see Theorem 6), and every case of Theorem 7 can be posed as a subspace pair problem (with* $[A\ B] = M_V^0$*,* $[I\ \ -I] = M_I^0$*.)*

4. Survey of Formulations

We survey various formulations and results about them to seek relations and generalizations. An assortment of mathematical tools have been used to prove existence theorems:

- Fill in here: Minty.
- Global implicit function theorems and approximation of continuous non-decreasing functions by differential strictly increasing function: DeSoer and Wu[**DW74**], Sandberg and Willson[**SW72**]
- Induction: Hasler and Neirynck [**HN86**], Sandberg and Willson[**SW69**, ???], our work.
- Convex complementarity and optiomization: Rockafellar[**Roc67**].

4.1. Pairs with a distinguished complementary base pair, "Laplacians", Elastic and Electrical problems in terms of nodes. We demonstrate here how the subspace pair model applies to the reduced nodal admittance matrix formulation of the resistive electrical network analysis problem.

The Laplacian of the connected graph with nodes $\{0, 1, \ldots, n\}$, edges E and edge weight g_e on each $e \in E$ is the $n \times n$ matrix of coefficients from the following equations with variables $(v_i)_{i=1,\ldots,n}$ and constants $v_0 = 0$ and $(y_i)_{i=1,\ldots,n}$:

$$\sum_{j:e=\{i,j\}\in E} g_e(v_i - v_j) = y_i \quad \text{for} \quad i = 1, \ldots, n. \tag{1}$$

Recall that the Laplacian matrix can be factored as MGM^T where M is the incidence matrix with row 0 deleted (*reduced*) and G is the diagonal matrix with entries $(g_e)_{e\in E}$. In electrical network theory, the Laplacian is called the *nodal admittance matrix* when the v_i denote node voltages relative to the ground node 0 (hence $v_0 = 0$), the g_e denote edge conductance and each y_i denotes current through an external path flowing from node 0 into node i. KVL implies that the voltage across edge $e = ij$ is $v_i - v_j$ for some node potential $v : \{0, \ldots, n\} \to \mathbf{R}$. Ohm's law means the current through edge e from i to j is $g_e(v_i - v_j)$. Equation (1) expresses KCL at node i. The Laplacian can be shown to be non-singular when each $g_e > 0$ by means of the Matrix Tree Theorem[6]. The inverse of the Laplacian is called the *nodal resistance matrix* R. Given externally imposed currents y into the nodes $1, \ldots, n$ (so their sum flows out of node 0), $v = Ry$ gives the voltages at nodes $1, \ldots, n$ relative to node 0.

Voltage sources (batteries) are more common in elementary electrical problems. Suppose one or more voltages sources are connected between node 0 and other nodes which establish some *voltage input* nodes to be have particular electrical potential relative to one another. Quotients of minors of the Laplacian can also be used to

[6]The determinant of the Laplacian equals $\sum g_T$ where g_T is the product of the conductances g_{ij} of the edges in spanning forest T.

solve for the other node voltages. Note that for this problem, the external current into the voltage input nodes is unspecified and the external current into the other nodes is zero.

To model an electrical network interacting with its environment by means of currents and voltages at its nodes in both these situations, and to draw analogies to elastic frameworks, we take for the matrix M_V the graph's incidence matrix (not reduced, so it has n rows but its rank is $n - \#$connected components) with the columns of an identity matrix appended. We write $U = E \cup N$; N is the set of n nodes. (M_V is the reduced incidence matrix of the graph constructed by appending a new node plus a new edge directed from this node to each original node.) Writing $M_V = [M \ I]$, we can take $M_I = [I \ -M^T]$ so $L(M_V) = L(M_I)^\perp$ and both M_I and M_V have full row rank. Thus the Laplacian or nodal admittance matrix is $M_I^T G M_V$.

4.1.1. *Elastic Analog of the Indefinite Nodal Admittance Matrix.* The analogous matrices for a framework are constructed with M being the $dn \times |E|$ rigidity matrix, $M_V = [M \ I]$ and $M_I = [I \ -M^T]$. For brevity, we let N^d denote the row index set of M_V and so $U = E \cup N^d$. For the electrical network, $d = 1$.

Here are some observations about some vectors and covectors supported by subsets of E and N^d.

- Suppose $x \in L(M_V)$ and supp $X \subseteq N^d$ where covector $X = \sigma(x) \in \mathcal{L}_V$. Electrically, x is a node potential that is constant on connected components of the network graph. Mechanically, $x = \mathbf{v}$ is a combination of node velocities for which $0 = u_e = (\mathbf{v}(i) - \mathbf{v}(j)) \cdot (\mathbf{p}(i) - \mathbf{p}(j))$ for each $e = (i, j) \in E$.

  By duality, such $x \in V(M_I) = L(M_V)^\perp$, so $X \in \mathcal{V}_I$ is a vector. Electrically, x is a linear dependence among external node currents. When the network is connected, every such $x = c(1, \ldots, 1)$ for $c \in \mathbf{R}$. When $c \neq 0$ this expresses the result of KCL that the sum of node currents is 0. Mechanically, x is a linear dependence among externally supplied forces on the nodes necessary for equilibrium.
- Suppose $y \in L(M_I)$ and supp $Y \subseteq E$ where covector $Y = \sigma(y) \in \mathcal{L}_I$. Electrically, y is a current flow in the edges that satisfies KCL with zero external current. Mechanically, y is called a self-stress.

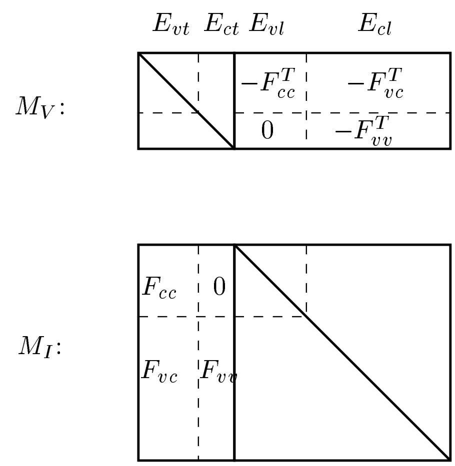
  By duality, such $y \in V(M_V) = L(M_V)^\perp$, so $Y \in \mathcal{V}_V$ is a vector. Electrically, y is a linear dependence among edge voltages that is a case of KVL. Mechanically, it is a dependence among first order relative changes of bar lengths $((\Delta l)/l)$ necessary to maintain geometric consistency of the framework embedding.

We can now explain how to modify the subspace pair (L_V, L_I) to exclude from $\mathcal{L}_V$ covectors for electrical potential offsets that are constant on connected components and mechanical node velocities that result in zero first order length changes in the bars. Choose a basis for E in the matroid $\mathcal{M}_V$ and independent set $T \subset N^d$ that extends the basis for E to a basis for S. The modified $(L'_V, L'_I) = (L_V, L_I)/T = (L_V/T, L_I \setminus T)$. Let $N^{d'}$ denote $N^d \setminus T$. We observe L'_V and L'_I are orthogonal (so $\mathcal{M}(L'_V) = \mathcal{M}'_V$ and $\mathcal{M}(L'_I) = \mathcal{M}'_I$ are duals) and $N^d \setminus T$ is both independent and coindependent in both $\mathcal{M}'_V$ and $\mathcal{M}'_I$.

Let us now introduce dn ports at the elements of N^d.

4.2. DeSoer and Wu's Formulation. Desoer and Wu presented results[**DW74**] on existance and uniqueness of solutions of electrical network problems with monotone non-decreasing but not necessarilly onto nonlinear resistance or conductance functions for the constitutive laws of two-terminal resistors. Their formulation led to a function $G : \mathbf{R}^m \times \mathbf{R}^r \to \mathbf{R}^m$. A solution $x \in \mathbf{R}^m$ for which $G(x, u) = y$ determines all the currents and voltages from Kirchhoff's laws. $u \in \mathbf{R}^r$ and $y \in \mathbf{R}^m$ are parameters. A global implicit function theorem was used to prove a solution x exists for all u, y from certain graph theoretic conditions on the resistors classified according to the directions ($x \to +\infty$ or $x \to -\infty$) in which the domain and range of the resistors' constitutive relations are bounded. Ports or outputs were not used explicitly. We will describe Desoer and Wu's formulation. It will illustrate how the equations of the formulation abstract to series and parallel extensions followed by port insertion.

The starting point is the matrices representing an orthogonally complementary pair of subspaces, illustrated below:



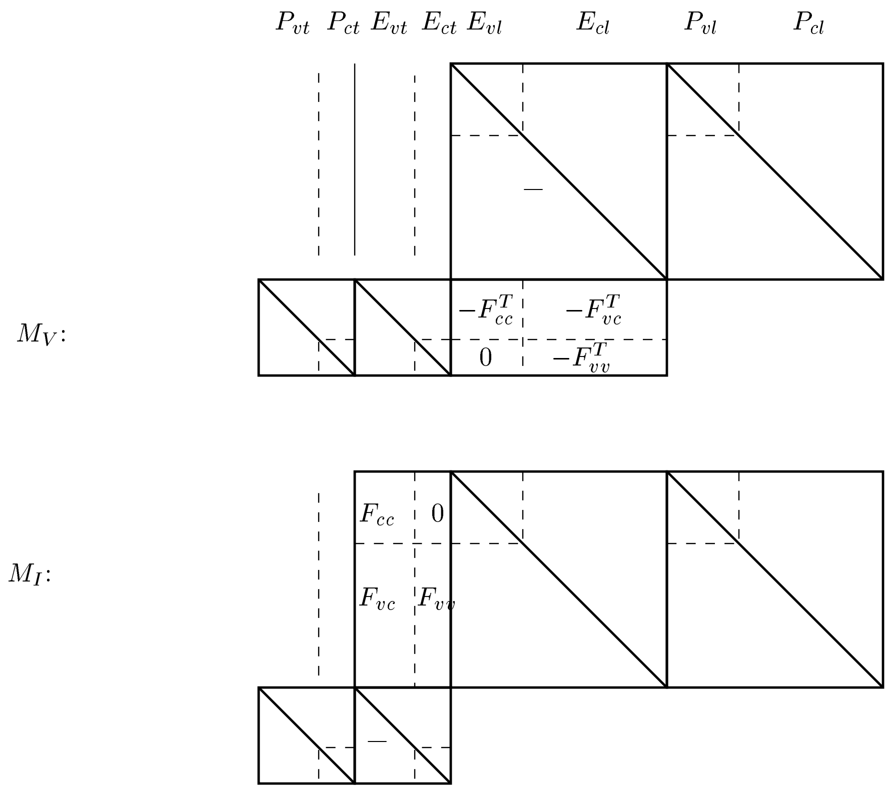
Graph theoretically, the distinguished base E_t in $\mathcal{M}_V$ is a spanning tree[7].

Each resistor is assumed to have a constitutive relation of the form either $v = r(i)$ or $i = g(v)$ (or both) where the functions are monotone non-decreasing. A resistor characterized by $v = r(i)$ or $i = g(v)$ is called *current controlled* (CC) or *voltage controlled* (VC) respectively. The spanning tree is chosen so E_t contains a maximum sized set of VC resistor edges E_{vt}. A matroid theoretic consequence is the other VC resistors E_{vl} are spanned in $\mathcal{M}_V$ by E_{vt}. So F is further partitioned:

Desoer and Wu use variables for each voltage across resistors in E_t and currents through E_l, and write a real linear equation for each row in this tableau. The equations and variables corresponding to $E_{ct} \uplus E_{vl}$ are then eliminated. The result is the equation $G(x, u) = y$ in variables x. The describe the parameters in terms of our subspace pair model, we do the following operations which result in a tableau description in figure ():

(1) Parallel extend E_t and series extend E_l in $(\mathcal{M}_V, \mathcal{M}_I)$ to P_t and P_l respectively. The result is illustrated in figure 4.2.
(2) Introduce ports at $P_t \uplus P_l$. The result is shown in figure 4.2.

[7]Assuming the graph is connected.



By lemma (), operations (1) preserve the complementary base and orthogonallity properties of the original subspace pair. By construction, $V - zeroing$ of P_l and $I - zeroing$ of P_t produces the original subspace pair.

Observation: $P = P_t \cup P_l$ is a coindependent base in both $\mathcal{M}_V{}'$ and $\mathcal{M}_I{}'$. (P is in fact a cobase in each). Note the similarity to the Laplace section.

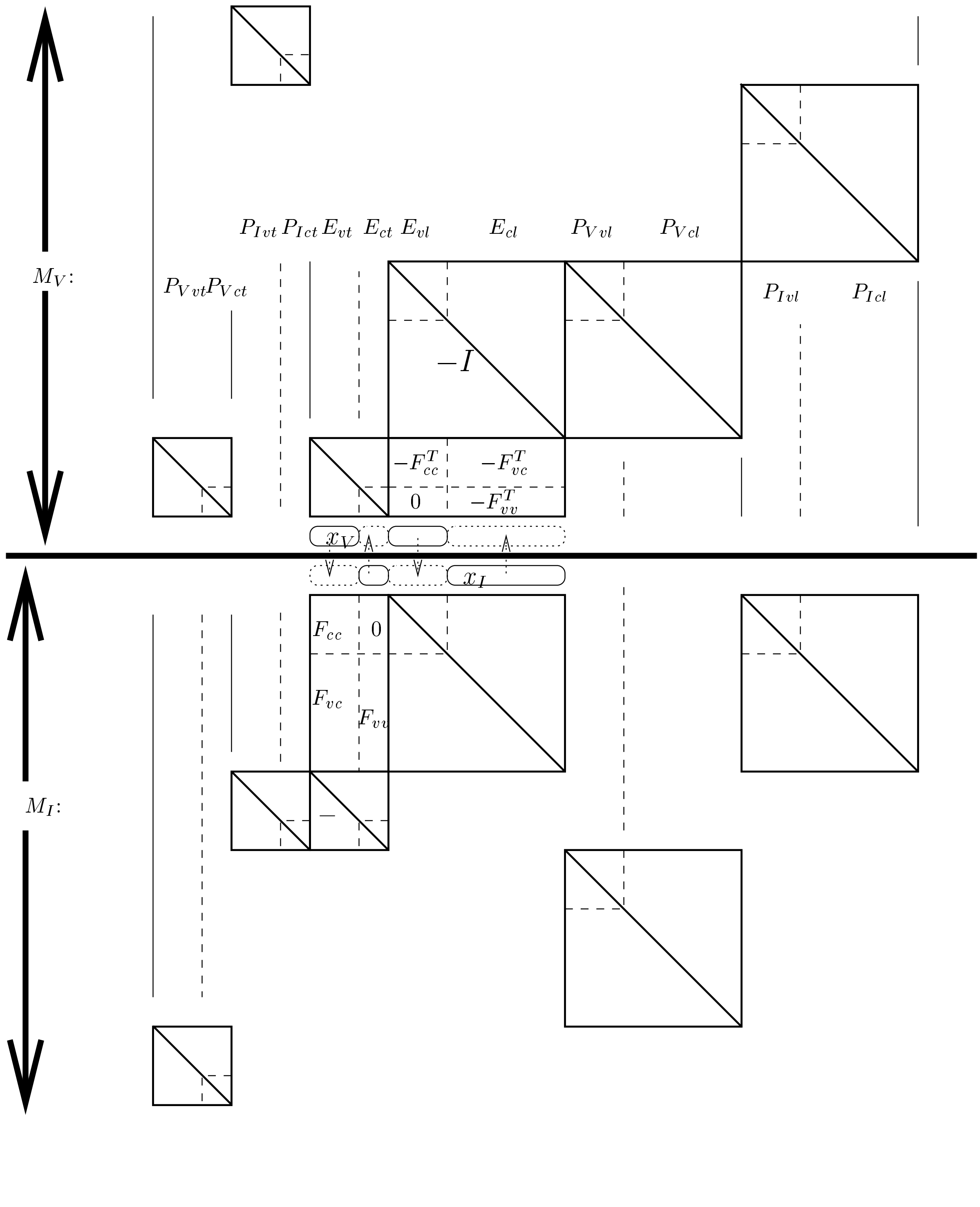
4.3. Rockefellar's Formulation.

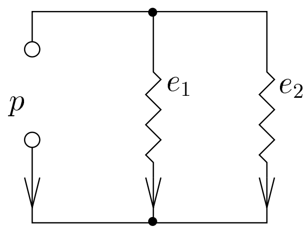
4.4. Sandberg and Willson.

4.5. Bott-Duffin Constrained Inverse.

5. Direct Existence Proof

We develop a direct proof of Theorem 6 by generalizing the work of [**HN86, Fos92**]. In this prior work, what we see now as oriented matroid pairs were pairs of graphic and cographic oriented matroids that were obtained from an original electrical network graph by nullator, norator and V-driven and I-driven port insertions along original edges. A monotonicity condition on the behavior of the system "looking into" a port that was obtained by replacing a resistor was used. In our more general context, the original subspace pair problem might be given in which the direction of monotonicty at an original port might be unpredicatable from the supplemental oriented matroid pair. Indeed, if a non-port element is reversed (reoriened) in one oriented matroid but not the other and then transformed to a port, the common covector property of the supplemental oriented matroid pair


M_V:
P_{Ivt} P_{Ict} E_{vt} E_{ct} E_{vl} E_{cl} P_{Vvl} P_{Vcl}
$P_{Vvt}P_{Vct}$
P_{Ivl} P_{Icl}
$-I$
$-F_{cc}^T$ $-F_{vc}^T$
0 $-F_{vv}^T$
x_V
x_I
F_{cc} 0
F_{vc}
F_{vv}
M_I:
$-$



$$\begin{array}{ccc} 1 & 0 & 0 \\ \hline p_I & e_1 & e_2 \\ \hline 0 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{array}$$

ZIR analysis for voltage source input: p_V contracted. Zero response (unique solution) even if negative resistances $\neq 0$ are allowed.

$$\begin{array}{cccc} \multicolumn{4}{c}{(M_V \text{ matrix})} \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ \hline\hline p_V & p_I & e_1 & e_2 \\ \hline\hline 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ \multicolumn{4}{c}{(M_I \text{ matrix})} \end{array}$$

$$\begin{array}{ccc} 0 & 0 & 0 \\ 1 & 1 & 1 \\ \hline p_V & e_1 & e_2 \\ \hline 1 & 0 & 0 \\ 0 & 1 & -1 \end{array}$$

ZIR analysis for current source input: p_I deleted. Zero response (unique solution) *unless* $g_1 = -g_2$.

FIGURE 6. Simple example that illustrates the solution is unique when the port is V-driven provided each $g_e \neq 0$, but the I-driven system has a unique solution provided each $g_e > 0$ because the resulting supplemental oriented matroid pair has complementary bases and no common covector.

will be unchanged, but the direction of any monotonicity at the inserted port will be reversed.

We will therefore need to give an induction proof which distinguishes between the port elements that were given in the original subspace pair model and the port elements that arose by port insertions at elements that remained in the original supplemental subspace pair. The original supplemental subspace pair is assumed by hypothesis of Theorem 6 to have no common covector. We will see that this hypothesis is preserved by new port insertions.

We begin by stating the theorem in a stronger form that supports the inductive proof.

Recall that a subspace pair model with sources $|S|$ satisfies $|S| = |P|$ and for each pair $\{p_V, p_I\} \subset P$, exactly one of p_V or p_I is in S. For a "V-driven" port p, $p_V \in S$, $p_I \notin S$, and u_{pV} is called the "input variable". Let u_{pI} be called the *corresponding output variable.* For an "I-driven" port, u_{pV} is called the corresponding output variable.

THEOREM 9. *Let* $\mathbf{M}$ *be a subspace pair model with sources where that satisfies the following properties:*

(1). *There are bases* $B_V \in \mathcal{B}(L_V)$, $B_I \in \mathcal{B}(L_I)$ *for which all V-driven ports p satisfy* $p_V \in B_V$ *(note* p_I *must be in* B_V *since it's an isthmus in* $\mathcal{M}_V$*) and* $p_I \notin B_I$*, and all I-driven ports p satisfy* $p_I \in B_I$ *and* $p_V \notin B_V$*.*

(2). $B_V \cup B_I = U$*, for the bases in* ***(1)****.*

(3). *The oriented matroid pair of supplemental pair* (L_V^0, L_I^0) *have no common*

(nonzero) covector. (i.e., $\mathcal{L}(L_V^0) \cap \mathcal{L}(L_I^0) = \{0\}$.)
(4). All constitutive laws are increasing monotone onto functions $\mathbf{R} \to \mathbf{R}$.

Let $P_{new} \subset E$, $P_{new_V} = P_{new} \cap B_V$, $P_{new_I} = P_{new} \cap B_I$, and $\mathbf{M}_{new}$ be the subspace pair problem with sources $S \cup P_{new}$ obtained by inserting into $\mathbf{M}$ the new I-driven ports P_{new_I} and V-driven ports P_{new_V}.

Suppose that $|E \setminus P_{new}| \leq n$.

Then, for every input assignment to $\mathbf{M}_{new}$ there is a unique solution that depends continuously on the input assignment. Furthermore, whenever one new input variable's assigned value u increases with the other inputs constant, the corresponding output variable's value u' does not increase.

We get Theorem 6's existence conclusion by taking $P_{\text{new}} = \emptyset$ above.

5.1. Some Lemmas.

LEMMA 5.1. *Under the assumptions and terminology of Theorem 9, let $B_V^0 = B_V \cap E$ and $B_I^0 = B_I \cap E$. Then $B_V^0 \cap B_I^0 = \emptyset$ and $B_V^0 \cup B_I^0 = E$. (Denoted $E = B_V^0 \uplus B_I^0$ for brevity.)*

LEMMA 5.2. *Under the assumptions and terminology of Theorem 9, $\{p_V | p$ is V-driven$\}$ is an independent set in the matroid $\mathcal{M}(L_V)$, $\{p_I | p$ is I-driven$\}$ is an independent set in the $\mathcal{M}(L_I)$, $\{p_I | p$ is V-driven$\}$ is coindependent in $\mathcal{M}(L_I)$, and $\{p_V | p$ is I-driven$\}$ is coindependent in $\mathcal{M}(L_V)$.*

LEMMA 5.3. *Suppose $e \in B_V \cap E$. Let $\mathbf{M}'$ be obtained by inserting e as a V-driven port in $\mathbf{M}$. Then the supplemental subspace pair $(L_V'^0, L_I'^0)$ of $\mathbf{M}'$ is $(L_V^0, L_I^0)/e$. Similarly, if $e \in B_I \cap E$ and e is inserted as an I-driven port, then $(L_V'^0, L_I'^0) = (L_V^0, L_I^0) \setminus e$.*

LEMMA 5.4. *Suppose (M_V^0, M_I^0) have complementary base pair $B_V^0 \cup B_I^0 = E$, $B_V^0 \cap B_I^0 = \emptyset$ and no common covector. Let $e \in E$. If $e \in B_V^0$ then $(M_V^0, M_I^0)/e$ also has a complementary base pair $((B_V^0 \setminus e) \cup B_I^0)$ and no common covector. Dually, if $e \in B_I^0$ then $(M_V^0, M_I^0) \setminus e$ has complementary base pair and no common covector.*

This lemma expresses the combinatorial argument on which the induction is based. It is an application of Theorem 3.

PROOF. **Proof of Theorem 9, basis**

$n = 0$, so all elements of $\mathbf{M}_{\text{new}}$ are ports. Lemma 5.2 shows the input values can be assigned arbitrarilly. Each output value is in fact independent of the corresponding input; but it depends on other inputs or could be identically 0. Lemma 5.2 also shows the output values are unique. Notice this argument holds for any number of port elements. Thus, each new output value is a non-increasing function of the corresponding input because all such functions are constant.

Proof of Theorem 9, induction

Assume Theorem 9 for given n. Let $\mathbf{M}$ be a subspace pair problem that satisfies the hypotheses, and $P_{\text{new}} \subset E$ be given as specified with $|E \setminus P_{\text{new}}| \leq n+1$.

Suppose $e \in B_V \setminus P_{\text{new}}$ (the dual case of $e \in B_I$ is similar). (In the terminology of [**HN86, Fos92**], "resistor e is current-controlled.") Let $\mathbf{M}_{\text{new}}'$ be obtained by inserting a new V-driven port at e together with new ports P_{new}.

By induction, for each choice of u_{Ve} and other input values U, there is a unique solution to the subspace pair problem $\mathbf{M}_{\text{new}}'$ that depends continuously on u_{Ve} and other input values U and the function for the new output value $u_{Ie} = \phi(u_{Ve}, U)$ is non-increasing in u_{Ve} for fixed U.

The constitutive law Γ_e is a monotone increasing function $u_{Ve} = r_e(u_{Ie})$. Hence $f(u_{Ie}) = \phi(r_e(u_{Ie}), U) - u_{Ie}$ is a continuous decreasing function of u_{Ie} with $f(-\infty) = +\infty$ and $f(+\infty) = -\infty$. Therefore $f(u) = 0$ has a solution. Thus, the unique solution to $\mathbf{M}_{\text{new}}{}'$ with new input value $u_{Ie} = u$ and other input values U is a solution to $\mathbf{M}_{\text{new}}$ with input values U.

It remains to show that for each choice of *other* input values (new and old, for elements $\neq e$) of an $\mathbf{M}_{\text{new}}$ in Theorem 9 (which might have $|E \setminus P_{\text{new}}| = n + 1$), the solution is unique, it depends continuously on each input value, and each new output value depends non-increasingly on its corresponding input variable when the other inputs are constant.

The uniqueness is proven from Lemma 5.3 followed by Lemma 5.4 and then Proposition 2.

Continuity follows from the fact that ϕ is continuous...

Finally, suppose $p \in P_{\text{new}}$, u_i is the corresponding input variable and u_o is the corresponding output variable, and for particular values of the other inputs, there are u_i, u_o values satisfying $(u_i - u_i')(u_o - u_o') > 0$. We can construct a supplemental pair (L_V^p, L_I^p) by zeroing all the sources except for p. An argument based on the signs of the solutions with giving u_i, u_i', u_o, and u_o' similar to that of Proposition 2 demonstrates the existence of a common covector of (L_V^p, L_I^p). On the other hand, Lemma 5.3 and Lemma 5.4 combined with hypotheses **(3)** and **(4)** demonstrate (L_V^p, L_I^p) have no common covector. Hence $(u_i - u_i')(u_o - u_o') > 0$ is impossible. □

Heuristic Comment ("Fuzz") 3. **6. Other Stuff (Orphaned)**

??? Hence we do not assume any rank or orthogonality conditions on subspace pairs in the definitions below.

Theorem 5 applies to the subspace pair obtained from all declarations of nullators, norators, opens and shorts.

There is a subtle difference between declaring a V-source with 0 input value and contracting the same element. If a set S of k such elements is not independent in $\mathcal{M}(L_V)$, then the rank of $\mathcal{M}(L_V/S)$ will be more than $\text{rank}(\mathcal{M}(L_V)) - k$ but the given combination of input values will still be feasible. If they are not coindependent in $\mathcal{M}(L_I)$ (which will certainly be true when there are no nullors). then the rank of $\mathcal{M}(L_I \backslash S)$ will be less than $\text{rank}(\mathcal{M}(L_I))$ but there will be a non-zero combination of output only variables. (Physically, that corresponds to non-zero current circulating in a loop of ideal wires; or a non-zero self-stress in an overbraced subframework of rigid bars. The dual physical situation is that is possible for the disconnected parts of an electrical network to differ in electrical potential when a cut-set of branches are removed; mechanically, more flexes of the framework can exist when some edges are removed. For this reason, we are careful to distinguish deletion/contraction from port insertion.)

It's yet to be done to handle simutaneous application of force to more than 2 vertices....

6.0.1. *Topological Formulas.* They can come out of the subspace pair formulation three ways:

- Besides the "g" or "r" variables, do not insert ports but do use an extra variable "x" to relate the voltage to current of one port. Then the equation $\det(M_V G; M_I) = 0$ has the form $Ax + B = 0$, so $x = -B/A$.

- Use the Plücker coordinate formulation of the intersection subspace to identify a minor of a hybrid or other description matrix of as the determinant of the solution submatrix for a system of equations; then use the Cramer's rule generalization to find the ratios of minors of the equation matrix to analyzed. This was done for my ISCAS 98 paper.
- Use Rota's Grassmann-Cayley algebra meet formula to extract expansion directly from subspace pair matrices, together with the previous way to identify Plücker coordinate ratios with description matrix minors.

6.0.2. *Series-Parallel.* For some (L_V, L_I), the common covectors are unique up to multiplication by -1. Such subspace pairs are from electrical networks in which the direction of each current flow and voltage drop is predictable from the network structure alone: These are networks whose graph is *series-parallel*.

???coordinate w. Theorem (5) A subspace pair model with sources S is called *structurally feasible* if $S \cap P_V$ is an independent set in $\mathcal{M}_V$ and $S \cap P_I$ is an independent set in $\mathcal{M}_I$.

We say a subspace pair problem with sources S is *well-posed* when for all input assignments there is a unique solution.

re zero-ing a port: I'M NOT SURE THIS IS RIGHT...WHAT IF say there is a response to $x_{p_V} = 0$ in which the only non-zero value is x_{p_I} ??? I think the topological conditions (no cocircuits on port response elements) exclude this, but the point must be made clear.

Reduced nodal admittance matrix. Nodal resistance matrix. Interaction with a physical framework with it's environment. A framework is first order rigid iff it "resolves all applications of static equilibrium forces". However, every physical bar has some elasticity: An ideal rigid bar is analogous to an ideal voltage source. Hence, given an elastic framework, for every application of static equilibrium forces on the vertices, the vertex positions will change as the bars stretch or shrink under the forces they now carry to resolve the applied force. These first order vertex position changes are given by $Z\mathbf{f}$.

The environment might interact by "forcing" some vertices to change position relative to one another. Intuitively, the framework will "push back". The other vertices are free to move as adjacent vertices move and incident bars change length in response to the forces developed in them to resolve the forces required to hold the framework in its new position. The position changes of the free vertices V can be calculated by solving for the unknown position changes in the system of equations $(Y\mathbf{v}_V)(V) = 0$.

For our purposes, we insert port elements in order to make interactions with the environment explicit. This enables a coordinate of an environmental interaction quantity to correspond to an oriented matroid element, so that its sign can be read off from the corresponding entry in a covector.

(3) OMP from "leaky electricity" in digraph. No common covector property proven graph theoretically. Citations of my ISCAS-95 paper and Tutte's triangulation of a triangle aapplication.

(5) Port insertion on a initially dual pair.

(7) Finish researching applicability to DeSoer and Wu's formulation of monotone non-linear systems.

(8) Application to decomposing a rectangle into rectangles with non-linear aspect ratio functions..

7. Old Examples

7.1. Using an oriented matroid to reason about inequalities. We illustrate an application of Definition 1 together with the linear algebra operation of (1) combining members in L and (2) defining new a coordinate to be a specific linear combination of existing coordinates to derive inequalities on parameters necessary for known conditions on a covector set. Willson ([**Wil75**], Theorem 5) gave the following conditions for a (common ground node) 2-port to be a no-gain element when it is described by continuous functions $i_1 = F_1(v_1, v_2)$ and $i_2 = F_2(v_1, v_2)$:

$$v_1 F_1(v_1, v_2) > 0 \text{ for } \frac{v_2}{v_1} < 1$$

$$v_2 F_2(v_1, v_2) > 0 \text{ for } \frac{v_1}{v_2} < 1$$

$$(v_1 + v_2)(F_1(v_1, v_2) + F_2(v_1, v_2)) > 0 \text{ for } v_1 v_2 > 0$$

Notice these conditions can be expressed as restrictions on the combinations of the signs $\sigma(v_1), \sigma(v_2), \sigma(i_1), \sigma(i_2), \sigma(v_1 - v_2)$ and $\sigma(i_1 + i_2)$, since the third condition is relevant when $\sigma(v_1) = \sigma(v_2) \neq 0$ and in that case, $\sigma(v_1 + v_2) = \sigma(v_1)$. We will derive conditions on the g_{ij} of a linear admittance matrix description when the F_i are linear. First, we write a matrix whose rows span the space of $(v_1, v_2, g_{11}v_1 + g_{12}v_2, g_{21}v_1 + g_{22}v_2)$:

v_1	v_2	i_1	i_2
1	0	g_{11}	g_{21}
0	1	g_{12}	g_{22}

We append two more columns, for $(v_1 - v_2)$ and $(i_1 + i_2)$:

v_1	v_2	$v_1 - v_2$	i_1	i_2	$i_1 + i_2$
1	0	1	g_{11}	g_{21}	$g_{11} + g_{21}$
0	1	-1	g_{12}	g_{22}	$g_{12} + g_{22}$

Suppose $v_1 << 0$, which we abbreviate "$v_1 --$" and v_2 is "small" and positive, abbreviated v_2+. In that case, the second condition applies. However relevant covector is
$(-, +, *, *, \sigma(-g_{21}) \circ \sigma(g_{22}), *) = (-, +, *, *, +, *)$ (where "$*$" denotes "we don't care"), so $\sigma(-g_{21}) \circ \sigma(g_{22}) = +$, and we conclude $g_{21} \leq 0$ (and if $g_{21} = 0$ then $g_{22} > 0$). We conclude $g_{12} \leq 0$ from the first condition and covector $(+, -, *, +, *)$ in similar fashion.

On the other hand, if $v_1 + +$ and v_2+, we get the composition of the two covectors (from the first two rows of the matrix) $(\sigma(1), 0, \sigma(1), \sigma(g_{11}), \sigma(g_{21}), \sigma(g_{11} + g_{21})) \circ$
$(0, \sigma(1), \sigma(-1), \sigma(g_{12}), \sigma(g_{22}), \sigma(g_{12} + g_{22}))$ which is
$(+, +, +, \sigma(g_{11}) \circ \sigma(g_{12}), \ldots)$. Note the first condition applies since $\sigma(v_1 - v_2) = + \circ - = +$. Therefore, $\sigma(i_1) = \sigma(g_{11}) \circ \sigma(g_{12})$ must be $+$. We conclude that $g_{11} > 0$ because we have already concluded $g_{12} \leq 0$. In the same way, we conclude $g_{22} > 0$ from the composition of the same covectors in the opposite order and the second condition.

Let us analyze the sign of $i_1 + i_2$ in the same two compositions: $(+, +, +, \ldots, \sigma(g_{11} + g_{21}) \circ \sigma(g_{22} + g_{12}))$ and $(+, +, -, \ldots, \sigma(g_{22} + g_{12}) \circ \sigma(g_{11} + g_{21}))$. The third condition tells us that the last sign in both covectors must be $+$. We conclude $g_{11} + g_{21} \geq 0$ and $g_{22} + g_{12} \geq 0$ and $g_{11} + g_{21} + g_{22} + g_{12} > 0$.

To investigate the situations when v_1 and v_2 are close, we do exact calculation of the matrix rows to extend the matrix to:

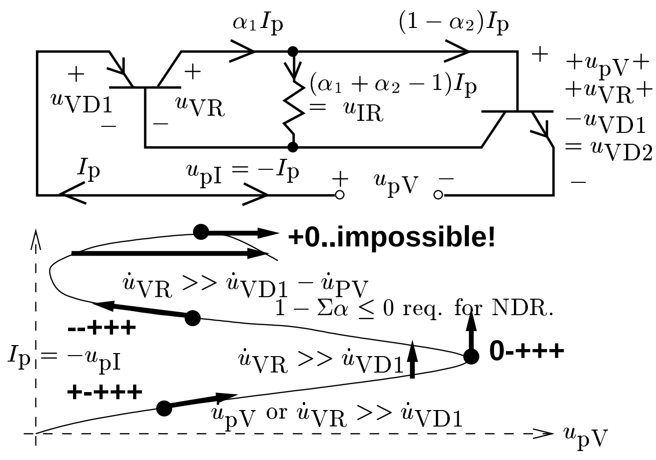
v_1	v_2	$v_1 - v_2$	i_1	i_2	$i_1 + i_2$
1	0	1	g_{11}	g_{21}	$g_{11} + g_{21}$
0	1	-1	g_{12}	g_{22}	$g_{12} + g_{22}$
1	1	0	$g_{11} + g_{12}$	$g_{21} + g_{22}$	$g_{11} + g_{21} + g_{12} + g_{22}$

The covector obtained by composing the covector from the third row with the first produces a covector to which the first condition applies. We conclude $g_{11} + g_{12} \geq 0$. Similarly, the second condition implies $g_{22} + g_{21} \geq 0$.

7.2. A feedback structure case of Trajkovič and Willson [TW90]. We illustrate the oriented matroid approach by reproducing the result of [**TW90**] that a particular configuration of a "feedback structure" with two Ebers-Moll transistors and one port cannot exhibit negative differential resistance by itself, and it can exhibit NDR if one resistor is added and $\alpha_1 + \alpha_2 - 1 > 0$. Under KVL, the voltages across the port, resistor and two Ebers-Moll diodes are given by the row space member of M_V when the three rows are multiplied by the 3 independent voltages V_1, V_2 and the port voltage V. The space of current values feasible in the same 4 elements under KCL and the two Ebers-Moll linear CCCS's is only one dimensional; it is spanned by the one row of M_I. Beginning with the signatures of the rows of the matrices, we can apply the covector axioms to explore what common covectors are possible under several variations.

We used the rules of Definition 1 to figure out common covectors in $\mathcal{L}(M_V)$, $\mathcal{L}(M_I)$ with their first two signs, of port elements pV, pI given, for the tangents and differences drawn boldly on the $I_{\text{p}}/V_{\text{p}}$ curve. (The algorithm ideas used appear in [**Cha96**].) In three cases, (unique) extensions of the given signs exist when $1 - \Sigma\alpha \leq 0$ but there is only one such case otherwise, verifying a condition for NDR from [**TW90**]. Multiple operating points with an I-driven port are impossible. The magnitude inequalities come from "$\epsilon << 1$" in explaining "$\circ$" above Definition 1. Fig. 1: Example. M_V/M_I are the upper/lower right arrays.

(row mults.)u_{pI}	0	1	0	0	0
u_{pV}	1	0	0	0	$+1$
u_{VR}	0	0	1	0	$+1$
u_{VD1}	0	0	0	1	-1
Γ coeffs. $\rightarrow$	1	1	g_{R}	g_{D1}	g_{D1}
elements $\rightarrow$	pV	pI	R	D1	D2
$u_{\text{pI}} = -I_{\text{p}}$	0	1	$1 - \alpha_1 - \alpha_2$	$\alpha_1 - 1$	$\alpha_2 - 1$
u_{pV}	1	0	0	0	0



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