Tutte Polynomials and Electrical Networks

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Outline

1. Spanning trees and equivalent (linear) resistance.
2. An exterior algebra (extensor) Tutte function and a (linear) resistance network's behavior projected on distinguished coordinates.
3. Rayleigh's inequalities.
4. Tutte polynomials on pairs and (linear) amplifier networks.
5. Distinguished graph vertices and splitting formulas.

Equiv. Resistance $:\equiv -(v_p/i_p)$ observed at a port p by the environment (Port edge p locates the *2 terminals*.)

Theorem

$$-(\frac{v_p}{i_p}) = \frac{WTS(G/p)}{WTS(G\backslash p)} = \frac{Matrix\text{-}Tree\ Det(G/p)}{Matrix\text{-}Tree\ Det(G\backslash p)}$$

This "Maxwell's rule" is proved via the Matrix Tree Thm. on 2 DIFFERENT GRAPHS G/p and $G\backslash p$.

Theorem

Weighted Tree Sum (WTS) is a colored Tutte function:

$$WTS(G') = g_e WTS(G'/e) + r_e WTS(G'\backslash e) \text{ for all } e \notin P$$

$$WTS(coloop(e)) = g_e$$

$$WTS(loop(e)) = r_e$$

Next steps

1. One (terminal-pair) port $\rightarrow$ set of ports P.
2. 1-dim subspace of homogeneous coordinates of solutions $((v_p, i_p)) \rightarrow$ p-dim subspace of $k^{2|P|}$.
3. p-dim subspace $\rightarrow$ EXTENSOR (decomposible exterior algebra, i.e., anti-symmetric tensor) with $\binom{2p}{p}$ Plucker coordinates (determinants).

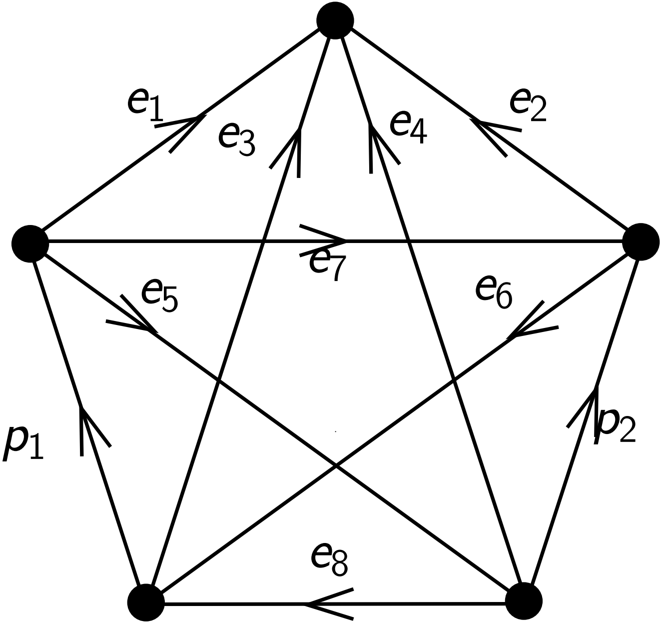
Theorem

(After careful definitions...) For fixed P,

each Plucker coordinate	*and*	*this extensor in the exterior algebra*

satisfy weighed Tutte recursion, when $/e$ and $\backslash e$ are restricted to $e \notin P$.

Example



Example

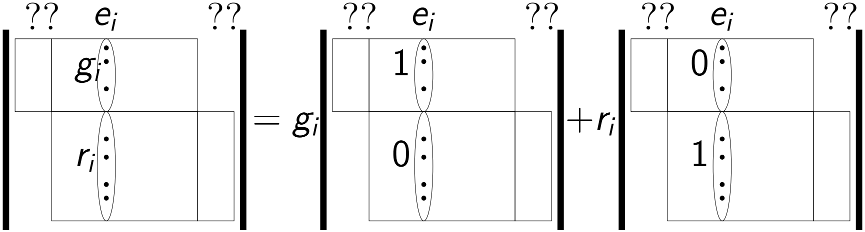
Here's M of the electrical network equations $Mx = 0$. Kirchhoff's laws apply to all cycles and cocyles with $r_i x_{e_i}$ as voltage and and $g_i x_{e_i}$ as current of resistor (not port) edges. TWO SEPARATE voltage and current variables are used for each port edge.

ip_1	ip_2	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	vp_1	vp_2
1	0	0	0	$+g_3$	0	0	$-g_6$	0	$-g_8$		
-1	0	$-g_1$	0	0	0	$+g_5$	0	$+g_7$	0		
0	$+1$	0	0	0	$+g_4$	$-g_5$	0	0	$+g_8$		
0	-1	0	$-g_2$	0	0	0	g_6	$+g_7$	g_8		
		$+r_1$	0	$-r_3$	0	0	0	0	0	1	0
		0	$+r_2$	0	$-r_4$	0	0	0	0	0	1
		$-r_1$	0	0	$+r_4$	$+r_5$	0	0	0	0	0
		0	$-r_2$	$+r_3$	0	0	$+r_6$	0	0	0	0
		$-r_1$	$+r_2$	0	0	0	0	$+r_7$	0	0	0
		0	0	$+r_3$	$+r_4$	0	0	0	$+r_8$	0	0

Top 4 rows: Basis for cocycle space. Represents graphic matroid.
Bot 6 rows: Basis for cycle space. Represents cographic matroid.

The Tutte Decomposition

For all choices denoted by ?? of the $\binom{2|P|}{|P|}$ size $|P|$ subsets of the $2|P|$ columns $\{ip_k, vp_k\}$, the matrices in the equation below are square.
So the elementary multilinearity of determinants means Tutte decompostion holds for all $e_i \notin P$:



(Technical detail: Define the Tutte function on all graphs with distinguished or port subset P so the det. signs are consistent with the decomposition.)

Rayleigh Identity which $\Rightarrow$ inequality, "Neg. Spanning Tree Correlation"

$\Gamma_e(G)$ is equivalent conductance across e. Rayleigh: $0 \leq \dfrac{\partial \Gamma_p}{\partial g_f} = \dfrac{\partial \frac{T_G}{T_{G/e}}}{\partial g_f}$

is equivalent to

$$0 \leq \frac{\partial T_G}{\partial g_f} T_{G/e} - T_G \frac{\partial T_{G/e}}{\partial g_f} = T_{G/f} T_{G/e} - T_G T_{G/e/f}$$

Theorem

$$T_{G/f} T_{G/e} - T_G T_{G/e/f} = \left(T^{+}_{G/e \ \& \ G/f} - T^{-}_{G/e \ \& \ G/f} \right)^2$$

$T^{\pm}_{G/e \ \& \ G/f}$ *enumerate the* $\pm$ *common spanning trees.*

Choe, Cibulka, Hladky, Lacroix and Wagner gave bijective proofs; we give det. based proofs and generalizations.

Linear Alg./Oriented Matroid Proof of Rayleigh's Identity

Let R be the transfer resistance matrix for 2 ports across e and f. Our result implies that

$$\det R = \begin{vmatrix} R_{ee} & R_{ef} \\ R_{fe} & R_{ff} \end{vmatrix} = + \frac{T_{G/e/f}}{T_G}$$

It and better-known results tell us

$$R_{ee} = \frac{T_{G/e}}{T_G}; \quad R_{ff} = \frac{T_{G/f}}{T_G}; \quad R_{ef} = R_{fe} = \frac{T^+_{G/e\ \&\ G/f} - T^-_{G/e\ \&\ G/f}}{T_G}$$

$T_{G/f}\,T_{G/e} - T_G\,T_{G/e/f} = \left(T^+_{G/e\ \&\ G/f} - T^-_{G/e\ \&\ G/f}\right)^2$ is immediate after substituting these into

$$\det R = R_{ee}R_{ff} - (R_{ef})^2$$

The $+$ follows from physical grounds if the $g_e, r_e \geq 0$. Our characterization and proof are combinatorial.

New Rayleigh's Identities!

The same method generates identities and inequalities from

$$\begin{vmatrix} R_{ee} & R_{ef} & R_{eg} \\ R_{fe} & R_{ff} & R_{fg} \\ R_{ge} & R_{gf} & R_{gg} \end{vmatrix} = +\frac{T_{G/e/f/g}}{T_G} \geq 0$$

when all $r_{..}, g_{..} \geq 0$, ETC...

(Applications???)

Might the same methods address a much harder problem: The same inequality for forests instead of spanning trees?

Pairs: The Common Covector Model

The cycle space of G_I GENERATES the covectors of an **oriented matroid** over $(E \cup P_I)$.

0

(signs indexed by E) (by P_I)

Non-linear monotone resistors CONSTRAIN SIGNS of voltage drops (from ↓) and flows (from ↑) TO BE EQUAL

(signs indexed by E) (by P_V)

The cocycle space of G_V GENERATES the covectors of an oriented matroid over $E \cup P_V$.

G_V
SOMETIMES EQUALS
G_I

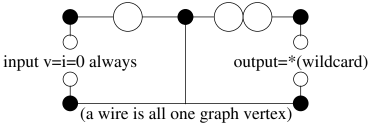
0

Voltage and Current graphs G_V, G_I

"Voltage graph" G_V (EE [8, 13], NOT Gross, ...) represents KVL $\mathbf{v} \in$ Cocycles W/ SOME $v_e \equiv 0$

"Current graph" G_I represents KCL $\mathbf{i} \in$ Cycles WITH SOME FLOWS $\equiv 0$

- They are EQUAL GRAPHS for resistor networks.
- For networks with idealized amplifiers, they are not equal.



The output voltage and current are whatever makes the input voltage and current BOTH BE zero.

- (More) realistic amp. model = idealized amp. + resistors.

open

$$G_v = G \backslash e$$

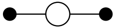
$$G_I = G \backslash e$$

short

$$G_v = G / e$$

$$G_I = G / e$$

nullator



$$G_v = G / e$$

$$G_I = G \backslash e$$

norator

$$G_v = G \backslash e$$

$$G_I = G / e$$

Distinguished graph vertices and splitting formulas

Let Q be a set of distinguished, labelled graph VERTICES, analogous to the distinguished port edges P

Theorem

Given graph $G(V \cupdot Q, E \cupdot P)$ let $T(G, P, Q)$ be the Tutte polynomial determined by restricting $/e$ and $\backslash e$ to $e \notin P$ AND carrying along the partition of Q defined by the components of the contracted edges.
Construct $G^Q(V \cupdot Q, E \cupdot P \cupdot P_Q)$ by adding to G a new vertex Z and the $|Q|$ new port edges from Z to each vertex in Q.
Then $T(G, P, Q)$ and $T(G^Q, P \cupdot P_Q)$ (the ported Tutte polynomial) determine each other by substitutions.

So we can use ported Tutte polys to express splitting formulas for Tutte polynomials of graph, beginning with Crapo [5] and continuing with Andrzejak [1], Bonin and de Meir [4], and Narayanan [12, 14].

etc

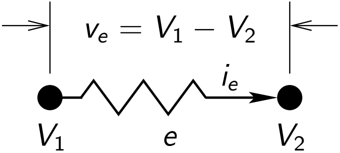
Extra slides...

Why Electricity, EE?

- Scholarly topic suggested by G.-C. Rota $\approx$ 1980?.
- $\approx$ 100 yrs. geometry-like intuition of circuit configurations known by engineers, EE books: "Intuitive Analog Circuit Design (2013)" [15]; "Non-linear Circuits" [8] translates to our Oriented Matroid pair model.
- Geometry of linear spaces and oriented matroids; Tutte decomp. w/ techniques from Barnabi, Brini and Rota's Exterior Calculus [2])
- Real behavior $\approx$ ideal plus perturbations, ideal constraints predict intended real behavior,
- Interesting, accessible, intuitively understandable intentential designs, applicable, easy to both simulate and build physically, dimension $\approx$ 12 or 24, depending on formulation
- Analogs to chemical (and real algebraic geometry [11]), biological, elastic/tensegrity strs. etc., random walks ...
- Merely one scalar non-linearity can cause chaos.

Kirchhoff (1847) [9] Maxwell (1891) [10] The equivalent resistance PROBLEM IS SOLVED by the Matrix Tree Theorem. (1) POSE! the VARIABLES or COORDINATES

$\boxed{v_e, i_e : e \in E}$

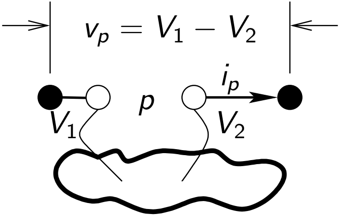


(Use **voltage drops along the flow**, not potentials V_1, V_2.)

ordinary, resistor edge $e \in E$

(IF linear Ohm's law use $|E|$ variables ($g_e v_e = r_e i_e$) ELSE use $2|E|$ variables.)

$\boxed{v_p, i_p : p \in P}$



DISTINGUISHED, PORT edge $p \in P$

The interface to an environment is modelled with $2|P|$ variables.

(math, not EE sign convention)

(2) POSE: EQUATIONS. Preview the consequences.

- (KCL) $(i_e)_{e \in S}$ is a cycle (a flow).
- (KVL) $(v_e)_{e \in S}$ is a cocycle
- (constituitive Law) $i_e = g_e(v_e)$ non-linear, usually monotonic increasing $R \to R$.
  (Sometimes use Ohm's approximation $i_e = g_e v_e$)

Combinatorics!

The signs $\{+, -, 0\}$ have a DUAL-PAIR ORIENTED MATROID structure (combinatorial, geometric, topological).

Engineering with amplifiers!

There's good unique solvablility due to STRUCTURE, when the NON-DUAL PAIR (for voltages and currents) is ALMOST DUAL: No common covectors.

Multiple Ports. (your stereo: 3=power plug & 2 speakers)

- One formula expresses $\binom{2|P|}{|P|}$ different Matrix Tree Theorems...
- ... long vertex-based proofs are shortened; Rayleigh inequalities too.
- Interesting **non-commutative ranges** of new ORIENTED MATROID Tutte invariants with pattern:

  $$TF(N(P \mathbin{\dot\cup} E)) = F(N(P \mathbin{\dot\cup} E)/E)$$

  (They distinguish DIFFERENT ORIENTATIONS of the SAME MATROID.)
- Ported/Relative OM Tutte Poly. terms embed SPECIFIC MINORS as variables, making proofs just with $\partial T/\partial x_e$ easier.
- Formalize composition of systems [12], Tutte poly. splitting formulas.
- Model practical devices (transistors, op amps); Label variables to observe.
- Align EE applications with knots [6] (Ported = "Relative") and combinatorial geometry [17] (Ported = "Set Pointed").

Constraint/Generator Duals and 2 Results.

- (Part 1) Technique:

$$\text{Solution Space} = \bigcap \text{Constraint Subspaces}$$

- Result: An exterior algebraic algebraic Tutte function: Each of its $\binom{2|P|}{|P|}$ Plücker coordinates satisfies a Matrix Tree Theorem.
  This and det. formulas easily prove Rayleigh inequalities.

- (Part 2) Combine with:

$$\text{Solution Space} = \text{Closure(Set of Generators)}$$

- To apply: An oriented matroid's COVECTOR SET encodes ALL POSSIBLE $(+,-,0)$ coordinate behaviors or δs.
- Result: An oriented matroid pair model for some non-linear problem (AMPLIFIER!) well-posedness. (How? Sign contradictions $\Rightarrow$ a KERNEL$=\{(0)\}$.)

Part 1) Use Matrix M in CONSTRAINTS $MX = 0$ to get...

The Tutte-like function $\mathbf{M}_E()$: Extensor $\mathbf{N} \to$ Extensor $\mathbf{M_E}(\mathbf{N})$.
(STUDENT NOTE: An EXTENSOR represents the row-space of an $r \times s$ $r-$rank matrix M by the $\binom{s}{r}$-TUPLE of the DETERMINANTS of M's $r \times r$ submatrices. Plücker coords.)
Given N (matrix), construct $N^{\perp}$ with orthog. comp. row space.
Construct: $(G = \text{diag}(g_e),\ R = \text{diag}(r_e))$

$$M = \left[\begin{array}{c|c|c} N(P) & 0 & N(E)G \\ \hline 0 & N^{\perp}(P) & N^{\perp}(E)R \end{array} \right]$$

with columns labelled by $P_I \uplus P_V \uplus E$.
Extensor $\mathbf{M}$ over $k[g_e, r_e](P_V \uplus P_I \uplus E)$ is the $\wedge$-product of M's **row vectors**. The contraction result $\mathbf{M}_E(\mathbf{N}) = \mathbf{M}/E$ appears:

$$\mathbf{M} = \mathbf{M}_E(\mathbf{N})\mathbf{e_1}\mathbf{e_2}\cdots\mathbf{e}_{|E|} + (\cdots)$$

$\mathbf{M}_E(\mathbf{N})$ is our Tutte function $\mathbf{N} \to$ Ext. Alg.

Contracting means "Eliminate variables"

ELIMINATE the variables indexed by E, leaving $2|P|$ variables labelled by P_I and P_V. ie, CONTRACT E. **Answer $\mathbf{M_E}$ IS:**

$$\mathbf{M}_E = \bigwedge_{\text{JOIN over rows}}^{\text{Exterior}} \left[\begin{array}{c|c} A_{I,I} & A_{I,V} \\ \hline A_{V_I} & A_{V,V} \end{array}\right] [\mathbf{p_{I_1}}, \cdots, \mathbf{p_{I_p}}; \mathbf{p_{V_1}}, \cdots, \mathbf{p_{V_p}}]^{\mathbf{t}}$$

$$= \ldots + C_i\mathbf{XXX} + \ldots; \text{Equiv. Resistance} = \text{certain } \ C_i/C_j$$

All the other C_k's have similar interpretations.
$\binom{2|P|}{|P|}$ **Matr. Tree Theorems:** Each $C_k(N)$ (a PRINCIPAL MINOR of MATRIX A ABOVE!) $= g_e C_k(N/e) + r_e C_k(N\backslash e)$ ($e \notin P$, e not (co)loop).
Each C_k is a signed weighted enumerator of forests satisfying **conditions ...**

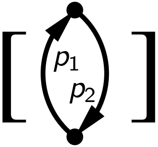
Conditions (what sets F are enumerated by one det. C_i)

The **conditions ...** are on the rank, nullity of F and, WHAT ORIENTED MINOR is $G/F \setminus (E \setminus F)$, the minor with ONLY PORT EDGES from contracting F and deleting the other resistor edges, leaving the ports.

The conditions for a given C_k *sometimes* make all the signs the same (eg: C_i and C_j in 1-port equivalent resistance $R = C_i/C_j$)

Othertimes, the oriented **P-minors** in the completed Tutte decomposition of C_k determine some $+$ and some - signs.

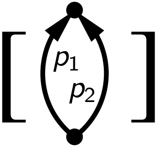
When $[G/F|P]$ is



the term is

$+ g_F r_{E \setminus F}$

When $[G/F|P]$ is



the term is

$- g_F r_{E \setminus F}$

Application: Rayleigh Identity, "Neg. Spanning Tree Correlation"

$\Gamma_e(G)$ is equivalent conductance across e. Rayleigh: $0 \leq \frac{\partial \Gamma_p}{\partial g_f} = \frac{\partial \frac{T_G}{T_{G/e}}}{\partial g_f}$

is equivalent to

$$0 \leq \frac{\partial T_G}{\partial g_f} T_{G/e} - T_G \frac{\partial T_{G/e}}{\partial g_f} = T_{G/f} T_{G/e} - T_G T_{G/e/f}$$

In fact,

$$T_{G/f} T_{G/e} - T_G T_{G/e/f} = \left(T^{+}_{G/e \ \& \ G/f} - T^{-}_{G/e \ \& \ G/f} \right)^2$$

$T^{\pm}_{G/e \ \& \ G/f}$ enumerate the $\pm$ common spanning trees.

Known Partial and Full Combinatorial Proofs

$$T_{G/f}\,T_{G/e} - T_G\,T_{G/e/f} = \left(T^{+}_{G/e\ \&\ G/f} - T^{-}_{G/e\ \&\ G/f}\right)^2$$

$T^{\pm}_{G/e\ \&\ G/f}$ enumerate the $\pm$ common spanning trees.

Choe (2004) proved essentially this using the vertex-based all-minors matrix tree theorem, combinatorial cases and Jacobi's theorem relating the minors of a matrix to the minors of its inverse..

Cibulka, Hladky, Lacroix and Wagner (2008) gave a completely bijective proof that utilizes some natural 2:2 and 2:1 correspondances.

Difficulty: Some terms on the left cancel and some reduce to terms with coefficients ± 2.

“Colors” are parameters on every Tutte decomposition step

The Bollobos/Riordan/Zaslavsky [3, 18], Traldi-Ellis-Monaghan [16, 7], (sdc unpub) BRZ theory for well-definedness of “Relative Tutte Polynomials for Colored Graphs” ALL GOES THROUGH (Diao and Hetyei [6]): The 3 BRZ conditions on (colors,initial values) GENERALIZE TO 5; activity theory WORKS TOO, when based on linear orders on the non-port-elements.

In a nutshell

The 5 conditions $\implies$ activities define an unambiguous Tutte function from the deletion/contraction and initial value formulas. Additional conditions $\implies$ the Tutte function has a rank-nullity expansion.

(The rank-nullity conditions are satisfied in our application.)

To specify the activity/deletion-contraction linear order GLOBALLY is UNNECESSARY.

The Gordon/McMahon computation-tree-based activity theory also generalizes. (sdc).

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