Ported Tutte Functions of Extensors and Oriented Matroids

Seth Chaiken *

Computer Science Department
University at Albany, State Univ. of New York, USA
sdc@cs.albany.edu

August, 2 2006
Mathematics Subject Classifications: 52C40, 15A75, 05C50

*Some of this research was performed during my 2001 academic year Sabbatical at the University at Albany.

Abstract

The Tutte equations are ported (or set-pointed) when the equations $F(\mathcal{N}) = g_e F(\mathcal{N}/e) + r_e F(\mathcal{N} \setminus e)$ are omitted for elements e in a distinguished set called ports. The solutions F, called ported Tutte functions, can distinguish different orientations of the same matroid. A ported extensor with ground set is a (fully) decomposable element in the exterior algebra (of antisymmetric tensors) over a vector space with a given basis, called the ground set, containing a distinguished subset called ports. A ported extensor is one way to present a linearly representable ported matroid or oriented matroid. There are extensor operations corresponding to oriented matroid dualization, and to deletions and contractions.

We define a ported extensor function by means of dualization, port element renaming, exterior multiplication, and then contraction of all non-port elements. The main result is that this function satisfies a sign-corrected variant of the Tutte equations in which deletion and contraction are extensor operations, and addition and the anticommutative multiplication belong to an exterior algebra rather than to a commutative ring.

For extensors representing unimodular, i.e., regular matroids with an empty port set, our function reduces to the basis generating function; and then, for graphs, to the Laplacian (or Kirchhoff) determinant. On graphs with port edges, the function value, as an extensor, signifies the space of solutions to Kirchhoff's and Ohm's electricity equations after projection to the voltage and current variables associated to the ports. In particular, the Laplacian matrix with the identity matrix appended presents one example of our extensor function's value. Combinatorial interpretation of various determinants (the Plücker coordinates) generalize the matrix tree theorem and forest enumeration expressions for electrical resistance.

We also demonstrate how the corank-nullity polynomial, basis expansions with activities, and a geometric lattice expansion generalize to ported Tutte functions of oriented matroids. The ported Tutte functions are parametrized, which raises the problem of how to generalize known characterizations of parametrized non-ported Tutte functions.

Contents

1 Introduction **3**
1.1 Exterior Algebra . . . 6
1.2 New Tutte-Like Invariants . . . 7
1.3 Structure of Solutions . . . 8
1.4 Computational Complexity . . . 9
1.5 Additional Context . . . 9

2 Preliminaries **10**
2.1 Exterior Algebra . . . 10
2.2 Extensors with Ground Set . . . 12
2.3 Ground Set Orientation and Duality . . . 16
2.4 Identities . . . 17

3 An Extensor Tutte Function **19**
3.1 Main Result . . . 22

4 Corank-Nullity Polynomials **27**

5 Basis, Set and Flat Expansions **29**
5.1 Boolean Interval Expansion . . . 33
5.2 Geometric Lattice Flat Expansion . . . 33

6 Electrical Networks **34**
6.1 Discrete Laplacian . . . 35
6.2 Electrical Network Equations in terms of Edges . . . 36
6.3 Maxwell's Rules . . . 38
6.4 Deriving Maxwell's Rule . . . 39
6.5 Signed Contributions of Spanning Forests . . . 41

7 Additional Background and Directions **42**
7.1 Ground Set Orientation . . . 42
7.2 Computational Complexity . . . 42
7.3 Other Directions . . . 43

8 Acknowledgments **43**

1 Introduction

A ported unoriented or oriented matroid $\mathcal{N} = \mathcal{N}(P, E)$ has its ground set $S(\mathcal{N}) = P \mathbin{\dot\cup} E$ given with a distinguished subset P of elements which we call **ports**; $P \cap E = \emptyset$. The following definition combines the idea of Tutte invariants of set-pointed matroids studied by Las Vergnas [35–37] with the idea of the parametrized Tutte equations and functions studied by Zaslavsky [67] (who calls our Tutte functions "strong"), Bollobas and Riordan [5] and Ellis-Monaghan and Traldi [23]. Let two parameters g_e and r_e be given for each $e \in E$.

Definition 1.1. A function F is a **(ported and parametrized) Tutte function** if the domain of F is a minor closed class of ported unoriented or oriented matroids and F satisfies the following **(ported and parametrized) Tutte Equations** for each $\mathcal{N}$ in the class:

When $e \in E$ is a non-separating element, i.e., e is neither a port nor loop nor a coloop (i.e., isthmus):

$$F(\mathcal{N}) = g_e F(\mathcal{N}/e) + r_e F(\mathcal{N} \setminus e). \tag{1}$$

When $\mathcal{N} = \mathcal{N}_1 \oplus \mathcal{N}_2$ with ground sets $S(\mathcal{N}_1) \cap S(\mathcal{N}_2) = \emptyset$:

$$F(\mathcal{N}_1 \oplus \mathcal{N}_2) = F(\mathcal{N}_1)F(\mathcal{N}_2). \tag{2}$$

In the following, a Tutte function shall be assumed to be both ported and parametrized as above unless otherwise indicated. The letters g_e and r_e which we use for parameters, and the terminology "set of ports" for the distinguished ground set elements, are suggested by the electrical network theory application which motivates our research. Briefly, g_e (for conductance or admittance) and r_e (for resistance or impedance[1]) appear in the homogeneous expression $r_e : g_e$ of a resistance value (i.e., the resistance value is either r_e/g_e ohms if $g_e \neq 0$, or infinity if $g_e = 0$ and $r_e \neq 0$.) A port designates one interface (a pair of terminal vertices and two variables, one for current and one for voltage) between the physical electrical network and an external environment. (Think of an ordinary 2-pin electrical plug or receptacle.) Other physical applications of parametrized Tutte functions of graphs are surveyed in [48]. Details and references for the electrical application are presented but the logical background, results and proofs (§2-5) are independent of these details.

Our main topic is a generalization of one construction of a linear subspace from the graph (with port edges and parameters) that occurs in the application. The constructed subspace consists of the space of network solutions projected to the port variables. We generalize by replacing the graph's coboundary (cocycle) space by a finite dimensional linear subspace. The underlying space has a particular ground set basis, whose elements generalize graph edges, and this ground set has the distinguished subset of port elements.

Our formulation with ports subsumes (see §6.1) the classical equation $L(g,1)\phi = J$ on the node voltages $\{\phi_i\}$ and external currents flowing into nodes $\{J_i\}$. Here, $L(g,1)$ denotes the edge-weighted Laplacian matrix; the edge weights g_e are the conductances (or admittances) of edges and we take each $r_e = 1$. Assuming the graph is connected, each principal cofactor of $L(g,1)$ equals (according to the famous Matrix Tree Theorem) the weighted spanning tree polynomial

$$T(g) = \sum_{\substack{T \subseteq E \\ T \text{ a spanning tree}}} \prod_{e \in T} g_e = \sum_T g_T.$$

[1]Electrical engineers customarily use the words admittance and impedance for complex values.

Choe, Oxley, Sokal and Wagner [66] prove that this and similar basis generating polynomials for some but not all other matroids have the **half-plane property**: For complex values g_e, if $\Re g_e > 0$ for all $e \in E$ then $T(g) \neq 0$. The half-plane property characterizes those electrical networks that cannot generate energy. These authors show that the monomials of every non-zero homogeneous multiaffine polynomial with the half-plane property enumerate the bases of a matroid. After noting that the half-plane property is a strengthening of the property that all the coefficients have the same phase, they raise the open question of whether, for every matroid, coefficient values exist so the basis enumerating polynomial with these coefficients has the half-plane property. It is noted that such coefficients can always be assumed to be positive.

We treat polynomials (including $T(g)$ when $r_e = 1$) whose ratios, for graphs, signify externally observable numerical characteristics of an electrical network that pertain to several port current or voltage values together. These quantities include the "transpedances" used by Brooks, Smith, Stone and Tutte [7] in their work on finding dissections of squares into squares by means of solving electrical network equations. Indeed, these authors' presentation of the Matrix Tree Theorem is in the context of combinatorial proofs of solution properties. Our treatment describes the coefficient signs in terms of oriented matroids. Some of the polynomials have the half-plane property and some clearly do not because they have terms with different signs. We therefore hope that our work may contribute insight into the questions posed in [66], as well as more recent questions raised by Wagner [58] pertaining to inequalities among differences between polynomials that enumerate certain trees or forests. This work combines Rayleigh's inequality (motivated by the physics of electrical networks with positive edge conductances, see also [17]) and analysis of correlations where ratios of edge parameters signify probabilities.

It may help some readers to know that the generalization of the coboundary space and the parameters r_e and g_e, which together determine our constructed solution subspace, are specified separately—This separation corresponds to the distinction between two kinds of constraints (exact or geometric versus approximate) that is explained and modeled with matroid theory by Murota [43].

Our work distinguishes the polynomial determinants where all terms (in the parameters) have the same sign from those where differing signs occur. So, a non-zero determinant with terms of differing signs might vanish for critical combinations of parameter values; whereas a determinant with terms all the same sign will never vanish. In Murota's model, this distinction would depend on the exact or geometric constraints; we express the distinction in terms of its oriented matroid properties.

Some of our previous work [13] applied oriented matroids to distinguish the case of a vanishing determinant for critical parameter value combinations. The approach did not apply Tutte function theory. Instead, we investigated the property of a pair of oriented matroids with a common ground set have a common non-zero covector. The electrical network applications treated were more general than in the present paper. We began with a dual oriented matroid pair (graphic and cographic), deleted or contracted certain elements in each, and then evaluated the above common covector property in the resulting oriented matroid pair with a common ground set. See [62] for Tutte theory developed for paired matroids. We are currently investigating how to generalize paired matroids by adding port elements and relate the theories of the present paper with [13] and [62].

1.1 Exterior Algebra

Exterior algebra is used to represent and operate on linear subspaces. A (fully or completely) decomposable element in an exterior algebra is either a field element or the exterior product of vectors. For the sake of brevity, such decomposable will be called **extensors**. An extensor is **ported** when an underlying space is given with a ground set basis and a distinguished port subset. We remind the reader that each non-zero extensor corresponds to the unique subspace whose bases are the sets of vectors whose exterior product equals the given extensor, up to a non-zero field element multiple (see Theorem 2.1.)

For the electrical network application, a graph's coboundary space is represented by extensor $\mathbf{N}$. In our theory, $\mathbf{N}$ would represent an arbitrary linear subspace. We will define some functions of $\mathbf{N}$ or of certain matroids with values within an exterior algebra. Each function value represents the solution subspace for a generalization of the electrical network problem. The domain of the first such function (see Definition 12) consists of ported extensors. The second function applies to the class of ported unimodular (i.e., regular) matroids. The second function is defined by specializing the first function to the extensors that represent unimodular matroids. See Definition 4.3. When the latter function is specialized to graphic oriented matroids with no port elements, it reduces to the Laplacian determinant which, according to the famous Matrix Tree Theorem, equals the spanning tree polynomial discussed above.

The additive Tutte equation pertains to a graph or matroid and its two minors obtained by deletion and contraction. We will define algebraic operations on $\mathbf{N}$. Their values are denoted by $\mathbf{N}/e$ and $\mathbf{N}\backslash e$ for each $e \in E$. We also adapt the Hodge star operation to define an operation corresponding to oriented matroid dualization. Our main results (Theorems 3.3 and 4.5) are that these two functions obey equations similar to the Tutte equations above, taken in an exterior algebra rather than in a commutative ring. The dualization operation is used in defining these functions and in proving the main results. We hope the reader will bear with us in using the terms Tutte equations and functions in this context before the precise equations can be presented. If not, one can skip to the definitions and theorems in §2 and §3 without loss of logical continuity. Sign factors are required in our Tutte equation variants to accommodate the anticommutative multiplication.

It is well-known and easy to verify that the spanning tree count is a Tutte invariant of graphs. Our results further elucidate the relationships between Tutte functions, the Matrix Tree Theorem theorem and enumeration methods for resistive electrical network solutions pioneered by Kirchhoff [33] and Maxwell [42]. These methods were introduced into combinatorial theory by Brooks, Smith, Stone and Tutte [7,53] who attributed them to Kirchhoff, see §6.1. They continue to be applied within some electrical engineering computer aided design tools [24]. Our two points of departure from [7] are to replace analysis in terms of graph vertices and incidences by analysis of functions on the graph edges, and then to express the relevant equations (§6.2) in exterior algebra.

The extensor functions that we study are generalizations of the determinant of the (reduced) edge-weighted Laplacian matrix of a graph. The homogeneous form of this determinant, according to the Matrix Tree Theorem, equals

$$r_S \det(NDN^t) = r_S \sum_{X \subseteq S} \frac{g_X}{r_X} N[X]^2 = r_S \sum_B \frac{g_B}{r_B} = \sum_B g_B r_{\overline{B}}; \quad D = \operatorname{diag}(\frac{g_e}{r_e}, e \in E) \quad (3)$$

where N is the reduced signed incidence matrix of the graph and the sum is over all spanning spanning trees B. We remind the reader that every extensor can be represented by a sequence of determinants which is called its **Plücker coordinates**. Of course, the maximal forests are the bases of the graphic matroid.

In summary, our theorems about extensor functions satisfying sign-corrected Tutte equations generalize the graphic matroid case of the easy-to-prove fact that the basis generating function is a (non-ported) Tutte function defined for all matroids.

1.2 New Tutte-Like Invariants

However, somewhat deeper theory involving the Laplacian, matroid invariants and exterior algebra is involved. First, consider Tutte invariants of matroids, such as the number of bases. The universal Tutte invariant, the well-known Tutte polynomial, is defined for all matroids; hence no Tutte invariant depends on the orientation of an oriented matroid. Let us extend the definition as follows: A **ported Tutte invariant** of ported oriented matroids is a Tutte function f where all the parameters $r_e = g_e = 1$ that is invariant under oriented matroid isomorphisms f that preserve each port element. Specifically, $f(p) = f^{-1}(p) = p$ for every port p in the domain or range of f. Las Vergnas [35–37] developed the analogous ideas for ported matroids (without orientation), and called them "set-pointed." This work on invariants of matroid morphisms, i.e., strong maps, applies the universal ported invariant called the **big Tutte polynomial**. We had applied this idea to study the behavior of such polynomials under the restricted matroid union operation [12] where the composed matroids can only have port elements in common, which is one generalization of matroid series connection.

It is easy to develop the corresponding universal ported *oriented* matroid invariant. Since the latter has one variable for each connected oriented matroid whose ground set contains port elements exclusively, we conclude that some ported Tutte invariants of oriented matroids *do* distinguish different orientations of the same matroid. We mention that $\mathcal{N}$ is unoriented or oriented in the definition of a ported Tutte function because the big Tutte polynomial never distinguishes different orientations of the same matroid but the oriented variant of it, which is defined for ported oriented matroids, does distinguish some orientations of the same matroid.

Here is a simple example: Let the set of ports be $P = \{p_1, p_2\}$. Let $\mathcal{N}_1^-(P, \emptyset)$ be the oriented matroid with ground set P and oriented circuit collection $\{++, --\}$. $\mathcal{N}_1^-$ is one of the two orientations of the rank 1 uniform matroid with ground set P. Let $\mathcal{N}_1^+(P, \emptyset)$ be the other orientation—Its oriented circuit collection is $\{+-, -+\}$. Since $\mathcal{N}_1^+$ and $\mathcal{N}_1^-$ are decomposable under ported Tutte decomposition, any ported Tutte invariant F of oriented matroids for which $F(\mathcal{N}_1^+) \neq F(\mathcal{N}_1^-)$ distinguishes different orientations of the same matroid. More interesting examples are given in §6.3.

The basic theory of non-ported matroid Tutte invariants proves that the Tutte polynomial has various combinatorial interpretations, i.e., different expansions over subsets or other structures. The generalizations of these for ported Tutte functions and invariants are discussed in §5—They all include variables identified with (connected) matroids on port elements as in Las Vergnas' big Tutte polynomial, except that these matroids are oriented. One such expansion generalizes the corank-nullity polynomial. One of our results is that our extensor-valued Tutte function of ported unimodular oriented matroids can be expressed by replacing each oriented matroid monomial by this function's value, an extensor, on the oriented matroid which the monomial identifies. Since the expression also requires $u = v = 0$ where u, v are the corank-nullity polynomial's variables, we see that our extensor-valued Tutte function generalizes the basis enumerator.

Our extensor-valued Tutte function provides one example of a ported Tutte invariant of graphic oriented matroids (and more generally, unimodular oriented matroids). Given a linear representation N of a graphic or other unimodular oriented matroid, the invariant's value is defined by our extensor function applied to the extensor presenting N, when we take all the parameters $r_e = g_e = 1$. We will see that this generalizes the fact that the reduced Laplacian determinant equals the number of bases in the corresponding graphic matroid, which is the number of spanning trees when the graph is connected.

1.3 Structure of Solutions

The second involvement between Laplacians, ported Tutte functions and exterior algebra beyond the matrix tree theorem grows out from the discrete (i.e., matrix) variant of Laplace's equation, the problems in classical physics that it models, and the structure and solution methods for those problems. Among many other analogs, the parametrized, discrete equation of Laplace combines Kirchhoff's current law (of flow conservation), Kirchhoff's voltage law (electromotive force in direct current electrical systems is determined by a potential function), and Ohm's law (current flow is proportional to potential difference). Given a resistive electrical network, the voltages (i.e., potentials) at all vertices can be determined, by solving Laplace's equation, from given voltages at certain vertices and/or the currents into other vertices flowing from the environment. Our work might contribute a few new insights of matroid theory into this situation. One of our starting points for developing this application is well-known in both matroid theory and electrical network theory: Certain insights are obtained when one can replace analysis in terms of graph vertices by analyses involving graph edges and matroids on them. Hence, we model the interactions with the environment by currents through, and voltage drops across port edges, instead of currents into vertices and potential values at vertices.

Consider all electrical networks with a fixed set of port edges P. Consider two networks to be the same members of this class if they have the same edges and the same cycle spaces. Such equivalent networks will have the same electrical behavior. For us, this class is the same as the class of graphic oriented matroids with a fixed set of port edges for which two parameters, r_e and g_e are given for each non-port edge. Each network determines the ported extensor that represents the graph's graphic oriented matroid. When our extensor function is applied to such an extensor, the function value is an extensor corresponding to the linear subspace of solutions projected to the port variables. For the electrical network application, the significance of our main theorem is that the function of networks with port edges P that gives each network's projected solution space *is a ported Tutte function.* Routine network analysis to solve for port behavior can use the Laplacian determinant and cofactors (or equivalent elimination methods) to solve for the constraints between port variables. But our results show that not only is the Laplacian determinant a Tutte function, but when an extensor is used to express the solution space, the entire solution is a Tutte function. As a corollary, the network solution *when expressed in exterior algebra* can be written as linear combination of solutions of networks with *no resistor edges at all—only port edges.* The coefficients are homogeneous multilinear products of the parameters.

The extensor value is can be calculated in three ways. One is by recursive application of the Tutte equations, i.e., Tutte decomposition. The second is by substitution into a polynomial that extends the parametrized Tutte polynomial to ported, parametrized oriented matroids. The third is by Gaussian elimination. That calculation generalizes the evaluation of the Laplacian determinant. For graphs it is equivalent to finding a suitable representation of the solution space of an electrical network projected onto the variables associated to the port edges. Of course, our extensor-valued invariant is calculated after setting all $r_e = g_e = 1$.

1.4 Computational Complexity

An additional motivation for our work comes from theoretical computer science. The number of bases in unimodular matroids is virtually the only non-trivial Tutte invariant whose computation is tractable. For such matroids, the computation input is a totally unimodular matrix N and the output value is $\det NN^t$. The more general problem, to compute the number of bases a matroid represented by an arbitrary matrix, is $\#\mathcal{P}$-complete (see §7.2 for details.) Our generalization of this $\det NN^t$ is an extensor; a succinct matrix representation of it can be computed by Gaussian elimination (like a determinant). Therefore, when the notion of Tutte invariant is generalized with ports and with exterior algebraic values (and the Tutte equations are modified with sign corrections), new computationally tractable invariants are obtained.

1.5 Additional Context

See [43, 46] for elements of electrical network theory from the point of view of matroids. Electrical networks with non-linear but monotone resistance functions were studied using graphs by [28, 29] in a way that led us to apply oriented matroids to this topic [13]; these publications may help orient the reader our point of view. Our current work and [12] extend to sets of more than one distinguished element results and ideas about series and parallel connection of both networks and matroids. In particular, the four-variable Tutte polynomial for a pointed matroid, which was defined and applied to these topics by Brylawski [8], is a special case of our Tutte function. Our previous work covered solutions [11] by energy minimization [39] of electrical networks (and generalizations) with multiple ports, applications of enumeration [15] combined with oriented matroid ideas [14] to network analysis, and a (non-oriented) matroid abstraction [10] of the solutions.

We proceed to the details about exterior algebra and Plücker coordinates pertaining to realizable non-oriented and oriented matroids. They include algebraic operations and identities which correspond to some elementary matroid relationships. Our main construction, an extensor valued function of ported extensors, and the ported Tutte equation variant that it satisfies, is presented in §3. A variant of the corank-nullity polynomial in §4 is used to express our function restricted to unimodular extensors. The variant differs from Las Vergnas' big Tutte polynomial so (1) it applies to oriented matroids instead of unoriented matroids, and (2) it includes parameters as in Definition 1.1. Extensions to ported matroids of known results about expressing Tutte functions as set, basis and flat expansions, and the related open questions follow in §5. Further discussion of the electrical context is given in §6 and brief remarks on peripheral topics appear in §7.3.

2 Preliminaries

Throughout, K denotes a field, either the reals, rationals, or their extensions generated by the parameters g_e, r_e.

2.1 Exterior Algebra

We refer the reader to basic texts such as [31, §7.1-7.2, on associative and exterior algebras over fields] for complete development and proofs. The following is a synopsis with the emphasis on the facts we will need. It also explains certain notational conventions which help to mimic oriented matroid theory in exterior algebra. We use a combinatorial approach to adapt the operations of Hodge star (for duality) and tensor contraction (for matroid contraction).

An associative algebra $\mathcal{A}$ over K is a ring that is also a vector space over K, for which addition and 0 are the same in both the ring and the vector space, and for which the ring and scalar multiplications are compatible: $a(xy) = (ax)y = x(ay)$ where $a \in K$ and $x, y \in \mathcal{A}$.

Let V be the vector space KS where finite set S is a basis. Thus V consists of the all $\sum_{e\in S} a_e e$, $a_e \in K$, where $\sum a_e e = 0$ if and only if $a_e = 0$ for every $e \in S$. The associative algebra over K generated by V consists of all finite K-linear combinations of $\mathbf{1}$ (the ring identity) and formal finite (non-commuting) products of elements of S.

The **exterior algebra** $\mathcal{E}(V)$ over V is the quotient of the associative algebra over K generated by V modulo the algebra ideal I generated by products v^2, $v \in V$. The image of each non-zero $v \in V$ under the map $v \to v + I \in \mathcal{E}(V)$ is denoted by $\mathbf{v}$. These $\mathbf{v}$ will also be called **vectors**. Thus, for $v_1, v_2 \in V$, $(\mathbf{v}_1 + \mathbf{v}_2)(\mathbf{v}_1 + \mathbf{v}_2) = \mathbf{0}$, $\mathbf{v}_i\mathbf{v}_i = \mathbf{0}$ and so $\mathbf{v}_1\mathbf{v}_2 = -\mathbf{v}_2\mathbf{v}_1$ in $\mathcal{E}(V)$. This and the associativity law imply that each product of a sequence of vectors vanishes if the sequence has repeated elements. Indeed the product vanishes if and only if there is a linear dependency among the vectors being multiplied. Note that a non-zero product of two or more vectors is not a vector.

A particular basis of $\mathcal{E}(V)$ is constructed from the basis $S = \{s_1, \ldots, s_n\}$ of V. This basis consists of $\mathbf{1}$ together with the $2^n - 1$ products of vectors in distinct non-empty subsets of $\{\mathbf{s}_1, \ldots, \mathbf{s}_n\}$, each product written in a particular order. A formula for exterior product expressed in terms of this basis is used in [31] to prove that the product is associative. The formula expresses the following fact for products of basis vectors which is true for all products: Given any sequence of vectors $v_1 \cdots v_k$ and permutation $\sigma \in \mathfrak{S}_k$ with sign $\epsilon(\sigma)$, the exterior product satisfies the **alternating law**

$$\mathbf{v}_1 \cdots \mathbf{v}_k = \epsilon(\sigma)\mathbf{v}_{\sigma_1} \cdots \mathbf{v}_{\sigma_k}.$$

As a result, $\mathcal{E}(V)$ is an associative algebra that has dimension 2^n when viewed as a vector space over K. It is customary to use increasing order of subscripts, so each $X \in \mathcal{E}(V)$ can be expressed by

$$X = x_\emptyset \mathbf{1} + \sum_{\substack{\emptyset \neq A \subseteq S: \\ A = \{s_{i_1}, \ldots, s_{i_k}\},\ i_1 < \cdots < i_k}} x_A \mathbf{s}_{i_1} \cdots \mathbf{s}_{i_k}$$

with 2^n unique coefficients x_A, $A \subseteq S$.

We follow a different convention which mimics the one used with the chirotope cryptomorphism for oriented matroids given in [4]. For us, A will denote an arbitrary sequence of elements of S. The value in K symbolized by coefficient x_A will depend on the order as well as the elements of A, but these values will satisfy

$$x_{A_\sigma} = \epsilon(\sigma) x_A$$

where $A_\sigma = (a_1 \cdots a_k)_\sigma = a_{\sigma 1} \cdots a_{\sigma k}$ is A permuted by σ. In general, a function like $A \to x_A$ is called **alternating** if it has this property. Our convention allows A to have repeated elements but the alternating property requires $x_A = 0$ for such A.

We follow a related convention for subset expansions and formulas within them. When necessary, a set symbol like A in $A \subseteq S$ will denote distinct elements written in an arbitrary sequence. But the expansion or formula will be written only if its value is independent of the sequence chosen for each symbol. Furthermore, when A is a sequence of distinct basis vectors, the corresponding product of their images in $\mathcal{E}(V)$ will be denoted by $\mathbf{A}$. The empty sequence $\emptyset$ corresponds to $\mathbf{1} \in \mathcal{E}(V)$. No sequence of linearly independent vectors corresponds to $\mathbf{0} \in \mathcal{E}(V)$. The above 2^n term basis expansion is thus written

$$X = \sum_{\substack{A \subseteq S \\ A = \{a_1, \ldots, a_{|A|}\}}} x_A \mathbf{a}_1 \cdots \mathbf{a}_{|A|} = \sum_{A \subseteq S} x_A \mathbf{A}.$$

This expansion follows our convention because $x_{A_\sigma}\mathbf{A}_\sigma = \epsilon^2(\sigma) x_A \mathbf{A} = x_A \mathbf{A}$. Note that $X = \mathbf{0}$ if and only if every $x_A = 0$.

The concatenation of sequences A, B, C, $\ldots$ is denoted by $ABC \ldots$

2.2 Extensors with Ground Set

The exterior algebra $\mathcal{E}(V)$ is a powerful tool to explore, in a coordinate free way, relationships between linear subspaces of V [2, 65]. These relationships are, in other words, the theorems about the projective geometry whose points are the rank 1 (zero-dimensional) subspaces. The geometric flats (the empty set, points, lines, planes, etc., and the whole space) correspond to these linear subspaces. One way to present a K-realizable matroid is to map each ground set element to either a point or to the empty flat in this projective geometry. The matroid structure is then expressed in terms of incidence of the images of the ground set elements with the geometric flats. The formulation is coordinate-free because these relationships do not change under a change of the basis S for V. We mention this formulation to contrast it with our application of exterior algebra.

In our application, each K-realizable matroid with ground set S will be presented by *a separate* (fully decomposable) non-zero element $\mathbf{N} \in \mathcal{E}(V)$, where $V = KS$. Each such element $\mathbf{N}$ will determine a linear subspace $L = L(\mathbf{N})$ of KS. Consider the family of matrices with columns indexed by S whose row space equals L. The matroid is presented by the linear dependencies among the columns of any such matrix. When L is viewed as a linear subspace of functions from S to K, the matroid is the "function space geometry (or chain-group geometry) $G(S, L)$" discussed in [63, §1.1.C]. As such, each $e \in S$ corresponds to the linear functional given by evaluation of $f \in L$ on e. Therefore these functionals, as a finite subset of the dual space of L, comprise a vector representation of the matroid.

The members of L present an oriented matroid $\mathcal{N}(\mathbf{N})$ by a set of **covectors** $\mathcal{L}$. Each covector is the function $l : S \to \{+, -, 0\}$ determined by some $f \in L$ by $l(e) = \operatorname{sign}\big(f(e)\big)$ for all $e \in S$. In other words, if L is viewed as the row space of a matrix, then each $l \in \mathcal{L}$ is the signature (i.e., the sequence of $+$, $-$ or 0s) that indicates the signs in one row $f \in L$. The signed cocircuits are the covectors with minimal non-empty support. See [1] for an exposition of oriented matroids that begins with linear subspace presentations including the cycle and coboundary (or cocycle) spaces of graphs. Our topic utilizes exterior algebra and the chirotope given by the signs of the Plücker coordinates of $\mathbf{N}$ to present oriented matroids. Theorem 2.1 states the needed details. Deeper discussions appear in [4, especially §2.4 on stratifications of the Grassmann variety and chap. 8 on realizations].

The following theorem summarizes the facts we will need. It characterizes those elements of $\mathcal{E}(V)$ (as fully decomposable) that determine linear subspaces and present the K-realizable matroids with ground set S. We will call such elements extensors.

Theorem 2.1. *Given non-zero element $\mathbf{N} \in \mathcal{E}(V)$ where $V = KS$, the following conditions are equivalent:*

1. *There exist r linearly independent vectors $v_i \in V$ for which*

$$\mathbf{N} = \mathbf{v}_1 \cdots \mathbf{v}_r.$$

(When $r = 0$, the condition is $\mathbf{N} = \alpha\mathbf{1}$ for some $\alpha \in K$, $\alpha \neq 0$.)

L is the subspace spanned by $\{v_1, \ldots, v_r\}$.

2. *There exists r, $0 \leq r \leq |S|$, such that the only non-zero coefficients $\mathbf{N}[A]$ in*

$$\mathbf{N} = \sum_{A \subseteq S} \mathbf{N}[A]\mathbf{A}$$

satisfy $|A| = r$, and the function $\mathbf{N}[A]$ from sequences A to K is alternating and satisfies the Grassmann-Plücker *relationships:*

For all length r sequences $A = a_1 \cdots a_r$ and $B = b_1 \cdots b_r$ over S,

$$\mathbf{N}[A]\mathbf{N}[B] = \sum_{i=1}^{r} \mathbf{N}[b_i a_2 \cdots a_r]\mathbf{N}[b_1 \cdots a_i\hat{b_i} \cdots b_r].$$

(Here, $a_i\hat{b_i}$ means b_i within sequence B is replaced by a_i.)

3. *There exists a rank r matrix N with r rows and with columns indexed by S for which the coefficients of $\mathbf{N}$ with $|A| = r$ satisfy*

$$\mathbf{N}[A] = \det N(A)$$

where $N(A)$ is the submatrix of N with columns A, and the other coefficients are 0*. (For $r = 0$ the condition is $\mathbf{N}[\emptyset] \neq 0$ and all other $\mathbf{N}[A] = 0$.)*

L is the subspace spanned by the rows of N.

If $r \neq 0$ then N and the $v_1, \ldots v_r$ in $\mathbf{N} = \mathbf{v_1} \cdots \mathbf{v}_r$ can be chosen so row i of N holds the coefficients for writing v_i as a linear combination of vectors from basis S.

Definition 2.2 (Extensor)**.** When the conditions in Theorem 2.1 are true about $\mathbf{N}$, we say $\mathbf{N}$ is decomposable[2], and the coefficients denoted by $\mathbf{N}[A]$ are called the Plücker coordinates of $\mathbf{N}$. A decomposable element of an exterior algebra is called an **extensor**. The integer r is called its rank and is denoted by $\rho\mathbf{N}$.

Remark: We do not define Plücker coordinates to equivalence classes of homogeneous coordinates, so extensors that differ by a non-zero scalar multiple have different Plücker coordinates, even though they represent the the same subspace.

Remark: $\mathbf{N}[A]$ is defined as 0 for all A with $|A| \neq \rho\mathbf{N}$.

The fact that each rank r extensor is the exterior product of r vectors, and $\mathbf{v}_1\mathbf{v}_2 = -\mathbf{v_2}\mathbf{v}_1$ for vectors implies that extensor multiplication satisfies the following **anticommutative law**:

$$\mathbf{N}_1\mathbf{N}_2 = (-1)^{\rho\mathbf{N}_1\rho\mathbf{N}_2}\mathbf{N}_2\mathbf{N}_1. \tag{4}$$

Theorem 2.3. *Suppose r, subspace L, matrix N, extensor $\mathbf{N}(S)$ and its Plücker coordinates are as described in Theorem 2.1.*

[2]We follow [31] and other authors who omit the qualifier "fully" in this context.

1. *There exists a rank $|S|-r$ matrix $N^\perp$ with columns indexed by S and with $|S|-r$ rows for which coefficients with $|A|=r$ satisfy*

$$\mathbf{N}[A] = \det N^\perp(\overline{A})\epsilon(A\overline{A})$$

   where $\overline{A} = S \setminus A$ expressed in an arbitrary sequence and ϵ is some non-zero alternating sign function of sequences over S.

2. *L consists of all $x \in KS$ that satisfy the equations*

$$N^\perp x = 0.$$

   In other words, L (the row space of N) and the row space of $N^\perp$ are orthogonal complements in KS.

Proof. See [31, chap. 7]. An elementary proof of theorem 2.3 related to theorem 2.1 appears in [30, VII.3 Theorem I]. The equivalence of the Grassmann-Plücker relationships to the other conditions is proved in [30, VII.6 Theorem II]. □

An **extensor N with ground set** T is the finite set T paired with an extensor in $\mathcal{E}(KT)$. We use the notation $\mathbf{N} = \mathbf{N}(T)$ and $T = S(\mathbf{N})$ to indicate that $\mathbf{N}$ has ground set T.

We need the ground set for the same reason that a ground set is necessary to define the dual of a matroid with coloops. Independent sets, or the collection of sequences B for which chirotope $\chi(B) = +$ is not sufficient because a loop doesn't appear in any of these objects. Furthermore, in our calculations and proofs we find it very helpful to combine the signs of sequences (i.e., permutations of subsets) from several different sets within one analysis without having to relabel any elements. Identities like the Tutte equations relate function values for objects with different ground sets. It is not sufficient for an extensor or chirotope to be defined up to sign for certain identities to be valid (not just up to sign). This validity may facilitate the use of the identities in computer programs.

Theorem 2.4. *Given $\mathbf{N}(S)$, let N be a matrix satisfying Theorem 2.1.*

1. *The collection of those $B \subseteq S$ for which $\mathbf{N}[B] \neq 0$ is the collection of bases of a matroid with ground set S.*

   The same matroid is presented by the independent sets of columns of N.

2. *The function χ of sequences over S into $\{+1, -1, 0\}$ for which $\chi(B)$ is the sign of $\mathbf{N}[B]$ is the chirotope function of the oriented matroid $\mathcal{N}$ denoted by $\mathcal{N} = \mathcal{N}(S) = \mathcal{N}(\mathbf{N})$.*

   The covectors of $\mathcal{N}$ are presented by the signatures of N's row space elements; the signed circuits (i.e., oriented matroid "vectors" with minimal support sets) are presented by the signatures of the minimal linear dependencies among the columns of N.

3. *If $\mathbf{N}[B] \in \{0, \pm 1\}$ for all B, then $\mathcal{N}(\mathbf{N})$ is the unimodular (or regular) oriented matroid whose chirotope function satisfies $\chi(B) = \mathbf{N}[B]$. Furthermore, every unimodular oriented matroid can be presented by such an $\mathbf{N}$.*

Proof. See [4] or [1, chap. 5]. Details pertaining to the unimodular matroids including several characterizations are given in [64, Theorem 3.1.1, p. 41]. □

Definition 2.5. If $\mathbf{N}(S) \neq 0$ and $e \in S$ then

- e is called a loop if $\mathbf{N}[B] = 0$ for all B with $e \in B$, and
- e is called a coloop if every B such that $\mathbf{N}[B] \neq 0$ satisfies $e \in B$.

Remark: e is therefore a loop or coloop in $\mathbf{N}$ if and only if it is a loop or coloop respectively in the matroid presented by $\mathbf{N}$.

Definition 2.6. Each $\mathbf{N}(S) \neq 0$ defines the function $\rho_{\mathbf{N}}$ on subsets $A \subseteq S$ where $\rho_{\mathbf{N}}(A)$ is the rank of A in the matroid presented by $\mathbf{N}(S)$.

Theorem 2.7. *Given $\mathbf{N}(S) \neq 0$, $e \in S$ and $S' = S \setminus e$:*

1. *The Plücker coordinate function for $\mathbf{N}$ restricted to sequences $B \subseteq S'$ is the Plücker coordinate function for an extensor denoted by $(\mathbf{N} \setminus e)(S')$. This operation $\mathbf{N} \to \mathbf{N} \setminus e$ is called **deletion** of e.*

   $(\mathbf{N} \setminus e) \neq \mathbf{0}$ if and only if e is not a coloop in the matroid presented by $\mathbf{N}$. In this case, the unoriented or oriented matroid minor $\mathcal{N} \setminus e$ is presented by $\mathbf{N} \setminus e$ and $\rho(\mathbf{N} \setminus e) = \rho\mathbf{N}$.

2. *The function defined by $\mathbf{N}[Be]$ for sequences $B \subseteq S'$ is the Plücker coordinate function for an extensor denoted by $(\mathbf{N}/e)(S')$. This operation $\mathbf{N} \to \mathbf{N}/e$ is called **contraction** of e.*

   $(\mathbf{N}/e) \neq \mathbf{0}$ if and only if e is not a loop in the matroid presented by $\mathbf{N}$. In this case, the unoriented or oriented matroid minor $\mathcal{N}/e$ is presented by $\mathbf{N}/e$ and $\rho(\mathbf{N}/e) = \rho\mathbf{N} - 1$.

Remarks: If $\mathbf{N} = \mathbf{0}$ then $\mathbf{N}/e = \mathbf{N} \setminus e = \mathbf{0}$. The zero extensor $\mathbf{0}$ does not present any matroid. All rank 0 (empty S or loops only) matroids have only one basis $\emptyset$; they are presented by the non-zero extensors $\alpha\mathbf{1}(S)$ of rank 0.

Proof. Let N be a matrix representing $\mathbf{N}$ in Theorem 2.1.

$(\mathbf{N} \setminus e)(S')$ is the extensor known from Theorem 2.1 when the column labeled by e is deleted from N. Note that if e is a coloop then this reduces the rank of N and so $(\mathbf{N} \setminus e)(S') = \mathbf{0}$.

If e is a loop in $\mathbf{N}$ then $(\mathbf{N}/e) = \mathbf{0}$. Otherwise, the $\mathbf{N}[Be]$ are fixed non-zero multiples of the Plücker coordinates from a matrix obtained from the N by row operations to make all but one entry in column e zero and then deleting the row and column with that non-zero entry.

See [4, §3.5] for oriented matroid minors and other structures in terms of chirotopes. □

Theorem 2.8. *Given* $\mathbf{N}(S)$ *and* $e \in S$,

$$\mathbf{N}(S) = (\mathbf{N}/e)\mathbf{e} + (\mathbf{N} \setminus e)\mathbf{1}(\{e\})$$

The multiplication by $\mathbf{1}(\{e\})$ makes the ground set of the second term be S instead of $S \setminus e$. It will be omitted in contexts where the ground set is clear.

Proof. Let $B \subseteq S$. We prove that each Plücker coordinate $\mathbf{N}[B]$ equals the sum of the corresponding Plücker coordinates of the extensors on the right.

If $e \in B$ we can write $B = B'e$. $\mathbf{N}[B'e] = (\mathbf{N}/e)[B'] = (\mathbf{N}/e)\mathbf{e}[B'e]$, and $(\mathbf{N}\setminus e)[B] = 0$.

If $e \notin B$ then $(\mathbf{N}/e)\mathbf{e}[B] = 0$ and $\mathbf{N}[B] = (\mathbf{N} \setminus e)[B]$. □

It is convenient to let $\mathbf{N}/A$ denote $\mathbf{N}/a_k/\cdots/a_1$ where $A = a_1 \cdots a_k$, and similarly for $\mathbf{N} \setminus A$. It follows that $\mathbf{N}/A[X] = \mathbf{N}[XA]$ for all X. We note that for $\sigma \in \mathfrak{S}_k$,

$$\begin{aligned} \mathbf{N}/A_\sigma &= \epsilon(\sigma)\mathbf{N}/A, \text{ but} \\ \mathbf{N} \setminus A_\sigma &= \mathbf{N} \setminus A. \end{aligned} \tag{5}$$

2.3 Ground Set Orientation and Duality

Definition 2.9 (Ground set orientation)**.** An orientation of the ground set ϵ is an alternating function into $\{+1, -1, 0\}$ of sequences of ground set elements that is non-zero on sequences of distinct elements, and which satisfies $\epsilon(\emptyset) = 1$.

One family of ground set orientations is derived from fixed linear orders on all ground set elements using the rule that $\epsilon(X) = (-1)^v$ where v is the number of inversions in X (where an inversion is (i, j) with $i < j$ and $x_i > x_j$). A permutation $\sigma \in \mathfrak{S}_n$ of $\{1, \ldots, n\}$ is always considered a sequence $\sigma_1\sigma_2 \ldots \sigma_n$ of natural numbers with ground set orientation derived from their usual ordering. Hence, $\epsilon(\sigma)$ is the usual sign of permutation σ. However, ground set orientations of matroid elements or graph edges will *not* be assumed to derive from a linear order.

Since permutations $\sigma \in \mathfrak{S}_n$ and sequences of ground set elements will not be confused, we use the same symbol ϵ for permutation sign and ground set orientation.

Given a length n sequence $X = x_1 \ldots x_n$ and $\sigma \in \mathfrak{S}_n$, let X_σ denote $x_{\sigma_1} \ldots x_{\sigma_n}$. The following routine facts will be used in our proofs: Of course, F is alternating means $F(X_\sigma) = \epsilon(\sigma)\ F(X)$ for all sequences X and $\sigma \in \mathfrak{S}_{|X|}$.

Lemma 2.10. *Suppose* ϵ_1 *and* ϵ_2 *are arbitrary alternating functions of sequences.*

1. *If* $n = |X| = |Y|$, $\sigma \in \mathfrak{S}_n$, *and* A, X, B, C, Y, D *are sequences then*

$$\begin{aligned} \epsilon_1(AXB)\epsilon_2(CYD) &= \epsilon(\sigma)\epsilon_1(AX_\sigma B)\epsilon_2(CYD) \\ &= \epsilon(\sigma)\epsilon_1(AXB)\epsilon_2(CY_\sigma D) \\ &= \epsilon_1(AX_\sigma B)\epsilon_2(CY_\sigma D) \end{aligned}$$

2. $\epsilon_i(XY) = (-1)^{|X||Y|}\epsilon_i(YX)$.

With a ground set orientation ϵ in hand, we define:

Definition 2.11 (Canonical Dual)**.** Given $\mathbf{N}(S)$, $\mathbf{N}^{\perp}[X] = \mathbf{N}^{\perp_\epsilon}[X] = \mathbf{N}[\overline{X}]\epsilon(\overline{X}\ X)$, where $\overline{X}$ is any sequence of the distinct elements in $S \setminus X$.

The symbol $\perp_\epsilon$ will be abbreviated by $\perp$ when ϵ is irrelevant or doesn't require emphasis.

Remark: Each ground set S determines with ϵ a sign choice from among the two that both provide a presentation of the oriented matroid dual.

The demonstration in [4, end of §3.6] of oriented matroid chirotope dualization has a similar formula, whose right hand side is independent of an arbitrarily chosen sequence. It follows that our extensor dualization corresponds to the oriented matroid dualization of the oriented matroid presented by $\mathbf{N}$. Theorem 2.3 justifies this for realizable matroids. (Dualization is also the Hodge star operator [30] when S is identified with the corresponding basis of the dual space. Also see §7.1.)

2.4 Identities

Our main proof uses some identities on extensors that correspond to well-known relationships among matroid operations. These identities involve extensors with ground sets for which a ground set orientation is used to define dualization. The union of disjoint sets is denoted by $\uplus$.

Theorem 2.12.

$$\begin{aligned}(\mathbf{N}_1 + \mathbf{N}_2)^{\perp} &= \mathbf{N}_1^{\perp} + \mathbf{N}_2^{\perp} \\ (\alpha\mathbf{N})^{\perp} &= \alpha\mathbf{N}^{\perp}\end{aligned} \tag{6}$$

$$\mathbf{N}^{\perp\perp}(S) = (-1)^{\rho\mathbf{N}\ (|S|-\rho\mathbf{N})}\ \ \mathbf{N}(S) = (-1)^{\rho\mathbf{N}\ \rho\mathbf{N}^{\perp}}\ \ \mathbf{N}(S) \tag{7}$$

Given $\mathbf{N}(S)$, and sequences $X \subseteq S$ and $S' = S \setminus X$,

$$(\mathbf{N} \setminus X)^{\perp} = \epsilon(S')\epsilon(S'X)\ \ (\mathbf{N}^{\perp}/X) \tag{8}$$

$$(\mathbf{N}/X)^{\perp} = \epsilon(S')\epsilon(S'X)(-1)^{|X|\ (|S|-\rho\mathbf{N})}\ \ (\mathbf{N}^{\perp} \setminus X) \tag{9}$$

Given $\mathbf{N}_i(S_i)$ with $S_1 \cap S_2 = \emptyset$, the extensor product $\mathbf{N}_1\mathbf{N}_2(S_1 \uplus S_2)$ presents the (oriented) matroid direct sum and

$$(\mathbf{N}_1\mathbf{N}_2)^{\perp} = \epsilon(S_1)\epsilon(S_2)\epsilon(S_1S_2)(-1)^{\rho\mathbf{N}_1^{\perp}\rho\mathbf{N}_2}\ \mathbf{N}_1^{\perp}\mathbf{N}_2^{\perp} \tag{10}$$

Proof. Linearity (6) is immediate from the Plücker coordinate definition. It will be repeatedly used with $\alpha = \pm 1$ below.

To prove Theorem 2.12 (7), write

$$\mathbf{N}^{\perp\perp}[A] = (\mathbf{N}^{\perp})^{\perp}[A] = (\mathbf{N}^{\perp})[\overline{A}]\ \epsilon(\overline{A}A),\ = \mathbf{N}[\overline{\overline{A}}]\ \epsilon(\overline{\overline{A}}\ \overline{A})\epsilon(\overline{A}A) = \mathbf{N}[A]\ \epsilon(A\overline{A})\epsilon(\overline{A}A)$$

where in the last equation we chose the sequence order $\overline{\overline{A}} = A$. Therefore the sign correction is $(-1)^{|A|\,|\overline{A}|}$. For non-zero coordinates this is $(-1)^{\rho\mathbf{N}\,(|S|-\rho\mathbf{N})} = (-1)^{\rho\mathbf{N}\,\rho\mathbf{N}^\perp}$.

To prove Theorem 2.12 (8), write

$$(\mathbf{N} \setminus X)^\perp[A] = (\mathbf{N} \setminus X)[\overline{A}]\ \epsilon(\overline{A}A) = \mathbf{N}[\overline{A}]\ \epsilon(\overline{A}A),$$

where $\overline{A} = S' \setminus A$. But in

$$(\mathbf{N}^\perp/X)[A] = \mathbf{N}^\perp[AX] = \mathbf{N}[\overline{AX}]\ \epsilon(\overline{AX}AX)$$

the elements in sequence $\overline{AX}$ are $S' \setminus A$, the same as in the sequence symbolized by $\overline{A}$ in the previous equation. We can therefore choose $\overline{AX} = \overline{A}$ and write

$$(\mathbf{N}^\perp/X)[A] = \mathbf{N}[\overline{A}]\ \epsilon(\overline{A}AX).$$

Combining the two sign corrections gives $\epsilon(\overline{A}A)\epsilon(\overline{A}AX)$. That equals $\epsilon(S')\epsilon(S'X)$ for all reorderings S' of $\overline{A}A$.

We can get (9) from (7) and (8). Specifically, $(\mathbf{N}/X)^\perp = (\mathbf{L}^\perp/X)^\perp(-1)^{\rho\mathbf{N}\rho\mathbf{N}^\perp}$ with $\mathbf{L} = \mathbf{N}^\perp$. This equals

$$\begin{aligned} &(\mathbf{L} \setminus X)^{\perp\perp}\ \epsilon(S')\epsilon(S'X)(-1)^{\rho\mathbf{N}\rho\mathbf{N}^\perp} \\ &= (\mathbf{N}^\perp \setminus X)\ (-1)^{\rho(\mathbf{N}^\perp\setminus X)\rho(\mathbf{N}/X)}\epsilon(S')\epsilon(S'X)(-1)^{\rho\mathbf{N}\rho\mathbf{N}^\perp} \end{aligned}$$

As usual, we can restrict attention to non-zero coordinates. Let $\rho\mathbf{N} = r$, $|S| = s$, and $|X| = x$ so $\rho(\mathbf{N}/X) = r - x$ and $\rho(\mathbf{N}^\perp \setminus X) = \rho\mathbf{N}^\perp = s - r$. The sign correction is therefore

$$\epsilon(S')\epsilon(S'X)(-1)^{(s-r)(r-x)+r(s-r)} = \epsilon(S')\epsilon(S'X)(-1)^{(s-r)(2r-x)} = \epsilon(S')\epsilon(S'X)(-1)^{(s-r)x}.$$

To prove Theorem 2.12 (10), take $\overline{A_1A_2} = \overline{A_1}\ \overline{A_2}$ with each $\overline{A_i} = S_i \setminus A_i$ in

$$\begin{aligned} &(\mathbf{N}_1\mathbf{N}_2)^\perp[A_1A_2] = \\ &(\mathbf{N}_1\mathbf{N}_2)[\overline{A_1A_2}]\ \epsilon(\overline{A_1A_2}A_1A_2) \\ &= \mathbf{N}_1[\overline{A_1}]\ \mathbf{N}_2[\overline{A_2}]\ \epsilon(\overline{A_1}\ \overline{A_2}A_1A_2) \\ &= \mathbf{N}_1^\perp[A_1]\ \epsilon(\overline{A_1}A_1)\ \mathbf{N}_2^\perp[A_2]\ \epsilon(\overline{A_2}A_2)\ \epsilon(\overline{A_1}(\overline{A_2}A_1)A_2) \\ &= \mathbf{N}_1^\perp[A_1]\ \mathbf{N}_2^\perp[A_2]\ \epsilon(\overline{A_1}A_1)\epsilon(\overline{A_2}A_2)\epsilon(\overline{A_1}(A_1\overline{A_2})A_2)(-1)^{|A_1||\overline{A_2}|} \\ &= \mathbf{N}_1^\perp[A_1]\ \mathbf{N}_2^\perp[A_2]\ \epsilon(S_1)\epsilon(S_2)\epsilon(S_1S_2)(-1)^{\rho\mathbf{N}_1^\perp\rho\mathbf{N}_2} \end{aligned}$$

where in the last equation, we applied permutations σ and τ, each twice, for which $(\overline{A_1}A_1)_\sigma = S_1$ and $(\overline{A_2}A_2)_\tau = S_2$. We then substituted the correct ranks for cases where the coordinate is not 0. $\square$

3 An Extensor Tutte Function

Recall that a ported extensor or matroid is one whose ground set has a distinguished subset of port elements.

Given a ported extensor $\mathbf{N}(P, E)$ (the notation means P is the set of ports and the ground set is $P \uplus E$), we will define a parametrized extensor $\mathbf{M}_E(\mathbf{N})$ using extensor operations. We will illustrate its construction with extensors and equivalent matrices. We then write and prove parametrized identities satisfied by the function $\mathbf{N}(P, E) \to \mathbf{M}_E(\mathbf{N})$ which are analogous to the ported Tutte equations. Our identities however apply to extensors rather than to commutative ring values. The identities include *sign-correction factors* that depend on the particular ground set orientation ϵ that was used to define $\mathbf{M}_E(\mathbf{N})$.

The definition of $\mathbf{M}_E(\mathbf{N})$ below applies to all extensors $\mathbf{N}$ over $K(E \uplus P)$. The main result therefore belongs to exterior algebra. Section 4 shows how $\mathbf{M}_E$ defines the extensor valued function on the minor closed class of ported unimodular oriented matroids $\mathcal{N}(P, E)$ by $\mathbf{M}_E(\mathcal{N}) = \mathbf{M}_E(\pm\mathbf{N})$ where $\mathcal{N}$ is presented by either unimodular extensor $\pm\mathbf{N}$.

Each linear map on V can be extended to a unique exterior algebra map on the exterior algebra $\mathcal{E}(V)$ [31, Theorem 7.1]. Given ground set $P \uplus E$, let P_υ and P_ι be two disjoint copies of P, both also disjoint from E. For each $p \in P$, let $p_\upsilon \in P_\upsilon$ and $p_\iota \in P_\iota$ be the corresponding elements in the respective copies. We define the following maps from $K(P \uplus E)$ to $K[r_e, g_e](P_\upsilon \uplus P_\iota \uplus E)$ and extend them to the exterior algebra. If necessary, the field K is extended with the parameters g_e, r_e, $e \in E$.

$$\begin{gathered}\upsilon_r(\mathbf{e}) = r_e\mathbf{e} \text{ for } e \in E \text{ and } \upsilon_r(\mathbf{p}) = \mathbf{p}_\upsilon \text{ for } p \in P. \\ \iota_g(\mathbf{e}) = g_e\mathbf{e} \text{ for } e \in E \text{ and } \iota_g(\mathbf{p}) = \mathbf{p}_\iota \text{ for } p \in P.\end{gathered} \tag{11}$$

In terms of matrices, ι_g signifies multiplying column labeled e by g_e for each $e \in E$ and renaming column p by p_ι for each $p \in P$. Likewise, υ_r signifies multiplying column e by r_e and renaming column p by p_υ.

The parameter subscript symbols in ι_g and υ_r will sometimes be omitted for brevity. For subset $Q \subseteq P$, Q_υ denotes $\{q_\upsilon : q \in Q\} \subseteq P_\upsilon$ and Q_ι denotes $\{q_\iota : q \in Q\} \subseteq P_\iota$. Recall that set symbols denote sequences. The sequences of Q_υ and Q_ι correspond to the sequence of Q.

Definition 3.1. Given a ported extensor $\mathbf{N}(P, E)$, a ground set orientation ϵ and dual operator $\perp_\epsilon$, parameters g_e and r_e for each $e \in E$, and ϵ-preserving functions υ_r and ι_g defined above, let

$$\begin{gathered}\mathbf{M}(\mathbf{N}) = \iota_g(\mathbf{N})\ \upsilon_r(\mathbf{N}^{\perp_\epsilon}) \text{ and} \\ \mathbf{M}_E(\mathbf{N}) = \mathbf{M}(\mathbf{N})/E\end{gathered} \tag{12}$$

Hence, $\mathbf{M}_E(\mathbf{N})$ is defined as a ported extensor $\mathbf{M}(\mathbf{N}) = \mathbf{M}(\mathbf{N})(P_\iota \uplus P_\upsilon, E)$ contracted by the sequence of non-port elements E. Therefore, by Theorem 2.7 it is an extensor. Each pair of sequences $I \subseteq P$, $V \subseteq P$ with $|I| + |V| = |P|$ specifies the Plücker coordinate of $\mathbf{M}_E(\mathbf{N})$ with index $I_\iota V_\upsilon$ and value

$$\mathbf{M}_E(\mathbf{N})[I_\iota V_\upsilon] = \big(\iota(\mathbf{N})\ \ \upsilon(\mathbf{N}^\perp)\big)[I_\iota V_\upsilon E]. \tag{13}$$

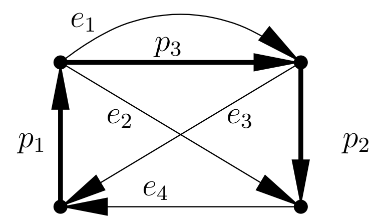


Figure 1: Graph defining the graphic oriented matroid $\mathcal{N}$

Proposition 3.2. *For* $\alpha \in K$, $\mathbf{M}_E(\alpha\mathbf{N}) = \alpha^2\mathbf{M}_E(\mathbf{N})$.

Proof. $\mathbf{M}(\alpha\mathbf{N}) = \alpha^2\mathbf{M}(\mathbf{N})$ is immediate from the definition. Contraction $\mathbf{M}/E$ is linear in $\mathbf{M}$. ☐

We can express (12) in matrix terms. Let N be some full row rank matrix with columns indexed by $P \cup E$ that presents $\mathbf{N}(P \cup E)$. Similarly, let $N^\perp$ denote a matrix presentation of $\mathbf{N}^\perp$.

Example. We show one totally unimodular matrix representation N of the ported graphic oriented matroid with $P = \{p_1, p_2, p_3\}$ and $E = \{e_1, e_2, e_3, e_4\}$ for the graph in figure 1. The rows code 3 oriented cutsets which determine a basis for the 1-coboundary (or cocycle) space. We also express $\mathbf{N}$ by the exterior product of the vectors given by the rows of this matrix.

$$N = \begin{array}{c} \begin{array}{ccccccc} p_1 & p_2 & p_3 & e_1 & e_2 & e_3 & e_4 \end{array} \\ \left[\begin{array}{ccc|cccc} -1 & 0 & +1 & +1 & +1 & 0 & 0 \\ 0 & +1 & -1 & -1 & 0 & +1 & 0 \\ -1 & -1 & +1 & +1 & 0 & 0 & +1 \end{array}\right] \end{array}$$

$$\mathbf{N} = \begin{array}{c} (-\mathbf{p}_1 + \mathbf{p}_3 + \mathbf{e}_1 + \mathbf{e}_2)\cdot \\ (\mathbf{p}_2 - \mathbf{p}_3 - \mathbf{e}_1 + \mathbf{e}_3)\cdot \\ (-\mathbf{p}_1 - \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{e}_1 + \mathbf{e}_4) \end{array}$$

Next, we write one totally unimodular matrix $N^\perp$ for the canonical dual. We have checked that the sign satisfies Definition 2.11 with ϵ chosen so $\epsilon(p_1p_2p_3e_1e_2e_3e_4) = 1$ by verifying $\mathbf{N}^\perp[e_1e_2e_3e_4] = \mathbf{N}[p_1p_2p_3]\epsilon(p_1p_2p_3e_1e_2e_3e_4)$.

$$N^\perp = \begin{array}{c} \begin{array}{ccccccc} p_1 & p_2 & p_3 & e_1 & e_2 & e_3 & e_4 \end{array} \\ \left[\begin{array}{ccc|cccc} 0 & 0 & +1 & -1 & 0 & 0 & 0 \\ +1 & +1 & +1 & 0 & 0 & 0 & +1 \\ 0 & +1 & +1 & 0 & -1 & 0 & 0 \\ +1 & 0 & +1 & 0 & 0 & +1 & 0 \end{array}\right] \end{array}$$

Continuing the general discussion, let G and R be the diagonal matrices of the g_e and r_e. The matrix

$$M(N) = \left[\begin{array}{ccc} N(P) & 0 & N(E)G \\ 0 & N^\perp(P) & N^\perp(E)R \end{array}\right] \tag{14}$$

has order $(p+e)\times(2p+e)$, columns indexed by sequence $P_\iota P_\upsilon E$ it and presents $\mathbf{M}(\mathbf{N})$.

Example continued. We abbreviate labels $p_{\iota 1}$ and $p_{\upsilon 1}$ by i_1 and v_1, etc.

$$M(N) = \begin{array}{c} \begin{array}{cccccccccc} i_1 & i_2 & i_3 & v_1 & v_2 & v_3 & e_1 & e_2 & e_3 & e_4 \end{array} \\ \left[\begin{array}{ccc|ccc|cccc} -1 & 0 & +1 & 0 & 0 & 0 & g_1 & g_2 & 0 & 0 \\ 0 & +1 & -1 & 0 & 0 & 0 & -g_1 & 0 & g_3 & 0 \\ -1 & -1 & +1 & 0 & 0 & 0 & g_1 & 0 & 0 & g_4 \\ \hline 0 & 0 & 0 & 0 & 0 & +1 & -r_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & +1 & +1 & 0 & 0 & 0 & r_4 \\ 0 & 0 & 0 & 0 & +1 & +1 & 0 & -r_2 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 & +1 & 0 & 0 & r_3 & 0 \end{array}\right] \end{array}$$

Generally, $\mathbf{M}(\mathbf{N})(P_\iota \mathbin{\dot\cup} P_\upsilon \mathbin{\dot\cup} E)$ is the exterior product of the vectors in $K(P_\iota \mathbin{\dot\cup} P_\upsilon \mathbin{\dot\cup} E)$ corresponding to the rows of this matrix. $\mathbf{M}_E(\mathbf{N})(P_\iota \mathbin{\dot\cup} P_\upsilon)$ appears in the expression

$$\mathbf{M}(\mathbf{N}) = \big(\mathbf{M}_E(\mathbf{N})\big)\mathbf{E} + \cdots$$

where the initial term is the only one with factor $\mathbf{E}$.

Example continued. We calculate $\mathbf{M}_E(\mathbf{N})$ by doing ring operations on rows to eliminate all but one non-zero entry in each E column in $M(N)$. The result is that

$$g_1 g_2 g_3 g_4 r_1^6 r_2 r_3 r_4 \mathbf{M}(\mathbf{N})$$

is equal to the following extensor in matrix form:

$$\begin{array}{c} \begin{array}{cccccccccc} i_1 & i_2 & i_3 & v_1 & v_2 & v_3 & e_1 & e_2 & e_3 & e_4 \end{array} \\ \left[\begin{array}{ccc|ccc|cccc} -r_1r_2 & 0 & r_1r_2 & 0 & g_2r_1 & g_1r_2 + g_2r_1 & 0 & 0 & 0 & 0 \\ 0 & r_1r_3 & -r_1r_3 & -g_3r_1 & 0 & -g_1r_3 - g_3r_1 & 0 & 0 & 0 & 0 \\ -r_1r_4 & -r_1r_4 & r_1r_4 & -g_4r_1 & -g_4r_1 & g_1r_4 - g_4r_1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & g_1 & -g_1r_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & g_4r_1 & g_4r_1 & g_4r_1 & 0 & 0 & 0 & g_4r_1r_4 \\ 0 & 0 & 0 & 0 & g_2r_1 & g_2r_1 & 0 & -g_2r_1r_2 & 0 & 0 \\ 0 & 0 & 0 & g_3r_1 & 0 & g_3r_1 & 0 & 0 & g_3r_1r_3 & 0 \end{array}\right] \end{array}$$

After some cancellation, we can read off the answer from the 3×6 upper left submatrix, which is a matrix presentation of the extensor $r_1^2 \mathbf{M}_E(\mathbf{N})$:

$$\begin{array}{c} \begin{array}{cccccc} i_1 & i_2 & i_3 & v_1 & v_2 & v_3 \end{array} \\ \left[\begin{array}{ccc|ccc} -r_1r_2 & 0 & r_1r_2 & 0 & g_2r_1 & g_1r_2 + g_2r_1 \\ 0 & r_1r_3 & -r_1r_3 & -g_3r_1 & 0 & -g_1r_3 - g_3r_1 \\ -r_1r_4 & -r_1r_4 & r_1r_4 & -g_4r_1 & -g_4r_1 & g_1r_4 - g_4r_1 \end{array}\right] \end{array}$$

Remark: Each Plücker coordinate of $\mathbf{M}_E(\mathbf{N})$ is a homogeneous polynomial of degree $|E|$ in the g_e, r_e. However, this example demonstrates that there sometimes doesn't exist a matrix expression for $\mathbf{M}_E(\mathbf{N})$ all of whose entries are polynomials. The reader can verify that each order 3 minor of the above matrix is a multiple of r_1^2.

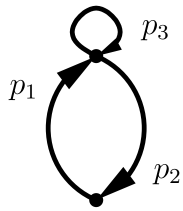
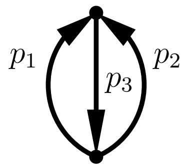
Graph Minor	Extensor Minor	Term in $\mathbf{M}_E(\mathbf{N})[v_1 i_1 v_3]$
	$\mathbf{N}/\{e_1, e_4\} \mid P$	$-g_1 g_4 r_2 r_3$
	$\mathbf{N}/\{e_2, e_3\} \mid P$	$+g_2 g_3 r_1 r_4$

Figure 2: The two graph and extensor minors with corresponding terms in $\mathbf{M}_E(\mathbf{N})[v_1 i_1 v_3]$.

Example continued: Here are examples of Plücker coordinates, which can be calculated from the above matrix as order 3 minors divided by r_1^2. See §6.3.

$$\begin{aligned}
\mathbf{M}_E(\mathbf{N})[v_1 v_2 v_3] &= \quad g_1 g_2 g_3 r_4 + g_1 g_2 g_4 r_3 + g_1 g_3 g_4 r_2 + g_2 g_3 g_4 r_1 \\
\mathbf{M}_E(\mathbf{N})[i_1 v_2 v_3] &= \quad (g_1 r_3 + g_3 r_1)(g_2 r_4 + g_4 r_2) \\
\mathbf{M}_E(\mathbf{N})[v_1 i_1 v_3] &= \quad -g_1 g_4 r_2 r_3 + g_2 g_3 r_1 r_4
\end{aligned}$$

Observe $\mathbf{M}_E(\mathbf{N})[v_1 v_2 v_3]$ is the basis enumerator for $\mathcal{N}(\mathbf{N}) \setminus P$. The graph and extensor minors corresponding to the terms of $\mathbf{M}_E(\mathbf{N})[v_1 i_1 v_3]$ are shown in figure 2.

3.1 Main Result

Theorem 3.3. *The parametrized extensor-valued function $\mathbf{M}_E(\mathbf{N})(P_\upsilon \cup P_\iota)$ of ported extensor $\mathbf{N} = \mathbf{N}(P, E)$ has the following properties:*

1. *Given $\mathbf{N}_1(P_1, E_1)$ and $\mathbf{N}_2(P_2, E_2)$ with $E = E_1 \cup E_2$ and $P = P_1 \cup P_2$,*
$$\mathbf{M}_E(\mathbf{N}_1 \ \ \mathbf{N}_2)(P, E) = \epsilon(P_1 P_2 E)\epsilon(P_1 E_1)\epsilon(P_2 E_2)\, \mathbf{M}_{E_1}(\mathbf{N}_1) \ \ \mathbf{M}_{E_2}(\mathbf{N}_2). \tag{15}$$

2. *If $e \in E$ and $E' = E \setminus e$ then*
$$\mathbf{M}_E(\mathbf{N}) = \epsilon(PE)\epsilon(PE')\, \big(g_e \mathbf{M}_{E'}(\mathbf{N}/e) + r_e \mathbf{M}_{E'}(\mathbf{N} \setminus e)\big). \tag{16}$$

3. *Let $E = \emptyset$. The Plücker coordinates of $\mathbf{M}_\emptyset(\mathbf{N})(P_\iota \cup P_\upsilon)$ satisfy*
$$\mathbf{M}_\emptyset(\mathbf{N})[I_\iota V_\upsilon] = \mathbf{M}[I_\iota V_\upsilon] = \epsilon(\overline{V} \ \ V)\ \mathbf{N}[I]\mathbf{N}[\overline{V}].$$
for all $I \subseteq P$ and $V \subseteq P$.

4. $\mathbf{M}_E(\mathbf{0}) = \mathbf{0}$.

Properties 1. and 2. in terms of Plücker coordinates are:

$$\begin{aligned} &\mathbf{M}_E(\mathbf{N}_1\ \ \mathbf{N}_2)[I_{1I}V_{1V}I_{2I}V_{2V}] = \\ \epsilon(P_1P_2E)\epsilon(P_1E_1)\epsilon(P_2E_2)\ &\mathbf{M}_{E_1}(\mathbf{N}_1)[I_{1I}V_{1V}]\ \mathbf{M}_{E_2}(\mathbf{N}_2)[I_{2I}V_{2V}]. \end{aligned} \tag{17}$$

$$\begin{aligned} &\mathbf{M}_E(\mathbf{N})[I_\iota V_\upsilon] = \\ \epsilon(PE)\epsilon(PE')\ &\big((g_e\mathbf{M}_{E'}(\mathbf{N}/e)[I_\iota V_\upsilon] + r_e\mathbf{M}_{E'}(\mathbf{N}\setminus e)[I_\iota V_\upsilon]\big). \end{aligned} \tag{18}$$

Remarks:

1. Property 2. implies that every linear combination $g_e\mathbf{M}_{E'}(\mathbf{N}/e) + r_e\mathbf{M}_{E'}(\mathbf{N}\setminus e)$ is decomposable, i.e., an extensor.

2. Proposition 3.2 with $\alpha = \pm 1$ implies $\mathbf{M}_E(\mathbf{N}_1\mathbf{N}_2) = \mathbf{M}_E(\mathbf{N}_2\mathbf{N}_1)$. We can also verify this from the right hand side of property 1 using Lemma (2.10.3) and noting the rank of extensor $\mathbf{M}_{E_i}(\mathbf{N}_i)$ is $|P_i|$.

3. If $\mathbf{N} \neq \mathbf{0}$, one but not both of $\mathbf{N}/e$ and $\mathbf{N}\setminus e$ will be the $\mathbf{0}$ extensor if and only if e is a loop or a coloop in the matroid of $\mathbf{N}$. If $\mathbf{N}' = \mathbf{0}$ then $\mathbf{N}'^{\perp} = \mathbf{0}$ and $\mathbf{M}_E(\mathbf{N}') = \mathbf{0}$. We therefore write property 2 without restricting e to a non-separator.

4. Property 1 except for signs is immediate from direct sum of subspaces and their corresponding extensors. Property 2 except for the signs follows immediately from the fact that minor $[I_\iota V_\upsilon E]$ of matrix (14) equals a linear combination with coefficients g_e and r_e because the column e belongs to this minor no matter which $e \in E$ is specified for the identity.

Proof. From the definition, $\mathbf{M}(\mathbf{N}_1\mathbf{N}_2) = \iota(\mathbf{N}_1\mathbf{N}_2)\ \upsilon\big((\mathbf{N}_1\mathbf{N}_2)^{\perp}\big)$ which equals

$$\epsilon(S_1)\epsilon(S_2)\epsilon(S_1S_2)(-1)^{\rho\mathbf{N}_1^{\perp}\rho\mathbf{N}_2}\ \iota(\mathbf{N}_1\mathbf{N}_2)\ \upsilon(\mathbf{N}_1^{\perp}\mathbf{N}_2^{\perp})$$

by Theorem 2.12(7). $\iota(\mathbf{N}_1\mathbf{N}_2)\ \upsilon(\mathbf{N}_1^{\perp}\mathbf{N}_2^{\perp}) = \iota(\mathbf{N}_1)\iota(\mathbf{N}_2)\upsilon(\mathbf{N}_1^{\perp})\upsilon(\mathbf{N}_2^{\perp})$ which equals (by (4))

$$(-1)^{\rho\mathbf{N}_1^{\perp}\rho\mathbf{N}_2}\ \iota(\mathbf{N}_1)\ \upsilon(\mathbf{N}_1^{\perp})\ \iota(\mathbf{N}_2)\ \upsilon(\mathbf{N}_2^{\perp}).$$

Therefore $\mathbf{M}(\mathbf{N}_1\mathbf{N}_2) =$

$$\epsilon(S_1)\epsilon(S_2)\epsilon(S_1S_2)\ \iota(\mathbf{N}_1)\ \upsilon(\mathbf{N}_1^{\perp})\ \iota(\mathbf{N}_2)\ \upsilon(\mathbf{N}_2^{\perp}). \tag{19}$$

Therefore, $\mathbf{M}(\mathbf{N}_1\mathbf{N}_2)/E_1E_2 =$

$$\epsilon(S_1)\epsilon(S_2)\epsilon(S_1S_2)\ \big((\mathbf{M}(\mathbf{N}_1)/E_1)\ \mathbf{E_1}\ (\mathbf{M}(\mathbf{N}_2)/E_2)\ \mathbf{E_2}\big)/E_1E_2$$

because the only Plücker coordinates of the form $\mathbf{M}(\mathbf{N}_i)[X_i]$ for $i = 1$ or 2 that contribute to (19) when it is contracted by E_1E_2 satisfy $E_i \subseteq X_i$. The anticommutativity law (4) then implies that $\mathbf{M}(\mathbf{N}_1\mathbf{N}_2)/E_1E_2 =$

$$\begin{aligned} \epsilon(S_1)\epsilon(S_2)\epsilon(S_1S_2)(-1)^{|E_1||P_2|}\ &\big(\mathbf{M}_{E_1}(\mathbf{N}_1)\ \mathbf{M}_{E_2}(\mathbf{N}_2)\ \mathbf{E_1}\ \mathbf{E_2}\big)/E_1E_2 \\ &= \epsilon(S_1)\epsilon(S_2)\epsilon(S_1S_2)(-1)^{|E_1||P_2|}\ \mathbf{M}_{E_1}(\mathbf{N}_1)\ \mathbf{M}_{E_2}(\mathbf{N}_2). \end{aligned}$$

Since the sequence orders of the S_i are arbitrary, let $S_i = P_iE_i$ for $i = 1$ and 2. According to equation (5), $\mathbf{M}_E(\mathbf{N}_1\mathbf{N}_2) = \mathbf{M}(\mathbf{N}_1\mathbf{N}_2)/E = \epsilon(\sigma)\mathbf{M}(\mathbf{N}_1\mathbf{N}_2)/E_1E_2$ where $(E_1E_2)_\sigma = E$. Therefore, $\mathbf{M}_E(\mathbf{N}_1\mathbf{N}_2) = \pm\mathbf{M}_{E_1}(\mathbf{N}_1)\ \mathbf{M}_{E_2}(\mathbf{N}_2)$ with the sign equal to

$$\begin{aligned}\epsilon(\sigma)\epsilon(P_1E_1)\epsilon(P_2E_2)\epsilon(P_1E_1P_2E_2)(-1)^{|E_1||P_2|}\\ = \epsilon(\sigma)\epsilon(P_1E_1)\epsilon(P_2E_2)\epsilon(P_1P_2E_1E_2)\\ = \epsilon^2(\sigma)\epsilon(P_1E_1)\epsilon(P_2E_2)\epsilon(P_1P_2(E_1E_2)_\sigma)\\ = \epsilon(P_1E_1)\epsilon(P_2E_2)\epsilon(P_1P_2E),\end{aligned}$$

which proves property 1 of the theorem.

Now for property 2. Let us apply Theorem 2.8 to $\mathbf{N}$ and $\mathbf{N}^\perp$, and apply ι and υ respectively.

$$\mathbf{N} = (\mathbf{N}/e)\mathbf{e} + (\mathbf{N}\setminus e)\mathbf{1}(e).$$

$$\mathbf{N}^\perp = (\mathbf{N}^\perp/e)\mathbf{e} + (\mathbf{N}^\perp\setminus e)\mathbf{1}(e).$$

$$\begin{aligned}\iota(\mathbf{N}) =& \iota\big((\mathbf{N}/e)\mathbf{e}\big) + \iota\big((\mathbf{N}\setminus e)\mathbf{1}(e)\big)\\ =& \iota(\mathbf{N}/e)g_e\mathbf{e} + \iota(\mathbf{N}\setminus e)\mathbf{1}(e).\end{aligned}$$

$$\begin{aligned}\upsilon(\mathbf{N}^\perp) =& \upsilon\big((\mathbf{N}^\perp/e)\mathbf{e}\big) + \upsilon\big((\mathbf{N}^\perp\setminus e)\mathbf{1}(e)\big)\\ =& \upsilon(\mathbf{N}^\perp/e)r_e\mathbf{e} + \upsilon(\mathbf{N}^\perp\setminus e)\mathbf{1}(e).\end{aligned}$$

The exterior product of $\iota(\mathbf{N})$ and $\upsilon(\mathbf{N}^\perp)$ is therefore

$$\begin{aligned}& g_e\iota(\mathbf{N}/e)\ \mathbf{e}\ \upsilon(\mathbf{N}^\perp\setminus e) + r_e\iota(\mathbf{N}\setminus e)\ \upsilon(\mathbf{N}^\perp/e)\ \mathbf{e}\\ &+ g_er_e\ \iota(\mathbf{N}/e)\ \mathbf{e}\ \upsilon(\mathbf{N}^\perp/e)\ \mathbf{e}\\ &+ \iota(\mathbf{N}\setminus e)\ \upsilon(\mathbf{N}^\perp\setminus e).\end{aligned}$$

$\mathbf{M}_E(\mathbf{N})$ is the result of contracting the above by E. The third term above is $\mathbf{0}$ because $\mathbf{e}$ is a repeated factor. The last term will vanish when contracted by E because none of its non-zero Plücker coordinates have an index that contains e. So we will omit them in the following. By Theorem 2.12(9)

$$\upsilon(\mathbf{N}^\perp\setminus e) = \upsilon\big((\mathbf{N}/e)^\perp\big)\ \epsilon(S')\epsilon(S'e)(-1)^{1\cdot(|S|-\rho\mathbf{N})}$$

and

$$\upsilon(\mathbf{N}^\perp/e) = \upsilon\big((\mathbf{N}\setminus e)^\perp\big)\ \epsilon(S')\epsilon(S'e).$$

Notice that $\rho(\mathbf{N}/e^\perp) = |S'| - (\rho\mathbf{N}-1) = |S| - \rho\mathbf{N}$. So $\mathbf{e}\ \upsilon(\mathbf{N}/e)^\perp = \upsilon(\mathbf{N}/e)^\perp\ \mathbf{e}(-1)^{|S|-\rho\mathbf{N}}$. Therefore when the above substitutions are made we get

$$\mathbf{M}_E(\mathbf{N})\epsilon(S')\epsilon(S'e) = \big(g_e\iota(\mathbf{N}/e)\ \upsilon((\mathbf{N}/e)^\perp) + r_e\iota(\mathbf{N}\setminus e)\ \upsilon((\mathbf{N}\setminus e)^\perp)\big)\ \mathbf{e}/E.$$

Lemma 2.10 used with σ such that $E_\sigma = E'e$ shows that the right hand side is

$$\epsilon(\sigma)((\cdots)\mathbf{e})/E'e = \epsilon(E)\epsilon(E'e)((\cdots)\mathbf{e})/E'e.$$

So the right hand side is

$$\epsilon(E)\epsilon(E'e)\big(g_e\mathbf{M}_{E'}(\mathbf{N}/e) + r_e\mathbf{M}_{E'}(\mathbf{N}\setminus e)\big).$$

Since the sequence order of S' is arbitrary, we can choose $S' = PE'$. The sign correction is then

$$\epsilon(S')\epsilon(S'e)\epsilon(E)\epsilon(E'e) = \\ \epsilon(PE')\epsilon(PE'e)\epsilon(E)\epsilon(E'e).$$

Applying the permutation τ for which $(E'e)_\tau = E$ to the two appearances of subsequence $E'e$ does not change this expression's value. Hence the sign correction is

$$\epsilon(PE')\epsilon(PE)\epsilon(E)\epsilon(E) = \epsilon(PE')\epsilon(PE)$$

and property 2 of the theorem is verified.

The definition of $\mathbf{M}_E$ immediately gives property 4, and, together with the definition of extensor dual, gives property 3. ☐

Corollary 3.4. *The set of $\mathbf{M}_E(\mathbf{N})$ obtained as the g_e, r_e range over $\mathbb{R}$ for each $e \in E$ represents the points in a projective subspace of a Grassmannian (which consists of all the linear subspaces over $\mathbb{R}(P_\iota \cup P_\upsilon)$ with dimension $|P|$).*

Proof. Induction: Use Theorem 3.3 property 2 for when $|E| > 0$ and property 3 for the the basis. ☐

Proposition 3.5. *Given $\mathbf{N} = \mathbf{N}(P, E)$, and sequences $I \subseteq P$, $V \subseteq P$, and $\overline{V} = P \setminus V$,*

$$\epsilon(\overline{V}\ V)\epsilon(PE)\mathbf{M}_E(\mathbf{N})[I_\iota V_\upsilon] = \epsilon(P)\sum_{A\subseteq E}\mathbf{N}[IA]\mathbf{N}[\overline{V}A]g_A r_{\overline{A}}.$$

Remark: The only non-zero terms in this sum are those for which both $A \cup I$ and $A \cup \overline{V}$ are bases in the matroid of $\mathbf{N}$.

Proof. Recalling that E symbolizes a sequence $e_1 \cdots e_n$, let $E_i = e_iE_{i+1}$ so $E_1 = E$ and $E_{n+1} = \emptyset$. When property (2) of Theorem 3.3 is applied successively for $e := e_i$, $E := E_i$ and $E' := E_{i+1}$ for $i = 1, 2, \ldots, n$ and the products are expanded, the result is a sum of 2^n terms, one for each subset $A \subseteq E$. For each fixed i, $1 \le i \le n$, the instances of the symbols E and $E' = E \setminus e_i$ within all applications of property (2) each denote the same sequences. Therefore, sign cancellation occurs and we can write

$$\mathbf{M}_E(\mathbf{N})[I_\iota V_\upsilon] = \epsilon(PE)\sum_{A\subseteq E}\epsilon(P)g_A r_{\bar{A}}\mathbf{M}_\emptyset(\mathbf{N}/A\setminus\bar{A})[I_\iota V_\upsilon]. \tag{20}$$

Property (3) combined with the definitions of extensor deletion, contraction and dualization demonstrate that within each term

$$\mathbf{M}_\emptyset(\mathbf{N}/A\setminus\bar{A})[I_\iota V_\upsilon] = \epsilon(\bar{V}V)\mathbf{N}[IA]\mathbf{N}[\bar{V}A],$$

and the conclusion follows. ☐

Corollary 3.6. *If* $\mathbf{N} \neq \mathbf{0}$ *and the parameters are generic or are all positive, then* $\mathbf{M}_E(\mathbf{N}) \neq \mathbf{0}$.

Proof. $\mathbf{N}$ has some basis, i.e., $B \subseteq P \cup E$ for which $\mathbf{N}[B] \neq 0$. (N.B. $B = \emptyset$ is possible.) Take $I = B \cap P$ and $V = P \setminus I$. Proposition 3.5 indicates $\mathbf{M}_E(\mathbf{N})[I_\iota V_\upsilon] \neq \mathbf{0}$ with $A = B \setminus I$. □

Corollary 3.7. *Plücker coordinate* $\mathbf{M}_E(\mathbf{N})[I_\iota V_\upsilon]$ *is a homogeneous polynomial in the* g_e, r_e *whose terms are square-free and have degree* $\rho\mathbf{N} - |I|$ *in the* g_e *and degree* $|E| - \rho\mathbf{N} + |I| = |E| + |P| - \rho\mathbf{N} - |V| = \rho\mathbf{N}^\perp - |V|$ *in the* r_e.

Proof. Immediate from Proposition 3.5, matroid duality and the fact $\mathbf{N}[X] \neq 0$ only if $|X| = \rho\mathbf{N}$. □

Corollary 3.8.

$$\epsilon(PE)\mathbf{M}_E(\mathbf{N}) = \epsilon(P) \sum_{\substack{A \subseteq E\,:\, \rho_{\mathbf{N}}A = |A|, \\ \rho\mathbf{N} - \rho(\mathbf{N}/A|P) - \rho_{\mathbf{N}}A = 0}} \mathbf{M}_\emptyset(\mathbf{N}/A|P) g_A r_{\overline{A}}. \tag{21}$$

Proof. The definition of extensor deletion indicates $\mathbf{N}/A|P$ is an alternative notation for $\mathbf{N}/A \setminus \bar{A}$ when $A \subseteq E$, $\bar{A}$ means $E \setminus A$ and the ground set of $\mathbf{N}$ is $P \cup E$. The conditions stated in Corollary 3.7 allow us to restrict the sum as indicated. Hence formula (20) for each Plücker coordinate is equivalent to the given expression for the extensor. □

The following definition and consequence of Proposition 3.5 clarify some of the sign behavior resulting from the definitions.

Definition 3.9. A function $F = F^\epsilon(X)$ whose value might depend on the ground set orientation ϵ and on the sequence X is said to be

1. **alternating in** X if $F^\epsilon(X_\sigma) = \epsilon(\sigma)F^\epsilon(X)$, for all $\sigma \in \mathfrak{S}_{|X|}$; and

2. **alternating in** ϵ if $F^{-\epsilon}(X) = -F^\epsilon(X)$.

Corollary 3.10. *Let* $Q \subseteq P_\iota \cup P_\upsilon$ *with* $|Q| = |P|$.

1. $\mathbf{M}^\epsilon_E(\pm\mathbf{N})[Q]$ *is constant under sign change of* $\pm\mathbf{N}$, *and is alternating in* E, ϵ *and* Q.

2. $\epsilon(PE)\mathbf{M}^\epsilon_E(\pm\mathbf{N})[Q]$ *is constant under sign change of* $\pm\mathbf{N}$ *and under changes or reorderings of* ϵ *or* E; *it is alternating in* P *and in* Q.

3. $\epsilon(PE)\mathbf{M}^\epsilon_E(\pm\mathbf{N})[P_\iota]$ *enumerates the bases of* $\mathcal{N}(\mathbf{N}/P)$, *assuming* P *is independent in the matroid* $\mathcal{N}(\mathbf{N})$, *by*

$$\epsilon(PE)\mathbf{M}^\epsilon_E(\pm\mathbf{N})[P_\iota] = \sum_{B \subseteq E} g_B r_{\overline{B}} \mathbf{N}^2[BP],$$

4. *and* $\epsilon(PE)\mathbf{M}^{\epsilon}_{E}(\pm\mathbf{N})[P_v]$ *enumerates the bases of* $\mathcal{N}(\mathbf{N}\setminus P)$*, assuming* P *is coindependent in* $\mathcal{N}(\mathbf{N})$*, by*

$$\epsilon(PE)\mathbf{M}^{\epsilon}_{E}(\pm\mathbf{N})[P_v] = \sum_{B\subseteq E} g_B r_{\overline{B}} \mathbf{N}^2[B].$$

Remark: Properties 3. and 4. express the Matrix Tree Theorem.

4 Corank-Nullity Polynomials

The well-known corank-nullity (or rank) polynomial is easily generalized to ported matroids. We did [12] this with a definition that differs from Las Vergnas' big Tutte polynomial [37] only in notation and in our applications. We will now generalize it by including parameters and modify it for oriented matroids by reinterpreting the symbols for minors. In the definition below the symbol $\mathcal{N}/A|P$ represents the oriented minor of the oriented matroid $\mathcal{N}$ obtained by contracting A and restricting to P. Minors that are different orientations of the same unoriented matroid are deemed different objects.

Definition 4.1 (Parametrized and Ported Corank-Nullity Polynomial)**.**

$$R(\mathcal{N}(P,E)) = \sum_{A\subseteq E} [\mathcal{N}/A|P]\; g_A r_{\overline{A}}\; u^{\rho\mathcal{N}-\rho[\mathcal{N}/A|P]-\rho A}\; v^{|A|-\rho A}.$$

In this formula, the bracketed oriented matroid $[\mathcal{N}/A|P] = [\mathcal{N}_{i1}\oplus\ldots\oplus\mathcal{N}_{ic}]$ denotes the (commutative) product of the *variables* $[\mathcal{N}_{i1}]$, . . . , $[\mathcal{N}_{ic}]$, where each variable signifies a connected component of $\mathcal{N}/A|P$. If $P=\emptyset$ then $[\mathcal{N}/A|P] = [\emptyset] = 1$; so R reduces to the corank-nullity polynomial, parametrized.

The formula therefore defines a polynomial in parameters g_e, r_e, whose (other) variables are u, v together with a distinct variable for every connected component of every minor of $\mathcal{N}$ obtained by contracting some subset $A\subseteq E$ and deleting $\overline{A} = E\setminus A$. The latter variables only occur in monomials that signify direct sums of one or more minors.

It is readily verified that $R(\mathcal{N}(P,E))$ satisfies the ported Tutte equations below. The details published in [12] can be immediately adapted to the changes we described above. We state these results without proof:

Proposition 4.2. *1. If* $e\in E$ *is neither a port nor a loop nor a coloop in* $\mathcal{N}(P,E)$*,*

$$R(\mathcal{N}(P,E)) = g_e R(\mathcal{N}/e) + r_e R(\mathcal{N}\setminus e). \tag{22}$$

2. $R(\mathcal{N}_1\oplus\mathcal{N}_2) = R(\mathcal{N}_1)R(\mathcal{N}_2) = R(\mathcal{N}_2)R(\mathcal{N}_1)$.

3. $R(\mathcal{N}_1(e)) = g_e + r_e u$ *and* $R(\mathcal{N}_0(e)) = r_e + g_e v$*, for the coloop and loop matroids* $\mathcal{N}_1(e)$ *and* $\mathcal{N}_0(e)$ *on* $E=\{e\}$*,* $P=\emptyset$*.*

4. $R(\mathcal{N}(P,\emptyset)) = [\mathcal{N}]$ *(i.e. when* $E=\emptyset$*.)*

Let us take $\mathcal{N}$ to be the oriented matroid presented by extensor $\mathbf{N}$. The reader can now verify that the expansion in corollary 3.8 for $\mathbf{M}_E(\mathbf{N})$ is obtained from $R(\mathcal{N}(P, E))$ by substituting $u = 0$, $v = 0$ and the extensor $\epsilon(P)\epsilon(PE)\mathbf{M}_\emptyset(\mathbf{N}/A|P)$ for the monomial $[\mathcal{N}/A|P]$ in the term with factor $g_A r_{\overline{A}}$, for each A. Note that $|A| - \rho A = 0$ and $\rho\mathcal{N} - \rho[\mathcal{N}/A|P] - \rho A = \rho\mathcal{N} - \rho(P \cup A) = 0$ imply that A is independent and $P \cup A$ is spanning in $\mathcal{N}$. Therefore $\mathbf{N}/A|P \neq \mathbf{0}$ for those terms where the exponents of u and v are both zero.

With arbitrary $\mathbf{N}$, the substitution of $\epsilon(PE)\epsilon(P)\mathbf{M}_\emptyset(\mathbf{N}/A_i|P)$ for monomial $[\mathcal{N}/A_j|P] = [\mathcal{N}/A_i|P]$ in $R(\mathcal{N}(P, E))$ is not well-defined. The reason is that the same oriented matroid $\mathcal{N}/A_j|P$ might be represented by different extensors all with the form $\mathbf{N}/A_i|P$, for various $A_i \neq A_j$. They may differ by representing different subspaces (in the same oriented matroid stratification [4, §2.4] layer) of KP. Therefore, different values $\mathbf{M}_E(\mathbf{N}/A_i|P)$ must be substituted in $[\mathcal{N}/A_i|P]g_{A_i}r_{\overline{A_i}}$ with different A_i even though these $[\mathcal{N}/A_i|P]$ all denote the same oriented matroid.

The one general situation where $R(\mathcal{N})$ with $u = v = 0$ determines $\mathbf{M}_E(\mathbf{N})$ is when $\mathbf{N}(P, E)$ is a unimodular extensor, i.e., one that represents the unimodular oriented matroid $\mathcal{N}$. [64, Theorem 3.1.1, p. 41] provides this among other equivalent characterizations of unimodular (also called regular) matroids. One of these characterizations is that bracket values from $\{+1, -1, 0\}$ may be assigned so the Grassmann-Plücker relationships hold over $\mathbb{Q}$.

Definition 4.3. The extensor-valued function $\mathcal{N}(P, E) \to \mathbf{M}_E(\mathcal{N})$ is defined on the minor-closed class of ported unimodular oriented matroids by

$$\mathbf{M}_E(\mathcal{N}) = \mathbf{M}_E(\mathbf{N})$$

where $\pm\mathbf{N}(P, E)$ are the two unimodular presentations of the ported oriented matroid $\mathcal{N}(P, E)$.

The simplest case to demonstrate that the unimodular matroids must be oriented in order for monomial substitution in $R\big(\mathcal{N}(P, E)\big)$ to produce $\mathbf{M}_E(\mathcal{N})$ is the two orientations $\mathcal{N}_1$, $\mathcal{N}_2$ of the 2-circuit matroid on two ports. Here, $E = \emptyset$ and $\mathbf{M}_\emptyset(\mathcal{N}_1) \neq \mathbf{M}_\emptyset(\mathcal{N}_2)$.

We conclude:

Theorem 4.4. *For unimodular oriented matroid $\mathcal{N} = \mathcal{N}(P, E)$, $\mathbf{M}_E(\mathcal{N})$ is the result of evaluating $R(\mathcal{N})$ (in the exterior algebra) after the substitutions $u = 0$, $v = 0$ and extensor $\epsilon(P)\epsilon(PE)\mathbf{M}_\emptyset\big(\mathcal{N}_i(P)\big)$ for each monomial $[\mathcal{N}_i(P)]$ (which symbolizes an oriented unimodular matroid with ground set P).*

Proof. Immediate from the above remarks and Corollary 3.8. □

Theorem 4.5. *The function $\mathbf{M}_E(\mathcal{N})$ defined above on ported unimodular oriented matroids satisfies the properties: (Symbols like E and P denote sequences and $\mathbf{M}_E(\mathcal{N})$ depends on the g_e, r_e and ϵ.)*

1. *If $e \in E$ is neither a separator nor a port, and $E' = E \setminus E$, then*

$$\mathbf{M}_E(\mathcal{N}) = \epsilon(PE)\epsilon(PE')\ \big(g_e\mathbf{M}_{E'}(\mathcal{N}/e) + r_e\mathbf{M}_{E'}(\mathcal{N} \setminus e)\big).$$

2. If $\mathcal{N}_1(P_1, E_1)$ and $\mathcal{N}_2(P_2, E_2)$ have disjoint ground sets and $E = E_1 \cup E_2$, then

$$\mathbf{M}_E(\mathcal{N}_1 \oplus \mathcal{N}_2) = \epsilon(P_1 P_2 E)\epsilon(P_1 E_1)\epsilon(P_2 E_2)\, \mathbf{M}_{E_1}(\mathcal{N}_1)\, \mathbf{M}_{E_2}(\mathcal{N}_2).$$

3. If $P = \emptyset$ and $\mathcal{B}(\mathcal{N})$ denotes the collection of bases of $\mathcal{N}$, then

$$\mathbf{M}_E(\mathcal{N}) = \epsilon(E) \sum_{B \in \mathcal{B}(\mathcal{N})} g_B r_{\overline{B}}.$$

Proof. Immediate from Theorem 3.3 and the above remarks. □

5 Basis, Set and Flat Expansions

Theorem 4.5 shows that when $\mathcal{N}(P, E)$ is a unimodular matroid, $\mathbf{M}_E(\mathcal{N})$ is a substitution of extensors and $u = v = 0$ into $R_P(\mathcal{N})$ of Definition 4.1. Proposition 4.2 demonstrates R_P is a ported Tutte function. This motivates general study of ported and parametrized Tutte functions of matroids. Below, we extend some known general expressions for Tutte functions and invariants. Our results are more clearly expressed and no harder to prove than if the parameters were omitted. Furthermore, we identify an unsolved problem due to combining parameters with ports.

Evidently, each Tutte function value $F(\mathcal{N})$ is determined via the Tutte equations from parameter values and from values of F on decomposable minors of $\mathcal{N}$. One way to study a Tutte function is to present the definition independently of the Tutte equations and then prove that the function so defined satisfies the Tutte equations. A second way is to specify the parameters and the function values on indecomposable minors and then prove a solution exists and is unique. In other words, one proves that all the Tutte equations are consistent with the given parameters and values, and that the solution is unique. (This extends to our Tutte functions remarks of Pak [34].) The issue that arbitrary parameters and values on indecomposable sometimes fail to be consistent or sometimes result in multiple solutions was studied in [5, 23, 67]. Fortunately, the Tutte functions (of matroids, oriented matroids and of extensors) studied in this paper are all defined the first way. However, our work suggests the open problem to extend these studies to the ported Tutte equations. The indecomposables will then include connected matroids or oriented matroids with more than one element, i.e., not just loops and coloops.

Zaslavsky [67] defines the normal class of parametrized (but not ported) Tutte functions as those Tutte functions for which there exist u and v for which, for all e, the point value on coloop e is $r_e u + g_e$ and the point value on loop e is $g_e v + r_e$. The normal Tutte functions are exactly those obtained by substitutions into the parametrized corank-nullity polynomial. All Tutte invariants are normal Tutte functions and non-normal Tutte functions do not express much of the matroid structure—See [5, 23, 67] for details about how parametrization complicates Tutte invariant theory.

Our Theorem 4.4 expresses how $\mathbf{M}_E(\mathcal{N})$ fits into the natural ported generalization of the normal class. Our extensor Tutte function of ported oriented unimodular matroids and its invariant specialization is expressed by a substitution into the ported corank-nullity polynomial of oriented matroids.

While only the normal Tutte functions have corank-nullity polynomial expressions, they all have basis expansion expressions [67]. In the rest of this section, we discuss these and other expansion expressions for ported unoriented and oriented matroids.

The basis expansion originated by Tutte [51] for graphs and Crapo [21] for matroids depends on a particular but arbitrary ground set element order O. Each basis determines a term from the internal and external activities of elements with respect to that basis according to the ordering O. Our way to generalize is to restrict O to orders in which every port element is ordered before each $e \in E$. (We use the convention that the deleted/contracted element is the last, i.e., greatest element under order O eligible for reduction.)

Gordan and McMahon define [26] a "computation tree" to formalize the application of a subset of Tutte equations to a matroid and some of its minors. Each (Tutte) computation tree determines a polynomial in the parameters and point values. Therefore, when $\mathcal{N}$ is in the domain of a Tutte function, each of these computation trees determine the same value. Computation trees are a way to give a basis expansion expression in terms of a more general definition of internal and external activities of elements with respect to a basis. The expansion is more general because it is based on any Tutte equation computation rather than on an element order O. We will extend to ported computation trees the classification [26] of elements as internally or externally, active or passive with respect to each path down the tree. In each case, the result is an interval partition of the boolean subset lattice of E.

Definition 5.1. Given $\mathcal{N}(P, E)$, a P**-subbasis** $F \in \mathcal{B}_P(\mathcal{N})$ is an independent set with $F \subseteq E$ (so $F \cap P = \emptyset$) for which $F \cup P$ is a spanning set for $\mathcal{N}(P, E)$ (in other words, F spans $\mathcal{N}/P$, see [37].)

Proposition 5.2. *For every P-subbasis F there exists an independent set $Q \subseteq P$ that extends F to a basis $F \cup Q \in \mathcal{B}(\mathcal{N})$. Conversely, if $B \in \mathcal{B}(\mathcal{N})$ then $F = B \cap E = B \setminus P$ is a P-subbasis.*

Proof. Immediate. $\square$

Definition 5.3 (Activities with respect to a P-subbasis and an element ordering O)**.** Let ordering O have every $p \in P$ before every $e \in E$. Let F be a P-subbasis. Let B be any basis for $\mathcal{N}$ with $F \subseteq B$.

- Element $e \in F$ is internally active if e is the least element within its principal cocircuit with respect to B. Thus, this principal cocircuit contains no ports. The reader can verify this definition is independent of the B chosen to extend F. Elements $e \in F$ that are not internally active are called internally inactive.

- Dually, element $e \in E$ with $e \notin F$ is externally active if e is the least element within its principal circuit with respect to B. Thus, each externally active element is spanned by F. Elements $e \in E \setminus F$ that are not externally active are called externally inactive.

Definition 5.4 (Computation Tree, following [26]). A ported (Tutte) computation tree for $\mathcal{N}(P, E)$ is a binary tree whose root is labeled by $\mathcal{N}$ and which satisfies:

1. If $\mathcal{N}$ has non-separating elements not in P, then the root has two subtrees and there exists one such element e for which one subtree is a computation tree for $\mathcal{N}/e$ and the other subtree is a computation tree for $\mathcal{N} \setminus e$.

   The branch to $\mathcal{N}/e$ is labeled with "e contracted" and the other branch is labeled "e deleted".

2. Otherwise (i.e., every element in $S(\mathcal{N}) \setminus P$) is separating) the root is a leaf.

An immediate consequence is

Proposition 5.5. *Each leaf of a P-ported computation tree for $\mathcal{N}(P, E)$ is labeled by the direct sum of some minor of $\mathcal{N}$ on P (oriented if $\mathcal{N}$ is oriented) summed with loop and/or coloop matroids with ground sets $\{e\}$ for various distinct $e \in E$ (possibly none).*

Definition 5.6 (Activities with respect to a leaf). For a ported computation tree for $\mathcal{N}(P, E)$, a given leaf, and the path from the root to this leaf:

- Each $e \in E$ labeled "contracted" along this path is called **internally passive**.
- Each coloop $e \in E$ in the leaf's matroid is called **internally active**.
- Each $e \in E$ labeled "deleted" along this path is called **externally passive**.
- Each loop $e \in E$ in the leaf's matroid is called **externally active**.

Proposition 5.7. *Given a leaf of a ported computation tree for $\mathcal{N}(P, E)$: The set of internally active or internally passive elements constitute a P-subbasis of $\mathcal{N}$ which we say* ***belongs to the leaf****. Furthermore, every P-subbasis F of $\mathcal{N}$ belongs to a unique leaf.*

Proof. For the purpose of this proof, let us extend Definition 5.6 so that, given a computation tree with a given node i labeled by matroid $\mathcal{N}_i$, $e \in E$ is called internally passive when e is labeled "contracted" along the path from root $\mathcal{N}$ to node i. Let IP_i denote the set of such internally passive elements.

It is easy to prove by induction on the length of the root to node i path that (1) $IP_i \cup S(\mathcal{N}_i)$ spans $\mathcal{N}$ and (2) IP_i is an independent set in $\mathcal{N}$. The proof of (1) uses the fact that elements labeled deleted are non-separators. The proof of (2) uses the fact that for each non-separator $f \in \mathcal{N}/IP_i$, $f \cup IP_i$ is independent in $\mathcal{N}$.

These properties applied to a leaf demonstrate the first conclusion, since each $e \in E$ in the leaf's matroid must be a separator by Definition 5.4.

Given a P-subbasis F, we can find the unique leaf as follows: Beginning at the root, descend the tree according to the rule: At each branch node, descend along the edge labeled "e-contracted" if $e \in F$ and along the edge labeled "e-deleted" otherwise (when $e \notin F$). (This algorithm also operates on arbitrary $F' \subseteq E$.) ☐

The above definitions and properties enable us to conclude:

Proposition 5.8. *Given element ordering O in which every $p \in P$ is ordered before each $e \notin P$, suppose we construct the unique P-ported computation tree $\mathcal{T}$ in which the greatest non-separator $e \in E$ is deleted and contracted in the matroid of each tree node.*

The activity of each $e \in E$ relative to ordering O and P-subbasis $F \subseteq E$ is the same as the activity of e defined with respect to the leaf belonging to F in $\mathcal{T}$.

Definition 5.9. Given a computation tree for (oriented) matroid $\mathcal{N}(P, E)$, each P-subbasis $F \subseteq E$ is associated with the following subsets of non-port elements defined according to Definition 5.6 from the unique leaf determined by the algorithm given above.

- $IA(F) \subseteq F$ denotes the set of internally active elements,
- $IP(F) \subseteq F$ denotes the set of internally passive elements,
- $EA(F) \subseteq E \setminus F$ denotes the set of externally active elements, and
- $EP(F) \subseteq E \setminus F$ denotes the set of externally passive elements.

Proposition 5.10. *Given a computation tree for $\mathcal{N}(P, E)$, the boolean lattice of subsets of E is partitioned by the collection of intervals $[IP(F), F \cup EA(F)]$ (note $F \cup EA(F) = IP(F) \cup A(F)$) determined from the collection of P-subbases F, which correspond to the leaves.*

Proof. Every subset $F' \subseteq E = S(\mathcal{N}) \setminus P$ belongs to the unique interval corresponding to the unique leaf found by the tree descending algorithm given at the end of the previous proof. □

Dualizing, we obtain:

Proposition 5.11. *Given a computation tree for $\mathcal{N}(P, E)$, the boolean lattice of subsets of E is also partitioned by the collection of intervals $[EP(F), E \setminus F \cup IA(F)]$ (note $E \setminus F \cup IA(F) = EP(F) \cup A(F)$).*

Proof. The dual of the tree descending algorithm is to descend along the edge labeled "e-deleted" if $e \in F$. □

The following generalizes the basis expansion expression given in [67] to ported (oriented) matroids, as well as Theorem 8.1 of [37].

Definition 5.12. Given parameters g_e, r_e, point values x_e, y_e, and (oriented) $\mathcal{N}(P, E)$ the Tutte polynomial expression determined by the sets in Definition 5.9 from a computation tree is equal to

$$\sum_{F \in \mathcal{B}_P} [\mathcal{N}/F|P]\, x_{IA(F)}\, g_{IP(F)}\, y_{EA(F)}\, r_{EP(F)}. \tag{23}$$

Each Tutte polynomial expression is constructed by applying some of the Tutte equations. Therefore, if $\mathcal{N}(P, E)$ is in the domain of Tutte function f, then $f(\mathcal{N})$ is given by any Tutte polynomial expression with $f(\mathcal{N}/F|P)$ substituted for each oriented or unoriented matroid monomial $[\mathcal{N}/F|P]$. (This generalizes the expression used in [67] to define the Tutte polynomial under the condition that all expansions yield the same expression.)

5.1 Boolean Interval Expansion

The following proposition expresses the ported corank-nullity polynomial in terms of a P-subbasis expansion. It is obtained by substituting binomials $x_e = g_e{+}r_e u$, $y_e = r_e{+}g_e v$ and leaving the matroid variables unchanged in Definition 5.12. The different expansions from different element orderings and Tutte computation trees all express the same polynomial because Proposition 4.2 demonstrates that R_P is a ported Tutte function and the values of R_P on coloop, loop and indecomposable matroids are readily verified to be given by these substitutions.

Proposition 5.13. *The polynomial $R_P(\mathcal{N})$ is given by the following activities and boolean interval expansion formula:*

$$R_P(\mathcal{N}) = \sum_{F\in\mathcal{B}_P} [\mathcal{N}/F|P]\Big(\sum_{\substack{IP(F)\subseteq K\subseteq F\\ EP(F)\subseteq L\subseteq E\setminus F}} g_{K\cup(E\setminus F\setminus L)}\ v^{|E\setminus F\setminus L|}\ r_{L\cup(F\setminus K)}\ u^{|F\setminus K|}\ \Big) \tag{24}$$

Proof. Let $A = K\cup(E\setminus F\setminus L)$ within the above expansion. We can verify $\overline{A} = E\setminus A = L\cup(F\setminus K)$. For each $A\subseteq E$ a unique P-subbasis F, and two tree leaves are determined, one by the tree descending algorithm and the other leaf by the dual algorithm. Thus A and $\overline{A}$ respectively belong to intervals within the boolean lattice partitions of Propositions 5.10 and 5.11. In particular, $A\in[IP(F), F\cup EA(F)]$ and $\overline{A}\in[EP(F), E\setminus F\cup IA(F)]$. Therefore the terms in the above sum are equal one by one to the terms in the corank-nullity polynomial's subset expansion (Definition 4.1). □

For the purposes of this paper it was sufficient to recognize that our extensor valued ported parametrized Tutte function of unimodular oriented matroids belongs to the natural generalization of Zaslavsky's normal class. As such, it has, for arbitrary parameters, expressions obtained by substitutions into (1) computation trees, (2) ported parametrized Tutte polynomials from such trees, (3) various P-subbasis expansions, and (4) the ported parametrized corank-nullity polynomial.

We state here the open problem to include ports into the results of Zaslavsky, Bollobas and Riordan, and Ellis-Monaghan and Traldi: Can we classify with universal forms all of the ported parametrized Tutte functions according to their parameters, non-port point values, and the values on oriented or unoriented minors on port ground sets?

5.2 Geometric Lattice Flat Expansion

A formula for the unparametrized ported Tutte (or corank-nullity) polynomials of non-oriented matroids in terms of the lattice of flats (closed sets) and its Mobius function was given in [12]. We generalize: (1) The expansion's monomials $[\mathcal{Q}]$ can signify either oriented matroid minors, when $\mathcal{N}$ is oriented, or non-oriented minors when $\mathcal{N}$ is not oriented. (2) The polynomial is parametrized with r_e, g_e for each $e\in E$. The derivation relies on the fact that the oriented or non-oriented matroid minor $[\mathcal{N}/A|P]$ (according to whether $\mathcal{N}$ is oriented or not) depends only on the flat spanned by $A\subseteq E$.

Proposition 5.14. *Let $\mathcal{N}(P,E)$ be an oriented or unoriented. Let $R_P(\mathcal{N})$ be given from Definition 4.1. In the formula below, F and G range over the geometric lattice of flats contained in E.*

$$R_P(\mathcal{N})(u,v) = \sum_{\mathcal{Q}} [\mathcal{Q}] \sum_{\substack{F \le E \\ [\mathcal{N}/F|P]=[\mathcal{Q}]}} u^{\rho\mathcal{N}-\rho\mathcal{Q}-\rho F} v^{-\rho F} \sum_{G \le F} \mu(G,F) \prod_{e \in G} (r_e + g_e v) \tag{25}$$

Proof. It follows the steps for theorem 8 in [12]. □

Remark: The chirotope values for the oriented matroid minor $\mathcal{N}' = \mathcal{N}/F|P$ are $\chi_{\mathcal{N}'}(X) = \chi_{\mathcal{N}}(XB_F)$ where X is restricted to sequences over P and B_F is any basis for the flat spanned by F. While this formula defines a chirotope function only up to a constant sign factor, the oriented matroid (which is what the monomial $[\mathcal{N}/F|P]$ denotes) is uniquely defined. We mention this because when we evaluate the corank-nullity polynomial to obtain $\mathbf{M}_E(\mathcal{N})$ for the unimodular oriented matroid $\mathcal{N}$ substitute an extensor for each $[\mathcal{N}/A|P]$. However, the object we substitute is $\mathbf{M}_\emptyset(\mathbf{N}/A|P)$, not $\mathbf{N}/A|P$. It is the unique extensor defined by equation (13) applied to one of the chirotopes that present $[\mathcal{N}/A|P]$ (or to any other representation of $[\mathcal{N}/A|P]$ for that matter). We already remarked that equation (13) is unchanged when its argument changes sign!

6 Electrical Networks

The most common way to formulate the electrical network problem that occurs in our subject is to use the discrete and parametrized Laplace's equation. The insights from matroid theory appear with greater clarity when the analyses of the solution, which is in terms of the electrical potential values at graph vertices, are replaced by analyses in terms of voltage and current variables directly associated with edges. After we review the Laplacian, we move on to edge-based formulations. Some details of relationships between our results and electrical network problems are then demonstrated.

The following observations underlie the combination of ideas about electrical networks and ported oriented matroids: First, the basis enumerator is a Tutte function that happens to be a determinant in the case of unimodular (or regular) matroids, such as the graphic matroids. Second, topics involving determinants, including the chirotope presentation of realizable oriented matroids, can profitably be studied with exterior algebra. Third, non-trivial electrical networks (see §6.2) must have port elements for supplying power or for connecting to an external environment, in addition to the resistor elements usually modeled by graph edges. We are interested in their combinatorial properties beyond spanning tree counts [9–15]. Finally, electrical flows and potential differences are inherently directional. The patterns of their directions (expressed by sign functions on graph edges) feasible under Kirchhoff's two laws are precisely the vector and covector families, respectively, of the graphic oriented matroid. Indeed, the duality between these laws, current and voltage, is characterized by oriented matroid theory.

6.1 Discrete Laplacian

The combinatorial (or discrete) Laplacian is the matrix of coefficients in the equations (26) below in variables ϕ_i, $1 \leq i \leq n$. These discrete Laplace equations model (among other situations) a resistive electrical network when ϕ_i represents the electrical potential (or voltage) at vertex i and constant I_i represents the current flowing into vertex i.

$$\sum_{\{j|e=ij\in E\}} g_e(\phi_i - \phi_j) = I_i \quad 1 \leq i \leq n \tag{26}$$

If each conductance g_e is non-negative, or is either zero or generic, then the rank of the $n \times n$ Laplacian matrix is $n-k$, where k is the number of connected components in the n vertex undirected graph whose edges are the $e = ij$ with $g_e \neq 0$. Each order $n-k$ non-singular diagonal submatrix is called a reduced Laplacian. (In §6.2, we will express Laplace's equations with an $\mathbf{M}_E(\mathbf{N})$ independent of k.) The reduced Laplace equations, together with $\phi_i = 0$ for each vertex i corresponding to a deleted column, model a network where each such i is contracted into a single **grounded** vertex whose potential is fixed to zero and whose external current is unrestricted[3]. Since the graph is undirected, equations (26) imply that the current into the grounded vertex equals the sum of the I_j for the non-grounded vertices.

The inverse of a reduced Laplacian matrix is called the discrete Green's function in [20]. This inverse matrix's elements are each expressed (using Cramer's rule) by a ratio of an order $n-k-1$ minor to a common order $n-k$ minor denominator. The list of all minors, of all orders, is an example of Plücker coordinates—Here, these are the $\binom{2n}{n}$ maximal minors of the matrix obtained by appending the $n \times n$ identity matrix to the side of the Laplacian.

The Matrix Tree Theorem asserts that each $n-1$ order minor equals $\pm \sum g_T$, the enumeration of spanning trees T by products of edge parameters $g_T = \prod_{e\in T} g_e$. See [9] for similar interpretations of all the minors and for generalizations to directed graphs. The formulas we call "Maxwell's rules" were given without proof by Maxwell [41], and the dual forms of them were proved by Kirchhoff [33]. Maxwell also described the static equilibrium solution for stressed linear elastic framework in terms of enumerations over minimally rigid subframeworks [42]; this enumerated set is the basis set for the rigidity matroid [27]. The one-dimensional case is analogous to the electrical problem. The survey by Biggs [3] covers the discrete Laplacian, the Matrix Tree Theorem, and the use of spanning tree enumeration to solve the discrete Laplace equations, and many additional topics, including the asymmetric discrete Laplacian. Biggs presents the Kirchhoff's solution method. and Nerode and Shank [44], also used by Bott and Duffin [6], Smith [47] and Maurer [40]. This method constructs a symmetric projection matrix from a sum of fundamental cocycle matrices, one for each spanning tree. Analysis of basis exchange, i.e., the pivot calculation implies the appropriately weighted matrix sum is symmetric. We plan to present the generalization of this argument to extensors in a future publication.

[3]Do not confuse with ground set.

Tree counting, the discrete Laplacian and electrical network models with parameters have a spectrum of applications including electrical circuit theory, knot theory, random walks and the analysis of Markov chains (see for example [3,19,20,22]). Their application to square dissections is described in [7, 53]; Tutte gives a Laplacian based "barycentric embedding" proof of Kuratowski's Theorem in [52].

It is generally known among electrical engineers in circuit theory that the same kinds of homogeneous rational polynomial functions that appear in Maxwell's rule occur generally as the coefficients (and minors of them) in all of the linear relationships between the port quantities that define the externally observable characteristics of a linear resistive network. Our results display this principle within the mathematical contexts of the enumerative combinatorics of graphs, oriented matroids and exterior algebra: See Corollary 3.7. Some electrical network analysis software actually enumerates trees and related structures to do "symbolic analysis." See for example [14, 24, 49, 50].

6.2 Electrical Network Equations in terms of Edges

Consider a graph with two kinds of edges, called ports P and resistors E. The graph is directed with an arbitrary edge orientation. Let unimodular extensor $\mathbf{N}(P, E)$ present its ported graphic oriented matroid. Let r be the rank of this matroid.

Let g_e, r_e be parameters for each $e \in E$. The extensors $\iota(\mathbf{N})(P_\iota \uplus P_\upsilon \uplus E)$ and $\upsilon(\mathbf{N}^\perp)(P_\iota \uplus P_\upsilon \uplus E)$ defined in §3 determine the electrical network equations in the way expressed by Theorem 2.3 applied to $\mathbf{N}^\perp$.

To be specific, these equations are a linear system on the $|E| + 2|P|$ variables $\{x_e, \ldots;\ v_p, \ldots;\ i_p, \ldots\}$. Each $e \in E$ is associated to variable x_e. Each $p \in P$ is associated to two variables, v_p called the voltage and i_p called the current. Let matrix K with r rows be any matrix that presents $\iota(\mathbf{N})$. Let C be any matrix with $|P| + |E| - r$ rows that presents $\upsilon(\mathbf{N}^\perp)$. These matrices express Kirchhoff's equations combined with a homogeneous expression of Ohm's law. K determines the following **current** equations:

$$\sum_{p \in P} K_{j,p_\iota} i_p + \sum_{e \in E} K_{j,e} x_e = 0 \text{ for } j = 1, \ldots, r.$$

C determines the following **voltage** equations:

$$\sum_{p \in P} C_{j,p_\upsilon} v_p + \sum_{e \in E} C_{j,e} x_e = 0 \text{ for } j = 1, \ldots, |E| + |P| - r.$$

It is helpful to see the electrical network equations in terms of $\mathbf{N}$ directly. Let N be any matrix presentation of $\mathbf{N}$; for example, a reduced oriented incidence matrix of the graph. Let $N^\perp$ present $\mathbf{N}^\perp$; the rows of $N^\perp$ comprise a basis for the cycle space of the graph. The current equations can be written:

$$\sum_{p \in P} N_{j,p} i_p + \sum_{e \in E} N_{j,e} g_e x_e = 0 \text{ for } j = 1, \ldots, r.$$

The voltage equations can be written:

$$\sum_{p\in P} N^{\perp}_{j,p} v_p + \sum_{e\in E} N^{\perp}_{j,e} r_e x_e = 0 \text{ for } j = 1, \ldots, |E| + |P| - r.$$

The equations which $\mathbf{M}_E(\mathbf{N})$ presents are obtained by eliminating all the variables x_e, $e \in E$ in the voltage and current equations taken together. Corollary 3.6 indicates the rank of the resulting system of $|P|$ equations on $2|P|$ variables is $|P|$, provided that the parameters are generic or all positive.

The above analysis illustrates the role for the port element distinction in modeling a physical system. Each non-port element models a completely defined subsystem. The "proto-voltage [47]" x_e parametrizes the state of one electrical resistor, for example. The behavior of this resistor is thus defined by Ohm's law: When g_e and r_e are both non-zero, the current is $g_e x_e$ if and only if the voltage is $r_e x_e$. The entire model (the graph, for example) specifies all the interactions (via Kirchhoff's laws, for electricity) between its subsystems. Each port element models an interface pertaining to an interaction of the system with an unspecified environment, for observing the system behavior of interest to the application, and to help specify how certain larger systems are composed out of previously entire subsystems. For us, the environment is assumed, for each port, to constrain the currents into one terminal and out of the other terminal to be equal. Environmental constraints between voltages at terminals belonging to distinct ports are forbidden as well. (Engineering models encompass multiport elements, whose behavior is specified using multiple port elements [16, 18, 46]. For example, a linear multiport element is specified a linear constraints among the variables associated with its ports; this generalizes Ohm's law to so-called multi-terminal resistors. Each of our ported objects can model a single multiport element within a larger model. A topic for future research is to abstract this along the lines given here.)

Let a graph on vertices $\{1, \ldots, n\}$ be given with conductance parameters g_e for each edge. We now derive Laplace's equations from the voltage and current equations. Let us append a new vertex 0 (which will be grounded) and n port edges $p_i \in P$, with each p_i directed from vertex 0 to vertex i. For simplicity take parameter $r_e = 1$ for each edge e from the original graph. We can choose N so that the current equations are

$$i_p = \sum_{e\in E} J_{p,e} g_e x_e, \quad p \in P$$

and the voltage equations are

$$x_e = \phi_{p_r} - \phi_{p_s} = \sum_{i=1}^{n} J_{p_i,e} \phi_{p_i}, \text{ where } e = rs, e \in E.$$

where we used potential (relative to vertex 0) ϕ_i in place of v_{p_i} and J is the oriented vertex-edge incidence matrix of the original graph. Laplace's equation is obtained by eliminating the variables x_e which represent differences of potential across resistor edges. Indeed, one presentation for $\mathbf{M}_E(\mathbf{N})$ in this case is the $n \times 2n$ matrix $[I_n \Delta]$ formed by concatenating the identity matrix with the Laplacian matrix Δ. One manifestation of Theorem 3.3 is therefore that each of the forest enumerating polynomials given by an arbitrary minor, of any order, of the Laplacian is a (non-ported) Tutte function of graphic oriented matroids.

6.3 Maxwell's Rules

Kirchhoff [33] first described the solution of a resistive electrical network problem in terms of enumerations of spanning trees and certain forests (or dual forests). One such description is the following surprising yet classical result, one of the "easily remembered" rules stated without proof by Maxwell [41].

Let's discuss Maxwell's rule for a network with one port edge p in terms of graph edges. Let all edges $\neq p$ represent unit resistors in an electrical network. Port edge p is to demarks two terminal vertices. Maxwell's rule asserts that the equivalent resistance between the two terminal vertices equals the quotient of the count A of spanning trees that contain edge p divided by the count B of spanning trees that omit edge p. Note that whenever a two-tree spanning forest F for which $F \uplus p$ is a tree counted by A is contracted and the other non-p edges deleted, p becomes a coloop. When this is done for B, p becomes a loop. A and B are the two Plücker coordinates of the extensor we construct in this paper. They are the coefficients in the linear equation $Ai_p + B(-v_p) = 0$ that relates the port current and the (negated) port voltage. The reader can verify that each of A and B satisfy the additive ported Tutte equation; so (A, B) satisfies it. (It is instructive to calculate how (A, B) behaves under the multiplicative ported Tutte equation.) The spanning tree count resistance expression (A, B) generalizes with weighted spanning tree generating function when conductance parameters g_e are given for $e \neq p$.

Suppose two ports, p_1 and p_2 are given. One, say p_1, denotes the terminal pair between which a specified current value i_{p_1} is constrained to flow (assuming this is feasible.) The other, p_2, denotes the terminals between which to observe or measure the potential difference v_{p_2} (voltage) that results when the specified current constraint is the only electrical power source (assuming this voltage value is unique.) Maxwell's rule for one port is the specialization with $p_1 = p_2 = p$.

The two-port Maxwell's rule holds that the port variables are related by $Ai_{p_1} + B(-v_{p_2}) = 0$ were A and B are again generating functions, except that some of the monomials in A might be negative. The spanning trees that omit p_1 and p_2 are enumerated by B. Term A enumerates forests F for which each of $F \cup p_1$ and $F \cup p_2$ is a spanning tree are enumerated by A. For each such F, $F \cup \{p_1, p_2\}$ contains a unique circuit, which of course contains both p_1 and p_2. The relative directions in which p_1 and p_2 are traversed determines the sign of $\pm g_F$ in A.

It is amusing go back to the one port p case and derive the Maxwell's rule coefficients (A, B) from the principles we just illustrated. The port behavior of the network consisting of one coloop p is defined by $1 \cdot i_p + 0 \cdot v_p = 0$: This constraint can be expressed by Plücker coordinates $(1, 0)$. The port current is 0 and the port voltage is unconstrained since the graph has no cycles. Dually, the port behavior of the one port loop network is expressed by $(0, 1)$. The port voltage is 0 but the current is unconstrained. One consequence of our main theorem proves that $A(1,0)+B(0,1)=(A,B)$ gives the Plücker coordinates of the constraint that the original graph imposes on its port current and (negated) port voltage variables, in accordance with Maxwell's rule $Ai_p + B(-v_p) = 0$. Our theory justifies the choices of $(1,0)$ and $(0,1)$ rather than any other multiple $(\alpha,0)$ or $(0,\beta)$ in $A(\alpha,0)+B(0,\beta)$.

6.4 Deriving Maxwell's Rule

See [16] for an elementary derivation that includes the version of Maxwell's rule [41] that applies to to a pair of ports that do not share a common vertex. Graphic matroid orientation becomes relevant in this situation. Explicit port edges have proven their usefulness in electrical network analysis [18, §13.6]. In this situation, some Plücker coordinates of $\mathbf{M}_E(\mathbf{N})$, as polynomials in g_e, r_e, have *terms of opposite sign* only when two port edges do not share a vertex. Our contribution is to characterize the signs within the theory of oriented matroids and Tutte functions. It is true that such polynomials can be expressed in terms of minors of the Laplacian; this was done by manipulation of solutions to Laplace's equation in [7]; see also [53]. However, our extensor and oriented matroid formulation enables the analysis to be done without the introduction of vertices.

Our derivation of Maxwell's rule for two ports (of which the one port version is a special case) begins with the electrical network equations with $P = \{p_1, p_2\}$. Let $\mathbf{M} = \mathbf{M}_E(\mathbf{N})$ as defined in §3 be as discussed above, and let M be any 2×4 matrix presentation of $\mathbf{M}$. The two equations

$$M \begin{bmatrix} i_1 \\ i_2 \\ v_1 \\ v_2 \end{bmatrix} = 0$$

are obtained by eliminating the variables x_e, $e \in E$ from the electrical network equations. The currents i_1, i_2 in edges $p_1 = ab$ and $p_2 = cd$ flow from vertices a to b, and c to d respectively. The voltage (drop) v_1 across edge p_1 is the potential at a minus the potential at b; the corresponding convention defines the voltage v_2 across edge p_2.

We will assume that all the $r_e = 1$ and that $\mathbf{M}[p_{1v}p_{2v}] \neq 0$. The latter is assured from Corollary 3.10 (4.) provided that E contains a spanning tree and all the g_e are either positive or generic. Under these conditions, the **transfer resistance** ρ_{21} given by $(-v_2)/i_1$ when $i_2 = 0$ and $i_1 \neq 0$ is well-defined. (These conventions are used so that when p_1 and p_2 are identical or parallel, ρ signifies the familiar equivalent resistance which is always positive when E is connected and all $g_e > 0$.)

Proposition 6.1 (Maxwell's Rule). *Given the electrical network graph model described above, let $\mathcal{B}$ denote the collection of edge sets $T \subseteq E$ of trees that span the vertex set V, and assume $\sum_{T\in\mathcal{B}} g_T \neq 0$.*

For vertices i, j, k, l, let $\mathcal{B}_{ik,jl}$ be the collection of all $F \subseteq E$ for which the subgraph (V, F) is a forest with exactly two trees where vertices i and k are in one tree and j and l are in the other tree.

The transfer resistance ρ_{21}, where $p_1 = ab$ and $p_2 = cd$, is well-defined and is given by:

$$\rho_{21} = \frac{\sum_{F \in \mathcal{B}_{ac,bd}} g_F - \sum_{F \in \mathcal{B}_{ad,bc}} g_F}{\sum_{T \in \mathcal{B}} g_T}. \tag{27}$$

Proof. We will abuse the notation slightly by using v_k and i_k for ground set elements $p_{k\upsilon}$ and $p_{k\iota}$, and $\mathbf{v}_k$, $\mathbf{i}_k$ for the corresponding extensors, $k = 1, 2$.

Corollary 3.10 (4.) shows that $\mathbf{M}[v_1 v_2]$ $(= \mathbf{M}[p_{1\upsilon} p_{2\upsilon}]) = \sum_{T \in \mathcal{B}} g_T \neq 0$, so by Cramer's rule,

$$\rho_{21} = -\frac{v_2}{i_1} = -\Big(-\frac{\mathbf{M}_E(\mathbf{N})[v_1 i_1]}{\mathbf{M}_E(\mathbf{N})[v_1 v_2]} \Big).$$

Let us apply Corollary 3.8 to the numerator and denominator. To do this, we first calculate $\mathbf{M}_\emptyset(\mathbf{N})$ for 6 unimodular extensors $\mathbf{N}(\{p_1, p_2\}, \emptyset)$ that present the 6 oriented matroids on $P = \{p_1, p_2\}$, which are all graphic. For each of the 6, we can then determine the Plücker coordinate values $\mathbf{M}_\emptyset(\mathbf{N})[v_1 i_1]$ and $\mathbf{M}_\emptyset(\mathbf{N})[v_1 v_2]$. Those oriented matroid minors for which one of these values is non-zero will characterize, together with the rank conditions in Corollary 3.8, which forests or trees contribute to each sum. These characterizations of the forest or tree terms, and of their signs, analyzed for each Plücker coordinate, will complete the derivation of Maxwell's rule.

Four of the 6 oriented matroids are the direct sums of either the loop $\mathcal{N}_0(p_1)$ or coloop $\mathcal{N}_1(p_1)$ with either the loop or coloop on p_2. The other two oriented matroids are the orientations of the 2-circuit matroid on $\{p_1, p_2\}$. Let $\mathcal{N}_1^+$ denote the oriented circuit $\pm(+-)$; graphically, p_1 and p_2 are parallel. So $\mathcal{N}_1^-$ denotes the oriented matroid of antiparallel p_1 and p_2. Table 1 lists these six distinct ported oriented matroids $\mathcal{N}(\{p_1, p_2\}, \emptyset)$, their unimodular extensor presentations $\mathbf{N}(\{p_1, p_2\})$, and the corresponding extensor values $\mathbf{M}_\emptyset(\mathbf{N})$. The $\mathbf{M}_\emptyset(\mathbf{N})$ values are easily found up to sign. The signs are given in Proposi-

Table 1: The six oriented matroids on $\{p_1, p_2\}$.

matroid	$\mathbf{N}$	$\mathbf{M}_\emptyset(\mathbf{N})$
$\mathcal{N}_0(p_1) \oplus \mathcal{N}_0(p_2)$	± 1	$\mathbf{v}_1 \mathbf{v}_2$
$\mathcal{N}_0(p_1) \oplus \mathcal{N}_1(p_2)$	$\pm \mathbf{p}_2$	$\mathbf{i}_2 \mathbf{v}_1$
$\mathcal{N}_1(p_1) \oplus \mathcal{N}_0(p_2)$	$\pm \mathbf{p}_1$	$\mathbf{i}_1 \mathbf{v}_2$
$\mathcal{N}_1(p_1) \oplus \mathcal{N}_1(p_2)$	$\pm \mathbf{p}_1 \mathbf{p}_2$	$\mathbf{i}_1 \mathbf{i}_2$
$\mathcal{N}_1^-$	$\pm(\mathbf{p}_1 - \mathbf{p}_2)$	$(\mathbf{i}_1 - \mathbf{i}_2)(\mathbf{v}_1 + \mathbf{v}_2) = \mathbf{i}_1 \mathbf{v}_1 + \mathbf{i}_1 \mathbf{v}_2 - \mathbf{i}_2 \mathbf{v}_1 - \mathbf{i}_2 \mathbf{v}_2$
$\mathcal{N}_1^+$	$\pm(\mathbf{p}_1 + \mathbf{p}_2)$	$(\mathbf{i}_1 + \mathbf{i}_2)(\mathbf{v}_2 - \mathbf{v}_1) = \mathbf{i}_1 \mathbf{v}_2 - \mathbf{i}_1 \mathbf{v}_1 + \mathbf{i}_2 \mathbf{v}_2 - \mathbf{i}_2 \mathbf{v}_1$

tion 3.5. One way to calculate is to analyze the corresponding electrical network with 2 ports and no resistors. For example, the network with oriented matroid $\mathcal{N}_0(p_1) \oplus \mathcal{N}_1(p_2)$ constrains its port p_1, a loop, to have voltage drop $v_1 = 0$ but its port p_2, a coloop, to have current $i_2 = 0$. The current in the loop and voltage across the coloop are unconstrained. The solution subspace corresponds to equations $(v_1 = 0;\, i_2 = 0)$. The extensors representing these equations are $\alpha \mathbf{v}_1 \mathbf{i}_2$, $\alpha \neq 0$.

Similarly, the network of two parallel ports (case $\mathcal{N}_1^+$) constrains the sum of voltage drops going around the oriented circuit to be 0, so Kirchhoff's voltage law is expressed by $v_1 - v_2 = 0$. Kirchhoff's current law in the same network is expressed $i_1 + i_2 = 0$. Hence the corresponding extensor is $\pm(\mathbf{v}_1 - \mathbf{v}_2)(\mathbf{i}_1 + \mathbf{i}_2)$.

We complete the derivation. First for the denominator. From table 1 the only terms in (21) of Corollary 3.8 that might contribute to $\mathbf{M}_E(\mathbf{N})[v_1v_2]$ are those for which $\mathcal{N}/A|P$ is the matroid of two loops $\mathcal{N}_0(p_1) \oplus \mathcal{N}_0(p_2)$ because the only appearance of $\mathbf{v}_1\mathbf{v}_2$ is in that matroid's row. The rank conditions further restrict the contributing A to spanning trees.

Finally, for the numerator $\mathbf{M}_E(\mathbf{N})[v_1i_1]$, we locate $\pm\mathbf{v}_1\mathbf{i}_1$ in the bottom two rows. These appearances have *opposite* sign. For $A \subseteq E$ with $\mathcal{N}/A|P = \mathcal{N}_1^-$, the contribution is $-g_A$. The sign is opposite when $\mathcal{N}/A|P = \mathcal{N}_1^+$, so the distinct orientations of the 2-circuit obtained when contracting A account for the opposite signs in (27). We can again verify from the rank conditions that the $F = A$ contributing to the numerator of (27) are the spanning forests with 2 trees containing the indicted vertices as claimed.

Note that the sign dependence of $\mathbf{M}_E(\mathbf{N})$ on ϵ and the order of $P = p_1p_2$ cancels in the ratio ρ_{21}. $\square$

While Theorem 6.1 can be proved by elementary arguments as in [16], the above proof demonstrates how it can be derived from the forgoing theory using algebraic calculations.

Remark: The one port version is immediately derived using a graph where p_1 and p_2 are parallel edges because our proof puts no special conditions on the two ports.

6.5 Signed Contributions of Spanning Forests

In some cases of graphs given with two or more port edges, some of these Plücker coordinates will equal the difference between the counts of two kinds of spanning forests. Such a coordinate pertains to the coefficient that relates a quantity observed at one port to of a voltage or a current quantity at a different port. Our results show how oriented matroid properties determine the sign by which each forest F contributes to this coordinate (Corollary 3.8). In particular, that sign is determined by the graphic oriented matroid on the (directed) port edges obtained by contracting F and deleting the remaining non-port edges. The simplest case where distinct signs do occur is when both orientations of the same 2-circuit matroid on two ports appear in this process. The contribution of F to the Plücker coordinate with a given index X is calculated in a particularly simple way: We solve the electrical network *of port edges only* (with no resistors!) that resulted from this deletion and contraction, after checking matroid rank conditions necessary for at least one coordinate to be non-zero. The contribution *equals* the Plücker coordinate with the same index X from the solution of the latter, port-edge-only network, weighted by $g_F r_{E\setminus F}$. It should be noted that the sign of the contribution is determined by the oriented graphic matroid on ports only, independent of the particular F contracted to obtain this oriented matroid. Details appeared in our proof of Maxwell's rule for two ports, section 6.3.

7 Additional Background and Directions

7.1 Ground Set Orientation

The ground set orientation and its role in defining a canonical dual of an extensor, and our $\mathbf{M}_E(\mathbf{N})$ are motivated by the idea of orientations of orientable manifolds and the definition of pseudo-forms (or "forms of odd-kind" attributed to de Rham in [25]) in the mathematics of physics. A pseudo-form is an antisymmetric multilinear operator $f = f_\epsilon$ that is parametrized by the orientation ϵ and for which ϵf_ϵ is independent of the orientation [25]. So, ϵf_ϵ is a well-defined form. In physics, an orientation specifies one's convention, say by a right-handed coordinate system, for how one defines a positive volume or other naturally unsigned physical quantity in terms of an exterior algebra form. The orientation specifies which ordered bases determine right handed coordinate systems. In this context, the orientation ϵ is a ± 1 function for which $\epsilon(B_1)\epsilon(B_2)$ is the sign of the determinant of the local Jacobian matrix which relates the ordered bases B_1, B_2.

7.2 Computational Complexity

Among the non-trivial Tutte invariant functions of succinctly presented graphs or matroids, only two (unless $\mathcal{P} = \#\mathcal{P}$) are polynomial time computable [32, 57]. One such function, the number of bases, is computable by the Matrix Tree Theorem for graphs and its extension to unimodular matroids. This number is well-known as the evaluation $T(\mathcal{N}, 1, 1)$ of the Tutte polynomial function $T(\mathcal{N}, x, y)$ of matroids $\mathcal{N}$. However, computing $T(\mathcal{N}, 1, 1)$ is $\#\mathcal{P}$-complete for arbitrary non-unimodular matroids [56]; this follows because counting the perfect matchings in a bipartite graph is a $\#\mathcal{P}$-complete problem [55].

The other easy-to-compute invariant is determined by the dimension of the intersection of a linear subspace and its orthogonal complement over a finite field [57]. The cited papers prove that all of the other Tutte matroid invariants are either trivial or $\#\mathcal{P}$-complete. More recently, analogous computational hardness results have been proven for Tutte functions of graphs (thus implying their hardness for matroids). Among these results, is that evaluating the parametrized Tutte polynomial for given matroids is a **VNP**-complete problem [38]. Here, Valiant's non-uniform algebraic complexity model [54] is used, which counts as one deterministic step each evaluation of a polynomial on constants, variables or previously computed values. **VNP** is this model's class that is analogous to **NP** in the Turing machine model. (See the references in [38]).

A full account of the computational complexity of Tutte invariants of graphs and matroids is given in [32, 45, 57, 59–61].

We remark for computationally-inclined readers that:

1. The Tutte equations describe non-unique recursive algorithms to compute Tutte functions that generally require $2^{|E|}$ steps.

2. A $|P| \times 2|P|$ matrix representing our extensor can be computed from a graph or

locally or totally unimodular matrix presentation of a ported oriented unimodular matroid.

One suitable algorithm is simple matrix block manipulations followed by Gaussian elimination. Such elimination-based algorithms use polynomial bounded numbers of field operations. Therefore, computation of our extensor generalization of the basis enumerator on graphic and other unimodular matroids is a polynomial time problem when all $r_e = g_e = 1$.

7.3 Other Directions

It is natural to generalize the current and voltage equations so their respective solutions subspaces (taken to be within KS) are not orthogonal [10]. This leads to the directed graph version of the Matrix Tree Theorem. It did lead as well to a "oriented matroid pair" model for combinatorial conditions for certain equations with monotone non-linearities to be uniquely solvable [13]. These conditions were stated in terms of two oriented matroids with a common ground set having complementary rank and no common non-zero covector; the current paper provides the insight that these two were obtained by deletion/contractions to eliminate port elements. Investigations of a generalization of the Tutte polynomial to two matroids with a common ground set were also begun in [62].

The computation tree formalism was used in greedoid generalizations [26]. of the Tutte polynomial because those generalizations do not always have an activities expansions based on element orders. We leave investigation of "ported greedoids" to the future.

8 Acknowledgments

The author wishes to thank Prof. Ariel Caticha, Physics Department of the University at Albany, for his invitation to participate in his graduate Geometry of Physics course; Joanna Ellis-Monaghan, Michel Las Vergnas, Gary Gordon, Alan D. Sokal, Richard P. Stanley, Lorenzo Traldi and David G. Wagner for helpful discussions, correspondence and encouragement; and Thomas Zaslavsky for his assistance leading to a much improved presentation of the subject.

References

[1] A. Bachem and W. Kern. *Linear Programming Duality, An Introduction to Oriented Matroids.* Springer-Verlag, Berlin, 1992.

[2] M. Barnabei, A. Brini, and G.-C. Rota. On the exterior calculus of invariant theory. *J. Algebra*, 96:120–160, 1985.

[3] N. Biggs. Algebraic potential theory on graphs. *Bull. London Math. Soc.*, 29:641–682, 1997.

[4] A. Björner, M. Las Vergnas, B. Sturmfels, N. White, and G. M. Ziegler. *Oriented matroids*, volume 46 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, second edition, 1999.

[5] B. Bollobas and O. Riordan. A Tutte polynomial for colored graphs. *Combin. Probab. Computat.*, 8(1–2):45–93, 1999.

[6] R. Bott and R. J. Duffin. On the algebra of networks. *Trans. Amer. Math. Soc.*, 74:99–109, 1953.

[7] R. L. Brooks, C. A. B. Smith, A. H. Stone, and W. T. Tutte. The dissection of rectangles into squares. *Duke Math. J.*, 7:312–340, 1940.

[8] T. H. Brylawski. A combinatorial model for series-parallel networks. *Trans. Amer. Math. Soc.*, 154:1–22, 1971.

[9] S. Chaiken. A combinatorial proof of the all minors matrix tree theorem. *SIAM J. Alg. Disc. Meth.*, 3:319–329, 1982.

[10] S. Chaiken. A matroid abstraction of the Bott-Duffin constrained inverse. *SIAM J. Alg. Disc. Meth.*, 4:467–475, 1983.

[11] S. Chaiken. Extremal length and width of blocking polyhedra, Kirchhoff spaces and multiport networks. *SIAM J. Alg. Disc. Meth.*, 8(4):635–645, 1987.

[12] S. Chaiken. The Tutte polynomial of a P-ported matroid. *J. Combin. Theory Ser. B*, 46:96–117, 1989.

[13] S. Chaiken. Oriented matroid pairs, theory and an electric application. In J. E. Bonin, J. G. Oxley, and B. Servatius, editors, *Matroid Theory, AMS-IMS-SIAM Joint Summer Research Conference*, volume 197 of *Contemporary Math.*, pages 313–331. American Math. Society, 1996.

[14] S. Chaiken. Hybrid matrix minors from tableau applied to a multiport generalization of NDR related to stability. In *IEEE Int. Conf. Circ. Syst.*, 1998.

[15] S. Chaiken and P. Narendran. The all-minors VCCS matrix tree theorem, half-resistors and applications in symbolic simulation. In *IEEE Int. Conf. Circ. Syst.* IEEE, 1995.

[16] W. K. Chen. *Applied Graph Theory, Graphs and Electrical Networks*. North-Holland, Amsterdam, 2nd edition, 1976.

[17] Y.-B. Choe and D. G. Wagner. Rayleigh matroids. *Combin. Prob. Comput.*, to appear. also arXiv.org/abs/math.CO/0307096.

[18] L. O. Chua, C. A. Desoer, and E. S. Kuh. *Linear and nonlinear circuits*. McGraw-Hill, New York, 1987.

[19] F. Chung and S.-T. Yau. Coverings, heat kernels and spanning trees. *Electronic J. Combinatorics*, 6((1)R 12), 1999.

[20] F. Chung and S.-T. Yau. Discrete Green's functions. *J. Comb. Theory A*, 91:191–214, 2000.

[21] H. H. Crapo. Mobius inversion in lattices. *Arch. Math. (Basel)*, 19:565–607, 1968.

[22] P. G. Doyle and J. L. Snell. *Random Walks and Electrical Networks*. Number 22 in Carus Monographs. Math. Assoc. America, 1984.

[23] J. A. Ellis-Monaghan and L. Traldi. Parametrized Tutte polynomials of graphs and matroids. *Combinatorics, Probability and Computing*, to appear.

[24] F. V. Fernandez, A. Rodriguez-Vazquez, J. L. Huertas, and G. Gielen, editors. *Symbolic Analysis Techniques: Applications to Analog Design Automation*. IEEE Press and John Wiley & Sons, Inc., 1998.

[25] T. Frankel. *Geometry of Physics, an Introduction*. Cambridge Univ. Press, Cambridge, 1997.

[26] G. Gordon and E. McMahon. Interval partitions and activities for the greedoid Tutte polynomial. *Adv. Appl. Math.*, 18(1):33–49, Jan. 1997.

[27] J. Graver, B. Servatius, and H. Servatius. *Combinatorial Rigidity*, volume 2 of *Graduate Studies in Math.* Amer. Math. Soc., 1993.

[28] M. Hasler, C. Marthy, A. Oberlin, and D. de Werra. A discrete model for studying existence and uniqueness of solutions in nonlinear resistive circuits. *Disc. Appl. Math.*, 50:169–184, 1994.

[29] M. Hasler and J. Neirynck. *Nonlinear Circuits*. Artech House, Norwood, Mass., 1986.

[30] W. V. D. Hodge and D. Pedoe. *Methods of Algebraic Geometry*. Cambridge Univ. Press, Cambridge, 1994 edition, 1947.

[31] N. Jacobson. *Basic Algebra I*. W. H. Freeman, San Francisco, 1974.

[32] F. Jaeger, D. L. Vertigan, and D. J. A. Welsh. On the computational complexity of the Jones and Tutte polynomials. *Math. Proc. Camb. Phil. Soc.*, 108:35–53, 1990.

[33] G. Kirchhoff. Uber die aufl osung der gleichungen, auf welshe man bei der untersuchung der linearen verteilung galvanischer str ome gefuhrt wird. *Ann. Physik Chemie*, 72:497–508, 1847. On the solution of the equations obtained from the investigation of the linear distribution of Galvanic currents, (J. B. O'Toole, tr.) *IRE Trans. Circuit Theory*, 5, 1958, pp. 238–249.

[34] M. Korn and I. Pak. Combinatorial evaluations of the tutte polynomial. Technical report, Math. Dept, MIT, 2003.

[35] M. Las Vergnas. Extensions normales d'un matroïde, polynôme de Tutte d'un morphisme. *C. R. Acad. Sci. Paris Sér. A-B*, 280(22):Ai, A1479–A1482, 1975.

[36] M. Las Vergnas. On the Tutte polynomial of a morphism of matroids. *Ann. Discrete Math.*, 8:7–20, 1980. Combinatorics 79 (Proc. Colloq., Univ. Montréal, Montreal, Que., 1979), Part I.

[37] M. Las Vergnas. The Tutte polynomial of a morphism of matroids. I. Set-pointed matroids and matroid perspectives. *Ann. Inst. Fourier (Grenoble)*, 49(3):973–1015, 1999. Symposium à la Mémoire de François Jaeger (Grenoble, 1998).

[38] M. Lotz and J. A. Makowsky. On the algebraic complexity of some families of colored Tutte polynomials. *Adv. Appl. Math.*, 32:327–349, February 2004.

[39] L. Lovasz. Energy of convex sets, shortest paths, and resistance. *J. Comb. Theory Ser. A*, 94:363–382, 2001.

[40] S. B. Maurer. Matrix generalizations of some theorems on trees, cycles and cocycles in graphs. *SIAM J. Appl. Math.*, 30:143–148, 1976.

[41] J. C. Maxwell. *A Treatise on Electricity and Magnetism*, volume 1, Part II, Appendix of Chapter VI, Mathematical Theory of the Distribution of Electric Currents, pages 409–410. Claredon Press and reprinted by Dover, New York (1954), 3rd edition, 1891.

[42] J. C. Maxwell. On the calculation of the equilibrium and stiffness of frames, page 598. In W. D. Niven, editor, *The Scientific Papers of James Clerk Maxwell.* Dover, New York, 1965. from the Philosophical Magazine, Vol. XXVII.

[43] K. Murota. *Matrices and matroids for systems analysis.* Springer, 2000.

[44] A. Nerode and H. Shank. An algebraic proof of Kirchhoff's network theorem. *Amer. Math. Monthly*, 68:244–247, 1961.

[45] J. G. Oxley and D. J. A. Welsh. Tutte polynomials computable in polynomial time. *Discrete Math.*, 109:185–192, 1992.

[46] A. Recski. *Matroid Theory and its Applications in Electric Network Theory and in Statics.* Springer-Verlag, 1989.

[47] C. A. B. Smith. Electric currents in regular matroids. In D. J. A. Welsh and D. R. Woodall, editors, *Combinatorics*, pages 262–284. Institute of Math. and its Appl., Southend-on-Sea, 1972.

[48] A. D. Sokal. The multivariate Tutte polynomial (alias Potts model) for graphs and matroids. Technical report, arXiv.org/abs/math.CO/0503607v1, March 25 2005.

[49] K. Thulasiraman and M. N. S. Swamy. *Graphs: Theory and Algorithms*. Wiley-Interscience, New York, 1992.

[50] L. Trajković and A. N. Willson, Jr. Computer–aided generation of circuit matrices. In *Proc. Modeling and Simulation Conf.*, pages 987–992, Univ. of Pittsburgh, PA, 1986.

[51] W. T. Tutte. A contribution to the theory of dichromatic polynomials. *Canad. J. Math.*, 6:80–91, 1954.

[52] W. T. Tutte. How to draw a graph. *Proc. London. Math. Soc.*, 13:743–767, 1963.

[53] W. T. Tutte. *Graph theory as I have known it*, volume 11 of *Oxford Lecture Series in Mathematics and its Applications*, chapter 1, 4 and 12, pages vi+156. The Clarendon Press Oxford University Press, New York, 1998. With a foreword by U. S. R. Murty.

[54] L. G. Valiant. Completeness classes in algebra. In *Conf. Record 11th Ann. ACM Symp. Theory Computing*, pages 249–261. ACM, 1979.

[55] L. G. Valiant. The complexity of enumeration and reliability problems. *SIAM J. Comput.*, 8(3):410–421, 1979.

[56] D. Vertigan. Counting bases in matroids. Reference in [57] to unpublished ms.

[57] D. Vertigan. Bicycle dimension and special points of the Tutte polynomial. *J. Combin. Theory Ser. B*, 74(2):378–396, 1998.

[58] D. G. Wagner. Negatively correlated random variables and Mason's conjecture. Technical report, arXiv.org/abs/math.CO/0602648v1, Feb. 28 2006.

[59] D. J. A. Welsh. *Complexity: knots, colourings and counting*, volume 186 of *London Math. Soc. Lecture Note Series*. Cambridge Univ. Press, Cambridge, 1993.

[60] D. J. A. Welsh. Randomised approximation schemes for Tutte-Gröthendieck invariants. In *Discrete probability and algorithms (Minneapolis, MN, 1993)*, volume 72 of *IMA Vol. Math. Appl.*, pages 133–148. Springer, New York, 1995.

[61] D. J. A. Welsh. Approximate counting. In R. A. Bailey, editor, *Surveys in Combinatorics*, volume 241 of *London Math. Soc. Lecture Notes*, pages 287–148. Cambridge Univ. Press, Cambridge, 1997.

[62] D. J. A. Welsh and K. K. Kayibi. A linking polynomial of two matroids. *Advances in Appl. Math.*, 32(1–2):391–419, 2004.

[63] N. White, editor. *Theory of Matroids*, volume 26 of *Encyclopedia of mathematics and its applications*. Cambridge Univ. Press, Cambridge, 1986.

[64] N. White, editor. *Combinatorial Geometries*, volume 29 of *Encyclopedia of mathematics and its applications*. Cambridge Univ. Press, Cambridge, 1987.

[65] N. L. White. Geometric applications of the Grassmann-Cayley algebra. In *Handbook of discrete and computational geometry*, Ser. Discrete Math. Appl., pages 881–892. CRC Press, Boca Raton, FL, 1997.

[66] A. D. S. Young-Bin Choe, James G. Oxley and D. G. Wagner. Homogeneous multivariate polynomials with the half-plane property. *Adv. Appl. Math.*, 32:88–187, 2004.

[67] T. Zaslavsky. Strong Tutte functions of matroids and graphs. *Trans. Amer. Math. Soc.*, 334(1):317–347, 1992.