Ported or Relative Oriented Matroids and Electric Circuits

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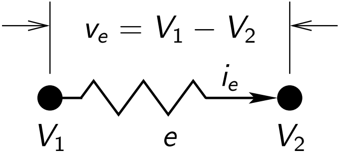
Why Electricity, EE?

- Scholarly topic suggested by G.-C. Rota $\approx$ 1980?.
- $\approx$ 100 yrs. geometry-like intuition of circuit configurations known by engineers, EE books: "Intuitive Analog Circuit Design (2013)" [11]; "Non-linear Circuits" [5] translates to our Oriented Matroid pair model.
- Geometry of linear spaces and oriented matroids; Tutte decomp. w/ techniques from Barnabi, Brini and Rota's Exterior Calculus [1])
- Real behavior $\approx$ ideal plus perturbations, ideal constraints predict intended real behavior,
- Interesting, accessible, intuitively understandable intentential designs, applicable, easy to both simulate and build physically, dimension $\approx$ 12 or 24, depending on formulation
- Analogs to chemical (and real algebraic geometry [8]), biological, mechanical, etc., random walks ...
- Merely one scalar non-linearity can cause chaos.

Kirchhoff (1847) [6] Maxwell (1891) [7] The equivalent resistance problem IS SOLVED by the Matrix Tree Theorem. (1) Let's POSE the problem: the VARIABLES

$\boxed{v_e, i_e : e \in E}$

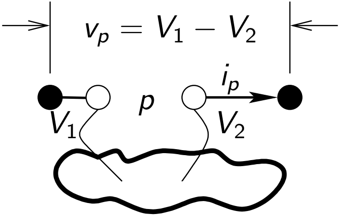


(Use **voltage drops along the flow**, not potentials V_1, V_2.)

ordinary, resistor edge $e \in E$

(IF linear Ohm's law use $|E|$ variables ($g_e v_e = r_e i_e$) ELSE use $2|E|$ variables.)

$\boxed{v_p, i_p : p \in P}$



DISTINGUISHED, PORT edge $p \in P$

The interface to an environment is modelled with $2|P|$ variables.

(math, not EE sign convention)

(2) Let's POSE the problem: EQUATIONS

- (KCL) $(i_e)_{e \in S}$ is a cycle (a flow).
- (KVL) $(v_e)_{e \in S}$ is a cocycle
- (constituitive Law) $i_e = g_e(v_e)$ non-linear, usually monotonic increasing $R \to R$.
  (Sometimes use Ohm's approximation $i_e = g_e v_e$)

Combinatorics!

The signs $\{+, -, 0\}$ have oriented matroid structure (combinatorial, geometric, topological).

SOLUTION: Equiv. Resistance $:\equiv -(v_p/i_p)$ observed at a port p by the environment EQUALS a Ratio of Spanning Tree Enumerators! (Port edge p locates the *2 terminals*.)

$$-\left(\frac{v_p}{i_p}\right) = \frac{\mathrm{WTS}(G/p)}{\mathrm{WTS}(G\backslash p)} = \frac{\text{Matrix-Tree Det}(G/p)}{\text{Matrix-Tree Det}(G\backslash p)}$$

- "Maxwell's rule" uses MatrTreeT on 2 DIFFERENT GRAPHS

  $(G/p$ and $G\backslash p)$

- Weighted Tree Sum (WTS) is a colored Tutte function:

$$\mathrm{WTS}(G') = g_e \mathrm{WTS}(G'/e) + r_e \mathrm{WTS}(G\backslash e) \text{ for all } e \notin P$$

$$\mathrm{WTS}(\mathrm{coloop}(e)) = g_e$$

$$\mathrm{WTS}(\mathrm{loop}(e)) = r_e$$

Benefits of Multiple Ports

- One formula expresses $\binom{2|P|}{|P|}$ different Matrix Tree Theorems...
- ... long vertex-based proofs are shortened; Rayleigh inequalities too.
- Interesting **non-commutative ranges** of new ORIENTED MATROID Tutte invariants with pattern:

  $$\mathrm{TF}(N(P \dot\cup E)) = F(N(P \dot\cup E)/E)$$

  (They distinguish DIFFERENT ORIENTATIONS of the SAME MATROID.)
- Formalize composition of systems [9], Tutte poly. splitting formulas.
- Label variables to observe
- Model practical devices (transistors, op amps)
- Align EE applications with knots (Ported = "Relative") and combinatorial geometry (Ported = "Set Pointed").

Constraint/Generator Duals and 2 Results.

- (Part 1) Technique:

$$\text{Solution Space} = \bigcap \text{Constraint Subspaces}$$

- Result: An exterior algebraic Tutte function: Every Pl ucker coordinate of it satisfies a Matrix Tree Theorem.
  This and det. formulas easily prove Rayleigh inequalities.

- (Part 2) Combine with:

$$\text{Solution Space} = \text{Closure(Set of Generators)}$$

- Result: An oriented matroid pair model for some non-linear problem well-posedness.

Part 1) Coef. Matrix M in CONSTRAINTS $MX = 0$

The Tutte-like function $\mathbf{M}_E(\mathbf{N})$: Extensors $\rightarrow$ Extensors:

Given N (matrix), construct $N^{\perp}$ with orthog. comp. row space.
Construct: $(G = \mathrm{diag}(g_e),\ R = \mathrm{diag}(r_e))$

$$M = \left[\begin{array}{c|c|c} N(P) & 0 & N(E)G \\ \hline 0 & N^{\perp}(P) & N^{\perp}(E)R \end{array}\right]$$

with columns labelled by $P_I \uplus P_V \uplus E$.
Extensor $\mathbf{M}$ over $k[g_e, r_e](P_V \uplus P_I \uplus E)$ is the product of M's **row vectors**. The contraction result $\mathbf{M}_E(\mathbf{N}) = \mathbf{M}/E$ appears:

$$\mathbf{M} = \mathbf{M}_E(\mathbf{N})\mathbf{e_1}\mathbf{e_2}\cdots\mathbf{e}_{|E|} + (\cdots)$$

$\mathbf{M}_E(\mathbf{N})$ is our Tutte function $\mathbf{N} \rightarrow$ Ext. Alg.

(Part 2) Common Covector Model

The cycle space of G_I GENERATES the covectors of an **oriented matroid** over $(E \cupdot P_I)$.

0

(signs indexed by E) (by P_I)

Non-linear monotone resistors CONSTRAIN SIGNS of voltage drops (from ↓) and flows (from ↑) TO BE EQUAL

(signs indexed by E) (by P_V)

The cocycle space of G_V GENERATES the covectors of an oriented matroid over $E \cupdot P_V$.

G_V
SOMETIMES EQUALS
G_I

0

Return to the Part 1 Equation $MX = 0$

$$M = \left[\begin{array}{c|c|c} N(P) & 0 & N(E)G \\ \hline 0 & N^{\perp}(P) & N^{\perp}(E)R \end{array}\right]; \mathbf{M} = \mathbf{M}_E(\mathbf{N})\mathbf{e_1}\mathbf{e_2}\cdots\mathbf{e}_{|E|} + (\cdots)$$

ELIMINATE the variables indexed by E, leaving $2|P|$ variables labelled by P_I and P_V. ie, CONTRACT E. **Answer** A IS:

$$\mathbf{M}_E = \bigwedge_{\text{JOIN over rows}}^{\text{Exterior}} \left[\begin{array}{c|c} A_{I,I} & A_{I,V} \\ \hline A_{V_I} & A_{V,V} \end{array}\right] [\mathbf{p_{I_1}}, \cdots, \mathbf{p_{I_p}}; \mathbf{p_{V_1}}, \cdots, \mathbf{p_{V_p}}]^{\mathbf{t}}$$

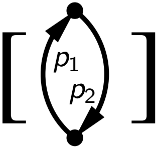
$$= \ldots + C_i\mathbf{XXX} + \ldots; \text{Equiv. Resistance} = \text{certain } \; C_i/C_j$$

All the other C_k's have similar interpretations.
$\binom{2|P|}{|P|}$ **Matr. Tree Theorems:** Each $C_k(N)$ (a PRINCIPAL MINOR of M ABOVE!) $= g_e C_k(N/e) + r_e C_k(N \backslash e)$ ($e \notin P$, e not (co)loop).
Each C_k is a signed weighted enumerator of forests satisfying **conditions ...**

Conditions (what sets F are enumerated by one det. C_i)

The **conditions ...** are on the rank, nullity of F and, WHAT ORIENTED MINOR is $G/F \setminus (E \setminus F)$, the minor with ONLY PORT EDGES from contracting F and deleting the other resistor edges, leaving the ports.
The conditions for a given C_k *sometimes* make all the signs the same (eg: C_i and C_j in 1-port equivalent resistance $R = C_i/C_j$)
Othertimes, the oriented **P-minors** in the completed Tutte decomposition of C_k determine some $+$ and some - signs.

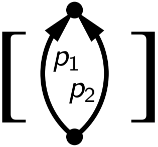
When $[G/F|P]$ is



the term is

$+g_F r_{E\setminus F}$

When $[G/F|P]$ is



the term is

$-g_F r_{E\setminus F}$

Application: Rayleigh Identity, "Neg. Spanning Tree Correlation"

$\Gamma_e(G)$ is equivalent conductance across e. Rayleigh: $0 \leq \dfrac{\partial \Gamma_p}{\partial g_f} = \dfrac{\partial \frac{T_G}{T_{G/e}}}{\partial g_f}$

is equivalent to

$$0 \leq \frac{\partial T_G}{\partial g_f} T_{G/e} - T_G \frac{\partial T_{G/e}}{\partial g_f} = T_{G/f} T_{G/e} - T_G T_{G/e/f}$$

In fact,

$$T_{G/f} T_{G/e} - T_G T_{G/e/f} = \left(T^{+}_{G/e \,\&\, G/f} - T^{-}_{G/e \,\&\, G/f} \right)^2$$

$T^{\pm}_{G/e \,\&\, G/f}$ enumerate the $\pm$ common spanning trees.

Known Partial and Full Combinatorial Proofs

$$T_{G/f}T_{G/e} - T_G T_{G/e/f} = \left(T^{+}_{G/e\ \&\ G/f} - T^{-}_{G/e\ \&\ G/f}\right)^2$$

$T^{\pm}_{G/e\ \&\ G/f}$ enumerate the $\pm$ common spanning trees.

Choe (2004) proved essentially this using the vertex-based all-minors matrix tree theorem, combinatorial cases and Jacobi's theorem relating the minors of a matrix to the minors of its inverse..

Cibulka, Hladky, Lacroix and Wagner (2008) gave a completely bijective proof that utilizes some natural 2:2 and 2:1 correspondances.

Difficulty: Some terms on the left cancel and some reduce to terms with coefficients ± 2.

Linear Alg./Oriented Matroid Proof of Rayleigh's Identity

Let R be the transfer resistance matrix for 2 ports across e and f. Our result implies that

$$\det R = \begin{vmatrix} R_{ee} & R_{ef} \\ R_{fe} & R_{ff} \end{vmatrix} = +\frac{T_{G/e/f}}{T_G}$$

It and better-known results tell us

$$R_{ee} = \frac{T_{G/e}}{T_G}; \quad R_{ff} = \frac{T_{G/f}}{T_G}; \quad R_{ef} = R_{fe} = \frac{T^+_{G/e\ \&\ G/f} - T^-_{G/e\ \&\ G/f}}{T_G}$$

$T_{G/f}\,T_{G/e} - T_G\,T_{G/e/f} = \left(T^+_{G/e\ \&\ G/f} - T^-_{G/e\ \&\ G/f}\right)^2$ is immediate after substituting these into

$$\det R = R_{ee}R_{ff} - (R_{ef})^2$$

The $+$ follows from physical grounds if the $g_e, r_e \geq 0$. Our characterization and proof are combinatorial.

New Rayleigh's Identities!

The same method generates identities and inequalities from

$$\begin{vmatrix} R_{ee} & R_{ef} & R_{eg} \\ R_{fe} & R_{ff} & R_{fg} \\ R_{ge} & R_{gf} & R_{gg} \end{vmatrix} = + \frac{T_{G/e/f/g}}{T_G} \geq 0$$

when all $r_{..}, g_{..} \geq 0$, ETC...

(Applications???)

Might the same methods address a much harder problem: The same inequality for forests instead of spanning trees?

(Part 2) Common Covector Model

The cycle space of G_I GENERATES the covectors of an **oriented matroid** over $(E \mathbin{\dot\cup} P_I)$.

0

(signs indexed by E) (by P_I)

Non-linear monotone resistors CONSTRAIN SIGNS of voltage drops (from ↓) and flows (from ↑) TO BE EQUAL

(signs indexed by E) (by P_V)

The cocycle space of G_V GENERATES the covectors of an oriented matroid over $E \mathbin{\dot\cup} P_V$.

G_V
SOMETIMES EQUALS
G_I

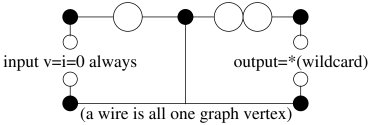
0

Voltage and Current graphs G_V, G_I

"Voltage graph" G_V (EE [5, 10], NOT Gross, ...) represents KVL $\mathbf{v} \in$ Cocycles W/ SOME $v_e \equiv 0$

"Current graph" G_I represents KCL $\mathbf{i} \in$ Cycles WITH SOME FLOWS $\equiv 0$

- They are EQUAL GRAPHS for resistor networks.
- For networks with idealized amplifiers, they are not equal.



  The output voltage and current are whatever makes the input voltage and current BOTH BE zero.
- (More) realistic amp. model = idealized amp. + resistors.

open

$$G_v = G \backslash e$$

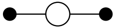
$$G_I = G \backslash e$$

short

$$G_v = G / e$$

$$G_I = G / e$$

nullator



$$G_v = G / e$$

$$G_I = G \backslash e$$

norator

$$G_v = G \backslash e$$

$$G_I = G / e$$

“Colors” are parameters on every Tutte decomposition step

The Bollobos/Riordan/Zaslavsky [2, 12], Traldi-Ellis-Monaghan [4], (sdc unpub) BRZ theory for well-definedness of “Relative Tutte Polynomials for Colored Graphs” ALL GOES THROUGH (Diao and Hetyei [3]): The 3 BRZ conditions on (colors,initial values) GENERALIZE TO 5; activity theory WORKS TOO, when based on linear orders on the non-port-elements.

In a nutshell

The 5 conditions $\Longrightarrow$ activities define an unambiguous Tutte function from the deletion/contraction and initial value formulas. Additional conditions $\Longrightarrow$ the Tutte function has a rank-nullity expansion.
(The rank-nullity conditions are satisfied in our application.)

To specify the activity/deletion-contraction linear order GLOBALLY is UNNECESSARY.

The Gordon/McMahon computation-tree-based activity theory also generalizes. (sdc).

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Our Tutte-like function $\mathbf{M}_E(\mathbf{N})$: Extensors $\to$ Extensors.
Given N (matrix), construct $N^\perp$ with orthog. comp. row space.
Construct: $(G = \mathrm{diag}(g_e),\ R = \mathrm{diag}(r_e))$

$$M = \left[\begin{array}{c|c|c} N(P) & 0 & N(E)G \\ \hline 0 & N^\perp(P) & N^\perp(E)R \end{array}\right]$$

with columns labelled by $P_I \cup\!\!\!\cdot\; P_V \cup\!\!\!\cdot\; E$.
Extensor $\mathbf{M}$ over $k[g_e, r_e](P_V \cup\!\!\!\cdot\; P_I \cup\!\!\!\cdot\; E)$ is the product of M's **row vectors**. The contraction result $\mathbf{M}_E(\mathbf{N}) = \mathbf{M}/E$ appears:

$$\mathbf{M} = \mathbf{M}_E(\mathbf{N})\mathbf{e_1}\mathbf{e_2}\cdots\mathbf{e}_{|E|} + (\cdots)$$

$\mathbf{M}_E(\mathbf{N})$ is our Tutte function $\mathbf{N} \to$ Ext. Alg.

What are the conditions like?

$$\text{Extensor } = \sigma(C)(a_1 \vee a_2 \vee \ldots \vee a_d)$$

$$C = \ldots + t + \ldots$$

$t \leftrightarrow$ sets of (“contractible” non-port edges E_1, E_2 for which

$$N_V/E_1 \backslash (E \setminus E_1) = \text{ certain OMs on } P$$

AND

$$N_I/E_2 \backslash (E \setminus E_2) = \text{ certain OMs on } P$$

Transfer resistance might be 0 and might be $\neq 0$ iff

$$\exists E_1, E_2 \text{so (diag) and (diag)}$$

(K_4 Wheatstone bridge diagram)

$$\frac{R_1}{R_2} > \frac{R_3}{R_4} \text{ neg } \frac{R_1}{R_2} < \frac{R_3}{R_4} \text{ pos}$$

Equivalent resistance is a coefficient ratio in an implicitly defined linear function

(diagram)
In other words

$$R_p i_p + v_p = 0$$

or dually,

$$\mathcal{B} = \{(i_p, v_p)\} = \{t(-1, R_p) | t \in R\}$$

The $2 \times d$ port variable constraint space, and its solution spaces, are d-dimensional.

We represent these spaces by carefully defined **extensors**, as Barnabei, Brini and Rota [1] term "decomposible antisymmetric tensors"
The solution extensor (not a ray) satisfies:

$$E(N) = \mathrm{sign}(...)(g_e E(N/e) + r_e E(N \backslash e) \text{ for } e \notin P$$

Coefs

$$E(N) = \ldots + C_i\mathbf{XXX} + \ldots$$

$$R = C_i/C_j$$

All the other C_k's have similar interpretations.
Each C_k is a determinant.
Each C_k is a signed weighted enumerator of forests satisfying **conditions ...**
Each C_k satisfies

$$C_k(N) = g_e C_k(N/e) + r_e C_k(N\backslash e) \text{ for } e \notin P$$

Conditions

What is the nature of the conditions? We state them using the network's graphic oriented matroid.
(diagram–glob w/ ports)