Resistive Networks, Linear Spaces and Tutte Polynomials for Systems of Lines: Applications of Algebraic Combinatorics Workshop Worcester Polytechnic Institute

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Old hat: A 1-dimensional linear subspace of $\mathbb{R}^n$ **is the line** $\mathbb{R}(a_1, a_2, \ldots a_n)$.
New hat: Explain exterior algebra and Grassmanians in 4 words?
Hint: The rank r grade of the exterior algebra over $\mathbb{R}^n$ has basis the $\binom{n}{r}$ (multilinear, alternating) products of the r coordinate vectors $e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_d}$ with $i_1 < i_2 < \cdots < i_d$.

Linear subspaces are (certain exterior algebra) lines.

extensors code subspaces

The subspace of $\mathbb{R}^n$ spanned by independent vectors $v_1, v_2, \ldots, v_d$ is coded by the **line** $\mathbb{R}(v_{i_1} \wedge v_{i_2} \wedge \ldots \wedge v_{i_d})$.

The row space of an $r \times n$ full rank matrix M is uniquely determined by the **line** of $\mathbb{R}$ multiples of the $\binom{n}{r}$-tuple of determinants $M[T]$ of the $r \times r$ submatrices.
(T is a sequence of r columns.)
Row operations multiply all by a common $\neq 0$ factor.
SPARSENESS:
The space of these lines which represent d-dim spaces is a $n(r-n)$-dimensional non-linear manifold called the Grassmannian.
The $\binom{n}{d}$ coordinates are constrained by the (quadratic) **Grassmann-Plücker relations**:

$$[s_1 s_2 \ldots s_d][t_1 t_2 \ldots t_d] = \sum_{i=1}^{d} [t_i s_2 \ldots s_d][t_1 t_2 \ldots \hat{t}_i s_1 \ldots t_d]$$

(swap column s_1 with column t_i for $i = 1, \ldots, n$)

A basis exchange axiom for matroids

$$[s_1 s_2 \ldots s_d][t_1 t_2 \ldots t_d] = \sum_{i=1}^{d} [t_i s_2 \ldots s_d][t_1 t_2 \ldots \hat{t}_i s_1 \ldots t_d]$$

Axiom

If $B_1 = \{s_1, s_2, \ldots, s_d\}$ and $B_2 = \{t_1, t_2, \ldots, t_d\}$ are bases, then there exists i, $1 \leq i \leq d$ for which both sets $B_1 \setminus s_i \cup t_1 = \{t_1, s_2, \ldots, s_d\}$ and $B_2 \setminus t_1 \cup s_i = \{t_1, t_2, \ldots, \hat{t}_i, s_i, \ldots, t_d\}$ are bases.

A Matroid Definition (one out of dozens)

A matroid coded by its bases is any finite collection of d-sized sets that satisfies the above axiom.

Exercise

Discover (chirotope) axioms that define **oriented matroids** by deriving the necessary conditions on the signs of the brackets (determinants) in the Grassmann-Plücker relation.

Electrical current is a network flow

The ± 1 vertex-edge incidence matrix M of a graph $\mathcal{N}$ fixes an arbitrary direction of each edge.

$$Mi = 0 \iff \{i_e\}_{e\in E} \text{ is a } \textbf{flow} \text{ (of conserved current)}$$

Let's make M full row rank by removing redundant rows.
Let $T \subset E$ be any spanning tree. $T^c = E \setminus T$ is a **cotree**.
Each $\{i_e\}_{e\notin T} \in \mathbb{R}^{T^c}$ extends to a unique flow $\{i_e\}_{e\in E}$.
How? Take the $\{i_e\}_{e\in T}$ to **balance the excess at each vertex**.
That is: Row operations transform M to

$$\left[\ I \ \middle|\ 0, \pm 1s\ \right]$$

if and only if T is a spanning tree.

M **represents** by its **columns** the (graphic) matroid whose bases are the spanning trees.

Electrical network problem: Solve for $i, v \in \mathbb{R}^n$

$$M_V i = 0 \Leftarrow \text{equivalently} \Rightarrow i \in \text{Row space}(M_I), \text{flows or 1-cycles.}$$

$$M_I v = 0 \Leftarrow \text{equivalently} \Rightarrow v \in \text{Row space}(M_V), \text{bonds or 1-cocycles.}$$

$$F(i, v) = 0 \ \text{ locally rank } \ n.$$

First two are Kirchhoff's two laws: Combinatorial, assumed exact in electrical network applications (geometrical for ≥ 1 dim. elastic networks). Total rank $= n$.
Second are the **constitutive** constraints.
Linear one-port network: For all but one edge, **Ohm's law** is written

$$r_e i_e - g_e v_e = 0$$

For the one **port edge** p, demarking a pair of **terminal vertices**, either $i_p = 1$ then solve for $v_p =$ **equivalent resistance** by say eliminating the v_e, i_e for the $n - 1$ resistors.
or $v_p = 1$ then solve for $i_p =$ **equivalent conductance** ... .

Equiv. Resistance $:\equiv -(v_p/i_p)$ observed at a port p

Theorem (Kirchhoff 1847, called "Maxwell's rule")

Let g_T denote $\prod_{e\in T} g_e$, etc.
G/p is G with edge p **contracted** *(vertices identified).*
$G \setminus p$ is G with edge p **deleted**.

$$-\left(\frac{v_p}{i_p}\right) = \frac{\sum_{T:\text{spanning tree of } (G/p)} g_T\, r_{T^c}}{\sum_{T:\text{spanning tree of } (G\backslash p)} g_T\, r_{T^c}} = \frac{\textit{Matrix-Tree Det}(G/p)}{\textit{Matrix-Tree Det}(G\backslash p)}$$

It's usually proved via the Matrix Tree Thm. on 2 DIFFERENT GRAPHS G/p and $G\backslash p$.

Theorem (Matrix Tree)

Every $n-1 \times n-1$ subdeterminant of a graph's Laplacian matrix enumerates the graph's spanning trees.

A liney consequence of Kirchhoff's solution

Pick any (resistor) edge e; factor the sums: $(T^c = E \setminus e \setminus T$ below$)$

$$\frac{\mathrm{WTS}(G/p)}{\mathrm{WTS}(G\backslash p)} =$$

$$\frac{r_e \sum_{T:\text{spanning tree of } ((G\backslash e)/p)} g_T r_{T^c} + g_e \sum_{T:\text{spanning tree of } ((G/e)/p)} g_T r_{T^c}}{r_e \sum_{T:\text{spanning tree of } ((G\backslash e)\backslash p)} g_T r_{T^c} + g_e \sum_{T:\text{spanning tree of } ((G/e)\backslash p)} g_T r_{T^c}}$$

So, when we express our ratio by **the line** of all non-zero $\mathbb{R}$ multiples of $(\mathrm{WTS}(G/p), \mathrm{WTS}(G \setminus p))$, carefully picked generators satisfy **Tutte decomposition**: For "ordinary" $e \neq p$

$$(\mathrm{WTS}(G/p), \mathrm{WTS}(G \setminus p)) =$$

$$r_e(\mathrm{WTS}((G \setminus e)/p), \mathrm{WTS}((G \setminus e) \setminus p)) +$$

$$g_e(\mathrm{WTS}((G/e)/p), \mathrm{WTS}((G/e) \setminus p))$$

Multiport Linear electrical network problem

Edge set $S = E \cup P$. $2|E| + 2|P|$ variables $i, e \in \mathbb{R}^S$. Flow (current) and bond (voltage) eqs. have rank $|E| + |P|$:

$$M_V i = 0 \Leftarrow \text{equivalently} \Rightarrow i \in \text{Row space}(M_I), \text{cycles or flows } \mathcal{C}.$$

$$M_I v = 0 \Leftarrow \text{equivalently} \Rightarrow v \in \text{Row space}(M_V), \text{cocycles } \mathcal{C}^\perp.$$

For the $|E|$ non-port, resistor edges,

$$g_e v_e = r_e i_e$$

So, the linear solution space has dim. $|P|$.
Project solutions into port voltage and current coordinate space $\mathbb{R}^{2|P|}$.

Our exterior algebra valued Tutte function

We've seen how a $|P|$-dim linear space of $\mathbb{R}^{2|P|}$ **is a line**.

Theorem

(After careful definitions...) For fixed P,

each Plücker coordinate	*and*	*this extensor in the exterior algebra, coding the row space of the $p \times 2p$ p-port network solution matrix*

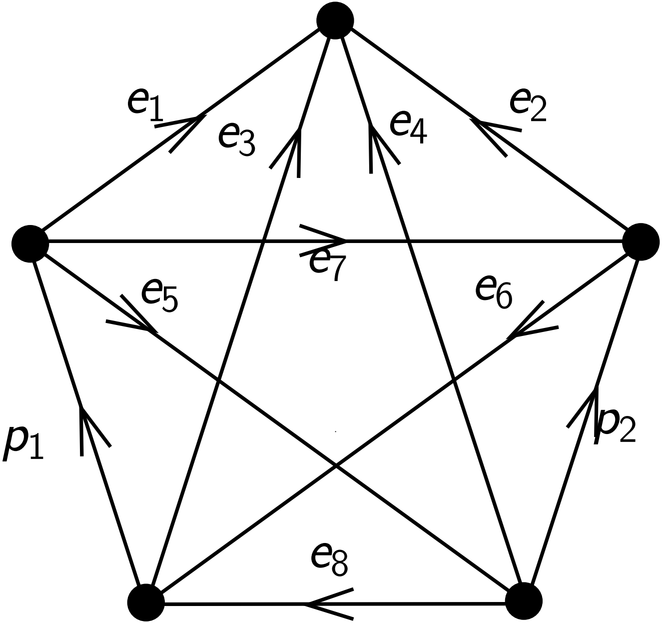
satisfy weighed Tutte recursion, when $/e$ and $\setminus e$ are restricted to $e \notin P$:

$$Sol(G) = r_e Sol(G \setminus e) + g_e Sol(G/e).$$

This result is about **a line of lines**.

As as one of the linear resistance values varies from 0 to ∞, the whole network solution ranges over a suitably defined **line**.

Example



Example

$|E| + |P|$ simplified electrical network equations $Nx = 0$.
Kirchhoff's laws apply to all cycles and cocyles with $r_i x_{e_i}$ as voltage and and $g_i x_{e_i}$ as current of resistor (not port) edges. TWO voltage vp and current ip variables are used for each port edge.

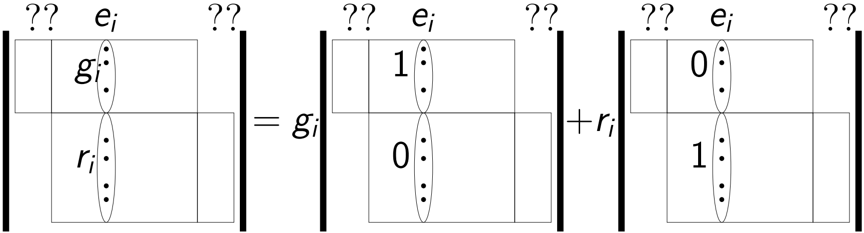
ip_1	ip_2	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	vp_1	vp_2
1	0	0	0	$+g_3$	0	0	$-g_6$	0	$-g_8$		
-1	0	$-g_1$	0	0	0	$+g_5$	0	$+g_7$	0		
0	$+1$	0	0	0	$+g_4$	$-g_5$	0	0	$+g_8$		
0	-1	0	$-g_2$	0	0	0	g_6	$+g_7$	g_8		
		$+r_1$	0	$-r_3$	0	0	0	0	0	1	0
		0	$+r_2$	0	$-r_4$	0	0	0	0	0	1
		$-r_1$	0	0	$+r_4$	$+r_5$	0	0	0	0	0
		0	$-r_2$	$+r_3$	0	0	$+r_6$	0	0	0	0
		$-r_1$	$+r_2$	0	0	0	0	$+r_7$	0	0	0
		0	0	$+r_3$	$+r_4$	0	0	0	$+r_8$	0	0

Top 4 rows: Basis for cocycle space. Represents graphic matroid.
Bot 6 rows: Basis for cycle space. Represents cographic matroid.

How Tutte Decomposition Emerges

For all choices denoted by ?? of the $\binom{2|P|}{|P|}$ size $|P|$ subsets of the $2|P|$ columns $\{ip_k, vp_k\}$, the matrices in the equation below are square.

So the elementary multilinearity of determinants means Tutte decompostion holds for all $e_i \notin P$:



(Technical detail: Define the Tutte function on all graphs with distinguished or port subset P so the det. signs are consistent with the decomposition.)

Non-linear situations

Relaxation oscillator built with a piecewise linear resistor. The lines
The (v, i) locus is contained in two Thevenin theorem (affine) lines. (v, i)
($v = -C \int idt$, circuit diagram)
(v waveform, dynamical system direction field)

Op. amplifier or comparator implementation

(circuit diagram, limit cycle diagram)
Positive feedback implements the some negative linear resistance.
The inductor models parasitic inductance and/or amplifier delay needed to resolve mathematically the dynamics at the **impasse** points.

Oriented matroid (= combinatorics of dependency signs) conditions

Why must the solution of a network with monotone increasing non-linear resistors be unique (Bott and Duffin, 1960's) ?
Immediately for linear resistors only: The Matrix Tree Theorem coefficients are all $+1$.
Suppose there are two solutions. On each resistor's curve there are two points joined by a positive slope line.
(figure with Δv_e and Δi_e)

A better reason:

$\Delta v \in$ linear cycle space $\mathcal{C}$, $\Delta i \in \mathcal{C}^{\perp}$, so $\Delta v \cdot \Delta i = 0$. Monotonicity $\Leftrightarrow$ $\text{sign}(\Delta i_e) = \text{sign}(\Delta v_e) \in \{0, +, -\}$ so $\Delta v \cdot \Delta i > 0$ for non-zero Δv or Δi, which is **impossible**.

What if an amplifier structure makes $\mathcal{C}_I$ and $\mathcal{C}_V$ be non-orthogonal? Uniquess is still guaranteed for there are no $\Delta i \in \mathcal{C}_I$ and $\Delta v \in \mathcal{C}_V$ with $\text{signs}(\Delta i) = \text{signs}(\Delta v) \neq (0, ..., 0)$

An amplifying subsystem simulates a negative resistance; how uniqueness fails

(diagram)

Interesting combinatorial theory emerges when quantities or relations of linear electrical network analysis are expressed as lines or more generally affine linear subspaces.

Our starting point is Thevenin's and Norton's theorems. They conclude that the voltage v_p and current i_p at a pair of terminals are characterized by the affine constraint $av_p + bi_p + c = 0$.

How the load line is used...

Outline

1. Spanning trees and equivalent (linear) resistance.
2. An exterior algebra (extensor) Tutte function and a (linear) resistance network's behavior projected on distinguished coordinates.
3. Rayleigh's inequalities.
4. Tutte polynomials on pairs and (linear) amplifier networks.
5. Distinguished graph vertices and splitting formulas.

Next steps

1. One (terminal-pair) port $\longrightarrow$ set of ports P.
2. 1-dim subspace of homogeneous coordinates of solutions $((v_p, i_p)) \longrightarrow$ p-dim subspace of $k^{2|P|}$.
3. p-dim subspace $\longrightarrow$ EXTENSOR (decomposible exterior algebra, i.e., anti-symmetric tensor) with $\binom{2p}{p}$ Plucker coordinates (determinants).

Rayleigh Identity which $\Rightarrow$ inequality, "Neg. Spanning Tree Correlation"

$\Gamma_e(G)$ is equivalent conductance across e. Rayleigh: $0 \leq \frac{\partial \Gamma_p}{\partial g_f} = \frac{\partial \frac{T_G}{T_{G/e}}}{\partial g_f}$

is equivalent to

$$0 \leq \frac{\partial T_G}{\partial g_f} T_{G/e} - T_G \frac{\partial T_{G/e}}{\partial g_f} = T_{G/f} T_{G/e} - T_G T_{G/e/f}$$

Theorem

$$T_{G/f} T_{G/e} - T_G T_{G/e/f} = \left(T^{+}_{G/e \,\&\, G/f} - T^{-}_{G/e \,\&\, G/f}\right)^2$$

$T^{\pm}_{G/e \,\&\, G/f}$ *enumerate the* $\pm$ *common spanning trees.*

Choe, Cibulka, Hladky, Lacroix and Wagner gave bijective proofs; we give det. based proofs and generalizations.

Linear Alg./Oriented Matroid Proof of Rayleigh's Identity

Let R be the transfer resistance matrix for 2 ports across e and f. Our result implies that

$$\det R = \begin{vmatrix} R_{ee} & R_{ef} \\ R_{fe} & R_{ff} \end{vmatrix} = +\frac{T_{G/e/f}}{T_G}$$

It and better-known results tell us

$$R_{ee} = \frac{T_{G/e}}{T_G}; \quad R_{ff} = \frac{T_{G/f}}{T_G}; \quad R_{ef} = R_{fe} = \frac{T^+_{G/e\ \&\ G/f} - T^-_{G/e\ \&\ G/f}}{T_G}$$

$T_{G/f}T_{G/e} - T_G T_{G/e/f} = \left(T^+_{G/e\ \&\ G/f} - T^-_{G/e\ \&\ G/f}\right)^2$ is immediate after substituting these into

$$\det R = R_{ee}R_{ff} - (R_{ef})^2$$

The $+$ follows from physical grounds if the $g_e, r_e \geq 0$. Our characterization and proof are combinatorial.

New Rayleigh's Identities!

The same method generates identities and inequalities from

$$\begin{vmatrix} R_{ee} & R_{ef} & R_{eg} \\ R_{fe} & R_{ff} & R_{fg} \\ R_{ge} & R_{gf} & R_{gg} \end{vmatrix} = +\frac{T_{G/e/f/g}}{T_G} \geq 0$$

when all $r_{..}, g_{..} \geq 0$, ETC...

(Applications???)

Might the same methods address a much harder problem: The same inequality for forests instead of spanning trees?

Pairs: The Common Covector Model

The cycle space of G_I GENERATES
the covectors of an
oriented matroid over $(E \,\dot\cup\, P_I)$.

0

(signs indexed by E) (by P_I)

Non-linear monotone resistors CONSTRAIN SIGNS of
voltage drops (from ↓) and flows (from ↑)
TO BE EQUAL

(signs indexed by E) (by P_V)

The cocycle space of G_V GENERATES
the covectors of an
oriented matroid over $E \,\dot\cup\, P_V$.

G_V
SOMETIMES EQUALS
G_I

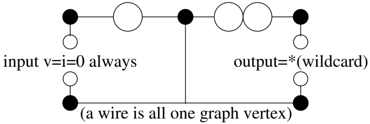
0

Voltage and Current graphs G_V, G_I

"Voltage graph" G_V (EE [8, 13], NOT Gross, ...) represents KVL $\mathbf{v} \in$ Cocycles W/ SOME $v_e \equiv 0$

"Current graph" G_I represents KCL $\mathbf{i} \in$ Cycles WITH SOME FLOWS $\equiv 0$

- They are EQUAL GRAPHS for resistor networks.
- For networks with idealized amplifiers, they are not equal.



The output voltage and current are whatever makes the input voltage and current BOTH BE zero.

- (More) realistic amp. model $=$ idealized amp. $+$ resistors.

open

$$G_v = G \backslash e$$

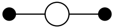
$$G_I = G \backslash e$$

short

$$G_v = G / e$$

$$G_I = G / e$$

nullator



$$G_v = G / e$$

$$G_I = G \backslash e$$

norator

$$G_v = G \backslash e$$

$$G_I = G / e$$

Distinguished graph vertices and splitting formulas

Let Q be a set of distinguished, labelled graph VERTICES, analogous to the distinguished port edges P

Theorem

Given graph $G(V \mathbin{\dot\cup} Q, E \mathbin{\dot\cup} P)$ let $T(G, P, Q)$ be the Tutte polynomial determined by restricting $/e$ and $\backslash e$ to $e \notin P$ AND carrying along the partition of Q defined by the components of the contracted edges.
Construct $G^Q(V \mathbin{\dot\cup} Q, E \mathbin{\dot\cup} P \mathbin{\dot\cup} P_Q)$ by adding to G a new vertex Z and the $|Q|$ new port edges from Z to each vertex in Q.
Then $T(G, P, Q)$ and $T(G^Q, P \mathbin{\dot\cup} P_Q)$ (the ported Tutte polynomial) determine each other by substitutions.

So we can use ported Tutte polys to express splitting formulas for Tutte polynomials of graph, beginning with Crapo [5] and continuing with Andrzejak [1], Bonin and de Meir [4], and Narayanan [12, 14].

etc

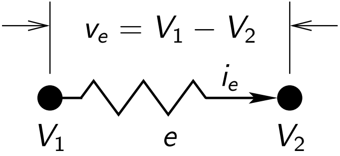
Extra slides...

Why Electricity, EE?

- Scholarly topic suggested by G.-C. Rota $\approx$ 1980?.
- $\approx$ 100 yrs. geometry-like intuition of circuit configurations known by engineers, EE books: "Intuitive Analog Circuit Design (2013)" [15]; "Non-linear Circuits" [8] translates to our Oriented Matroid pair model.
- Geometry of linear spaces and oriented matroids; Tutte decomp. w/ techniques from Barnabi, Brini and Rota's Exterior Calculus [2])
- Real behavior $\approx$ ideal plus perturbations, ideal constraints predict intended real behavior,
- Interesting, accessible, intuitively understandable intentential designs, applicable, easy to both simulate and build physically, dimension $\approx$ 12 or 24, depending on formulation
- Analogs to chemical (and real algebraic geometry [11]), biological, elastic/tensegrity strs. etc., random walks ...
- Merely one scalar non-linearity can cause chaos.

Kirchhoff (1847) [9] Maxwell (1891) [10] The equivalent resistance PROBLEM IS SOLVED by the Matrix Tree Theorem. (1) POSE! the VARIABLES or COORDINATES

$\boxed{v_e, i_e : e \in E}$

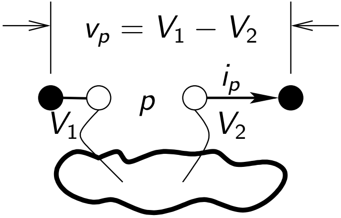


(Use **voltage drops along the flow**, not potentials V_1, V_2.)

ordinary, resistor edge $e \in E$

(IF linear Ohm's law use $|E|$ variables ($g_e v_e = r_e i_e$) ELSE use $2|E|$ variables.)

$\boxed{v_p, i_p : p \in P}$



DISTINGUISHED, PORT edge $p \in P$

The interface to an environment is modelled with $2|P|$ variables.

(math, not EE sign convention)

(2) POSE: EQUATIONS. Preview the consequences.

- (KCL) $(i_e)_{e \in S}$ is a cycle (a flow).
- (KVL) $(v_e)_{e \in S}$ is a cocycle
- (constituitive Law) $i_e = g_e(v_e)$ non-linear, usually monotonic increasing $R \to R$.
  (Sometimes use Ohm's approximation $i_e = g_e v_e$)

Combinatorics!

The signs $\{+, -, 0\}$ have a DUAL-PAIR ORIENTED MATROID structure (combinatorial, geometric, topological).

Engineering with amplifiers!

There's good unique solvablility due to STRUCTURE, when the NON-DUAL PAIR (for voltages and currents) is ALMOST DUAL: No common covectors.

Multiple Ports. (your stereo: 3=power plug & 2 speakers)

- One formula expresses $\binom{2|P|}{|P|}$ different Matrix Tree Theorems...
- ... long vertex-based proofs are shortened; Rayleigh inequalities too.
- Interesting **non-commutative ranges** of new ORIENTED MATROID Tutte invariants with pattern:

  $$\mathrm{TF}(N(P \mathbin{\dot\cup} E)) = F(N(P \mathbin{\dot\cup} E)/E)$$

  (They distinguish DIFFERENT ORIENTATIONS of the SAME MATROID.)
- Ported/Relative OM Tutte Poly. terms embed SPECIFIC MINORS as variables, making proofs just with $\partial T/\partial x_e$ easier.
- Formalize composition of systems [12], Tutte poly. splitting formulas.
- Model practical devices (transistors, op amps); Label variables to observe.
- Align EE applications with knots [6] (Ported = "Relative") and combinatorial geometry [17] (Ported = "Set Pointed").

Constraint/Generator Duals and 2 Results.

- (Part 1) Technique:

$$\text{Solution Space} = \bigcap \text{Constraint Subspaces}$$

- Result: An exterior algebraic algebraic Tutte function: Each of its $\binom{2|P|}{|P|}$ Plücker coordinates satisfies a Matrix Tree Theorem.
  This and det. formulas easily prove Rayleigh inequalities.

- (Part 2) Combine with:

$$\text{Solution Space} = \text{Closure(Set of Generators)}$$

- To apply: An oriented matroid's COVECTOR SET encodes ALL POSSIBLE $(+,-,0)$ coordinate behaviors or δs.
- Result: An oriented matroid pair model for some non-linear problem (AMPLIFIER!) well-posedness. (How? Sign contradictions $\Rightarrow$ a KERNEL$=\{(0)\}$.)

Part 1) Use Matrix M in CONSTRAINTS $MX = 0$ to get...

The Tutte-like function $\mathbf{M}_E()$: Extensor $\mathbf{N} \to$ Extensor $\mathbf{M_E}(\mathbf{N})$.
(STUDENT NOTE: An EXTENSOR represents the row-space of an $r \times s$ $r-$rank matrix M by the $\binom{s}{r}$-TUPLE of the DETERMINANTS of M's $r \times r$ submatrices. Plücker coords.)
Given N (matrix), construct $N^{\perp}$ with orthog. comp. row space.
Construct: $(G = \text{diag}(g_e),\ R = \text{diag}(r_e))$

$$M = \left[\begin{array}{c|c|c} N(P) & 0 & N(E)G \\ \hline 0 & N^{\perp}(P) & N^{\perp}(E)R \end{array}\right]$$

with columns labelled by $P_I \cup P_V \cup E$.
Extensor $\mathbf{M}$ over $k[g_e, r_e](P_V \cup P_I \cup E)$ is the $\wedge$-product of M's **row vectors**. The contraction result $\mathbf{M}_E(\mathbf{N}) = \mathbf{M}/E$ appears:

$$\mathbf{M} = \mathbf{M}_E(\mathbf{N})\mathbf{e_1}\mathbf{e_2}\cdots\mathbf{e}_{|E|} + (\cdots)$$

$\mathbf{M}_E(\mathbf{N})$ is our Tutte function $\mathbf{N} \to$ Ext. Alg.

Contracting means "Eliminate variables"

ELIMINATE the variables indexed by E, leaving $2|P|$ variables labelled by P_I and P_V. ie, CONTRACT E. **Answer $\mathbf{M_E}$ IS:**

$$\mathbf{M}_E = \bigwedge_{\text{JOIN over rows}}^{\text{Exterior}} \left[\begin{array}{c|c} A_{I,I} & A_{I,V} \\ \hline A_{V_I} & A_{V,V} \end{array}\right] [\mathbf{p_{I_1}}, \cdots, \mathbf{p_{I_p}}; \mathbf{p_{V_1}}, \cdots, \mathbf{p_{V_p}}]^{\mathbf{t}}$$

$$= \ldots + C_i\mathbf{XXX} + \ldots; \text{Equiv. Resistance} = \text{certain } \ C_i/C_j$$

All the other C_k's have similar interpretations.
$\binom{2|P|}{|P|}$ **Matr. Tree Theorems:** Each $C_k(N)$ (a PRINCIPAL MINOR of MATRIX A ABOVE!) $= g_e C_k(N/e) + r_e C_k(N \backslash e)$ ($e \notin P$, e not (co)loop).
Each C_k is a signed weighted enumerator of forests satisfying **conditions ...**

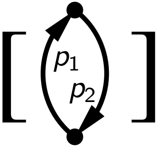
Conditions (what sets F are enumerated by one det. C_i)

The **conditions ...** are on the rank, nullity of F and, WHAT ORIENTED MINOR is $G/F \setminus (E \setminus F)$, the minor with ONLY PORT EDGES from contracting F and deleting the other resistor edges, leaving the ports.

The conditions for a given C_k *sometimes* make all the signs the same (eg: C_i and C_j in 1-port equivalent resistance $R = C_i/C_j$)

Othertimes, the oriented **P-minors** in the completed Tutte decomposition of C_k determine some $+$ and some - signs.

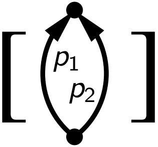
When $[G/F|P]$ is



the term is

$+g_F r_{E \setminus F}$

When $[G/F|P]$ is



the term is

$-g_F r_{E \setminus F}$

Application: Rayleigh Identity, "Neg. Spanning Tree Correlation"

$\Gamma_e(G)$ is equivalent conductance across e. Rayleigh: $0 \leq \dfrac{\partial \Gamma_p}{\partial g_f} = \dfrac{\partial \frac{T_G}{T_{G/e}}}{\partial g_f}$

is equivalent to

$$0 \leq \frac{\partial T_G}{\partial g_f} T_{G/e} - T_G \frac{\partial T_{G/e}}{\partial g_f} = T_{G/f} T_{G/e} - T_G T_{G/e/f}$$

In fact,

$$T_{G/f} T_{G/e} - T_G T_{G/e/f} = \left(T^{+}_{G/e \,\&\, G/f} - T^{-}_{G/e \,\&\, G/f} \right)^2$$

$T^{\pm}_{G/e \,\&\, G/f}$ enumerate the $\pm$ common spanning trees.

Known Partial and Full Combinatorial Proofs

$$T_{G/f}\,T_{G/e} - T_G\,T_{G/e/f} = \left(T^{+}_{G/e\ \&\ G/f} - T^{-}_{G/e\ \&\ G/f}\right)^2$$

$T^{\pm}_{G/e\ \&\ G/f}$ enumerate the $\pm$ common spanning trees.

Choe (2004) proved essentially this using the vertex-based all-minors matrix tree theorem, combinatorial cases and Jacobi's theorem relating the minors of a matrix to the minors of its inverse..

Cibulka, Hladky, Lacroix and Wagner (2008) gave a completely bijective proof that utilizes some natural 2:2 and 2:1 correspondances.

Difficulty: Some terms on the left cancel and some reduce to terms with coefficients ± 2.

“Colors” are parameters on every Tutte decomposition step

The Bollobos/Riordan/Zaslavsky [3, 18], Traldi-Ellis-Monaghan [16, 7], (sdc unpub) BRZ theory for well-definedness of “Relative Tutte Polynomials for Colored Graphs” ALL GOES THROUGH (Diao and Hetyei [6]): The 3 BRZ conditions on (colors,initial values) GENERALIZE TO 5; activity theory WORKS TOO, when based on linear orders on the non-port-elements.

In a nutshell

The 5 conditions $\Longrightarrow$ activities define an unambiguous Tutte function from the deletion/contraction and initial value formulas. Additional conditions $\Longrightarrow$ the Tutte function has a rank-nullity expansion.

(The rank-nullity conditions are satisfied in our application.)

To specify the activity/deletion-contraction linear order GLOBALLY is UNNECESSARY.

The Gordon/McMahon computation-tree-based activity theory also generalizes. (sdc).

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