

Restricted or Ported Tutte Decomposition and Analogues of All-Minors Laplacian Expansions

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What is a parametrized strong Tutte function?

Tutte equations are satisfied in a very general setup:

1. Elements $\{e\}$ each with parameters g_e, r_e .
2. A category \mathcal{N} of objects \mathbf{N} each with ground set $S = S(\mathbf{N})$ of elements.
3. For some *decomposable* \mathbf{N} , for one or more *separators* $e \in S(\mathbf{N})$, the *contraction* and *deletion* operations are defined with results \mathbf{N}/e and $\mathbf{N} \setminus e$ in \mathcal{N} , with ground sets $S(\mathbf{N}) \setminus \{e\}$
4. Some $\mathbf{N} = \mathbf{N}_1 \oplus \mathbf{N}_2$ are direct sums, where $S(\mathbf{N}_1) \cap S(\mathbf{N}_2) = \emptyset$.
5. For each indecomposable \mathbf{N} with no separators there is an additional parameter $i_{\mathbf{N}}$ called the *initial value*.

Tutte equations, functions and Good Questions

1. For all \mathbf{N} with separator $e \in S(\mathbf{N})$,

$$F(\mathbf{N}) = g_e F(\mathbf{N}/e) + r_e(\mathbf{N} \setminus e)$$

2. When $\mathbf{N} = \mathbf{N}_1 \oplus \mathbf{N}_2$,

$$F(\mathbf{N}) = F(\mathbf{N}_1)F(\mathbf{N}_2)$$

3. When \mathbf{N} is indecomposable,

$$F(\mathbf{N}) = i_{\mathbf{N}}$$

F is Tutte function when all the Tutte equations are satisfied.
This MEANS $F(\mathbf{N})$ is what is computed by applying Tutte equations *in any order they are applicable*.

Good Questions: When does \mathcal{N} and parameters ACTUALLY HAVE a Tutte function? If so, what is a *universal* Tutte function?

Some answers—for Graphs and Matroids

Only loops and coloops need initial values

The only \mathbf{N} with no separators and no $\mathbf{N} = \mathbf{N}_1 \oplus \mathbf{N}_2$ for $\mathbf{N}_i \neq \emptyset$ are **loop**(e) and **coloop**(e).

The famous Tutte Polynomial

Adding all $g_e = r_e = 1$, the Tutte polynomial $F(\mathbf{N})(x, y)$ obtained from $i_{\text{loop}(e)} = x$, $i_{\text{coloop}(e)} = y$ and $i_\emptyset = 1$. is a universal Tutte function.

Normal Tutte Functions for Matroids

(Zaslavsky, Bollobás/Riordan) With arbitrary g_e, r_e , and x, y , the *normal* Tutte functions for matroids are obtained with

$i_{\text{coloop}(e)} = g_e y + x$, $i_{\text{loop}(e)} = r_e x + y$ and $i_\emptyset = 1$. They are exactly the ones with a weighted rank-nullity generating function.

There's a big story about what relationships among the $g_e, r_e, i_{\text{coloop}(e)}, i_{\text{loop}(e)}, i_\emptyset$ give others.

Hopf Alg. from Minor Systems (Krajewski, Moffatt, Tanasa 2017)

Definition (Minor System)

- ▶ Finite combinatorial objects $\{N\}$ w/ ground sets $E(N)$, graded by $|E(N)|$; unique 1 with $E(1) = \emptyset$; $E(N)$ consists of objects at level $|E(N)|$.
- ▶ For distinct $e, f \in E(N)$, deletion & contraction ops so both $(\backslash\backslash e$ or $//e)$ commute with both $(\backslash\backslash f$ or $//f)$.

Tutte Functions using determinants: Our setup

- ▶ Matrices N_α, N_β^\perp ; full row rank, columns indexed by $P \amalg E$.
 $\text{rank}(N_\alpha) + \text{rank}(N_\beta^\perp) = |E| + |P|$.
 $P_\alpha, P_\beta \leftrightarrow P, P_\alpha \cap P_\beta = \emptyset$.
- ▶ Weight (parameter) matrices
 $G = \text{diag}\{g_e\}_{e \in E}, R = \text{diag}\{r_e\}_{e \in E}$.
- ▶ Matrix with columns $P_\alpha \amalg P_\beta \amalg E$

$$L = L \left(\begin{array}{c} N_\alpha \\ N_\beta^\perp \end{array} \right) = \left[\begin{array}{c|c|c} N_\alpha(P) & 0 & N_\alpha(E)G \\ \hline 0 & N_\beta^\perp(P) & N_\beta^\perp(E)R \end{array} \right]$$

Define

$$F(L) = \left(\binom{2p}{p} \right) - \text{tuple of determinants } L[Q_\alpha \overline{Q_\beta} E(\text{all of } E)]$$

indexed by length $p = |P|$ sequences $Q_\alpha \overline{Q_\beta} \subseteq P_\alpha P_\beta$ where
 $Q_\alpha \subseteq P_\alpha$ and $\overline{Q_\beta} \subseteq P_\beta$.

Column e of L when $e \notin P$ is

$$\begin{bmatrix} N_{\alpha,1,e}g_e \\ N_{\alpha,2,e}g_e \\ \dots \\ N_{\alpha,r_1,e}g_e \\ N_{\beta,1,e}^\perp r_e \\ N_{\beta,2,e}^\perp r_e \\ \dots \\ N_{\beta,r_2,e}^\perp r_e \end{bmatrix} = \begin{bmatrix} N_{\alpha,1,e} \\ N_{\alpha,2,e} \\ \dots \\ N_{\alpha,r_1,e} \\ 0 \\ 0 \\ \dots \\ 0 \end{bmatrix} g_e + \begin{bmatrix} 0 \\ 0 \\ \dots \\ 0 \\ N_{\beta,1,e}^\perp \\ N_{\beta,2,e}^\perp \\ \dots \\ N_{\beta,r_2,e}^\perp \end{bmatrix} r_e$$

So, for all $e \in E$, that is $e \notin P$:

$$F(L)_{Q_\alpha \overline{Q_\beta}} = L[Q_\alpha \overline{Q_\beta} E] = \\ g_e L \left(\begin{array}{c} N_\alpha / e \\ N_\beta^\perp \setminus e \end{array} \right) [Q_\alpha \overline{Q_\beta} E] + r_e L \left(\begin{array}{c} N_\alpha \setminus e \\ N_\beta^\perp / e \end{array} \right) [Q_\alpha \overline{Q_\beta} E].$$

Since deletion and contraction are done only for $e \notin P$

we get a **Ported** (sdc) or **Set-pointed** (Las Vergnas) or **restricted** (Dao and Hetyei) Tutte Function.

$|Q_\alpha \overline{Q_\beta}| = p$, so $\binom{2p}{p}$ determinants $L[Q_\alpha \overline{Q_\beta} E]$ make the tuple:

$$F(L) = g_e FL \begin{pmatrix} N_\alpha / e \\ N_\beta^\perp \setminus e \end{pmatrix} + r_e FL \begin{pmatrix} N_\alpha \setminus e \\ N_\beta^\perp / e \end{pmatrix}$$

where

N/e means remove the g_e or r_e but otherwise keep column e

$N \setminus e$ means replace column e by 0.

Plücker coordinates

These determinants can be considered an *affine* version of the (projective) Plücker coordinates for the row space of L projected into $K^{P_\alpha} \amalg P_\beta$. We need affine so Tutte's + identity makes sense.

$$FL \left(\begin{array}{c} N_\alpha \\ N_\beta^\perp \end{array} \right) = g_e FL \left(\begin{array}{c} N_\alpha/e \\ N_\beta^\perp \setminus e \end{array} \right) + r_e FL \left(\begin{array}{c} N_\alpha \setminus e \\ N_\beta^\perp/e \end{array} \right) \quad (*)$$

Real deletion/contraction removes e from the ground set of the matroid or other object, but $N/e, N \setminus e$ still have column e . But $(*)$ holds for all $e \in E$, so Laplace's expansion is a basis expansion:

$$L[Q_\alpha \overline{Q_\beta} E] = \sum_{A \subseteq E} g_A r_{\overline{A}} N_\alpha[Q_\alpha A] N_\beta^\perp[\overline{Q_\beta A}] \epsilon(Q_\alpha A, \overline{Q_\beta A})$$

The A term is $\neq 0$ iff $Q_\alpha A$ is a column basis for N_α and $\overline{Q_\beta A}$ is a column basis for N_β^\perp . So, for each $Q_\alpha \overline{Q_\beta}$

$$L[Q_\alpha \overline{Q_\beta} E] = \pm \sum_{A \subseteq E} g_A r_{\overline{A}} N_\alpha[Q_\alpha A] N_\beta^\perp[\overline{Q_\beta A}] \epsilon(A, \overline{A})$$

(The non-zero terms all have $|A| = \text{rank}(N_\alpha) - |Q_\alpha|$.)

Quick and dirty fix

1. Drag column e to the far right.
Changes sign of $F(L)$ by $\epsilon(E'e)$.
2. Left multiply by a determinant 1 matrix that sends the last column to $(0, \dots, 1g_e, 0, \dots, 1r_e)^t$ (if the top or bottom submatrix has just 1 row, do the hack: \mathbf{N}/e is number $\mathbf{N}_{1,e}$ that acts like a matrix with columns E' and no rows.)
3. Drag the row with the $1g_e$ to the bottom.
Changes sign of $F(L)$ by $(-1)^{r\mathbf{N}_\beta^\perp}$
4. With e deleted/contracted from the \mathbf{N} s defining L , define F by $FL_{Q_\alpha \overline{Q_\beta}} = L[Q_\alpha \overline{Q_\beta} E']$

Result

$$FL \left(\begin{array}{c} N_\alpha \\ N_\beta^\perp \end{array} \right) = \epsilon(E'e) \left(g_e (-1)^{r(N_\beta^\perp)} FL \left(\begin{array}{c} N_\alpha/e \\ N_\beta^\perp/e \end{array} \right) + r_e FL \left(\begin{array}{c} N_{\alpha \setminus e} \\ N_{\beta^\perp \setminus e} \end{array} \right) \right)$$

Simplify calculations /w minors via Exterior Algebra

Full r -row minors of matrix N with columns indexed by S :

$$\begin{array}{ccc} (e_1) & (e_2) & (e_3) \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ \hline N[e_1 e_3] = (a_1 b_3 - a_3 b_1) \end{array}$$

Coefficients when the exterior product of N 's row vectors \mathbf{N} are expressed in basis

$\{\mathbf{e}_{i_1} \wedge \mathbf{e}_{i_2} \cdots \mathbf{e}_{i_r} \mid i_1 < i_2 < \cdots < i_r\}$:

$$\begin{array}{c} (a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 + a_3 \mathbf{e}_3) \\ \wedge (b_1 \mathbf{e}_1 + b_2 \mathbf{e}_2 + b_3 \mathbf{e}_3) \\ \hline ((a_1 b_3 - a_3 b_1) \mathbf{e}_1 \mathbf{e}_3 + \cdots) \end{array}$$

We sometimes omit the \wedge and we can always write:

$$(\text{Exterior product})\mathbf{N} = \sum_{A \subseteq S; |E|=r} \mathbf{N}[A]\mathbf{A}$$

Each subset A is ordered $a_1 a_2 \dots a_r$ **arbitrarily** but \mathbf{A} denotes the exterior product of (row coordinate vectors) **in the same order**

$$\mathbf{A} = \mathbf{a}_1 \mathbf{a}_2 \dots \mathbf{a}_r$$

Catalogs of Oriented Matroid operations on the OM of matrix N and on $\mathbf{N} = \wedge(\text{rows}(N))$

domain D of operations:

D is which functions:

type of fun. value

chirotopes

$$\pm\chi : B \mapsto \text{sign}(N[B])$$

$$\text{sign} \in \{0, +, -\}$$

exterior products

$$\mathbf{N} : B \mapsto \mathbf{N}[B]$$

field value; number

deletion $\bullet \setminus A$

$$\pm\chi' : B \mapsto \chi(B)$$

$$\mathbf{N} \setminus A : B \mapsto \mathbf{N}[B]$$

contraction \bullet / A

$$\pm\chi' : B \mapsto \chi(BA)$$

$$\mathbf{N} / A : B \mapsto \mathbf{N}[BA]$$

duality \bullet^\perp

$$\pm\chi^\perp : B \mapsto \chi(\overline{B})\epsilon(\overline{B}B)$$

$$\mathbf{N}^\perp : B \mapsto \mathbf{N}[\overline{B}]\epsilon(\overline{B}B)$$

We must choose some global orientation ϵ in order to define duality as an exterior alg. operation!

ϵ is an alternating sign function on all finite sequences of elements.

This implies
commutations

$$(\mathbf{N} \setminus X)^\perp = \epsilon(S')\epsilon(S'X)(\mathbf{N}^\perp / X)$$

$$(\mathbf{N} / X)^\perp = \epsilon(S')\epsilon(S'X)(-1)^{|X|r}\mathbf{N}^\perp(\mathbf{N}^\perp \setminus X)$$

Our setup - again

- ▶ Matrices N_α, N_β^\perp ; full row rank, columns indexed by $P \coprod E$.
 $\text{rank}(N_\alpha) + \text{rank}(N_\beta^\perp) = |E| + |P|$.
 $P_\alpha, P_\beta \leftrightarrow P, P_\alpha \cap P_\beta = \emptyset$.
- ▶ Weight (parameter) matrices
 $G = \text{diag}\{g_e\}_{e \in E}, R = \text{diag}\{r_e\}_{e \in E}$.
- ▶ Matrix with columns $P_\alpha \coprod P_\beta \coprod E$

$$L \left(\begin{array}{c} N_\alpha \\ N_\beta^\perp \end{array} \right) = \left[\begin{array}{c|c|c} N_\alpha(P) & 0 & N_\alpha(E)G \\ \hline 0 & N_\beta^\perp(P) & N_\beta^\perp(E)R \end{array} \right]$$

Define

$$F(L) = \left(\binom{2p}{p} \right) - \text{tuple of determinants } L[Q_\alpha \overline{Q_\beta} E]$$

indexed by sequences $Q_\alpha \overline{Q_\beta} \subseteq P_\alpha P_\beta$ where $Q_\alpha \subseteq P_\alpha$,
 $\overline{Q_\beta} \subseteq P_\beta, |Q_\alpha \overline{Q_\beta}| = p = |P|$.

$$L \left(\begin{array}{c} N_\alpha \\ N_\beta^\perp \end{array} \right) = \left[\begin{array}{c|c|c} N_\alpha(P) & 0 & N_\alpha(E)G \\ \hline 0 & N_\beta^\perp(P) & N_\beta^\perp(E)R \end{array} \right] \quad F(L) = \text{tuple } (L[Q_\alpha \overline{Q_\beta} E])$$

Translate into exterior algebra definitions:

$$\begin{aligned} \mathbf{L} \left(\begin{array}{c} \mathbf{N}_\alpha \\ \mathbf{N}_\beta^\perp \end{array} \right) &:= (\iota(\mathbf{N}_\alpha)(P_\alpha) + \iota_G(\mathbf{N}_\alpha(E))) \wedge (v(\mathbf{N}_\beta^\perp)(P_\beta) + v_R(\mathbf{N}_\beta^\perp)(E)) \\ &= (\iota_G(\mathbf{N}_\alpha) \wedge v_R(\mathbf{N}_\beta^\perp)) \end{aligned}$$

$$\mathbf{F}_E(\mathbf{L}) := \mathbf{L}/E = \sum_{Q_\alpha, \overline{Q_\beta}} \mathbf{L}[Q_\alpha \overline{Q_\beta} E] \mathbf{Q}_\alpha \overline{\mathbf{Q}_\beta}$$

$$\begin{aligned} &= ((\iota(\mathbf{N}_\alpha) \setminus e(\text{no } \mathbf{e}) + g_e(\iota(\mathbf{N}_\alpha)/e) \wedge \mathbf{e}) \\ &\quad \wedge (v(\mathbf{N}_\beta^\perp) \setminus e(\text{no } \mathbf{e}) + r_e(v(\mathbf{N}_\beta^\perp)/e) \wedge \mathbf{e}))/E \end{aligned}$$

$$\text{2 of 4 terms} = \left(r_e \quad \iota(\mathbf{N}_\alpha) \setminus e \wedge (v(\mathbf{N}_\beta^\perp)/e) \wedge \mathbf{e} \right.$$

$$\text{vanish} \quad \left. + g_e(-1)^{r(\mathbf{N}_\beta^\perp)} (\iota(\mathbf{N}_\alpha)/e) \wedge (v(\mathbf{N}_\beta^\perp) \setminus e) \wedge \mathbf{e} \right) / E$$

$$L \left(\begin{array}{c} N_\alpha \\ N_\beta^\perp \end{array} \right) = \left[\begin{array}{c|c|c} N_\alpha(P) & 0 & N_\alpha(E)G \\ \hline 0 & N_\beta^\perp(P) & N_\beta^\perp(E)R \end{array} \right] \quad F(L) = \text{tuple } (L[Q_\alpha \overline{Q_\beta} E])$$

$$\begin{aligned} \mathbf{F}_E(\mathbf{L}) = \mathbf{L}/E = & \left(r_e \quad \iota(\mathbf{N}_\alpha \setminus e) \wedge (v(\mathbf{N}_\beta^\perp/e)) \wedge \mathbf{e} \right. \\ & \left. + g_e(-1)^{r(\mathbf{N}_\beta^\perp)} (\iota(\mathbf{N}_\alpha/e)) \wedge (v(\mathbf{N}_\beta^\perp \setminus e)) \wedge \mathbf{e} \right) / E \end{aligned}$$

$$= r_e \left(\mathbf{L} \left(\begin{array}{c} \mathbf{N}_\alpha \setminus e \\ \mathbf{N}_\beta^\perp/e \end{array} \right) \wedge \mathbf{e}/E \right) + g_e(-1)^{r(\mathbf{N}_\beta^\perp)} \left(\mathbf{L} \left(\begin{array}{c} \mathbf{N}_\alpha/e \\ \mathbf{N}_\beta^\perp \setminus e \end{array} \right) \wedge \mathbf{e}/E \right)$$

$$(\mathbf{N} \setminus e)^\perp = \epsilon(S')\epsilon(S'e)(\mathbf{N}^\perp/e) ; (\mathbf{N}/e)^\perp = \epsilon(S')\epsilon(S'e)(-1)^{|\{e\}|r\mathbf{N}^\perp}(\mathbf{N}^\perp \setminus e)$$

Result

$$= \epsilon(S)\epsilon(S'e) \left(r_e \left(\mathbf{L} \left(\begin{array}{c} \mathbf{N}_\alpha \setminus e \\ (\mathbf{N}_\beta \setminus e)^\perp \end{array} \right) \wedge \mathbf{e}/E \right) + g_e \left(\mathbf{L} \left(\begin{array}{c} \mathbf{N}_\alpha/e \\ (\mathbf{N}_\beta/e)^\perp \end{array} \right) \wedge \mathbf{e}/E \right) \right)$$

With $\mathbf{L}(\mathbf{N}_\alpha \ \mathbf{N}_\beta) = \mathbf{L} \left(\begin{array}{c} \mathbf{N}_\alpha \\ \mathbf{N}_\beta^\perp \end{array} \right)$, and more sign calculations:

Definition

For E, P sets written as ordered sequences,

$$\mathbf{F}_E(\mathbf{N}_\alpha \ \mathbf{N}_\beta) = \mathbf{L}(\mathbf{N}_\alpha \ \mathbf{N}_\beta)/E$$

Theorem

$$\begin{aligned} \epsilon(PE)\mathbf{F}_E(\mathbf{N}_\alpha \ \mathbf{N}_\beta) = \\ \epsilon(PE')(g_e\mathbf{F}_{E'}(\mathbf{N}_\alpha/e \ \mathbf{N}_\beta/e) + r_e\mathbf{F}_{E'}(\mathbf{N}_\alpha \setminus e \ \mathbf{N}_\beta \setminus e)) \end{aligned}$$

Corollary

1. $\mathbf{F} = \mathbf{F}_E(\mathbf{N}_\alpha \mid \mathbf{N}_\beta) = \pm \sum_{H \subseteq E} g_H r_{\overline{H}} \mathbf{F}_\emptyset(\mathbf{N}_\alpha / H \setminus \overline{H} \mid \mathbf{N}_\beta / H \setminus \overline{H})$
2. Componentwise, $\sum_{Q_\alpha, Q_\beta} \mathbf{F}_E[Q_\alpha \overline{Q}_\beta] Q_\alpha \overline{Q}_\beta =$

$$= \pm \sum_{Q_\alpha, Q_\beta} \sum_{H \in E} g_H r_{\overline{H}} \mathbf{N}_\alpha[Q_\alpha H] \mathbf{N}_\beta^\perp[\overline{Q}_\beta \overline{H}]$$

$$= \pm \sum_{Q_\alpha, Q_\beta} \sum_{H \in E} g_H r_{\overline{H}} \mathbf{N}_\alpha[Q_\alpha H] \mathbf{N}_\beta[Q_\beta H]$$

3. Two expr. for products of numbers $\mathbf{N}_\alpha[Q_\alpha H] \mathbf{N}_\beta[Q_\beta H]$:

$$(\mathbf{N}_\alpha / Q_\alpha)[H] \cdot (\mathbf{N}_\beta / Q_\beta)[H] = (\mathbf{N}_\alpha / H)[Q_\alpha] \cdot (\mathbf{N}_\beta / H)[Q_\beta]$$

4. It's non-zero iff H is a common basis (in the matroids of) $\mathbf{N}_\alpha / Q_\alpha$ and $\mathbf{N}_\beta / Q_\beta$
 iff Q_α is a basis in \mathbf{N}_α / H and Q_β is a basis in \mathbf{N}_β / H

Weighted Laplacian-like matrices

Generalize a graph's incidence matrix: Make P label the rows, E the columns of any matrices A_α, A_β . Take all $r_e \neq 0$. Then, $N_\alpha = (I(P) \ A_\alpha(E))$ and $N_\beta = (I(P) \ A_\beta(E))$, and

$$L \begin{pmatrix} N_\alpha \\ N_\beta^\perp \end{pmatrix} = \left[\begin{array}{c|c|c} I & 0 & A_\alpha G \\ \hline 0 & -A_\beta^t & IR \end{array} \right] = L(N_\alpha \ N_\beta). \text{ Do row ops:}$$

$$\begin{pmatrix} I & -A_\alpha GR^{-1} \\ 0 & R^{-1} \end{pmatrix} L = \begin{pmatrix} I & A_\alpha GR^{-1} A_\beta^t & 0 \\ 0 & -R^{-1} A_\beta^t & I \end{pmatrix}, \text{ and therefore}$$

$$\epsilon(Q_\alpha \overline{Q_\alpha}) \mathbf{F}_E(\mathbf{L})[Q_\alpha \overline{Q_\beta}] = \frac{1}{r_E} \sum_{B \in E} g_B r_{\overline{B}} A_\alpha[\overline{Q_\alpha} B] A_\beta[\overline{Q_\beta} B]$$

is the Cauchy-Binet expansion of any minor $(\overline{Q_\alpha}, \overline{Q_\beta})$ of the weighted graph Laplacian-like matrix $A_\alpha GR^{-1} A_\beta^t$.

(Note $\frac{1}{r_E} r_{\overline{B}} = (r^{-1})_B$.)

Examples

$N_\alpha = N_\beta = N$; $A =$ graph's incidence matrix w/ columns $(0, \dots, 0, 1, 0, \dots, -1, 0, \dots, 0)^t$ for each edge; reps. graphic matroid.

A_α, N_α as above. $A_\beta =$ only the $+1$ entries of A for a directed graph, so $+1$ is for an edge head on a vertex.

$N_\alpha = N_\beta = N$; $A =$ gain graph's incidence matrix w/ columns $(0, \dots, 0, 1, 0, \dots, -\gamma_e, 0, \dots, 0)^t$ for e with gain $\gamma_e \in \mathbf{C}$.

NB: Edge Gains γ_e are DIFFERENT ATTRIBUTES from weights/parameters g_e, r_e

The all-minors
Matrix Tree Theorem
for weighted undirected graphs

The all-minors
Matrix Tree Theorem
for weighted directed graphs

All-minors expansions of
the gain graph's Laplacian

Known to EEs: Linear electrical networks with IDEAL AMPLIFIERS

$N_\alpha i(P, E) = 0$ expresses Kirchhoff's current law on currents i_e in the network edges (along edge direction) and currents i_p into vertices from external connections.

$N_\beta^\perp v(P, E) = 0$ expresses Kirchhoff's voltage law: The voltage rise along a network edge $v_e = v_h - v_t$ is the difference of the head and tail vertex potentials. (Sometimes the vertex potentials are imposed by external connections.)

$N_\alpha = N_\beta$ in ordinary resistor networks.

Different Graphs for N_α and N_β

W. K. Chen models networks with ideal amplifiers by N_α by one graph on (P, E) called the **Current Graph** and another graph also on (P, E) called the **Voltage Graph**.