An Exterior Algebra Valued Tutte Function on Linear Matroids or their Pairs

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Summary

1. Usual parametrized Tutte functions F are valued in comm. rings.
2. Matrix Tree Theorem: The tree enumerator Tutte function is a determinant.
3. Our generalization of the represented matroid basis enumerating determinant is a restricted Tutte function valued in exterior algebras (ie., anti-symmetric tensor spaces.)
4. Restricted (against set P) aka set P pointed, P-"ported" F
$$F(M,P) = r_e F(M \setminus e) + g_e F(M/e)$$
only when non-loop non-coloop $e \notin P$.
5. P will play the role of graph vertices.
6.
$$F \rightarrow$$

Catalogs of Oriented Matroid operations on $\mathbf{OM}(N)$ of matrix N and on $\mathbf{N} = \wedge(\mathbf{rows}(N))$

Op is on:	chirotopes	exterior products
which are:	$\chi : B \rightarrow \{0, \pm\}$	decomposibles in $\wedge$
case we use:	$\chi : B \mapsto \text{sign}(N[B])$	$\mathbf{N} : B \mapsto \mathbf{N}[B]$
OPERATION		
deletion $\bullet \backslash A$	restriction	restriction
contraction $\bullet /A$	$\pm\chi' : B \mapsto \chi(BA)$	$\mathbf{N}/A : B \mapsto \mathbf{N}[BA]$
duality $\bullet^{\perp}$	$\pm\chi^{\perp} : B \mapsto \chi(\overline{B})\epsilon(\overline{B}B)$	$\mathbf{N}^{\perp} : B \mapsto \mathbf{N}[\overline{B}]\epsilon(\overline{B}B)$

We must choose some global orientation ϵ in order to define duality as an exterior alg. operation! ϵ is an alternating sign function on all finite sequences of elements. This implies commutations

$$(\mathbf{N}\backslash X)^{\perp} = \epsilon(S')\epsilon(S'X)(\mathbf{N}^{\perp}/X)$$
$$(\mathbf{N}/X)^{\perp} = \epsilon(S')\epsilon(S'X)(-1)^{|X|r\mathbf{N}^{\perp}}(\mathbf{N}^{\perp}\backslash X)$$

Setup and Theorem

- Matrices N_α, $N_\beta^\perp$; full row rank, columns indexed by $P \coprod E$. $\text{rank}(N_\alpha) + \text{rank}(N_\beta^\perp) = |E| + |P|$.
  $P_\alpha, P_\beta \leftrightarrow P$, $P_\alpha \cap P_\beta = \emptyset$.
- Weight (parameter) matrices
  $G = \text{diag}\{g_e\}_{e\in E}$, $R = \text{diag}\{r_e\}_{e\in E}$.
- Matrix with columns $P_\alpha \coprod P_2 \coprod E$

$$L\begin{pmatrix} N_\alpha \\ N_\beta^\perp \end{pmatrix} = \left[\begin{array}{c|c|c} N_\alpha(P) & 0 & N_\alpha(E)G \\ \hline 0 & N_\beta^\perp(P) & N_\beta^\perp(E)R \end{array}\right]$$

Define

$$F(L) = ((\binom{2p}{p})-\text{tuple of determinants } L[Q_\alpha\overline{Q_\beta}E])$$

indexed by sequences $Q_\alpha\overline{Q_\beta} \subseteq P_\alpha P_\beta$ where $Q_\alpha \subseteq P_\alpha$, $\overline{Q_\beta} \subseteq P_\beta$, $|Q_\alpha\overline{Q_\beta}| = p = |P|$.

Translate into exterior algebra definitions:

$$\mathbf{L}\begin{pmatrix} \mathbf{N}_\alpha \\ \mathbf{N}_\beta^\perp \end{pmatrix} := (\iota(\mathbf{N}_\alpha)(P_\alpha) + \iota_G(\mathbf{N}_\alpha(E))) \wedge (\upsilon(\mathbf{N}_\beta^\perp)(P_\beta) + \upsilon_R(\mathbf{N}_\beta^\perp)(E))$$
$$= (\iota_G(\mathbf{N}_\alpha) \wedge \upsilon_R(\mathbf{N}_\beta^\perp))$$

$$\mathbf{F}_E(\mathbf{L}) := \mathbf{L}/E = \sum_{Q_\alpha,\overline{Q_\beta}} \mathbf{L}[Q_\alpha\overline{Q_\beta}E]\mathbf{Q}_\alpha\overline{\mathbf{Q}_\beta}$$
$$= ((\iota(\mathbf{N}_\alpha)\backslash e(\text{no } \mathbf{e}) + g_e(\iota(\mathbf{N}_\alpha)/e) \wedge \mathbf{e})$$
$$\wedge (\upsilon(\mathbf{N}_\beta^\perp)\backslash e(\text{no } \mathbf{e}) + r_e(\upsilon(\mathbf{N}_\beta^\perp)/e) \wedge \mathbf{e}))/E$$
$$\text{2 of 4 terms} = \Big(r_e \qquad \iota(\mathbf{N}_\alpha)\backslash e \wedge (\upsilon(\mathbf{N}_\beta^\perp)/e) \wedge \mathbf{e}$$
$$\text{vanish} \quad + g_e(-1)^{r(\mathbf{N}_\beta^\perp)}(\iota(\mathbf{N}_\alpha)/e) \wedge (\upsilon(\mathbf{N}_\beta^\perp)\backslash e) \wedge \mathbf{e}\Big)/E$$

$$\mathbf{F}_E(\mathbf{L}) = \mathbf{L}/E = \Big(r_e \qquad \iota(\mathbf{N}_\alpha\backslash e) \wedge (\upsilon(\mathbf{N}_\beta^\perp/e)) \wedge \mathbf{e}$$
$$+g_e(-1)^{r(\mathbf{N}_\beta^\perp)}(\iota(\mathbf{N}_\alpha/e)) \wedge (\upsilon(\mathbf{N}_\beta^\perp\backslash e)) \wedge \mathbf{e}\Big)/E$$
$$= r_e\left(\mathbf{L}\begin{pmatrix} \mathbf{N}_\alpha\backslash e \\ \mathbf{N}_\beta^\perp/e \end{pmatrix} \wedge \mathbf{e}/E\right) + g_e(-1)^{r(\mathbf{N}_\beta^\perp)}\left(\mathbf{L}\begin{pmatrix} \mathbf{N}_\alpha/e \\ \mathbf{N}_\beta^\perp\backslash e \end{pmatrix} \wedge \mathbf{e}/E\right)$$

$(\mathbf{N}\backslash e)^\perp = \epsilon(S')\epsilon(S'e)(\mathbf{N}^\perp/e)$;
$(\mathbf{N}/e)^\perp = \epsilon(S')\epsilon(S'e)(-1)^{|\{e\}|r\mathbf{N}^\perp}(\mathbf{N}^\perp\backslash e)$

$$= \epsilon(S)\epsilon(S'e)(r_e\left(\mathbf{L}\begin{pmatrix} \mathbf{N}_\alpha\backslash e \\ (\mathbf{N}_\beta\backslash e)^\perp \end{pmatrix} \wedge \mathbf{e}/E\right)+$$
$$g_e\left(\mathbf{L}\begin{pmatrix} \mathbf{N}_\alpha/e \\ (\mathbf{N}_\beta/e)^\perp \end{pmatrix} \wedge \mathbf{e}/E\right))$$

With $\mathbf{L}(\mathbf{N}_\alpha \ \ \mathbf{N}_\beta) = \mathbf{L}\begin{pmatrix} \mathbf{N}_\alpha \\ \mathbf{N}_\beta^\perp \end{pmatrix}$, and more sign calculations:

Definition 1. *For E, P sets written as ordered sequences,*

$$\mathbf{F}_E(\mathbf{N}_\alpha \ \ \mathbf{N}_\beta) = \mathbf{L}(\mathbf{N}_\alpha \ \ \mathbf{N}_\beta)/E$$

Theorem 1.

$$\epsilon(PE)\mathbf{F}_E(\mathbf{N}_\alpha \ \ \mathbf{N}_\beta) = \epsilon(PE')\left(g_e\mathbf{F}_{E'}(\mathbf{N}_\alpha/e \ \ \mathbf{N}_\beta/e) + r_e\mathbf{F}_{E'}(\mathbf{N}_\alpha\backslash e \ \ \mathbf{N}_\beta\backslash e)\right)$$