RELIABILITY IN PORTED GRAPHS

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Abstract.

1. Introduction

In this note we propose an extension to the graph and matroid theoretic model for network reliability studied by Colbourn and others. The classical model characterizes $R(G)$, the reliability polynomial in p of graph G, by

$$R(G) = pR(G/e) + (1-p)R(G \setminus e) \ \text{ for } e \in E(G), \tag{1}$$

and

$$R(G) = \begin{cases} 1 & \text{if } |E(G)| = 0 \text{ and } |V(G)| = 1 \\ 0 & \text{if } |E(G)| = 0 \text{ and } |V(G)| > 1. \end{cases} \tag{2}$$

Here, $E = E(G)$ is the edge set, $V(G)$ is the vertex set, and $G/e = G \setminus e$ if $e \in E$ is a loop. It is immediate that $R(G)$ equals the probability that the random subgraph $(V(G), A)$ is path-connected when the edges $A \subseteq E$ are chosen each independently with probability p, $0 \le p \le 1$. (A is often called the *state*.) Indeed,

$$R(G) = \sum_{A \subseteq E} p^{|A|}(1-p)^{|E|-|A|}\chi(G, A) \tag{3}$$

where $\chi(G, A)$ is 1 if $(V(G), A)$ is a path-connected graph and 0 otherwise. $\chi(G, A) = 1$ when $G/A\backslash(E\backslash A)$ is a single vertex graph with no edges and $\chi(G, A) = 0$ when $G/A\backslash(E\backslash A)$ is a multiple vertex graph with no edges.

Our extension is to introduce a set P of distinguished edges (perhaps more precisely, edge names) so that equation (1) is restricted to apply only to $e \notin P$. We call P the *set of ports*. For convenience, we use $E(G)$ for the edges of G other than ports, and $S(G)$ for all the edges of G. Then, to unambigously define $R(G)$, we must combine (1) with *initial values* $I(Q)$, where each Q is a minor of G obtained by contracting some $A \subseteq E$ and deleting $E \setminus A$, leaving a multigraph whose edges all belong to P. The vertices of G are unlabelled, as are the the vertices of Q. The extension of (3) is

$$R(G) = \sum_{A \subseteq E} p^{|A|}(1-p)^{|E|-|A|} I(G/A|P), \tag{4}$$

where $G/A|P$ abbreviates $G/A \setminus (E \setminus A)$. When $0 \le I(G/A|P) \le 1$ is interpreted as a probability, we can discuss a probabilistic significance to our ported $R(G)$ value. However, after we give motivations, we will discuss another probablistic interpretation in addition to this one.

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2. Enter Matroid Theory

Assume for now $P = \emptyset$. It is impractical to compute $R(G)$, either as a polynomial or number for given p, even for small graphs of interest in practical applications. Upper and lower bounds on the coefficients are therefore desired because, for example,

$$\sum L_k p^k \leq R(G) \leq \sum U_k p^k$$

when

$$L_k \leq c_k \leq U_k \text{ and } R(G) = \sum c_k p^k.$$

For the same reason, bounds on the coefficients when $R(G)$ is expanded about numbers besides 0 are useful, as will as other information about coefficient behavior. Matroid theory has provided such information.

When G is a (non-empty) disconnected graph, $R(G) = 0$. Otherwise, $R(G)$ is related to $G's$ graphic matroid, specifically through its Tutte polynomial $T(M(G); X, Y)$ via

$$R(G) = p^{n-1}(1-p)^{m-n+1} T(M(G); 1, \frac{1}{1-p}). \tag{5}$$

where $m = |E(G)|$ and $n = |V(G)|$. Combinatorial theory of Tutte polynomial of matroids, and specifically graphic matroids, has provided information useful for approximating the coefficients in several expansions and other analyses.

The basics of Tutte polynomial theory has been extended to P-ported and P-ported parametrized Tutte functions, where restrictions like we gave already on reducing elements in P are applied to Tutte equations for matroids.

The goal of our project is to formulate ported extensions of the theory that has been successful in helping to approximate the non-ported reliability polynomial.

2.1. **Probability Variables.** Suppose a separate probability variable $0 \leq p_e \leq 1$ is given for each $e \in E$. Take $q_e = 1 - p_e$. Then (1) becomes

$$R(G) = p_e R(G/e) + q_e R(G \setminus e) \text{ for } e \in E(G), \tag{6}$$

and so, when e is a loop, $R(G) = 1 \cdot R(G \setminus e)$ and when e is a coloop, $R(G) = p_e R(G/e)$. In the notation of [sdc,emt], $x_e = p_e$, $y_e = q_e$, $X_e = p_e$ and $Y_e = 1$. The identities

$$x_e Y_f + y_e X_f = x_f Y_e + y_f X_e = p_e + p_f - p_f q_e,$$

$$x_e y_f + y_e X_f = x_f y_e + y_f X_e = p_e q_f + p_f q_e$$

and

$$x_e Y_f + y_e x_f = x_f Y_e + y_f x_e = p_e + p_f - p_e p_f$$

are satisfied. These imply the satisfaction of the identities shown in [sdc] to be necessary and sufficient for a ported parametrized Tutte polynomial to be well-defined by the equations

$$R(G) = x_e R(G/e) + y_e R(G \setminus e) \text{ if } e \notin P \text{ is a non-separator,}$$

$$R(G) = X_e R(G/e) \text{ if } e \notin P \text{ is a coloop, and}$$

$$R(G) = Y_e R(G \setminus e) \text{ if } e \notin P \text{ is a loop.}$$

One can also prove the well-definedness of the ported reliability polynomial with probability variables from the corank-nullity expansion

$$R(G) = \sum_{A \subseteq E} R(G/A|P) x_A u^{r(G) - r(G/A|P) - r(A)} y_{\overline{A}} 1^{|A| - r(A)}$$

with the values $u = 0$ and $v = 1$, i.e.,

$$R(G) \quad = \sum_{A \subseteq E, A \cup P \text{ spans } G} R(G/A|P) x_A y_{\overline{A}} \quad = \sum_{A \subseteq E, A \cup P \text{ spans } G} R(G/A|P) p_A q_{\overline{A}}$$

since $r(G) - r(G/A|P) - r(A) = r(G) - r(P \cup A)$. This expansion is also expresses probabilistic interpretations of the ported reliability polynomial with variable probabilities.

3. Applications

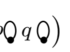
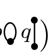
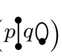
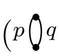
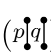
We know two disparate applications of our ported reliability function extension.

The first is to introduce ports for the purpose of expressing variations of basic (all-terminal) reliability, beginning with k-terminal reliability for $k = 2$ and for higher k. The second is to use ports to to model some edges whose joint state probability has a different distribution than that of the non-port edges E, possibly more complex than the product of independent Bernoulli distributions. Indeed, we can express the idea that if one of several edges fails, then the others are more likely to fail also, perhaps because the other ones are forced to carry more traffic. It is clear from (2.1) that when the $R(G/A|P) = R(Q)$ are assigned values each signifying the probability that at least one "functioning" subset $T \subseteq P$ is edge connected in graph Q (whose edge set is P), then $R(G)$ expresses the all-terminal reliablility.

Proposition 1. *Let $R(G)$ be the ported reliability function with $R(Q) = I(Q)$ for all P-minors Q of G. Let the p_e be probability values, $0 \leq p_e \leq 1$ and $q_e = 1 - p_e$ for all $e \in E$.*

1. *If $I(Q) = 1$ for all P-minors Q, then $R(G) = R(G/P)$, the non-ported reliability of G/P.*
2. *If $0 \leq I(Q) \leq 1$ for all P-minors Q, then $0 \leq R(G) \leq R(G/P) \leq 1$.*

We see that $R(G)$ is a probability value no matter what probabilities are assigned the $I(Q)$.

Example: $I($$) = I($$) = I($$) = I($$) = 1$ and $I($$) = 0$ models a situation that whenever both p and q in $P = \{p, q\}$ are needed to maintain connectivity, one or both will fail.

(???) Although the $I(Q)$ with Q ranging over P-minors are each probability values, I is *not* a probability distribution on P-minors. Rather, it is the *likelyhood function*

$$I(Q) = Pr(\text{ edges in } Q \text{ maintain connectivity } | \ \ G/A|P = Q \ \ ).$$

This suggests writing Bayes' formula:

$$Pr(\ \ G/A|P = Q \ \ | \text{ edges in } Q \text{ maintain connectivity }) = \frac{I(Q) \sum_{G/A|P=Q} p_A q_{\overline{A}}}{R(G)}.$$

4. Activities and Boolean Interval Expansions

Ported Tutte polynomials, including those with parameters, have activities and boolean interval expansions in which the the terms indexed by matroid bases are generalized to terms indexed by so-called P-subbases. In this section, we extend known expressions for $R(G)$ in terms of activities and interval partitions of facial simplicial complexes to the ported case. We thus extend the known h-vector expression for $R(G)$. It is convenient to work with variable probabilities and then to specialize to $p_e = p$.

A P-subbasis F is an independent subset of E for which $F \cup P$ is a spanning subset[sdc].

From [sdc],

$$T^{\mathfrak{C}}(M) = \sum_{F\in\mathcal{B}_P} [M/F|P]\Big(\sum_{\substack{IP(F)\subseteq K\subseteq F \\ EP(F)\subseteq L\subseteq E\setminus F}} x_{K\cup(E\setminus F\setminus L)}\, v^{|E\setminus F\setminus L|}\, y_{L\cup(F\setminus K)}\, u^{|F\setminus K|} \ \Big)$$

Substituting $u = 0$, $v = 1$, $x_e = p_e$, $y_e = q_e$, and $[Q] = R(Q)$, we get:

$$R(G) \ = \sum_{F\in\mathcal{B}_P} R(G/F|P)\Big(\sum_{EP(F)\subseteq L\subseteq E\setminus F} p_{E\setminus L}\, q_L\Big).$$

Let's write

$$R(G) = \sum_{A\subseteq E, A\cup P \text{ spans } G} R(G/A|P) x_A y_{\overline{A}} = \sum_{A\subseteq E, A\cup P \text{ spans } G} R(G/A|P) p_A q_{\overline{A}}$$

as

$$R(G) = \sum_{\text{coindependent } F\subseteq E} R(G/\overline{F}|P) p_{\overline{F}} q_F.$$

5. Beyond and Beyond

The lesson of the Tutte tree methodology applied to ZBR-type theorems for matroids is that the parametrized Tutte identities determine a class of computations and the identities have a solution when all computations in the class that start with the same input object have the same outcomes. The lesson of the methodology, underscored by its P-ported extension, applied to Tutte functions of oriented matroids and to graphs is that some useful functions have computations where the recursions are constrained by matroid structure but the initial values vary with additional structure. In the oriented matroid case, the additional structure is the orientation. In the non-ported graph case, the additional structure is the number of path-connected components.

This observation suggests we investigate values associated with a Tutte tree (equivalently, a choice of applications of Tutte equations to objects obtained recursively) in ways more general than the polynomial of monomials determined by the activity status of the elements along each root-to-leaf path. For example, each path may contribute a product (monomial) where the factor corresponding to a separator (loop or coloop) *depends on structure of the object on which a deletion/contraction reduction produces a separator in the resulting minor.* Here, the coloop becomes internally active along the path in which it occurs, or the loop becomes externally active along the path in which it occurs.

As before, the value would be well-defined if *all computation trees* produce the same value under the given set of rules. Are there rules leading to well-defined function for which the factor due to an active element varies with the object in which it becomes active?

Does the computation tree represent a sample in a random process? Might it correspond to a process in which network elements are chosen to be utilized (or reserved for use) and then found to be functioning or not?

Is there a notion of a random Tutte computation tree? I.E., a tree that evolved (grew) from a process in which one of the non-separators is picked for reduction at each node on the tree-growth boundary?

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