(PROPOSAL FOR ISCAS 2002) THE ORIENTED MATROID PAIR MODEL FOR MONOTONE DC ELECTRICAL AND ELASTIC NETWORK UNIQUE SOLVABILITY

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ABSTRACT

Resistive electrical networks and elastic mechanical systems such as trusses have a topological or geometric structure together with constitutive laws for the elements prior to their interconnection. Oriented matroids provide a common discrete mathematical model for such structure in which relationships on the signs of element quantities can be expressed. Pairing of oriented matroids enables non-linear monotone constitutive laws to be fit into the abstraction in a way that allows port and nullor insertions and provides discrete unique solvability conditions.

The resulting mathematical model clarifies some mechanical analogies for these circuit theory concepts, relates apparently dissimilar published theories for existence and uniqueness and shows how to handle elastic mechanical systems with small displacements. It also enables constraints on the signs of system quantities to be predicted from the structure when this is possible.

1. INTRODUCTION

Our topic in non-linear systems is DC equations, say for operating points/resistive circuits, whose only non-linearities are monotone increasing bijections $\mathbf{R} \to \mathbf{R}$. Special case conditions for existence and uniqueness of solutions for all such non-linear functions and additive constants were given by Duffin, Minty, and Rockafellar. The more general determinant based theory of $\mathcal{W}_0$ matrix pairs due to Sandberg and Willson [1] extended Fiedler and Pták's work [2]. Nielsen and Willson [3] used it to prove that disallowing the 2 transistor feedback structure in a transistor circuit is sufficient for uniqueness. Graph based theories that identified structures forbidden for general solvability and uniqueness were given by Hasler, Neirynck, and others [4, 5, 6] for nullor/resistor networks; and by Nishi and Chua [7, 8] for networks with all kinds of 2-port controlled sources, applied in [9] to reproduce Nielsen and Willson's result. Hasler *et. als.* conclusion is that a "pair of conjugate spanning trees" and the absence of a "non-trivial uniform partial orientation of the resistor [edges]" are necessary and sufficient for existence and uniqueness of solutions for all suitable functions and source values. Nishi and Chua's structures are "cactus graph" networks with negative determinants obtained by deletion/contraction operations particular to each kind of controlled source.

Our paper shows oriented matroid (OM) theory [10, 11] covers Hasler *et. als.* "conjugate spanning tree" and "orientation" concepts to provide a theory equivalent to Sandberg and Willson's theory of $\mathcal{W}_0$ pairs to solve the same problem. We also review the key oriented matroid concepts and demonstrate them on one feedback structure case of [12]. We believe distinguished port elements, pairings of oriented matroids with a common ground set, and common covectors (explained below) are crucial. Although oriented matroids can also be axiomatized with "chirotopes" which abstract determinant signs, (so suitable abstractions of graph orientation properties are mathematically equivalent to principles behind determinant signs) the oriented matroid pair common covector approach has an intuitive advantage for qualitative reasoning because the common covector displays precisely the signs of all state quantities or their differences.

The recognition of Minty's painting property[13] and other facts from OM theory used by Hasler *et. al.* [5, 6, 4] led us to generalize in [14] their graph model notion of a "pair of conjugate trees" to a "complementary pair of bases" in a pair of matroids (which abstract the "voltage and current graphs" of [15] and others); and of a "non-trivial uniform partial orientation of the resistors" to a "common (non-zero) covector in an oriented matroid pair". We showed that the graph model generalizes to a linear subspace pair model; the pair of linear subspaces defines a pair of oriented matroids. This OM pair is the discrete structure that generalizes a graph with designated resistor, source, nullator and norator edges. Topological conditions for the existence or uniqueness of solutions are expressed in it. Two real matrices, which can be easily generated from the system design, represent the OMs so that it is practical to work with them. (Some OMs, indeed most, are not representable by real matrices and would require much more space to store, but they do not occur in our application.) Theory valid for all OM pairs, not just those represented by a pair of linear subspaces, was presented in [14]. For lack of space, we omit determinant sign conditions shown equivalent in [14] to the cases of common covector conditions that we cover.

In many cases, interesting conclusions can be reached from the sign patterns of feasible subspace members by "calculations" in an "algebra of signs", a form of qualitative reasoning that uses easy, fundamental oriented matroid theoretic operations upon matrix sign patterns. In cases when the qualitative calculations show the outcome depends on numeric values, the numeric information can then be used, say to calculate a new matrix whose signs reveal better information; or else, inequalities on system parameters for each case of outcome can be derived. The pairing seems to be needed because the non-linear monotonicity constrains two quantities to only have a common sign. The list of those signs for one state or state difference is the common covector. (Our structural/constitutive law separation by OM pairing handles issues different from "imprecise constitutive law constants" of [16] and others.)

Part of this reseach was done during a Sabbatical from the University at Albany in 2001.

1.1. Single Oriented Matroids

See [11] for full details about OMs; [10] is a good introduction to our point of view. Other ways to apply matroids to electrical and other systems are given in [17, 16, 18] and EE literature on symbolic simulation. A full survey is omitted.

We think of the oriented matroid $\mathcal{M}(M)$ *represented* by matrix M as its finite set of *covectors* $\mathcal{L}(M)$, where each covector is the tuple of the *signs* $\{+,-,0\}$ or *signature* $X = \sigma(l)$ of the real coordinates of a member l of the *linear subspace* $L(M)$ in $\mathbf{R}^U$ spanned by the rows of M. (Italics denote a *term or symbol being defined*.) Hence $\mathcal{L}(M) = \mathcal{L}(L(M))$ has at most $3^{|U|}$ covectors. For example, when M is the signed incidence matrix of a network graph, each covector represents a combination of branch voltage drop signs feasible under Kirchhoff's voltage law; the finite *ground set* U labels the branches. One can call X a *signed set*, in which elements of subset X^+ occur with $+$ sign and those in X^- have $-$ sign. The *support* supp X is the subset of $e \in U$ for which $X_e \neq 0$, i.e., supp $X = X^+ \cup X^-$. Bachem and Kern's book [10] motivates oriented matroids from linear subspaces this way. For brevity's sake, we define oriented matroids using some "sign algebra" operation properties we will then use.

Given two sign tuples X^1, X^2, their *composition* $Z = X^1 \circ X^2$ has supp $Z =$ supp $X^1 \cup$ supp X^2 and for $e \in$ supp Z, $X_e = X_e^i$ where i is the smallest index for which $X_e^i \neq 0$. Note that if $X^i = \sigma(l^i)$ for $l^i \in \mathbf{R}^U$, then $X^1 \circ X^2 = \sigma(l^1 + \epsilon l^2)$ for some sufficiently small $\epsilon > 0$. Hence $\mathcal{L}(M)$ is closed under $\circ$.

Definition 1 *The collection $\mathcal{L}(\mathcal{M})$ of signed sets with ground set U is the set of covectors of an oriented matroid $\mathcal{M}$ if it satisfies:*
(L0) $0 \in \mathcal{L}$. ***(L1-2)*** *If $X, Y \in \mathcal{L}$ then $-X$ and $X \circ Y \in \mathcal{L}$.*
(L3) *For all X, $Y \in \mathcal{L}$ and $e \in X^+ \cap Y^-$ there is $Z \in \mathcal{L}$ such that $Z^+ \subset (X^+ \cup Y^+)\backslash\{e\}$, $Z^- \subset (X^- \cup Y^-)\backslash\{e\}$,*
and (supp $X \backslash$ supp Y) $\cup$ (supp $Y \backslash$ supp X) $\cup$
$(X^+ \cup Y^+) \cup (X^- \cup Y^-) \subset$ *supp Z.*

Property (L3) says $Z_e = 0$ and it predicts Z_g for all $g \neq e$ except those with $X_g Y_g = -$; i.e., g having opposite signs in X and Y. The logical equivalence of this definition to various apparently weaker axiomatizations is due to work of Edmonds, Fukada and Mandel cited and surveyed in [11].

Other oriented matroid notions such as orthogonality and independence can be expressed by properties of covector sets that are motivated by linear algebra. The covectors $\mathcal{L}(L^\perp)$ of the *orthogonal complement* $L^\perp$ of linear subspace $L \subset \mathbf{R}^U$ form another oriented matroid. We say $X \perp Y$ for signed sets X, Y when either supp $X \cap$ supp $Y = \emptyset$ or there are $e, f \in U$ with $X_f Y_f = -X_e Y_e \neq 0$. This abstracts a necessary condition for two real vectors to be orthogonal under the usual dot product. In fact, for every covector set $\mathcal{L}(\mathcal{M})$, the set $\mathcal{V} = \mathcal{L}^\perp$ defined by $\{Y | Y \perp X \text{ for all } X \in \mathcal{L}\}$ satisfies the covector axioms ([11], Prop. 3.7.12); $\mathcal{V}(\mathcal{M})$ is called the set of *vectors* of $\mathcal{M}$ and is the set of covectors of the *dual* or *orthogonal* oriented matroid $\mathcal{M}^\perp$. The OM vectors display all combinations of coefficient sign that occur among all linear dependencies of the columns of M, when $\mathcal{M} = \mathcal{M}(M)$. Abstractly, an *independent set* $I \subset U$ satisfies supp $V \not\subset I$ for all non-zero vectors $V \in \mathcal{V}(M)$.

KVL, KCL and analogous mechanical structural or geometric laws are each formulated by a constraint of the form $v \in L$ = row space(M) where $L \subset \mathbf{R}^U$. To reformulate this law by a system of linear equations, a maximal subset $B \subset U$ corresponding to a linearly independent set of columns of M is found. Such a B is a maximal independent set, called a *basis in the matroid* $\mathcal{M}(M)$. The collection of all bases in $\mathcal{M}$ is denoted by $\mathcal{B}(\mathcal{M})$. Row operations and possibly deletion of zero rows can transform M to $[I\ M^{\overline{B}}]$ (after column permutation) where I is the $r \times r$ identity matrix, where $r = \text{rank}(M) = \dim(L) = \text{rank}(L) = \text{rank}(\mathcal{M}(L))$. It is now clear that $v \in L$ is characterized by $v_{\overline{B}} = v_B M^{\overline{B}}$. For each independently chosen $v_B \in \mathbf{R}^B$, $v = (v_B; v_{\overline{B}}) \in L$ is unique with its B coordinates equal to v_B.

The *cocircuits* (resp. *circuits* $\mathcal{C}$) of an oriented matroid are the non-zero covectors (resp. vectors) whose support is minimal. Note $\mathcal{C}$ is the cocircuits of the orthogonal oriented matroid $\mathcal{M}^\perp$.

2. THE SUBSPACE PAIR MODEL AND APPLICATIONS

Some applications begin with basic structural laws modeled by an orthogonal complementary pair of linear subspaces (L_V, L_I), i.e., $L_I = L_V^\perp$. This pair is then modified by port insertion, deletion, contraction, and nullator/norator insertion operations which can destroy the original orthogonality and/or dim $(L_V) +$ dim $(L_I) = |U|$ properties. Other applications can begin with one or both of L_V, L_I expressing some constitutive laws too (see our example).

Electrical network structure is defined with the *network graph* $\mathcal{N}$ with *nodes* N and *arcs* U. The *incidence matrix* M_V has rows indexed by N, columns indexed by U, and $M_V(n, e) = +1$ when the tail of e is n, -1 if the head of e is n, and 0 if n and e are not incident. When $L_V = L(M_V)$ and $L_I = L_V^\perp$, each $u \in L_V$ is a combination of voltage drops in $\mathcal{N}$ feasible under KVL and each $w \in L_I$ is an arc current flow feasible under KCL. These facts restate Kirchhoff's laws and Tellegen's theorem.

The definition of mechanical elastic structure begins with the (undirected) *framework graph* $\mathcal{F}$ with *vertices* N and *edges* U. *Framework* $\mathcal{F}(\mathbf{p})$ is $\mathcal{F}$ and an *embedding* $\mathbf{p} : N \to \mathbf{R}^d$. The *rigidity matrix* M_V has $d|N|$ rows in groups of d corresponding to the vertices. For $e = (i, j) \in U$, column $M_V(e)$ is defined [18]
$(0, \ldots, 0, \mathbf{p}(n_i) - \mathbf{p}(n_j), 0, \ldots, \mathbf{p}(n_j) - \mathbf{p}(n_i), 0, \ldots, 0)^T$
where $\mathbf{p}(n_i) - \mathbf{p}(n_j)$ occupies n_i's group of positions and $\mathbf{p}(n_j) - \mathbf{p}(n_i)$ occupies n_j's. From [18], $u \in L_V = L(M_V) \subset \mathbf{R}^U$ iff for some combination of *vertex velocities* $\mathbf{v} : N \to \mathbf{R}^d$, $u_e = (\mathbf{v}(i) - \mathbf{v}(j)) \cdot (\mathbf{p}(i) - \mathbf{p}(j))$ for each $e = (i, j) \in U$. Also, the *self-stress subspace* $L_I = L_V^\perp$is shown to be all $w \in \mathbf{R}^U$ for which the framework is in static equilibrium when each edge $e = (i, j)$ exerts force $w_e(\mathbf{p}(j) - \mathbf{p}(i))$ on vertex i. By this convention, $w_e > 0$ means e is under tension and $w_e < 0$ means e is under compression. (The rigidity matrix in [18] is M_V^T, not M_V.)

Under this analogy, (1) KVL corresponds to geometric consistency of first order bar length changes under changes in the embedding, (2) KCL corresponds to Newton's laws of static equilibrium, and (3) Tellegen's theorem corresponds to a virtual work principle, that static equilibrium is characterized by the internal forces against every virtual embedding change doing zero virtual work.

Given (L_V, L_I) and $p \in U$ not already a port, the operation of *inserting a port at* p defines a new subspace pair (L'_V, L'_I) with $U' = U\backslash\{p\} \cup \{p_V, p_I\}$, $L'_V = L_V \oplus \mathbf{R}$ (direct sum) with p replaced by p_V; and the coordinate of the added $\mathbf{R}$ indexed by p_I, together with $L'_I = L_I \oplus \mathbf{R}$ with p now replaced by p_I and the added subspace indexed by p_V.

The *subspace pair model* $\mathbf{M} = (E, \Gamma, P, (L_V, L_I))$ consists of finite set E of *device elements*, *constitutive law relations* $\Gamma = \{\Gamma_e \subset \mathbf{R}\times\mathbf{R} | e \in E\}$, the finite set $P = P_V \cup P_I$ that results from inserting ports as defined above, and a subspace pair (L_V, L_I) over $\mathbf{R}^U$ with $U = E \cup P$.

The *variables* of $\mathbf{M}$ are $\{u_{Ve}, u_{Ie} | e \in E\} \cup \{u_{pV}, u_{pI} | p_I, p_V \in P\}$. (For brevity, subscript "pV" means port element $p_V \in P_V$, etc.) A *subspace pair model with sources* S is a subspace pair model $(E, \Gamma, P, (L_V, L_I), S)$ together with a subset S of exactly $|P|$ of the $2|P|$ elements in P. A V*-driven port* is one for which $p_V \in S$ and $p_I \notin S$, then u_{pV} is called an *input variable*. Reverse V and I to define an I*-driven port* and its input variable.

A *solution* of $\mathbf{M}$ with sources is a real valued extension to *all* variables of $\mathbf{M}$ of a given *input assignment* to the input variables that satisfies $(u_{PV}, u_{PI}, u_V) \in L_V$, $(u_{PV}, u_{PI}, u_I) \in L_I$ and $(u_{Ve}, u_{Ie}) \in \Gamma_e$ for all $e \in E$. The constraint $(u_{PV}, u_{PI}, u_V) \in L_V$ does not (by itself) imply any constraint on a I-driven port variable u_{pI}, similarly, u_{pV} is not constrained by $(u_{PV}, u_{PI}, u_V) \in L_I$. In matroid theory an element like p_I of $\mathcal{M}(L_V)$ that is independent of all others is called an *isthmus*.

Assume as usual no port is both I-driven and V-driven. The condition of no cycle of voltage source branches in the "voltage" graph nor a cutset of current source branches in the "current graph" (see, e.g., [15]) generalizes to:

Theorem 1 ***(1)*** *If all V input assignments are feasible under the L_V constraint then $\{p_V | p$ is V-driven$\}$ is an independent set in the matroid $\mathcal{M}(L_V)$.* ***(2)*** *If every solution is unique then $\{p_I | p$ is V-driven$\}$ must be coindependent in $\mathcal{M}(L_I)$.* ***(3)*** *If all V input assignments are feasible under the L_I constraint then $\{p_I | p$ is I-driven$\}$ is an independent set in the matroid $\mathcal{M}(L_I)$.* ***(4)*** *If every solution is unique then $\{p_V | p$ is I-driven$\}$ must be coindependent in $\mathcal{M}(L_V)$.*

Pf. (1,3): If set S of input variables is dependent, then there is some combination of input values that is not feasible. Pf. (2,4): S is not a *coindependent* set iff S contains a cocircuit, so there is a non-zero covector supported by S. Hence there is a feasible variable assignment that is non-zero on the some port output variables only.

Each port insertion increases $\text{rank}(L_V) + \text{rank}(L_I) - |U|$ by 1. When the constitutive laws are linear, the solutions of $\mathbf{M}$ are found from the *intersection* of two linear subspaces: Let G be the diagonal matrix with "conductances" g_e in its positions indexed by $e \in E$ (so $\Gamma_e = \{(v, g_e v) | v \in \mathbf{R}\}$ and 1 in its other diagonal positions. The solution set projected onto the u_I variables is $L_V G \cap L_I$. (Here, $L_V G$ means $L(M_V G)$.)

2.1. Example: a feedback structure case of Trajkovìc and Willson [12]

We used the rules of Definition 1 to figure out common covectors in $\mathcal{L}(M_V)$, $\mathcal{L}(M_I)$ with their first two signs, of port elements pV, pI given, for the tangents and differences drawn boldly on the Ip/Vp curve. (The algorithm ideas used appear in [14].) In three cases, (unique) extensions of the given signs exist when $1 - \Sigma\alpha \leq 0$ but there is only one such case otherwise, verifying a condition for NDR from [12]. Multiple operating points with an I-driven port are impossible. The magnitude inequalities come from "$\epsilon << 1$" in explaining "$\circ$" above Definition 1.

Fig. 1: Example. M_V/M_I are the upper/lower right arrays.

(row mults.)u_{pI}	0	1	0	0	0
u_{pV}	1	0	0	0	$+1$
u_{VR}	0	0	1	0	$+1$
u_{VD1}	0	0	0	1	-1
Γ coeffs. $\rightarrow$	1	1	g_{R}	g_{D1}	g_{D1}
elements $\rightarrow$	pV	pI	R	D1	D2
$u_{\text{pI}} = -I_{\text{p}}$	0	1	$1 - \alpha_1 - \alpha_2$	$\alpha_1 - 1$	$\alpha_2 - 1$
u_{pV}	1	0	0	0	0

$\alpha_1 I_{\text{p}}$ $(1-\alpha_2) I_{\text{p}}$

$+$ $+u_{\text{pV}}+$

$+$ $+$

u_{VD1} u_{VR} $(\alpha_1 + \alpha_2 - 1) I_{\text{p}}$ $+u_{\text{VR}}+$

$-$ $-$ $= u_{\text{IR}}$ $-u_{\text{VD1}}$

$= u_{\text{VD2}}$

I_{p} $u_{\text{pI}} = -I_{\text{p}}$ $+$ u_{pV} $-$ $-$

$\dot{u}_{\text{VR}} >> \dot{u}_{\text{VD1}} - \dot{u}_{\text{PV}}$

$1 - \Sigma\alpha \leq 0$ req. for NDR.

$I_{\text{p}} = -u_{\text{pI}}$ $\dot{u}_{\text{VR}} >> \dot{u}_{\text{VD1}}$

$\dot{u}_{\text{pV}}$ or $\dot{u}_{\text{VR}} >> \dot{u}_{\text{VD1}}$

u_{pV}

Given a subspace $L \subset \mathbf{R}^U$ and $e \in U$, the subspace "*L with e deleted*" is $L\backslash e = \{l(U\backslash e) | l \in L\} \subset \mathbf{R}^{U\backslash\{e\}}$, where $l(U\backslash e)$ denotes the $l \in \mathbf{R}^U$ with the component labeled by e dropped. Thus, $L\backslash e$ is the *projection* of L into $\mathbf{R}^{U\backslash\{e\}}$. If $L = L(M)$ then $L\backslash e = L(M(U\backslash\{e\}))$ is the row space of $M(U\backslash\{e\})$, which is M with column e deleted. The subspace "*L with e contracted*" $L/e = \{l(U\backslash e) | l(U) \in L \text{ and } l(e) = 0\}$. So, L/e is the intersection of L with the (hyperplane) subspace of $\mathbf{R}^U$ with $l(e) = 0$, projected into $\mathbf{R}^{U\backslash\{e\}}$.

We now define *deletion* and *contraction* on subspace pairs: $(L_V, L_I)\backslash e = (L_V\backslash e, L_I/e)$ and $(L_V, L_I)/e = (L_V/e, L_I\backslash e)$. Deleting e in the electrical application corresponds to *opening* the corresponding branch. Dually, contraction corresponds to *shorting*. Mechanically, deletion of an edge corresponds to "breaking" the corresponding bar: ignore any distance change between its ends and transmit no force. Contraction corresponds to declaring the bar to be inelastic, which rules out all (first order) distance changes between the endpoints and rules out any constitutive law referring to tension or compression in that bar.

A *nullator* element $e \in E$ expresses the ideal constitutive law $u_{Ve} = 0$ and $u_{Ie} = 0$ which approximates conditions at the input to a high-gain amplifier when a system is stabilized by feedback. Hence a nullator is declared by *contracting e in both L_V and L_I*. Ordinarily, this reduces both their ranks by 1. A *norator* element $e \in E$ indicates that the constitutive law puts no direct constraint on u_{Ve} or u_{Ie}; the amplifiers approximately adjust the output state so the feedback results in zero input. Hence a norator is declared by *deleting e in both L_V and L_I*. Ordinarily, their ranks don't change. Theorem 1 applies to the subspace pair obtained from all declarations of nullators, norators, opens and shorts.

The *supplemental subspace pair* is constructed by *zeroing* all the sources. For each V-driven (resp. I-driven) port p, p_V (resp. p_I) is contracted in both L_V and L_I, and p_I (resp. p_V) is deleted in both L_V and L_I. The resulting subspaces, etc. are denoted L_V^0, L_I^0, etc.

3. NO-COMMON-COVECTORS, $\mathcal{W}_0$ PAIRS

Theorem 2 uses Sandberg and Willson's $\mathcal{W}_0$ pairs to show every subspace pair model can be analyzed for unique solvability from the oriented matroid pair it generates. Theorem 3 shows how $(A, B) \in \mathcal{W}_0$ is characterized by a rank condition and a no-common-covector property. Theorem 4 is our unifying conclusion.

Theorem 2 *The subspace pair model has a unique solution for all input assignment values and positive monotone constitutive laws iff **(1)**, **(2)** and **(3)** are satisfied:*
(1). *There are bases $B_V \in \mathcal{B}(L_V)$, $B_I \in \mathcal{B}(L_I)$ for which all V-driven ports p satisfy $p_V \in B_V$ (note p_I must be in B_V since it's an isthmus in $\mathcal{M}_V$) and $p_I \notin B_I$, and all I-driven ports p satisfy $p_I \in B_I$ and $p_V \notin B_V$.*
(2). *$B_V \cup B_I = U$, for the bases in **(1)**.*
(3). *The oriented matroid pair of supplemental pair (L_V^0, L_I^0) have no common (nonzero) covector. (i.e., $\mathcal{L}(L_V^0) \cap \mathcal{L}(L_I^0) = \{0\}$.)*

Proof sketch, if: When (L_V^0, L_I^0) is constructed given **(1-2)**, we find rank (L_V^0)+rank $(L_I^0) \geq |E|$. By Theorem 1.3 of [14] and $|E| \geq$ rank of the matroid union ([19], 8.3) of $\mathcal{M}_V^0 \vee \mathcal{M}_I^0$, we must have equality else **(3)** would be contradicted. Form square matrices $A = \begin{pmatrix} M_V^0 \\ 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ -M_I^0 \end{pmatrix}$ and verify from **(2)** they satisfy the rank part of cond. **(2)** of Th. 3. $\mathcal{L}[A\ B] \cap \mathcal{L}[I\ \ -I] = \{0\}$. follows directly from hypothesis **(3)**. We can then formulate the monotonic subspace pair problem as a case of Theorem 3 part **(5)** to complete the proof.

Only if: Use Theorem 1 to verify **(1)** and **(2)**. If **(3)** were violated, positive linear constitutive laws could be constructed (as in [5, 1]) for which a non-zero solution for zero input exists.

Theorem 3 *(From [14]) For a pair of $n \times n$ matrices (A, B), the following conditions are equivalent.*
(1) *$(A, B) \in \mathcal{W}_0$ in the sense of Sandberg and Willson [1]; e.g., $|AD + B| \neq 0$ for all positive diagonal D, etc.*
(2) *$\operatorname{rank} \mathcal{M}[A\ B] = n$ and $\mathcal{L}[A\ B] \cap \mathcal{L}[I\ \ -I] = \{0\}$.*
(3) *$\operatorname{rank} \mathcal{M}[A\ B] = n$ and $\mathcal{V}[A\ B] \cap \mathcal{V}[I\ \ -I] = \{0\}$.*
(4) *(Fundamental theorem of Sandberg and Willson [1, 20]) For all functions $F : \mathbf{R}^n \to \mathbf{R}^n$ of the form $F(x)_k = f_k(x_k)$ where each f_k is a strictly monotone increasing function from $\mathbf{R}$ onto $\mathbf{R}$ and for all $c \in \mathbf{R}^n$, the equation $AF(x) + Bx = c$ has a unique solution x. [20].*
(5) *For all functions $G : \mathbf{R}^n \to \mathbf{R}^n$ of the form $G(w)_k = g_k(w_k)$ where each g_k is a strictly monotone increasing function from $\mathbf{R}$ onto $\mathbf{R}$ and for all $d', d'' \in \mathbf{R}^n$, the eqns. $u^t = z^t A + d'$, $w^t = z^t B + d''$, $u = -G(w)$ have a unique solution.*

A direct inductive proof generalizing [5, 6]'s has the advantage of revealing circuit theoretic concepts that occur. One step proves that if the no-common-covector property is true for (M_V, M_I), then it is true for the matrix pair from the system obtained by replacing one of the non-linear elements by a source.

Theorem 4 *With constitutive laws given by monotone increasing functions from $\mathbf{R}$ onto $\mathbf{R}$, every subspace pair problem can be posed as a case of Theorem 3 (see Theorem 2), and every case of Theorem 3 can be posed as a subspace pair problem (with $[A\ B] = M_V^0$, $[I\ \ -I] = M_I^0$.)*

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