Ported alias Set-Pointed and Non-scalar Tutte Functions

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Traditionally a **Tutte Function** F

$$F : \begin{cases} \text{Matroids} \\ \text{(or Graphs)} \end{cases} \to \text{Commutative Ring} \quad (R, +, \cdot)$$

$$F(\mathcal{N}) = g_e F(\mathcal{N}/e) + r_e F(\mathcal{N} \setminus e) \tag{A}$$

for all e not a loop or inthmus. g_e, r_e are parameters or 1.

$$F(\mathcal{N}_1 \oplus \mathcal{N}_2) = F(\mathcal{N}_1) \cdot F(\mathcal{N}_2) \tag{M}$$

We survey results where

(1) $(R, +, \cdot)$ is replaced by discrete or other algebraic structures (Matroids!), or "Matroids" is replaced by matroid presentations;

and (2) given a distinguished subset P (**ports** or **set of points**), (A) is restricted to $e \notin P$.

1. F : Matroids $\to$ Matroids.
2. F : Extensors $\to$ Extensors.
   The multiplication of exterior (aka Cayley, Grassmann) algebra is anticommutative: $\mathbf{N}_1\mathbf{N}_2 = (\mathbf{N}_1 \wedge \mathbf{N}_2) = \mathbf{N}_2\mathbf{N}_1(-1)^{r(\mathbf{N}_1)r(\mathbf{N}_2)}$
3. F : (Oriented) Matroids $\to$ Commutative Ring where a substitution expresses (2) in the unimodular (regular) oriented case.
4. Algebraic expressions of (3) for P-unions and generalized parallel connections over P.

In all cases, "ports", "set of points" P, for which deletion/contraction is forbidden, must satisfy $P \neq \emptyset$ for interesting results.

Result 1: (Construction of Recski, Weinberg 70's; new observation here)

Given: Matroid $\mathcal{N}(P, E)$ has ground set $P \mathbin{\dot\cup} E$.
P_V and P_I are two disjoint copies of P:
$P_V = \{p_V : p \in P\}$; $P_I = \{p_I : p \in P\}$.

$$\text{(The matroid!) } \mathcal{M}_E(\mathcal{N})(P_I \mathbin{\dot\cup} P_V) :\equiv \big(\mathcal{N}(P_I, E)) \cup \mathcal{N}^{\perp}(P_V, E)\big)/E$$

satisfies

$$\mathcal{M}_E(\mathcal{N}) = \mathcal{M}_{E'}(\mathcal{N}/e) \cup_{\mathcal{B}} \mathcal{M}_{E'}(\mathcal{N} \setminus e) \text{ if } e \notin P \text{ and } E' = E \setminus e,$$

where $\cup_{\mathcal{B}}$ denotes union of matroid basis collections; and

$$\mathcal{M}_E\big(\mathcal{N}_1(E_1, P_1) \oplus \mathcal{N}_2(E_2, P_2)\big) = \mathcal{M}_{E_1}\big(\mathcal{N}_1(E_1, P_1)\big) \oplus \mathcal{M}_{E_2}\big(\mathcal{N}_2(E_2, P_2)\big).$$

where $\oplus$ denotes matroid direct sum.

Proof: E is independent in $\mathcal{N}(P_I, E) \cup \mathcal{N}^{\perp}(P_V, E)$, so B is a basis of $\mathcal{M}_E(\mathcal{N})$ iff $A \subseteq E$, $B \mathbin{\dot\cup} A \mathbin{\dot\cup} (E \setminus A)$ is a basis of the union; thence either $e \in A$ or $e \in (E \setminus A)$.

Result 2. applies to a given **decomposible** ($\mathbf{N}(P, E) = \mathbf{v}_1\mathbf{v}_2 \cdots \mathbf{v}_r$ of vectors) in the **exterior algebra** $\mathcal{E}$ over vector space $k(P \uplus E)$ with distinguished basis $(P \uplus E)$.

We say an **extensor** is a decomposible element in $\mathcal{E}$.

Let $P \uplus E$ label the columns of matrix N (full row rank r).

The (oriented) matroid of column dependencies (rank r) is determined by the *row space* of N $\leftrightarrow$ k-multiples of one extensor $\mathbf{N}$, such as $\mathbf{N} = \wedge(\text{row vectors of } N) \in \mathcal{E}$.

$\mathcal{E}$ is quotient of the assoc. algebra generated by $k(P \uplus E)$ modulo the ideal generated by $\mathbf{v}^2$, $\mathbf{v} \in k(P \uplus E)$. $\mathcal{E}$ has dimension $2^{|P \uplus E|} = 2^n$.

$\mathcal{E}$ is graded: At rank r, $\mathcal{E}_r$ has dim. $\binom{n}{r}$, $r = 0, 1, \ldots, n$.

Natural coordinates of extensor $\mathbf{N}(P, E)$ are the $r \times r$ *determinants* in matrix N.

Exterior *sum* can be expressed by addition of the expansion coefficients under the basis of all 2^n subsets of $P \cupdot E$, (each subset with an element order fixed).

But, the sum of two extensors is not necessarilly *decomposible*, not necessarilly an *extensor*.

The exterior *product* of extensors for disjoint subspaces represents the **subspace join**.

When a ground set S is given (like $S = E \cupdot P$, distinguished basis for $k(E \cupdot P)$), **deletion** and **contraction** of $e \in S$, and **dualization**, known from multilinear algebra, represent the corresponding (oriented) matroid operations.

We define deletion/contraction so

$$\mathbf{N} = \mathbf{N} \setminus e + (\mathbf{N}/e)\mathbf{e}$$

Dualization: Copy the oriented matroid chirotope dualization formula (also known as Hodge star): Coefficients (determinants!) $\mathbf{N}^{\perp}[X] = \mathbf{N}[\overline{X}]\epsilon(\overline{X}, X)$.

Our Tutte-like function $\mathbf{M}_E(\mathbf{N})$: Extensors $\to$ Extensors.

Given N (matrix), construct $N^\perp$ so their row spaces are orthogonal complements ($N^\perp$ presents the (oriented) dual of the matroid from N).

Form the matrix: $(G = \mathrm{diag}(g_e),\ R = \mathrm{diag}(r_e))$

$$M = \left[\begin{array}{c|c|c} N(P) & 0 & N(E)G \\ \hline 0 & N^\perp(P) & N^\perp(E)R \end{array}\right]$$

with columns labelled by $P_I \mathbin{\dot\cup} P_V \mathbin{\dot\cup} E$.

Extensor $\mathbf{M}$ over $k[g_e, r_e](P_V \mathbin{\dot\cup} P_I \mathbin{\dot\cup} E)$ is the product of M's **row vectors**, and define $\mathbf{M}_E(\mathbf{N})$ by:

$$\mathbf{M} = \mathbf{M}_E(\mathbf{N})\mathbf{e_1}\mathbf{e_2}\cdots\mathbf{e}_{|E|} + (\cdots)$$

Result 2: (2003)

$$\epsilon(PE)\ \mathbf{M}_E(\mathbf{N}(P,E)) = \epsilon(PE')\ \big(g_e\mathbf{M}_{E'}(\mathbf{N}/e) + r_e\mathbf{M}_{E'}(\mathbf{N}\setminus e)\big)$$

$$\epsilon(P_1P_2E)\ \mathbf{M}_E(\ \mathbf{N}_1(P_1,E_1)\ \ \mathbf{N}_2(P_2,E_2)\ ) = \epsilon(P_1E_1)\epsilon(P_2E_2)\ \mathbf{M}_{E_1}(\mathbf{N}_1(P_1,E_1))\ \mathbf{M}_{E_2}(N_2(P_2,E_2))$$

Corollory:

$$\epsilon(PE)\mathbf{M}_E(\mathbf{N}) = \epsilon(P) \sum_{\substack{A\subseteq E\,:\,\mathrm{r}_{\mathbf{N}}A=|A|,\\ \mathrm{r}\mathbf{N}-\mathrm{r}(\mathbf{N}/A|P)-\mathrm{r}_{\mathbf{N}}A=0}} \mathbf{M}_\emptyset(\mathbf{N}/A|P)g_A r_{\overline{A}}. \quad (1)$$

Compare: (with $u=0$ and $v=0$)

$$R(\mathcal{N}(P,E)) = \sum_{A\subseteq E} [\mathcal{N}/A|P]\ g_A r_{\overline{A}} u^{\mathrm{r}\mathcal{N}-\mathrm{r}[\mathcal{N}/A|P]-\mathrm{r}A} v^{|A|-\mathrm{r}A}.$$

Result 3: The ported, parametrized corank-nullity polynomials of oriented and non-oriented matroids

$$R_P(\mathcal{N}(P,E)) = \sum_{A \subseteq E} [\mathcal{N}/A|P]\; g_A r_{\overline{A}} u^{\mathrm{r}\mathcal{N} - \mathrm{r}[\mathcal{N}/A|P] - \mathrm{r}A} v^{|A| - \mathrm{r}A}.$$

satisfy the ported Tutte equations.

(In the invariant case, this R is universal for Tutte P-invariants of oriented and non-oriented matroids)

$[\mathcal{N}/A|P]$ is a (commutative) *monomial* whose factors *are* **connected (oriented) matroids** over subsets of P.

(Las Vergnas "Big Tutte Polynomial" ('75,'99), oriented/parametrized by sdc.)

R_P can distinguish some orientations of the same matroid when $|P| \geq 2$.

Result 4 pertains to three matroid combination operations:

Given $\mathcal{N}_1(P, E_1)$ and $\mathcal{N}_2(P, E_2)$ with only elements P in common.

(1.) Ported matroid union: $\mathcal{B}\big(\mathcal{N}_1 \cup \mathcal{N}_2\big) = \{B_1 \cup B_2 : B_i \in \mathcal{B}(\mathcal{N}_i)\}$

(2.) Duality conjugate $\cup^*$ of $\cup$.

(3.) Given that P is a modular flat and a common submatroid in $\mathcal{N}_1$ and $\mathcal{N}_2$, the generalized parallel connection.

When $|P| = 1$, both dual $\cup$ and generalized parallel connection REDUCE to (one base point) parallel connection.

K=polynomial ring containing u and v. K_P=commutative K-module generated by monomials $[q_i]$ signifying matroids q_i over subsets of P, and $[\emptyset] = 1$.

Recall ported corank-nullity polynomial $R_P : \text{Matroids} \to K_P$.

Result 4: For each combination operation $(*_i, i = 1, 2, 3)$ there is a bilinear $M_i : K_P \times K_P \to K_P$ map and an $u^j v^k$-valued function f_i such that

$$R_P(\mathcal{N}_1 *_i \mathcal{N}_2) = f_i(\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_1 *_i \mathcal{N}_2) M_i\big(R_P(\mathcal{N}_1), R_P(\mathcal{N}_2)\big)$$

In other words, once $*_i$ is determined on pairs of matroids over subsets of P, the ported R_P for $\mathcal{N}_1 *_i \mathcal{N}_2$ can be calculated by formal multiplication

$$f_i(..) R_P(\mathcal{N}_1) *_i R_P(\mathcal{N}_2) = f_i(..) M_i\big(R_P(\mathcal{N}_1), R_P(\mathcal{N}_2)\big)$$

and substitutions of $\big(1/f_i(q_a, q_b, q_a *_i q_b)\big)[q_a *_i q_b] \leftarrow [q_a] *_i [q_b]$.

Splitting Formulas generalize $F(\mathcal{N}_1 \oplus \mathcal{N}_2) = F(\mathcal{N}_1)F(\mathcal{N}_2)$ and solve this type of problem:

Given $\mathcal{N}_1(P \uplus E_1)$, $\mathcal{N}_2(P \uplus E_2)$, can we calculate the Tutte polynomial of $\mathcal{N}_1 *_i \mathcal{N}_2(P \uplus E_1 \uplus E_2)$ from Tutte polynomials of minors of $\mathcal{N}_1$ and $\mathcal{N}_2$ gotten by deletion/contraction of subsets of P?

For one-point series and parallel connections, we can re-derive Brylawski's (1971) formulas by the above bilinear method (1989).

For generalized parallel connection, with P a modular flat and common restriction in both $\mathcal{N}_1$ and $\mathcal{N}_2$, a splitting formula was obtained by Bonin and de Mier (2004).

Their formula, for the Tutte polynomial, is in terms of:

1. The lattice of flats $F \leq P$,
2. characteristic polys. of P/F, and
3. Tutte polynomials of $\mathcal{N}_1/F$ and $\mathcal{N}_2/F$.

Our formulas are (1) for R_P (Big Tutte polynomial) and (2) are in terms of $R_P(\mathcal{N}_1) *_i R_P(\mathcal{N}_2)$. Our bilinear forms' coefficients and f in $R_P(\mathcal{N}_1 *_i \mathcal{N}_2) = f_i(...)R_P(\mathcal{N}_1) *_i R_P(\mathcal{N}_2)$:

1. $* = \cup$ (1989)

$$f(\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_1 \cup \mathcal{N}_2) = u^{\mathrm{r}\mathcal{N}_1 \cup \mathcal{N}_2 - \mathrm{r}\mathcal{N}_1 - \mathrm{r}\mathcal{N}_2}.$$

2. $* = \cup^*$ (1989) (nullity $n\mathcal{N} = |\mathcal{N}| - \mathrm{r}\mathcal{N}$)

$$f(\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_1 \cup^* \mathcal{N}_2) = v^{n(\mathcal{N}_1 \cup^* \mathcal{N}_2) - n\mathcal{N}_1 - n\mathcal{N}_2}$$

3. Generalized Parallel P-Connection $*$ (1991) Since P is a modular flat, quotients $Q_i = \mathcal{N}_i/A_i|P$ correspond to flats in P, and $[Q_i] * [Q_j] = (1/f)[Q']$ where Q' is the quotient corresponding to the join of those flats in P.

$$f(\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_1 * \mathcal{N}_2) = v^{n(\mathcal{N}_1 * \mathcal{N}_2) - n\mathcal{N}_1 - n\mathcal{N}_2}$$

Brylawski's splitting formulas for $|P| = 1$ (series/union and parallel/co-union conn.) can be derived from our $R_P(\mathcal{N}_1) * R_P(\mathcal{N}_2)$ formula.

Can Bonin and de Meir's generalized parallel connection splitting formula for all $|P|$ also be derived that way?

Are there other expressions for the splitting function?

Example $P = \{p\}$: The 2 matroids over P are $[\mathcal{P}_i]$, rank $i = 0, 1$.

Problem: express $R_P(\mathcal{N}_1 * \mathcal{N}_2)$ in terms of 4 polynomials $R(\mathcal{N}_j \setminus p)$, $R(\mathcal{N}_j/p)$, $j = 1, 2$.

Tool 1: for various $\mathcal{N}$, write $R_P(\mathcal{N}) = [\mathcal{P}_1]R^{(1)}(\mathcal{N}) + [\mathcal{P}_0]R^{(0)}(\mathcal{N})$ and try to express $R^{(0)}(\mathcal{N})$, $R^{(1)}(\mathcal{N})$ in terms of $R(\mathcal{N} \setminus p)$, $R(\mathcal{N}/p)$.

Solve the following equations:

$$R(\mathcal{N}/p) = R_P(\mathcal{N})\Big|_{\substack{[\mathcal{P}_0] \leftarrow u^0 v^1 \\ [\mathcal{P}_1] \leftarrow u^0 v^0}} = vR^{(0)} + R^{(1)}$$

$$R(\mathcal{N} \setminus p) = R_P(\mathcal{N})\Big|_{\substack{[\mathcal{P}_0] \leftarrow u^0 v^0 \\ [\mathcal{P}_1] \leftarrow u^1 v^0}} = R^{(0)} + uR^{(1)}$$

Solution: (Brylawski 1971, Cor. 6.14)

$$R^{(0)}(\mathcal{N}) = \frac{1}{1 - uv}\big(R(\mathcal{N} \setminus p) - uR(\mathcal{N}/p)\big)$$

$$R^{(1)}(\mathcal{N}) = \frac{1}{1 - uv}\big(R(\mathcal{N}/p) - vR(\mathcal{N} \setminus p)\big)$$

Tool 2: $R(\mathcal{N}_1 * \mathcal{N}_2) = R_P(\mathcal{N}_1 * \mathcal{N}_2)|_{[\mathcal{P}_0] \leftarrow 1+v; [\mathcal{P}_1] \leftarrow 1+u}$ before substitution

$$= v^{n\mathcal{N}_1 * \mathcal{N}_2 - n\mathcal{N}_1 - n\mathcal{N}_2}$$

$$\Big([\mathcal{P}_0]\, R^{(0)}(\mathcal{N}_1) + [\mathcal{P}_1]\, R^{(1)}(\mathcal{N}_1) \Big) * \Big([\mathcal{P}_0]\, R^{(0)}(\mathcal{N}_2) + [\mathcal{P}_1]\, R^{(1)}(\mathcal{N}_2) \Big)$$

$$= v^{-1} \Big([\mathcal{P}_0] * [\mathcal{P}_0]\, R^{(0)}(\mathcal{N}_1) R^{(0)}(\mathcal{N}_2)$$

$$+ [\mathcal{P}_0] * [\mathcal{P}_1] \left(R^{(0)}(\mathcal{N}_1) R^{(1)}(\mathcal{N}_2) + R^{(1)}(\mathcal{N}_1) R^{(0)}(\mathcal{N}_2) \right)$$

$$+ [\mathcal{P}_1] * [\mathcal{P}_1]\, R^{(1)}(\mathcal{N}_1) R^{(1)}(\mathcal{N}_2) \Big)$$

$$[\mathcal{P}_0] * [\mathcal{P}_0] \leftarrow v\, [\mathcal{P}_0] \leftarrow v(v+1)$$

$$[\mathcal{P}_0] * [\mathcal{P}_1] \leftarrow 1 \cdot [\mathcal{P}_0] \leftarrow (v+1)$$

$$[\mathcal{P}_1] * [\mathcal{P}_1] \leftarrow 1 \cdot [\mathcal{P}_1] \leftarrow (u+1)$$

now rederives splitting formula of Brylawski (1971, Thm. 6.15), using

$$R^{(0)}(\mathcal{N}_2) = \frac{1}{1-uv} \left(R(\mathcal{N}_2 \setminus p) - uR(\mathcal{N}_2 / p) \right) \text{ etc.}$$