Ported or Relative Oriented Matroids and Electric Circuits

Seth Chaiken
Dept. of Computer Science
Univ. at Albany
schaiken@albany.edu

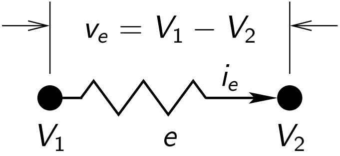
March 2, 2015

Why Electricity, EE?

- Scholarly topic suggested by G.-C. Rota $\approx$ 1980?.
- $\approx$ 100 yrs. geometry-like intuition of circuit configurations known by engineers, EE books: "Intuitive Analog Circuit Design (2013)" [11]; "Non-linear Circuits" [5] translates to our Oriented Matroid pair model.
- Geometry of linear spaces and oriented matroids; Tutte decomp. w/ techniques from Barnabi, Brini and Rota's Exterior Calculus [1])
- Real behavior $\approx$ ideal plus perturbations, ideal constraints predict intended real behavior,
- Interesting, accessible, intuitively understandable intentential designs, applicable, easy to both simulate and build physically, dimension $\approx$ 12 or 24, depending on formulation
- Analogs to chemical (and real algebraic geometry [8]), biological, mechanical, etc., random walks ...
- Merely one scalar non-linearity can cause chaos.

Kirchhoff (1847) [6] Maxwell (1891) [7] The equivalent resistance problem IS SOLVED by the Matrix Tree Theorem. (1) Let's POSE the problem: the VARIABLES

$\boxed{v_e, i_e : e \in E}$

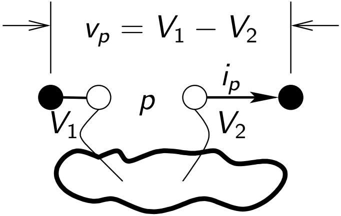


(Use **voltage drops along the flow**, not potentials V_1, V_2.)

ordinary, resistor edge $e \in E$

(IF linear Ohm's law use $|E|$ variables ($g_e v_e = r_e i_e$) ELSE use $2|E|$ variables.)

$\boxed{v_p, i_p : p \in P}$



DISTINGUISHED, PORT edge $p \in P$

The interface to an environment is modelled with $2|P|$ variables.

(math, not EE sign convention)

(2) Let's POSE the problem: EQUATIONS

- (KCL) $(i_e)_{e \in S}$ is a cycle (a flow).
- (KVL) $(v_e)_{e \in S}$ is a cocycle
- (constituitive Law) $i_e = g_e(v_e)$ non-linear, usually monotonic increasing $R \to R$.
  (Sometimes use Ohm's approximation $i_e = g_e v_e$)

Combinatorics!

The signs $\{+, -, 0\}$ have oriented matroid structure (combinatorial, geometric, topological).

SOLUTION: Equiv. Resistance $:\equiv -(v_p/i_p)$ observed at a port p by the environment EQUALS a Ratio of Spanning Tree Enumerators! (Port edge p locates the *2 terminals*.)

$$-(\frac{v_p}{i_p}) = \frac{\text{WTS}(G/p)}{\text{WTS}(G\backslash p)} = \frac{\text{Matrix-Tree Det}(G/p)}{\text{Matrix-Tree Det}(G\backslash p)}$$

- "Maxwell's rule" uses MatrTreeT on 2 DIFFERENT GRAPHS

$$(G/p \text{ and } G\backslash p)$$

- Weighted Tree Sum (WTS) is a colored Tutte function:

$$\text{WTS}(G') = g_e \text{WTS}(G'/e) + r_e \text{WTS}(G\backslash e) \text{ for all } e \notin P$$

$$\text{WTS}(\text{coloop}(e)) = g_e$$

$$\text{WTS}(\text{loop}(e)) = r_e$$

Benefits of Multiple Ports

- One formula expresses $\binom{2|P|}{|P|}$ different Matrix Tree Theorems...
- ... long vertex-based proofs are shortened; Rayleigh inequalies too.
- Interesting **non-commutative ranges** of new ORIENTED MATROID Tutte invariants with pattern:

$$\mathrm{TF}(N(P \mathbin{\dot\cup} E)) = F(N(P \mathbin{\dot\cup} E)/E)$$

  (They distinguish DIFFERENT ORIENTATIONS of the SAME MATROID.)
- Formalize composition of systems [9], Tutte poly. splitting formulas.
- Label variables to observe
- Model practical devices (transistors, op amps)
- Align EE applications with knots (Ported = "Relative") and combinatorial geometry (Ported = "Set Pointed").

Constraint/Generator Duals and 2 Results.

- (Part 1) Technique:

$$\text{Solution Space} = \bigcap \text{Constraint Subspaces}$$

- Result: An exterior algebraic Tutte function: Every Plucker coordinate of it satisfies a Matrix Tree Theorem.
  This and det. formulas easily prove Rayleigh inequalities.

- (Part 2) Combine with:

$$\text{Solution Space} = \text{Closure(Set of Generators)}$$

- Result: An oriented matroid pair model for some non-linear problem well-posedness.

Part 1) Coef. Matrix M in CONSTRAINTS $MX = 0$

The Tutte-like function $\mathbf{M}_E(\mathbf{N})$: Extensors $\rightarrow$ Extensors:

Given N (matrix), construct $N^\perp$ with orthog. comp. row space.
Construct: $(G = \operatorname{diag}(g_e),\ R = \operatorname{diag}(r_e))$

$$M = \left[\begin{array}{c|c|c} N(P) & 0 & N(E)G \\ \hline 0 & N^\perp(P) & N^\perp(E)R \end{array}\right]$$

with columns labelled by $P_I \cup P_V \cup E$.
Extensor $\mathbf{M}$ over $k[g_e, r_e](P_V \cup P_I \cup E)$ is the product of M's **row vectors**. The contraction result $\mathbf{M}_E(\mathbf{N}) = \mathbf{M}/E$ appears:

$$\mathbf{M} = \mathbf{M}_E(\mathbf{N})\mathbf{e_1}\mathbf{e_2}\cdots\mathbf{e}_{|E|} + (\cdots)$$

$\mathbf{M}_E(\mathbf{N})$ is our Tutte function $\mathbf{N} \rightarrow$ Ext. Alg.

(Part 2) Common Covector Model

The cycle space of G_I GENERATES
the covectors of an
oriented matroid over $(E \dot\cup P_I)$.

0

(signs indexed by E) (by P_I)

Non-linear monotone resistors CONSTRAIN SIGNS of
voltage drops (from ↓) and flows (from ↑)
TO BE EQUAL

(signs indexed by E) (by P_V)

The cocycle space of G_V GENERATES
the covectors of an
oriented matroid over $E \dot\cup P_V$.

G_V
SOMETIMES EQUALS
G_I

0

Return to the Part 1 Equation $MX = 0$

$$M = \left[\begin{array}{c|c|c} N(P) & 0 & N(E)G \\ \hline 0 & N^{\perp}(P) & N^{\perp}(E)R \end{array}\right]; \mathbf{M} = \mathbf{M}_E(\mathbf{N})\mathbf{e_1}\mathbf{e_2}\cdots\mathbf{e}_{|E|} + (\cdots)$$

ELIMINATE the variables indexed by E, leaving $2|P|$ variables labelled by P_I and P_V. ie, CONTRACT E. **Answer** A IS:

$$\mathbf{M}_E = \bigwedge_{\text{JOIN over rows}}^{\text{Exterior}} \left[\begin{array}{c|c} A_{I,I} & A_{I,V} \\ \hline A_{V_I} & A_{V,V} \end{array}\right] [\mathbf{p_{I_1}}, \cdots, \mathbf{p_{I_p}}; \mathbf{p_{V_1}}, \cdots, \mathbf{p_{V_p}}]^{\mathbf{t}}$$

$$= \ldots + C_i\mathbf{XXX} + \ldots; \text{Equiv. Resistance} = \text{certain } \; C_i/C_j$$

All the other C_k's have similar interpretations.
$\binom{2|P|}{|P|}$ **Matr. Tree Theorems:** Each $C_k(N)$ (a PRINCIPAL MINOR of M ABOVE!) $= g_e C_k(N/e) + r_e C_k(N\backslash e)$ ($e \notin P$, e not (co)loop).
Each C_k is a signed weighted enumerator of forests satisfying **conditions ...**

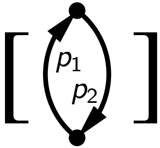
Conditions (what sets F are enumerated by one det. C_i)

The **conditions ...** are on the rank, nullity of F and, WHAT ORIENTED MINOR is $G/F \setminus (E \setminus F)$, the minor with ONLY PORT EDGES from contracting F and deleting the other resistor edges, leaving the ports.

The conditions for a given C_k *sometimes* make all the signs the same (eg: C_i and C_j in 1-port equivalent resistance $R = C_i/C_j$)

Othertimes, the oriented **P-minors** in the completed Tutte decomposition of C_k determine some $+$ and some - signs.

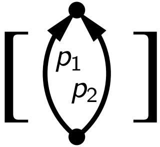
When $[G/F|P]$ is



the term is

$+ g_F r_{E \setminus F}$

When $[G/F|P]$ is



the term is

$- g_F r_{E \setminus F}$

Application: Rayleigh Identity, "Neg. Spanning Tree Correlation"

$\Gamma_e(G)$ is equivalent conductance across e. Rayleigh: $0 \leq \dfrac{\partial \Gamma_p}{\partial g_f} = \dfrac{\partial \frac{T_G}{T_{G/e}}}{\partial g_f}$

is equivalent to

$$0 \leq \frac{\partial T_G}{\partial g_f} T_{G/e} - T_G \frac{\partial T_{G/e}}{\partial g_f} = T_{G/f} T_{G/e} - T_G T_{G/e/f}$$

In fact,

$$T_{G/f} T_{G/e} - T_G T_{G/e/f} = \left(T^{+}_{G/e \ \& \ G/f} - T^{-}_{G/e \ \& \ G/f} \right)^2$$

$T^{\pm}_{G/e \ \& \ G/f}$ enumerate the $\pm$ common spanning trees.

Known Partial and Full Combinatorial Proofs

$$T_{G/f}T_{G/e} - T_G T_{G/e/f} = \left(T^{+}_{G/e\ \&\ G/f} - T^{-}_{G/e\ \&\ G/f}\right)^2$$

$T^{\pm}_{G/e\ \&\ G/f}$ enumerate the $\pm$ common spanning trees.

Choe (2004) proved essentially this using the vertex-based all-minors matrix tree theorem, combinatorial cases and Jacobi's theorem relating the minors of a matrix to the minors of its inverse..

Cibulka, Hladky, Lacroix and Wagner (2008) gave a completely bijective proof that utilizes some natural 2:2 and 2:1 correspondances.

Difficulty: Some terms on the left cancel and some reduce to terms with coefficients ± 2.

Linear Alg./Oriented Matroid Proof of Rayleigh's Identity

Let R be the transfer resistance matrix for 2 ports across e and f. Our result implies that

$$\det R = \begin{vmatrix} R_{ee} & R_{ef} \\ R_{fe} & R_{ff} \end{vmatrix} = +\frac{T_{G/e/f}}{T_G}$$

It and better-known results tell us

$$R_{ee} = \frac{T_{G/e}}{T_G}; \quad R_{ff} = \frac{T_{G/f}}{T_G}; \quad R_{ef} = R_{fe} = \frac{T^{+}_{G/e\ \&\ G/f} - T^{-}_{G/e\ \&\ G/f}}{T_G}$$

$T_{G/f}\,T_{G/e} - T_G\,T_{G/e/f} = \left(T^{+}_{G/e\ \&\ G/f} - T^{-}_{G/e\ \&\ G/f}\right)^2$ is immediate after substituting these into

$$\det R = R_{ee}R_{ff} - (R_{ef})^2$$

The $+$ follows from physical grounds if the $g_e, r_e \geq 0$. Our characterization and proof are combinatorial.

New Rayleigh's Identities!

The same method generates identities and inequalities from

$$\begin{vmatrix} R_{ee} & R_{ef} & R_{eg} \\ R_{fe} & R_{ff} & R_{fg} \\ R_{ge} & R_{gf} & R_{gg} \end{vmatrix} = +\frac{T_{G/e/f/g}}{T_G} \geq 0$$

when all $r_{..}, g_{..} \geq 0$, ETC...

(Applications???)

Might the same methods address a much harder problem: The same inequality for forests instead of spanning trees?

(Part 2) Common Covector Model

The cycle space of G_I GENERATES
the covectors of an
oriented matroid over $(E \mathbin{\dot\cup} P_I)$.

0

(signs indexed by E) (by P_I)

Non-linear monotone resistors CONSTRAIN SIGNS of
voltage drops (from ↓) and flows (from ↑)
TO BE EQUAL

(signs indexed by E) (by P_V)

The cocycle space of G_V GENERATES
the covectors of an
oriented matroid over $E \mathbin{\dot\cup} P_V$.

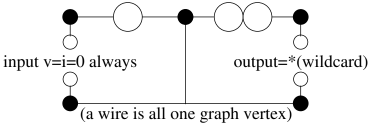
G_V
SOMETIMES EQUALS
G_I

0

Voltage and Current graphs N_V, N_I

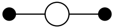
"Voltage graph" N_V (EE [5, 10], NOT Gross, ...) represents KVL $\mathbf{v} \in$ Cocycles W/ SOME $v_e \equiv 0$	"Current graph" N_I represents KCL $\mathbf{i} \in$ Cycles WITH SOME FLOWS $\equiv 0$

- They are EQUAL GRAPHS for resistor networks.
- For networks with idealized amplifiers, they are not equal.



The output voltage and current are whatever makes the input voltage and current BOTH BE zero.

- (More) realistic amp. model = idealized amp. + resistors.

open	short	nullator	norator
			
$N_v = N \backslash e$	$N_v = N / e$	$N_v = N / e$	$N_v = N \backslash e$
$N_I = N \backslash e$	$N_I = N / e$	$N_I = N \backslash e$	$N_I = N / e$

"Colors" are parameters on every Tutte decomposition step

The Bollobos/Riordan/Zaslavsky [2, 12], Traldi-Ellis-Monaghan [4], (sdc unpub) BRZ theory for well-definedness of "Relative Tutte Polynomials for Colored Graphs" ALL GOES THROUGH (Diao and Hetyei [3]): The 3 BRZ conditions on (colors,initial values) GENERALIZE TO 5; activity theory WORKS TOO, when based on linear orders on the non-port-elements.

In a nutshell

The 5 conditions $\implies$ activities define an unambiguous Tutte function from the deletion/contraction and initial value formulas. Additional conditions $\implies$ the Tutte function has a rank-nullity expansion.

(The rank-nullity conditions are satisfied in our application.)

To specify the activity/deletion-contraction linear order GLOBALLY is UNNECESSARY.

The Gordon/McMahon computation-tree-based activity theory also generalizes. (sdc).

References I

Marilena Barnabei, Andrea Brini, and Gian-Carlo Rota.
On the exterior calculus of invariant theory.
Journal of Algebra, 96:120–160, 1985.

B. Bollobas and O. Riordan.
A Tutte polynomial for colored graphs.
Combin. Probab. Computat., 8(1–2):45–93, 1999.

Yuanan Diao and Gábor Hetyei.
Relative Tutte polynomials for colored graphs and virtual knot theory.
Combin. Probab. Comput., 19(3):343–369, 2010.

J. A. Ellis-Monaghan and Lorenzo Traldi.
Parametrized Tutte polynomials of graphs and matroids.
Combinatorics, Probability and Computing, 15:835–854, 2006.

References II

M. Hasler and J. Neirynck.
Nonlinear Circuits.
Artech House, Norwood, Mass., 1986.

G. Kirchhoff.
Über die auflösung der gleichungen, auf welshe man bei der untersuchung der linearen verteilung galvanischer ströme gefuhrt wird.
Ann. Physik Chemie, 72:497–508, 1847.
On the solution of the equations obtained from the investigation of the linear distribution of Galvanic currents, (J. B. O'Toole, tr.) *IRE Trans. Circuit Theory*, 5, 1958, pp. 238–249.

References III

James Clerk Maxwell.
A Treatise on Electricity and Magnetism, volume 1, Part II, Appendix of Chapter VI, Mathematical Theory of the Distribution of Electric Currents, pages 409–410.
Claredon Press and reprinted by Dover, New York (1954), 3rd edition, 1891.

Stefan Müller, Elisenda Feliu, Georg Regensburger, Carsten Conradi, Anne Shiu, and Alicia Dickenstein.
Sign conditions for injectivity of generalized polynomial maps with applications to chemical reaction networks and real algebraic geometry.
Foundations of Computational Mathematics, 2015.

H. Narayanan.
On the decomposition of vector spaces.
Linear Algebra and its Applications, 79:61–98, 1986.

References IV

C. Sanchez-Lopez, F. V. Fernandez, E. Tlelo-Cuautle, and S. X. Tan.
Pathological element-based active device models and their application to symbolic analysis.
IEEE Transactions on Circuits and Systems-I-Regular Papers, 58(6):1382–1395, 2011.

Marc Thompson.
Intuitive Analog Circuit Design.
Newnes/Elsevier, 2nd edition, 2014.

Thomas Zaslavsky.
Strong Tutte functions of matroids and graphs.
Trans. Amer. Math. Soc., 334(1):317–347, 1992.

(15)

(16)

(17)

Our Tutte-like function $\mathbf{M}_E(\mathbf{N})$: Extensors $\to$ Extensors.
Given N (matrix), construct $N^\perp$ with orthog. comp. row space.
Construct: $(G = \mathrm{diag}(g_e),\ R = \mathrm{diag}(r_e))$

$$M = \left[\begin{array}{c|c|c} N(P) & 0 & N(E)G \\ \hline 0 & N^\perp(P) & N^\perp(E)R \end{array}\right]$$

with columns labelled by $P_I \cup P_V \cup E$.
Extensor $\mathbf{M}$ over $k[g_e, r_e](P_V \cup P_I \cup E)$ is the product of M's **row vectors**. The contraction result $\mathbf{M}_E(\mathbf{N}) = \mathbf{M}/E$ appears:

$$\mathbf{M} = \mathbf{M}_E(\mathbf{N})\mathbf{e_1}\mathbf{e_2}\cdots\mathbf{e}_{|E|} + (\cdots)$$

$\mathbf{M}_E(\mathbf{N})$ is our Tutte function $\mathbf{N} \to$ Ext. Alg.

What are the conditions like?

$$\text{Extensor } = \sigma(C)(a_1 \vee a_2 \vee \ldots \vee a_d)$$

$$C = \ldots + t + \ldots$$

$t \leftrightarrow$ sets of ("contractible" non-port edges E_1, E_2 for which

$$N_V/E_1 \backslash (E \setminus E_1) = \text{ certain OMs on } P$$

AND

$$N_I/E_2 \backslash (E \setminus E_2) = \text{ certain OMs on } P$$

Transfer resistance might be 0 and might be $\neq 0$ iff

$$\exists E_1, E_2 \text{so (diag) and (diag)}$$

(K_4 Wheatstone bridge diagram)

$$\frac{R_1}{R_2} > \frac{R_3}{R_4} \text{ neg } \frac{R_1}{R_2} < \frac{R_3}{R_4} \text{ pos}$$

Equivalent resistance is a coefficient ratio in an implicitly defined linear function

(diagram)
In other words

$$R_p i_p + v_p = 0$$

or dually,

$$\mathcal{B} = \{(i_p, v_p)\} = \{t(-1, R_p) | t \in R\}$$

The $2 \times d$ port variable constraint space, and its solution spaces, are d-dimensional.

We represent these spaces by carefully defined **extensors**, as Barnabei, Brini and Rota [1] term “decomposible antisymmetric tensors”
The solution extensor (not a ray) satisfies:

$$E(N) = \text{sign}(...)(g_e E(N/e) + r_e E(N \backslash e) \text{ for } e \notin P$$

Coefs

$$E(N) = \ldots + C_i \mathbf{XXX} + \ldots$$

$$R = C_i / C_j$$

All the other C_k's have similar interpretations.
Each C_k is a determinant.
Each C_k is a signed weighted enumerator of forests satisfying **conditions ...**
Each C_k satisfies

$$C_k(N) = g_e C_k(N/e) + r_e C_k(N \backslash e) \text{ for } e \notin P$$

Conditions

What is the nature of the conditions? We state them using the network's graphic oriented matroid.
(diagram–glob w/ ports)