$\mathbf{p}_{\alpha \mathbf{j}}, \mathbf{p}_{\beta \mathbf{j}}, \mathbf{e_j}$ : free generators, space basis elements

$$N_\alpha:\quad \begin{array}{|c|}\hline \cdot\, N_{\alpha ij}\mathbf{p}_{\alpha \mathbf{j}}\, \cdot \\ \hline\end{array} \quad \begin{array}{|c|}\hline \cdots 0 \cdots \\ \hline\end{array} \quad \begin{array}{|c|}\hline \cdots N_{\alpha ij}\widehat{\mathbf{e_j}} \cdots \\ \hline\end{array} \quad \longrightarrow \bigwedge \begin{pmatrix}\text{row}\\ \text{sums}\end{pmatrix}$$

matrices of coefs, representing matroids

$$N_\beta:\quad \begin{array}{|c|}\hline \cdots 0 \cdots \\ \hline\end{array} \quad \begin{array}{|c|}\hline N_{\beta_{ij}}\widehat{\mathbf{p}_{\beta \mathbf{j}}} \\ \hline\end{array} \quad \begin{array}{|c|}\hline \cdots N_{\beta_{ij}}\mathbf{e_j} \cdots \\ \hline\end{array}$$

With

$(\mathbf{p}_{\alpha \mathbf{i}} \dots | \widehat{\mathbf{p}_{\beta \mathbf{j}}} \dots) = \mathbf{p}_{\alpha \mathbf{i}} \wedge \widehat{\mathbf{p}_{\beta \mathbf{j}}} (\dots | \dots)$

$(\widehat{\mathbf{e}_{\mathbf{i_1}}}\widehat{\mathbf{e}_{\mathbf{i_2}}} \dots \widehat{\mathbf{e}_{\mathbf{i_k}}} | \mathbf{e}_{\mathbf{j_1}}\mathbf{e}_{\mathbf{j_2}} \dots \mathbf{e}_{\mathbf{j_k}}) = \delta^{i_1 i_2 \dots i_k}_{j_1 j_2 \dots j_k}$

$(\mathbf{N}_\alpha | \mathbf{N}_\beta)$ = def $\mathbf{L}_E(\mathbf{N}_\alpha; \mathbf{N}_\beta)$

$$\text{(1)} = \mathbf{L}_{E\setminus e}(\mathbf{N}_\alpha \setminus \mathbf{e}; \mathbf{N}_\beta \setminus \mathbf{e}) + \mathbf{L}_{E\setminus e}(\mathbf{N}_\alpha / \mathbf{e}; \mathbf{N}_\beta / \mathbf{e})$$

$\in \bigwedge P_\alpha \cup \widehat{P_\beta}$ : Exterior Algebra (anti-comm!)

$$\text{(2)} = \sum_{F\subseteq E}([\mathbf{N}_\alpha / F | P_\alpha] \quad | \quad [\mathbf{N}_\beta / F | \widehat{P_\beta}])$$

$$= \sum_{F\subseteq E}[\mathbf{N}_\alpha / F | P_\alpha] \wedge [\mathbf{N}_\beta / F | \widehat{P_\beta}]$$

Like $|\text{Graph Laplacian}| = |I\ I^t| = \sum_{F\subseteq E} |I(F)|^2$

$= \sum_{T:\text{span. tree}} 1$ (Cauchy-Binet expansion)

1. Distinguish matroid element “ports” associated with electric or elastic system parameter and solution variables of interest. (All vars are paired: (voltage, current), (force, displacement), etc. One gets a pair of submodels with dual matroids in elementary situations; not duals otherwise.)
2. Exterior algebra forms of deletion and contraction of a non-port yield a pair of simpler systems.
3. Cancelling non-port elements with a kind of bilinear pairing yields the parameter/solution variables of interest relation, in the form of an **exterior algebra valued function** of systems, that **is a Tutte function** (when the minor and direct sum operations are sign-consistent).

With the suitable incidence matrix form, we get the all-minors matrix tree theorem; but all the minors are packed into **one exterior algebra object** that is a Tutte function of graphs.
Seth Chaiken, Assoc. Prof. Emeritus, University at Albany.