

# Restricted or Ported Tutte Decomposition and Analogues of All-Minors Laplacian Expansions

Seth Chaiken

Assoc. Prof. Emeritus Dept. of Computer Science

Univ. at Albany

`schaiken@albany.edu`

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# What is a parametrized strong Tutte function?

Tutte equations are satisfied in a very general setup:

1. Elements  $\{e\}$  each with parameters  $g_e, r_e$ .
2. A category  $\mathcal{N}$  of objects  $\mathbf{N}$  each with ground set  $S = S(\mathbf{N})$  of elements.
3. For some *decomposable*  $\mathbf{N}$ , for one or more *separators*  $e \in S(\mathbf{N})$ , the *contraction* and *deletion* operations are defined with results  $\mathbf{N}/e$  and  $\mathbf{N} \setminus e$  in  $\mathcal{N}$ , with ground sets  $S(\mathbf{N}) \setminus \{e\}$
4. Some  $\mathbf{N} = \mathbf{N}_1 \oplus \mathbf{N}_2$  are direct sums, where  $S(\mathbf{N}_1) \cap S(\mathbf{N}_2) = \emptyset$ .
5. For each indecomposable  $\mathbf{N}$  with no separators there is an additional parameter  $i_{\mathbf{N}}$  called the *initial value*.

# Tutte equations, functions and Good Questions

1. For all  $\mathbf{N}$  with separator  $e \in S(\mathbf{N})$ ,

$$F(\mathbf{N}) = g_e F(\mathbf{N}/e) + r_e(\mathbf{N} \setminus e)$$

2. When  $\mathbf{N} = \mathbf{N}_1 \oplus \mathbf{N}_2$ ,

$$F(\mathbf{N}) = F(\mathbf{N}_1)F(\mathbf{N}_2)$$

3. When  $\mathbf{N}$  is indecomposable,

$$F(\mathbf{N}) = i_{\mathbf{N}}$$

$F$  is Tutte function when all the Tutte equations are satisfied.  
This MEANS  $F(\mathbf{N})$  is what is computed by applying Tutte equations *in any order they are applicable*.

Good Questions: When does  $\mathcal{N}$  and parameters ACTUALLY HAVE a Tutte function? If so, what is a *universal* Tutte function?

# Some answers—for Graphs and Matroids

## Only loops and coloops need initial values

The only  $\mathbf{N}$  with no separators and no  $\mathbf{N} = \mathbf{N}_1 \oplus \mathbf{N}_2$  for  $\mathbf{N}_i \neq \emptyset$  are **loop**( $e$ ) and **coloop**( $e$ ).

## The famous Tutte Polynomial

Adding all  $g_e = r_e = 1$ , the Tutte polynomial  $F(\mathbf{N})(x, y)$  obtained from  $i_{\text{loop}(e)} = x$ ,  $i_{\text{coloop}(e)} = y$  and  $i_\emptyset = 1$ . is a universal Tutte function.

## Normal Tutte Functions for Matroids

(Zaslavsky, Bollobás/Riordan) With arbitrary  $g_e, r_e$ , and  $x, y$ , the *normal* Tutte functions for matroids are obtained with

$i_{\text{coloop}(e)} = g_e y + x$ ,  $i_{\text{loop}(e)} = r_e x + y$  and  $i_\emptyset = 1$ . They are exactly the ones with a weighted rank-nullity generating function.

There's a big story about what relationships among the  $g_e, r_e, i_{\text{coloop}(e)}, i_{\text{loop}(e)}, i_\emptyset$  give others.

# Hopf Alg. from Minor Systems (Krajewski, Moffatt, Tanasa 2017)

## Definition (Minor System)

- ▶ Finite combinatorial objects  $\{N\}$  w/ ground sets  $E(N)$ , graded by  $|E(N)|$ ; unique 1 with  $E(1) = \emptyset$ ;  $E(N)$  consists of objects at level  $|E(N)|$ .
- ▶ For distinct  $e, f \in E(N)$ , deletion & contraction ops so both  $(\backslash\backslash e$  or  $//e)$  commute with both  $(\backslash\backslash f$  or  $//f)$ .

# Tutte Functions using determinants: Our setup

- ▶ Matrices  $N_\alpha, N_\beta^\perp$ ; full row rank, columns indexed by  $P \amalg E$ .  
 $\text{rank}(N_\alpha) + \text{rank}(N_\beta^\perp) = |E| + |P|$ .  
 $P_\alpha, P_\beta \leftrightarrow P, P_\alpha \cap P_\beta = \emptyset$ .
- ▶ Weight (parameter) matrices  
 $G = \text{diag}\{g_e\}_{e \in E}, R = \text{diag}\{r_e\}_{e \in E}$ .
- ▶ Matrix with columns  $P_\alpha \amalg P_\beta \amalg E$

$$L = L \left( \begin{array}{c} N_\alpha \\ N_\beta^\perp \end{array} \right) = \left[ \begin{array}{c|c|c} N_\alpha(P) & 0 & N_\alpha(E)G \\ \hline 0 & N_\beta^\perp(P) & N_\beta^\perp(E)R \end{array} \right]$$

Define

$$F(L) = ((\binom{2p}{p})) - \text{tuple of determinants } L[Q_\alpha \overline{Q_\beta} E(\text{all of } E)]$$

indexed by length  $p = |P|$  sequences  $Q_\alpha \overline{Q_\beta} \subseteq P_\alpha P_\beta$  where  
 $Q_\alpha \subseteq P_\alpha$  and  $\overline{Q_\beta} \subseteq P_\beta$ .

Column  $e$  of  $L$  when  $e \notin P$  is

$$\begin{bmatrix} N_{\alpha,1,e}g_e \\ N_{\alpha,2,e}g_e \\ \dots \\ N_{\alpha,r_1,e}g_e \\ N_{\beta,1,e}^\perp r_e \\ N_{\beta,2,e}^\perp r_e \\ \dots \\ N_{\beta,r_2,e}^\perp r_e \end{bmatrix} = \begin{bmatrix} N_{\alpha,1,e} \\ N_{\alpha,2,e} \\ \dots \\ N_{\alpha,r_1,e} \\ 0 \\ 0 \\ \dots \\ 0 \end{bmatrix} g_e + \begin{bmatrix} 0 \\ 0 \\ \dots \\ 0 \\ N_{\beta,1,e}^\perp \\ N_{\beta,2,e}^\perp \\ \dots \\ N_{\beta,r_2,e}^\perp \end{bmatrix} r_e$$

So, for all  $e \in E$ , that is  $e \notin P$ :

$$F(L)_{Q_\alpha \overline{Q_\beta}} = L[Q_\alpha \overline{Q_\beta} E] = \\ g_e L \left( \begin{array}{c} N_\alpha / e \\ N_\beta^\perp \setminus e \end{array} \right) [Q_\alpha \overline{Q_\beta} E] + r_e L \left( \begin{array}{c} N_\alpha \setminus e \\ N_\beta^\perp / e \end{array} \right) [Q_\alpha \overline{Q_\beta} E].$$

Since deletion and contraction are done only for  $e \notin P$

we get a **Ported** (sdc) or **Set-pointed** (Las Vergnas) or **restricted** (Dao and Hetyei) Tutte Function.

$|Q_\alpha \overline{Q_\beta}| = p$ , so  $\binom{2p}{p}$  determinants  $L[Q_\alpha \overline{Q_\beta} E]$  make the tuple:

$$F(L) = g_e FL \begin{pmatrix} N_\alpha / e \\ N_\beta^\perp \setminus e \end{pmatrix} + r_e FL \begin{pmatrix} N_\alpha \setminus e \\ N_\beta^\perp / e \end{pmatrix}$$

where

$N/e$  means remove the  $g_e$  or  $r_e$  but otherwise keep column  $e$

$N \setminus e$  means replace column  $e$  by 0.

## Plücker coordinates

These determinants can be considered an *affine* version of the (projective) Plücker coordinates for the row space of  $L$  projected into  $K^{P_\alpha} \amalg P_\beta$ . We need affine so Tutte's + identity makes sense.



$$FL \left( \begin{array}{c} N_\alpha \\ N_\beta^\perp \end{array} \right) = g_e FL \left( \begin{array}{c} N_\alpha/e \\ N_\beta^\perp \setminus e \end{array} \right) + r_e FL \left( \begin{array}{c} N_\alpha \setminus e \\ N_\beta^\perp/e \end{array} \right) \quad (*)$$

Real deletion/contraction removes  $e$  from the ground set of the matroid or other object, but  $N/e, N \setminus e$  still have column  $e$ . But  $(*)$  holds for all  $e \in E$ , so Laplace's expansion is a basis expansion:

$$L[Q_\alpha \overline{Q_\beta} E] = \sum_{A \subseteq E} g_A r_{\overline{A}} N_\alpha[Q_\alpha A] N_\beta^\perp[\overline{Q_\beta A}] \epsilon(Q_\alpha A, \overline{Q_\beta A})$$

The  $A$  term is  $\neq 0$  iff  $Q_\alpha A$  is a column basis for  $N_\alpha$  and  $\overline{Q_\beta A}$  is a column basis for  $N_\beta^\perp$ . So, for each  $Q_\alpha \overline{Q_\beta}$

$$L[Q_\alpha \overline{Q_\beta} E] = \pm \sum_{A \subseteq E} g_A r_{\overline{A}} N_\alpha[Q_\alpha A] N_\beta^\perp[\overline{Q_\beta A}] \epsilon(A, \overline{A})$$

(The non-zero terms all have  $|A| = \text{rank}(N_\alpha) - |Q_\alpha|$ .)

## Quick and dirty fix

1. Drag column  $e$  to the far right.  
Changes sign of  $F(L)$  by  $\epsilon(E'e)$ .
2. Left multiply by a determinant 1 matrix that sends the last column to  $(0, \dots, 1g_e, 0, \dots, 1r_e)^t$  (if the top or bottom submatrix has just 1 row, do the hack:  $\mathbf{N}/e$  is number  $\mathbf{N}_{1,e}$  that acts like a matrix with columns  $E'$  and no rows.)
3. Drag the row with the  $1g_e$  to the bottom.  
Changes sign of  $F(L)$  by  $(-1)^{r\mathbf{N}_\beta^\perp}$
4. With  $e$  deleted/contracted from the  $\mathbf{N}$ s defining  $L$ , define  $F$  by  $FL_{Q_\alpha \overline{Q_\beta}} = L[Q_\alpha \overline{Q_\beta} E']$

## Result

$$FL \left( \begin{array}{c} N_\alpha \\ N_\beta^\perp \end{array} \right) = \epsilon(E'e) \left( g_e (-1)^{r(N_\beta^\perp)} FL \left( \begin{array}{c} N_\alpha/e \\ N_\beta^\perp/e \end{array} \right) + r_e FL \left( \begin{array}{c} N_{\alpha \setminus e} \\ N_{\beta^\perp \setminus e} \end{array} \right) \right)$$

# Simplify calculations /w minors via Exterior Algebra

Full  $r$ -row minors of matrix  $N$  with columns indexed by  $S$ :

$$\begin{array}{ccc} (e_1) & (e_2) & (e_3) \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ \hline N[e_1 e_3] = (a_1 b_3 - a_3 b_1) \end{array}$$

Coefficients when the exterior product of  $N$ 's row vectors  $\mathbf{N}$  are expressed in basis

$\{\mathbf{e}_{i_1} \wedge \mathbf{e}_{i_2} \cdots \mathbf{e}_{i_r} \mid i_1 < i_2 < \cdots < i_r\}$ :

$$\begin{array}{c} (a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 + a_3 \mathbf{e}_3) \\ \wedge (b_1 \mathbf{e}_1 + b_2 \mathbf{e}_2 + b_3 \mathbf{e}_3) \\ \hline ((a_1 b_3 - a_3 b_1) \mathbf{e}_1 \mathbf{e}_3 + \cdots) \end{array}$$

We sometimes omit the  $\wedge$  and we can always write:

$$(\text{Exterior product})\mathbf{N} = \sum_{A \subseteq S; |E|=r} \mathbf{N}[A]\mathbf{A}$$

Each subset  $A$  is ordered  $a_1 a_2 \dots a_r$  **arbitrarily** but  $\mathbf{A}$  denotes the exterior product of (row coordinate vectors) **in the same order**

$$\mathbf{A} = \mathbf{a}_1 \mathbf{a}_2 \dots \mathbf{a}_r$$

# Catalogs of Oriented Matroid operations on the OM of matrix $N$ and on $\mathbf{N} = \wedge(\text{rows}(N))$

domain  $D$  of operations:

$D$  is which functions:

type of fun. value

chirotopes

$$\pm\chi : B \mapsto \text{sign}(N[B])$$

$$\text{sign} \in \{0, +, -\}$$

exterior products

$$\mathbf{N} : B \mapsto \mathbf{N}[B]$$

**field value; number**

deletion  $\bullet \setminus A$

$$\pm\chi' : B \mapsto \chi(B)$$

$$\mathbf{N} \setminus A : B \mapsto \mathbf{N}[B]$$

contraction  $\bullet / A$

$$\pm\chi' : B \mapsto \chi(BA)$$

$$\mathbf{N} / A : B \mapsto \mathbf{N}[BA]$$

duality  $\bullet^\perp$

$$\pm\chi^\perp : B \mapsto \chi(\overline{B})\epsilon(\overline{B}B)$$

$$\mathbf{N}^\perp : B \mapsto \mathbf{N}[\overline{B}]\epsilon(\overline{B}B)$$

We must choose some global orientation  $\epsilon$  in order to define duality as an exterior alg. operation!

$\epsilon$  is an alternating sign function on all finite sequences of elements.

This implies  
commutations

$$(\mathbf{N} \setminus X)^\perp = \epsilon(S')\epsilon(S'X)(\mathbf{N}^\perp / X)$$

$$(\mathbf{N} / X)^\perp = \epsilon(S')\epsilon(S'X)(-1)^{|X|r}\mathbf{N}^\perp(\mathbf{N}^\perp \setminus X)$$

## Our setup - again

- ▶ Matrices  $N_\alpha, N_\beta^\perp$ ; full row rank, columns indexed by  $P \amalg E$ .  
 $\text{rank}(N_\alpha) + \text{rank}(N_\beta^\perp) = |E| + |P|$ .  
 $P_\alpha, P_\beta \leftrightarrow P, P_\alpha \cap P_\beta = \emptyset$ .
- ▶ Weight (parameter) matrices  
 $G = \text{diag}\{g_e\}_{e \in E}, R = \text{diag}\{r_e\}_{e \in E}$ .
- ▶ Matrix with columns  $P_\alpha \amalg P_\beta \amalg E$

$$L \left( \begin{array}{c} N_\alpha \\ N_\beta^\perp \end{array} \right) = \left[ \begin{array}{c|c|c} N_\alpha(P) & 0 & N_\alpha(E)G \\ \hline 0 & N_\beta^\perp(P) & N_\beta^\perp(E)R \end{array} \right]$$

Define

$$F(L) = \left( \binom{2p}{p} \right) - \text{tuple of determinants } L[Q_\alpha \overline{Q_\beta} E]$$

indexed by sequences  $Q_\alpha \overline{Q_\beta} \subseteq P_\alpha P_\beta$  where  $Q_\alpha \subseteq P_\alpha$ ,  
 $\overline{Q_\beta} \subseteq P_\beta, |Q_\alpha \overline{Q_\beta}| = p = |P|$ .

$$L \left( \begin{array}{c} N_\alpha \\ N_\beta^\perp \end{array} \right) = \left[ \begin{array}{c|c|c} N_\alpha(P) & 0 & N_\alpha(E)G \\ \hline 0 & N_\beta^\perp(P) & N_\beta^\perp(E)R \end{array} \right] \quad F(L) = \text{tuple } (L[Q_\alpha \overline{Q_\beta} E])$$

Translate into exterior algebra definitions:

$$\begin{aligned} \mathbf{L} \left( \begin{array}{c} \mathbf{N}_\alpha \\ \mathbf{N}_\beta^\perp \end{array} \right) &:= (\iota(\mathbf{N}_\alpha)(P_\alpha) + \iota_G(\mathbf{N}_\alpha(E))) \wedge (v(\mathbf{N}_\beta^\perp)(P_\beta) + v_R(\mathbf{N}_\beta^\perp)(E)) \\ &= (\iota_G(\mathbf{N}_\alpha) \wedge v_R(\mathbf{N}_\beta^\perp)) \end{aligned}$$

$$\mathbf{F}_E(\mathbf{L}) := \mathbf{L}/E = \sum_{Q_\alpha, \overline{Q_\beta}} \mathbf{L}[Q_\alpha \overline{Q_\beta} E] \mathbf{Q}_\alpha \overline{\mathbf{Q}_\beta}$$

$$\begin{aligned} &= ((\iota(\mathbf{N}_\alpha) \setminus e(\text{no } \mathbf{e}) + g_e(\iota(\mathbf{N}_\alpha)/e) \wedge \mathbf{e}) \\ &\quad \wedge (v(\mathbf{N}_\beta^\perp) \setminus e(\text{no } \mathbf{e}) + r_e(v(\mathbf{N}_\beta^\perp)/e) \wedge \mathbf{e}))/E \end{aligned}$$

$$\text{2 of 4 terms} = \left( r_e \quad \iota(\mathbf{N}_\alpha) \setminus e \wedge (v(\mathbf{N}_\beta^\perp)/e) \wedge \mathbf{e} \right.$$

$$\text{vanish} \quad \left. + g_e(-1)^{r(\mathbf{N}_\beta^\perp)} (\iota(\mathbf{N}_\alpha)/e) \wedge (v(\mathbf{N}_\beta^\perp) \setminus e) \wedge \mathbf{e} \right) / E$$

$$L \left( \begin{array}{c} N_\alpha \\ N_\beta^\perp \end{array} \right) = \left[ \begin{array}{c|c|c} N_\alpha(P) & 0 & N_\alpha(E)G \\ \hline 0 & N_\beta^\perp(P) & N_\beta^\perp(E)R \end{array} \right] \quad F(L) = \text{tuple } (L[Q_\alpha \overline{Q_\beta} E])$$

$$\begin{aligned} \mathbf{F}_E(\mathbf{L}) = \mathbf{L}/E = & \left( r_e \quad \iota(\mathbf{N}_\alpha \setminus e) \wedge (v(\mathbf{N}_\beta^\perp/e)) \wedge \mathbf{e} \right. \\ & \left. + g_e(-1)^{r(\mathbf{N}_\beta^\perp)} (\iota(\mathbf{N}_\alpha/e)) \wedge (v(\mathbf{N}_\beta^\perp \setminus e)) \wedge \mathbf{e} \right) / E \end{aligned}$$

$$= r_e \left( \mathbf{L} \left( \begin{array}{c} \mathbf{N}_\alpha \setminus e \\ \mathbf{N}_\beta^\perp/e \end{array} \right) \wedge \mathbf{e}/E \right) + g_e(-1)^{r(\mathbf{N}_\beta^\perp)} \left( \mathbf{L} \left( \begin{array}{c} \mathbf{N}_\alpha/e \\ \mathbf{N}_\beta^\perp \setminus e \end{array} \right) \wedge \mathbf{e}/E \right)$$

$$(\mathbf{N} \setminus e)^\perp = \epsilon(S')\epsilon(S'e)(\mathbf{N}^\perp/e) ; (\mathbf{N}/e)^\perp = \epsilon(S')\epsilon(S'e)(-1)^{|\{e\}|r\mathbf{N}^\perp}(\mathbf{N}^\perp \setminus e)$$

Result

$$= \epsilon(S)\epsilon(S'e) \left( r_e \left( \mathbf{L} \left( \begin{array}{c} \mathbf{N}_\alpha \setminus e \\ (\mathbf{N}_\beta \setminus e)^\perp \end{array} \right) \wedge \mathbf{e}/E \right) + g_e \left( \mathbf{L} \left( \begin{array}{c} \mathbf{N}_\alpha/e \\ (\mathbf{N}_\beta/e)^\perp \end{array} \right) \wedge \mathbf{e}/E \right) \right)$$

With  $\mathbf{L}(\mathbf{N}_\alpha \ \mathbf{N}_\beta) = \mathbf{L} \left( \begin{array}{c} \mathbf{N}_\alpha \\ \mathbf{N}_\beta^\perp \end{array} \right)$ , and more sign calculations:

### Definition

For  $E, P$  sets written as ordered sequences,

$$\mathbf{F}_E(\mathbf{N}_\alpha \ \mathbf{N}_\beta) = \mathbf{L}(\mathbf{N}_\alpha \ \mathbf{N}_\beta)/E$$

### Theorem

$$\begin{aligned} \epsilon(PE)\mathbf{F}_E(\mathbf{N}_\alpha \ \mathbf{N}_\beta) = \\ \epsilon(PE')(g_e\mathbf{F}_{E'}(\mathbf{N}_\alpha/e \ \mathbf{N}_\beta/e) + r_e\mathbf{F}_{E'}(\mathbf{N}_\alpha \setminus e \ \mathbf{N}_\beta \setminus e)) \end{aligned}$$



## Corollary

1.  $\mathbf{F} = \mathbf{F}_E(\mathbf{N}_\alpha \mid \mathbf{N}_\beta) = \pm \sum_{H \subseteq E} g_H r_{\overline{H}} \mathbf{F}_\emptyset(\mathbf{N}_\alpha / H \setminus \overline{H} \mid \mathbf{N}_\beta / H \setminus \overline{H})$
2. Componentwise,  $\sum_{Q_\alpha, Q_\beta} \mathbf{F}_E[Q_\alpha \overline{Q_\beta}] Q_\alpha \overline{Q_\beta} =$

$$= \pm \sum_{Q_\alpha, Q_\beta} \sum_{H \in E} g_H r_{\overline{H}} \mathbf{N}_\alpha[Q_\alpha H] \mathbf{N}_\beta^\perp[\overline{Q_\beta H}]$$

$$= \pm \sum_{Q_\alpha, Q_\beta} \sum_{H \in E} g_H r_{\overline{H}} \mathbf{N}_\alpha[Q_\alpha H] \mathbf{N}_\beta[Q_\beta H]$$

3. Two expr. for products of numbers  $\mathbf{N}_\alpha[Q_\alpha H] \mathbf{N}_\beta[Q_\beta H]$ :

$$(\mathbf{N}_\alpha / Q_\alpha)[H] \cdot (\mathbf{N}_\beta / Q_\beta)[H] = (\mathbf{N}_\alpha / H)[Q_\alpha] \cdot (\mathbf{N}_\beta / H)[Q_\beta]$$

4. It's non-zero iff  $H$  is a common basis (in the matroids of)  $\mathbf{N}_\alpha / Q_\alpha$  and  $\mathbf{N}_\beta / Q_\beta$   
 iff  $Q_\alpha$  is a basis in  $\mathbf{N}_\alpha / H$  and  $Q_\beta$  is a basis in  $\mathbf{N}_\beta / H$

# Weighted Laplacian-like matrices

Generalize a graph's incidence matrix: Make  $P$  label the rows,  $E$  the columns of any matrices  $A_\alpha, A_\beta$ . Take all  $r_e \neq 0$ . Then,  $N_\alpha = (I(P) \ A_\alpha(E))$  and  $N_\beta = (I(P) \ A_\beta(E))$ , and

$$L \begin{pmatrix} N_\alpha \\ N_\beta^\perp \end{pmatrix} = \left[ \begin{array}{c|c|c} I & 0 & A_\alpha G \\ \hline 0 & -A_\beta^t & IR \end{array} \right] = L(N_\alpha \ N_\beta). \text{ Do row ops:}$$

$$\begin{pmatrix} I & -A_\alpha GR^{-1} \\ 0 & R^{-1} \end{pmatrix} L = \begin{pmatrix} I & A_\alpha GR^{-1} A_\beta^t & 0 \\ 0 & -R^{-1} A_\beta^t & I \end{pmatrix}, \text{ and therefore}$$

$$\epsilon(Q_\alpha \overline{Q_\alpha}) \mathbf{F}_E(\mathbf{L})[Q_\alpha \overline{Q_\beta}] = \frac{1}{r_E} \sum_{B \in E} g_B r_{\overline{B}} A_\alpha[\overline{Q_\alpha} B] A_\beta[\overline{Q_\beta} B]$$

is the Cauchy-Binet expansion of any minor  $(\overline{Q_\alpha}, \overline{Q_\beta})$  of the weighted graph Laplacian-like matrix  $A_\alpha GR^{-1} A_\beta^t$ .

(Note  $\frac{1}{r_E} r_{\overline{B}} = (r^{-1})_B$ .)

## Examples

$N_\alpha = N_\beta = N$ ;  $A =$  graph's incidence matrix w/ columns  $(0, \dots, 0, 1, 0, \dots, -1, 0, \dots, 0)^t$  for each edge; reps. graphic matroid.

$A_\alpha, N_\alpha$  as above.  $A_\beta =$  only the  $+1$  entries of  $A$  for a directed graph, so  $+1$  is for an edge head on a vertex.

$N_\alpha = N_\beta = N$ ;  $A =$  gain graph's incidence matrix w/ columns  $(0, \dots, 0, 1, 0, \dots, -\gamma_e, 0, \dots, 0)^t$  for  $e$  with gain  $\gamma_e \in \mathbf{C}$ .

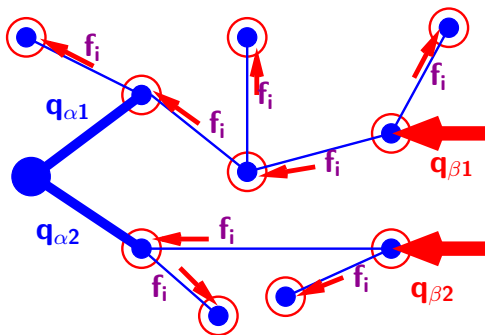
NB: Edge Gains  $\gamma_e$  are DIFFERENT ATTRIBUTES from weights/parameters  $g_e, r_e$

The all-minors  
Matrix Tree Theorem  
for weighted undirected graphs

The all-minors  
Matrix Tree Theorem  
for weighted directed graphs

All-minors expansions of  
the gain graph's Laplacian

## All-Minors Digraph Matrix Tree Theorem Example

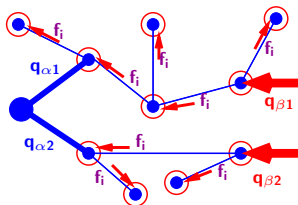


This contributes the term

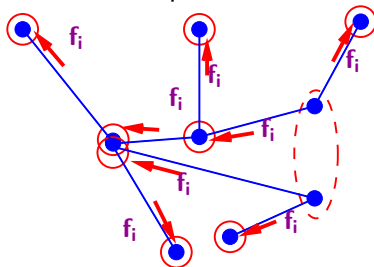
$$g_F r_{\overline{F}} \mathbf{N}_\alpha [Q_\alpha F] \mathbf{N}_\beta [Q_\beta F].$$

The  $\mathbf{q}_{\alpha 1}, \mathbf{q}_{\alpha 2}$  port edges  $\cup$  the  $f_i$  elements as edges in the graphic matroid comprise a spanning tree.

The  $\mathbf{q}_{\beta 1}, \mathbf{q}_{\beta 2}$  port arrows  $\cup$  the  $f_i$  elements as arrows in a partition matroid comprise a basis. Each part (a red circle) of the partition is the set of arrows incident to a vertex, except the star vertex.



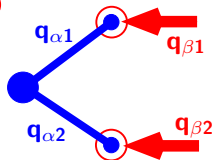
Contract the ports.



Count the bases in common.

$$gFr_{\overline{F}} \mathbf{N}_{\alpha} / Q_{\alpha}[F] \mathbf{N}_{\beta} / Q_{\beta}[F].$$

Contract the non-ports.



$\alpha$  and  $\beta$  ports are  
bases in the contracted  
 $N_{\alpha}$  and  $N_{\beta}$  matroids.

$$gFr_{\overline{F}} \mathbf{N}_{\alpha} / F[Q_{\alpha}] \mathbf{N}_{\beta} / F[Q_{\beta}].$$

# Chain Complexes View (Alg. Topology, Homological Alg.)

A graph is a  $k$ -dim simplicial complex  $X$  with  $k = 1$ .

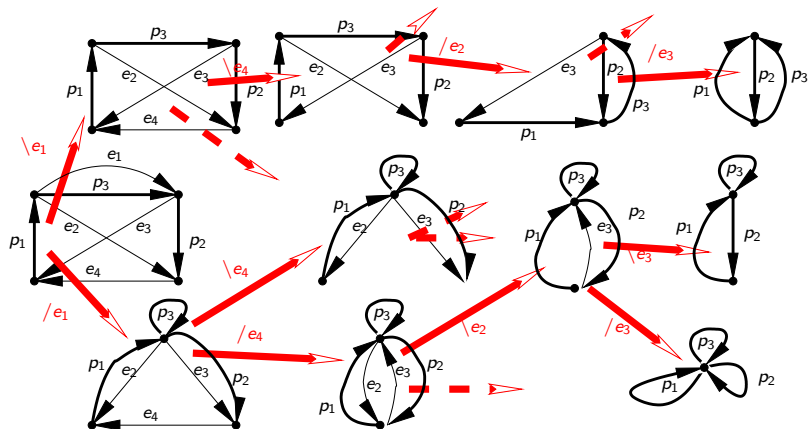
In general, for us, the  $k$ -chains  $C_k = Z[P \amalg E] = \{\sum_{x \in P \amalg E} c_x e\}$  are the free abelian group with basis  $P \amalg E$ .

The  $k$ -cochains  $C^k = \text{Hom}(C_k, \mathbb{R})$  is the  $\mathbb{R}$ -module of linear maps from  $C_k$  to a coefficient ring  $\mathbb{R}$ .

The  $k$ -complex  $X = \coprod_{j=0}^k X_j$  ( $X_j$  is the set of  $j$ -simplices) determines, (or the chain complex might just be subspaces given with) **boundary maps**  $\partial_j : C_j \rightarrow C_{j-1}$  for  $j = 0, \dots, k$  that satisfy  $\partial_{j-1} \circ \partial_j = 0$  for each  $j$ .

The dual  $\delta^j : C^{j-1} \rightarrow C^j$  is defined by  $(\delta^j(u^*))(v) = u^*(\partial_j(v))$ .

# Ported Tutte Decomposition (incomplete)



# Known to EEs: Linear electrical networks with IDEAL AMPLIFIERS

$N_\alpha i(P, E) = 0$  expresses Kirchhoff's current law on currents  $i_e$  in the network edges (along edge direction) and currents  $i_p$  into vertices from external connections.

$N_\beta^\perp v(P, E) = 0$  expresses Kirchhoff's voltage law: The voltage rise along a network edge  $v_e = v_h - v_t$  is the difference of the head and tail vertex potentials. (Sometimes the vertex potentials are imposed by external connections.)

$N_\alpha = N_\beta$  in ordinary resistor networks.

Different Graphs for  $N_\alpha$  and  $N_\beta$

W. K. Chen models networks with ideal amplifiers by  $N_\alpha$  by one graph on  $(P, E)$  called the **Current Graph** and another graph also on  $(P, E)$  called the **Voltage Graph**.