SOME PORTED, RELATIVE, OR SET POINTED PARAMETRIZED TUTTE FUNCTIONS

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Abstract. Tutte decompositions with deletion and contraction not done of elements in a fixed set of ports P, and the resulting polynomial expressions and functions for matroids, oriented matroids, graphs and an abstraction of labelled graphs are investigated. We give conditions on parameters x_e, y_e, X_e, Y_e for $e \notin P$, and on initial values $I(Q)$ for indecomposibles, that are necessary and sufficient for the following equations to have a well-defined solution: $T(M) = X_e T(M/e)$ for coloops $e \notin P$, $T(M) = Y_e T(M \setminus e)$ for loops $e \notin P$, and $T(M) = x_e T(M/e) + y_e T(M \setminus e)$ for other $e \notin P$. They generalize similar conditions given by Bollobás and Riordon, Zaslavsky, and Ellis-Monaghan and Traldi for Tutte functions defined without the $e \notin P$ restriction. We complete the generalization to matroids given Diao and Hetyei which was motivated by invariants for the virtual knots studied by Kauffman. Our motivations include electrical network analysis, oriented matroids, and negative correlation of edge appearances in spanning trees. The P-ported Tutte polynomials of oriented matroids express orientation information that ordinary Tutte polynomials cannot. The computation tree formalism of Gordon and McMahon gives activities expansions for P-ported parametrized Tutte polynomials more general than those just determined by the linear element orderings which originated in Tutte's dichromate.

The polynomials expressing conditions for the above Tutte matroid equations to have a solution all have one factor $I(Q_i)$. Since the elements are labelled, the methodology also applies to objects such as graphs with ports for which similar ZBR theorems can be proven. We abstract graphs to objects that have ported Tutte functions because they have matroids, but might have different Tutte function values on two objects with the same matroid. Two new ZBR-type theorems are given and are used to generalize the ZBR theorem to graphs with port edges. The abstraction is then used to characterize ported Tutte functions of an object combination, or a distinct object, whose matroid or oriented matroid is a direct matroid or oriented matroid sum. This extends with ports some known strong Tutte function and multiplicative Tutte function results.

1. Introduction

In his 1971 paper [3] "A Combinatorial Model for Series-Parallel Networks," Thomas H. Brylawski addressed series/parallel graphs, matroids, and series/parallel connections of matroids from the Tutte polynomial point of view. The rules for combining graphs or matroids in series or parallel, and thus for generating series/parallel graphs and matroids, refer to a basepoint edge or element in each. To study Tutte polynomials of series or parallel connections and so help characterize series/parallel matroids as having Tutte invariant $\beta = 1$, Brylawski developed a Tutte polynomial for "pregeometries with basepoint p_0" with the four variables z, x, z' and x'. His polynomial satisfies the well-known Tutte equations with

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deletion and contraction allowed only for $e \neq p_0$. Our first motivation is to generalize this to $e \notin P$ for a set P of elements which we call *ports*.

Brylawski developed this polynomial from a "class of polynomials whose variables are pointed and nonpointed pregeometries over the integers." Such variables, for matroids or oriented matroids having only distinguished elements, are present in our universal solution for Tutte equations with (1) Brylawsky's restriction extended from one p_0 to many distinguished elements, and (2) with parameters attached to the other elements. (See Corollary 6 and [6].) Additional references for this and the generalization to specify more than one element to forbid from being deleted or contracted are given in sec. 6.3.

When parameters are attached to elements (see (TA) and (TSSM) below), the equations are not consistant except if certain algebraic conditions on the parameters and values on indecomposibles are satisfied. (See 1.1 below.) Those conditions for $P = \emptyset$ were developed by Zaslavsky[26] and Bollobás and Riordon[2], and then expressed for a common generalization thereof by Ellis-Monaghan and Traldi[14], now called *separator-strong Tutte functions*. Our second motivation is to extend the theory, as formulate by the latter authors to nonempty P. We again find algebraic necessary and sufficient conditions on the parameters and values on indecomposibles. Now the indecomposibles are matroids or oriented matroids whose ground sets are contained in P, which we call the set of *ports*. Diao and Hetyei [11] characterize some solutions defined with the same restriction we do, and give a natural application to virtual knot invariants [19]. This led us to complete the generalization to all separator-strong parametrized P-ported Tutte functions. Here is the main definition:

Definition 1. *Let P be a set of* ports *and $\mathcal{C}$ be a class of matroids or oriented matroids closed under taking minors or oriented minors by, for $e \notin P$, deleting e if e is not a coloop or contracting e if e is not a loop. Such a $\mathcal{C}$ is called a P-family. $\mathcal{C}$ is given with four parameters in a commutative ring R, (x_e, y_e, X_e, Y_e), for each $e \notin P$ that is an element in some $M \in \mathcal{C}$.*

A separator-strong P-ported Tutte function T maps $\mathcal{C}$ to R or to an R-module and satisfies conditions (TA) *and* (TSSM) *below for all $M \in \mathcal{C}$ and all e in M.*

(TA)
$$T(M) = x_e T(M/e) + y_e T(M \setminus e)$$
if $e \notin P$ and e is a non-separator, i.e., neither a loop nor a coloop.

(TSSM)
If $e \notin P$ is a coloop in M then $T(M) = X_e T(M/e)$.
If $e \notin P$ is a loop in M then $T(M) = Y_e T(M \setminus e)$.

The $M \in \mathcal{C}$ for which neither (TA) *nor* (TSSM) *apply are called* indecomposibles *and the P-*quotients *in $\mathcal{C}$. The analogous definitions are used for graphs, directed graphs, and any other objects on which deletion and contraction are defined and act on matroids or oriented matroids associated to the objects.*

We follow many lines from Ellis-Monaghan and Traldi's paper[14]. First, we prove the generalization of their Zaslavsky, Bollobás and Riordon (ZBR) theorem for matroids; and find that some such Tutte functions depend on matroid orientation. Second, we develop an activities expansion (anticipated by Diao and Hetyei [11] and our earlier work[6]) for any recursive computation of the Tutte function value, not just those determined by a linear element order. The computations and expansions are nicely formalized by Gordon and MacMahon's (Tutte) *computation tree*[] which we adopt. It facilitates our enterprise to study the effects of distinguishing certain elements, those in P, so they are never deleted or contracted in the course of Tutte decompositions that carry the x_e, y_e, X_e, Y_e parameters into the resulting polynomials. Third, this activities expansion helps to derive the generalization of the corank-nullity polynomial for *normal* Tutte functions, introduced by Zaslavsky[]. We also give expansions for that based on boolean lattice interval partitions and on the matroid's geometric lattice. Finally, we use an abstraction of graphs called *objects with matroids or oriented matroids* to better understand P-ported Tutte functions of graphs and the generalizations of *strong* Tutte functions identified by Zaslavsky[] for matroids and graphs. We include examples to illustrate the complications. The results include two abstracted ZBR-type theorems which are applied to P-ported graphs. In one of them, one of the five ZBR-type conditions is essentially modified.

A "port" edge, "two terminals with the restriction that the terminal currents have the same magnitude but opposite sign"[9], as termed in the circuit theory used by electrical engineers, is also crucial for analyzing and generalizing the electrical resistance of a network. Brylawsky's 1977 work on "A Determinantal Identity for Resistive Networks" [4] helped underpin this and our earlier work (see for example [7] in addition to [6] and [8]) where one port is generalized to many. Kirchhoff first showed, essentially, that network resistance is the ratio of two parametrized spanning tree enumerating Tutte polynomials, one for the network with the port deleted and the other for the nework with the port contracted. Besides network resistance, other useful generalizations of Tutte polynomial theory defined by attaching parameters to elements are known[14, 12].

Equations (TA) and (TSSM) reduce to the well-known identities for the two-variable Tutte polynomial when the parameters are $x_e = y_e = 1$, $X_e = x$ and $Y_e = y$ for all e and the set $P = \emptyset$. The resulting Tutte polynomials or functions have been called *set-pointed* [22, 21], *ported* [6, 8] and *relative* [11]. We prefer the terms "port" and "P-ported equations or functions" because of our applications [7, 8] and because the P can be specified.

Besides (TA), classical Tutte polynomials and other so-called *strong Tutte functions* [26] satisfy the multiplicative identity

$$T(M_1 \oplus M_2) = T(M_1)T(M_2). \tag{TSM}$$

Different combinations of variations of Tutte equations determine different kinds of Tutte functions. For example, the additive identity (TA) alone (with $P = \emptyset$) characterizes *weak Tutte functions* [26, 27].

Evidently, (TA) and (TSSM) specify how $T(M)$ can be recursively calculated from the initial values $T(Q) = I(Q)$ on indecomposibles Q. The definition that T is a function means that all calculations of $T(M)$ for $M \in \mathcal{C}$ using (TA), (TSSM) and the initial values yield the same result. A simple induction on $|E(M)|$, the number of elements in M that are not in P, shows that if a Tutte function with specified values on the indecomposible matroids or oriented matriods exists for the given P-family and parameters, then the function is unique.

Restricting Tutte decomposition operations to $e \notin P$ gives some new information about oriented matroids and oriented (i.e., directed) graphs. The indecomposible matroids (or graphs) are minors of M that have all their elements (or edges) in P. We also call them P-quotients. If M is an oriented matroid, each indecomposible is an oriented matroid because the oriented minor $M/A \setminus B$ is well-defined when A, B partitions E. Hence, when $P \neq \emptyset$, P-ported Tutte functions can have different values on different orientations of the same orientable matroid. Other Tutte functions with this same domain can be defined after forgetting the orientations. Many of our results will be stated for "matroids or oriented matroids" because there are different indecomposibles and Tutte functions depending on whether or not the matroids carry an orientation. Analogous statements apply to graphs versus directed graphs and to any other objects with oriented matroids.

1.1. **Complications from Parametrization.** It is known that, even when $P = \emptyset$, (TA) fails to have a solution T except if certain algebraic relations are true about the parameters and initial values.

For example, if $M = U_1^{ef}$ is e, f in parallel, then applying (TA) for e gives the polynomial (U_0^e, U_1^e are the loop, coloop matroids on $\{e\}$, etc.)

$$x_e T(U_0^f) + y_e T(U_1^f)$$

whence applying it for f gives

$$x_f T(U_0^e) + y_f T(U_1^e).$$

Then, (TSSM) tells us $T(U_0^f) = Y_f I(\emptyset)$, $T(U_1^f) = X_f I(\emptyset)$, etc. The above are different polynomials in the parameters and initial values. The equation that says they are equal is an example of a relation that is necessary for a solution to exist. When the Tutte identities are parametrized, it is important to carefully distinguish between a solution value $T(M)$, where a solution T is a function that satisfies all the relevent identities, and a formal polynomial that results from using a subset of the identities to calculate $T(M)$ for one M[26].

We follow [2, 14] to say $T(M)$ is well-defined when the parameters are in ring R and the initial values are in R or an R-module for which the polynomial expressions for $T(M)$ obtained by all the recursive applications of the relevant Tutte equations are equal. The conditions are conveniently expressed as generators for the ideal I such that the universal Tutte function is into a quotient ring or module modulo I. See Corollary 6.

1.2. **Summary.** We first generalize the main result of [14], is called the Zaslavsky-Bollobás-Riordan (ZBR) theorem for matroids, to P-ported matroids or oriented matroids.

Theorem 2. *Let R be a commutative ring, let $\mathcal{C}$ be a minor-closed class of matroids defined on subsets of an R-parametrized class U, and let $\alpha \in R$. Then there is a parametrized Tutte polynomial on $\mathcal{C}$ with $T(\emptyset) = \alpha$ if and only if the following identities are satisfied.*

a *Whenever e and f are dyadic in $M \in \mathcal{C}$ (i.e., they constitute a two element circuit),*
$$\alpha \cdot (x_e Y_f + y_e X_f) = \alpha \cdot (x_f Y_e + y_f X_e).$$

b *Whenever e, f and g are triangular in $M \in \mathcal{C}$ (i.e. they constitute a three element circuit),*
$$\alpha \cdot X_g \cdot (x_e Y_f + y_e x_f) = \alpha \cdot X_g \cdot (Y_e x_f + x_e y_f).$$

c *Whenever e, f and g are triadic in $M \in \mathcal{C}$ (i.e. they constitute a three element cocircuit),*
$$\alpha \cdot Y_g \cdot (x_e Y_f + y_e x_f) = \alpha \cdot Y_g \cdot (Y_e x_f + x_e y_g).$$

The terminology "R-parametrized class U" (of matroid elements) was used in [14] to emphasize that the parameters (x_e, y_e, X_e, Y_e) are attached to elements e, not equations. The assumption that $\mathcal{C}$ is minor-closed implies that the three conditions can be restricted to the pair or triple being the only elements in M. Our generalization, Theorem 4 (sec. 3) is expressed that way.

The one indecomposible for the ZBR theorem is the empty matroid $\emptyset$. Loop or coloop matroids on e are decomposible; the values of the Tutte function on them equal respectively $Y_eT(\emptyset)$ and $X_eT(\emptyset)$. In our generalization of Theorem 2, the indecomposibles are those matroids Q_i in $\mathcal{C}$ whose ground set $S(Q_i) \subseteq P$. When $P \neq \emptyset$, instead of every monomial resulting from a Tutte decomposition containing the factor $T(\emptyset)$, every monomial resulting from a Tutte decomposition of M contains a factor $T(Q_i)$ where Q_i is some minor of M obtained by contracting or deleting every $e \notin P$. These *initial* values must be given for those Q_i in order to define a particular Tutte function. They generalize the $\alpha = T(\emptyset)$ value.

Our generalization of the ZBR theorem first replaces $T(\emptyset) = \alpha$ in its three equations with $I(Q_i)$. It also adds two more equations pertaining to $M \in \mathcal{C}$ in which series and parallel pairs $\{e, f\}$, $e \notin P$, $f \notin P$ are connected to one or more elements of P. Again, only one Q_i appears in each equation. See Theorem 4. We let some Tutte functions take values in modules to facilitate both defining universal Tutte functions with quotient modules and expressing formulas for Tutte functions of direct sums and related combinations.

An interesting consequence is that no relationships are required between $T(Q_i)$ and $T(Q_j)$ for different indecomposibles Q_i and Q_j for the Tutte function to be well-defined. This answers a question we raised in [8].

Second, we extend to $P \neq \emptyset$ the activities Tutte polynomial expressions and corresponding interval partitions of the Boolean lattice $2^{S(M) \setminus P}$. Like the activities expressions given by [17] for greedoid Tutte polynomials, the expressions we present are not just those obtained when a fixed linear ordering on the elements is used to determine for which element e to apply (TA) when two or more elements are eligible. Each of our activities expressions is based on a formal (Tutte) *computation tree* as defined in [17].

The remainder of the paper abstracts graphs with port edges and/or labelled vertices to objects with matroids or oriented matroids. We then prove ZBR-type theorems and characterize the behavior of P-ported Tutte functions when the matroid of an object, which might or might be a disjoint union or other combination of other object, be is a matroid or oriented matroid direct sum. Proofs as in [14] are based on the one indecomposible matroid being $\emptyset$, and on there being just one indecomposible graph (the edgeless graph E_k with k vertices) all with the same matroid $\emptyset$ in each minor closed subclass of graphs. Our abstraction and Tutte computation trees seem to make it easier to generalize these results. They also lead to new type of ZBR-theorem that pertains the situation illustrated by equation 7.

Finally, we specialize to graphs and so extend some of the graph results of [14]. Additional hypotheses are given for P-ported Tutte functions of graphs, including those with labelled vertices, for characterizations like Theorems 2 and 4 to be true.

2. Preliminaries

For a matroid or oriented matroid M, the ground set of *elements* is denoted by $S(M)$ and the rank function is denoted by r. Given *port* set P, $S(M) \setminus P = \{e \in S(M) \mid e \notin P\}$ is denoted by $E(M)$. A matroid, oriented matroid or other object with elements given with a set of ports P is be called *P-ported.* In graphs and directed graphs, the elements are the edges. Graphic matroids or oriented matroids always refer to the circuit matroids. Graph connectivity refers to path connectivity whereas connectivity in the graphic matroid refers to 2-edge-connectivity of the graph.

A *P-family* is a collection $\mathcal{C}$ of matroids or oriented matroids such that given $M \in \mathcal{C}$ and $e \in E(M)$, the contraction $M/e \in \mathcal{C}$ if e is not a loop in M and the deletion $M \setminus e \in \mathcal{C}$ if e is not a coloop in M. The set of non-port elements is $E(\mathcal{C}) = \{e \mid e \in E(M) \text{ for some } M \in \mathcal{C}\}$. It is straightforward to extend these definitions to families of objects, such as graphs, where each member object has an associated matroid on appropriate elements of the object, such as edges, and those elements can be deleted and/or contracted consistantly with the matroid.

The P-minors of M are obtained by deleting or contracting zero or more non-port elements. Thus, a P-family is a P-minor closed collection of matroids or oriented matroids. The P-minors Q_i for which $S(Q_i) \subseteq P$, i.e., those with no non-port elements, are called the P-quotients of M. We say a P-quotient belongs to $\mathcal{C}$ if it is a P-quotient of some $M \in \mathcal{C}$. Note that if P is finite, (and the objects are just matroids or oriented matroids) there are only a finite number of P-quotients because there are only a finite number of matroids or oriented matroids over subsets of P.

As usual, a *separator* is an element that is a loop, or is a coloop, i.e., an isthmus in a graph.

Let R be a commutative ring. We sometimes assume that a P-family $\mathcal{C}$ comes equipped with four parameters $x_e, y_e, X_e, Y_e \in R$ for each $e \in E(\mathcal{C})$ and for each P-quotient $Q \in \mathcal{C}$, one initial value $I(Q)$ either in R or in an R-module. (One can consider R to be the R-module generated by itself.) Note that whether or not the empty matroid $\emptyset$ is a P-quotient depends on $\mathcal{C}$. If $M \in \mathcal{C}$ with $S(M) \cap P = \emptyset$ then $\emptyset$ certainly is a P-quotient. In that case, (TSSM) specifies that $T(M) = X_e I(\emptyset)$ or $T(M) = Y_e I(\emptyset)$ if $e \notin P$ is a separator. Therefore, we consider the X_e and Y_e to be parameters because X_e and Y_e are values of the Tutte function only if $\emptyset \in \mathcal{C}$ and $I(\emptyset) = 1$. The Tutte equations (TA) and (TSSM) justify calling the P-quotients indecomposibles.

The uniform matroid with elements $\{e, f, \cdots\}$ and rank r is denoted by $U_r^{ef\cdots}$.

In the remainder, we consider only elements e, f, g none of which are in P.

Two distinct elements e, f in matroid M are *parallel* when every cocircuit that contains one of them also contains the other. This is equivalent to $\{e, f\}$ being a two-element circuit. They are *series* when every circuit that contains one of them also contains the other. This is equivalent to $\{e, f\}$ being a two-element cocircuit. They are called a *dyad* when they are both parallel and series. Note that every dyad is a matroid connected component of M.

Two distinct elements $\{e, f\}$ are a *parallel pair connected to* P when they are parallel and there is a cocircuit of the form $\{e, f\} \cup P'$ with $\emptyset \neq P' \subseteq P$.

Two distinct elements e, f are a *series pair connected to* P when they are series and there is a circuit of the form $\{e, f\} \cup P'$ with $\emptyset \neq P' \subseteq P$.

Three distinct elements e, f, g are called a *triangle* when they comprise a 3 element circuit U_2^{efg} that is a connected component of M.

Three distinct elements e, f, g are called a *triad* when they comprise a 3 element cocircuit U_1^{ef} that is a connected component of M.

The following is critical to the proof that the generalizations of identities in Theorem 2 all have the form $I(Q_i) \cdot r = 0$ where r is a polynomial in the x_e, y_e, X_e, Y_e parameters and Q_i is one P-quotient.

Proposition 3. *Suppose e, f are in series, or are in parallel, in matroid or oriented matroid M.*

1. *The minors $M/e \setminus f = M/f \setminus e$ are equal as matroids.*
2. *If M is oriented, the oriented minors $M/e \setminus f = M/f \setminus e$ are equal as oriented matroids.*

Proof. In the following, take all matroids as oriented or not depending on how M is given.

If e, f are in series, note that $M/e \setminus f = M \setminus f/e$. e is a coloop in $M \setminus f$, so $M \setminus f/e = M \setminus \{e, f\}$, which is clearly the same matroid or oriented matroid if e, f are interchanged. The relevant theory of minors of oriented matroids can be found in [1].

If e, f are in parallel, e, f are in series in the matroid or oriented matroid dual M^* of M. By the first case, $M^* \setminus e/f = M^* \setminus f/e$ as matroids or as oriented matroids. Thus $M/e \setminus f = (M^* \setminus e/f)^* = (M^* \setminus f/e)^* = M/f \setminus e$ as matroids or as oriented matroids. □

Part 1. Tutte Functions and Expansions

3. Parametrized Ported Tutte Functions

Let P be a set and $\mathcal{C}$ be a P-family of matroids or oriented matroids. We state, discuss and prove this generalization of Theorem 2 of [14]:

Theorem 4. *The following two statements are equivalent.*

(1) *T from $\mathcal{C}$ to R or an R-module is a P-ported separator-strong parametrized Tutte function with R-parameters (x, y, X, Y) whose values $T(Q_i)$ on P-quotients $Q_i \in \mathcal{C}$ are the initial values $I(Q_i)$.*

(2) (a) *For every $M = U_1^{ef} \oplus Q_j \in \mathcal{C}$ with P-quotient Q_j (U_1^{ef} is a dyad),*

$$I(Q_j)(x_e Y_f + y_e X_f) = I(Q_j)(x_f Y_e + y_f X_e).$$

(b) *For every $M = U_2^{efg} \oplus Q_j \in \mathcal{C}$ with P-quotient Q_j (U_2^{efg} is a triangle),*

$$I(Q_j)X_g(x_e y_f + y_e X_f) = I(Q_j)X_g(x_f y_e + y_f X_e).$$

(c) *For every $M = U_1^{efg} \oplus Q_j \in \mathcal{C}$ with P-quotient Q_j (U_1^{efg} is a triad),*

$$I(Q_j)Y_g(x_e Y_f + y_e x_f) = I(Q_j)Y_g(x_f Y_e + y_f x_e).$$

(d) *If $\{e, f\} = E(M)$ is a parallel pair connected to P,*

$$I(Q_j)(x_e Y_f + y_e x_f) = I(Q_j)(x_f Y_e + y_f x_e)$$

where P-quotient $Q_j = M/e \setminus f = M/f \setminus e$.

(e) *If $\{e, f\} = E(M)$ is a series pair connected to P,*

$$I(Q_j)(x_e y_f + y_e X_f) = I(Q_j)(x_f y_e + y_f X_e)$$

where P-quotient $Q_j = M/e \setminus f = M/f \setminus e$.

3.1. **Remarks.** Proposition 3 assures that the different expressions for P-quotients Q_i in Theorem 4 are in fact equal as matroid or as oriented matroids, depending on how M was given.

The first three cases are trivial extensions of the conditions in Theorem 2 [14]. The only difference is that our conditions have the factor $I(Q_j)$ in place of $\alpha = T(\emptyset)$. Just two new conditions are required by $P \neq \emptyset$. They are vacuous when $P = \emptyset$.

Our proof is the immediate result of adding considerations of ports to the proof in [14], there described as "a straightforward adaption of the proof of Theorem 3.3 of [26]."

As in [14], we rely on the hypothesis the $\mathcal{C}$ is closed under our P-minors in order to verify (1) that the conditions imply $T(M)$ is well-defined for $n = 0, 1$ and 2 and (2) that in a larger minimum n counterexample, the elements of $E(M)$ are either all in series or all in parallel, and then the conditions imply that all calculation orders give the same result. All the cases involve two different combinations of deleting and contracting of several elements in $E(M)$ where both combinations produce the same P-quotients.

The empty matroid $\emptyset$ is clearly the only indecomposible for the separator strong Tutte identities with $P = \emptyset$. Then $\emptyset \in \mathcal{C}$ is required, provided $\mathcal{C} \neq \emptyset$. When we generalize to the P-ported separator strong Tutte identities with $P \neq \emptyset$, the indecomposibles depend on $\mathcal{C}$ and $\emptyset \notin \mathcal{C}$ is possible.

3.2. **Proof.** We sketch the proof with a few details, pointing out differences from [14].

As in [14], the necessary relations are easy to deduce by applying (TA) and (TSSM) to the particular matroids or oriented matroids of $\mathcal{C}$ to which they apply. Now on to the converse.

Let $M \in \mathcal{C}$ be a counterexample with minimum $n = |E(M)|$, noting that n does not count $|P \cap S(M)|$. Therefore, whenever M' is a proper P-minor of M, $T(M')$ is well-defined. The Tutte conditions (TA) and (TSSM) have the property that given M and $e \in E(M)$, exactly one equation applies. Therefore, the induction hypothesis entails that calculations that yield different values for $T(M)$ must start with reducing by different elements of $E(M)$. Since $T(M)$ is given unambiguously by the initial value $I(M)$ when $n = 0$, we can assume $n \geq 2$.

M cannot contain a separator $e \in E(M)$. This is a consequence of the fact, applied to P-minors, that this e is a separator in every minor of M containing e. Therefore, as observed in [14], every computation has the same result $X_e T(M/e)$ or $Y_e T(M \setminus e)$ depending on whether e is a coloop or a loop.

Let e be one element in $E(M)$. Since no element in $E(M)$ is a separator, $V = x_e T(M/e) + y_e T(M \setminus e)$ is well-defined, and so is $x_{e'} T(M/e') + y_{e'} T(M \setminus e')$ for each other $e' \in E$. We follow [14] and define $D = \{e' \in E(M) \mid V = x_{e'} T(M/e') + y_{e'} T(M \setminus e')\}$. The induction hypothesis then tells us that there is at least one $f \in E(M) \setminus D$. (Recall $e, e', f \notin P$.)

Suppose that e is a separator in both $M \setminus f$ and M/f and f is a separator in both $M \setminus e$ and M/e. Then, T would be well-defined for all four of these P-minors and so we can write

$$T(M) = x_e x_f T(M/\{e, f\}) + x_e y_f T(M/e \setminus f) + y_e x_f T(M \setminus e/f) + y_e y_f T(M \setminus \{e, f\}).$$

Both computations give the same value because in this situation the reductions by e and f commute. So, for M to be a counterexample, there must be $e \in D$ and $f \notin D$ ($e, f \notin P$) to which one case of the following lemma applies:

Lemma 5. [26] *Let e, f be nonseparators in a matroid M. Within each column the statements are equivalent:*

(1) *e is a separator in $M \setminus f$.*	(1) *e is a separator in M/f.*
(2) *e is a coloop in $M \setminus f$.*	(2) *e is a loop in M/f.*
(3) *e and f are in series in M.*	(3) *e and f are in parallel in M.*
(4) *f is a separator in $M \setminus e$.*	(4) *f is a separator in M/e.*

We claim that one of the following five cases must be satisfied:

(1) $n = 2$ and $E(M) = \{e, f\}$ is a dyad.
(2) $n \geq 3$ and $E(M)$ is a circuit not connected to P.
(3) $n \geq 3$ and $E(M)$ is a cocircut not connected to P.
(4) $n \geq 2$ and for some $\emptyset \neq P' \subseteq P$, $P' \cup E(M)$ is a circuit.
(5) $n \geq 2$ and for some $\emptyset \neq P' \subseteq P$, $P' \cup E(M)$ is a cocircuit.

As in [14], we draw the conclusion that if $e \in D$ and $f \notin D$ then e, f are either series or parallel. It was further proven that a series pair and a parallel pair cannot have exactly one element in common. Therefore, the pairs e, f satisfying the conditions are either all series pairs or all parallel pairs. By minimality of n, $E(M)$ is either an n-element parallel class or an n-element series class. The last two cases are distinguished from the first three according to whether or not $E(M)$ is disconnected or not from elements of P in matroid M. We now use (TA) and (TSSM) to show that, in each case, the calculations that start with e and those that start with f have the same result, which contradicts $e \in D$ and $f \notin D$.

We give the details for case 4. By hypothesis, each of $T(M/e)$, $T(M \setminus e)$, $T(M/f)$, $T(M \setminus f)$, $T(M/f/e) = T(M/e/f)$, $T(M/e \setminus f)$ and $T(M/f \setminus e)$ is well-defined. Furthermore, by Proposition 3, $M/e \setminus f = M/f \setminus e)$ as matroids or oriented matroids depending on how M was given.

Starting with e and with f give the two expressions:

$$V = x_e x_f T(M/e/f) + x_e y_f T(M/e \setminus f) + y_e T(M \setminus e)$$

$$V \neq x_f x_e T(M/f/e) + x_f y_e T(M/f \setminus e) + y_f T(M \setminus f)$$

Let M' be the P-minor obtained by contracting each element in $E(M)$ except for e and f ($M' = M$ if $n = 2$.) Since $E(M') = \{e, f\}$, (TA) tells us that

$$I(Q)(x_e y_f + y_e X_f) = I(Q)(x_f y_e + y_f X_e),$$

where $Q = M'/e \setminus f = M'/f \setminus e$. The latter two matroids or oriented matroids are equal because e, f are in series in M' and so Proposition 3 applies. (When adapting this proof to Theorem 37, $Q = N'/e \setminus f$ and $Q' = N'/f \setminus e$ might differ and so might $I(Q)$ and $I(Q')$. Revising the calculations is left to the reader.) Since $A = E(M) \setminus \{e, f\}$ is a set of coloops ($\emptyset$ if $n = 2$) in $M/e \setminus f = M/f \setminus e$, we write $X_A = \prod_{a \in A} X_a$ (1 if $A = \emptyset$) by X_A and use (TSSM) to write

$$T(M/e \setminus f) = X_A I(Q).$$

and

$$T(M/f \setminus e) = X_A I(Q).$$
$$T(M \setminus e) = Y_f X_A I(Q).$$
$$T(M \setminus f) = Y_e X_A I(Q).$$

So

$$x_e y_f T(M/e \setminus f) + y_e T(M \setminus e) = x_f y_e T(M/f \setminus e) + y_f T(M \setminus f)$$

which contradicts $V \neq x_f x_e T(M/\{f, e\}) + x_f y_e T(M/f \setminus e) + y_f T(M \setminus f)$.

The remaining cases can be completed analogously. It might be noted that our proof differs slightly from [14] in that the cases of $n = 3$ and $n \geq 4$ are not distinguished.

3.3. **Universal Tutte Polynomial.** It is easy to follow [2, 14] to define a universal, i.e., most general P-ported parametrized Tutte function $T^{\mathcal{C}}$ for the P-minor closed class $\mathcal{C}$ given without parameters or initial values. To do this, we take indeterminates x_e, y_e, X_e, Y_e for each $e \in E(\mathcal{C})$ and an indeterminate $[Q_i]$ for each P-quotient $Q_i \in \mathcal{C}$. Let $\mathbb{Z}[x, y, X, Y]$ denote the integer polynomial ring generated by the x_e, y_e, X_e, Y_e indeterminates, define $\widetilde{\mathbb{Z}}$ to be the $\mathbb{Z}[x, y, X, Y]$-module generated by the $[Q_i]$. Let $I^{\mathcal{C}}$ denote the ideal of $\widetilde{\mathbb{Z}}$ generated by the identities of Theorem 4, comprising for example $[Q_i](x_e Y_f + y_e X_f - x_f Y_e - y_f X_e)$ for each subcase of case (a), etc. The universal Tutte function has values in the quotient module $\widetilde{\mathbb{Z}}/I^{\mathcal{C}}$. Finally, observe that the range of Tutte function T can be considered to be the R-module generated by the values $I(Q_i)$ where ring R contains the x, y, X, Y parameters. If the $I(Q_i) \in R$, consider the ring R to be the R-module generated by R. We follow [14] to write the corresponding consequence of Theorem 4:

Corollary 6. *Let $\mathcal{C}$ be a P-minor closed class of matroids or oriented matroids. Then there is a $\widetilde{\mathbb{Z}}/I^{\mathcal{C}}$-valued function $T^{\mathcal{C}}$ on $\mathcal{C}$ with $T^{\mathcal{C}}(Q_i) = [Q_i]$ for each P-quotient $Q_i \in \mathcal{C}$ that is a P-ported parametrized Tutte function on $\mathcal{C}$ where the parameters are the x, y, X, Y indeterminates. Moreover, if T is any R-parametrized Tutte function with parameters x'_e, y'_e, X'_e, Y'_e, then T is the composition of $T^{\mathcal{C}}$ with the homomorphism determined by $[Q_i] \to I(Q_i) = T(Q_i)$ for P-quotient and $x_e \to x'_e$, etc., for each $e \in E(\mathcal{C})$.*

4. Tutte Computation Trees and Activities

Several authors [13, 14] surveyed the two ways that the two-variable Tutte polynomial can be defined: It may be defined either as a universal solution to the recursive strong Tutte equations, or as a generating function. Further, two kinds of generating function definitions have been given. The first is what Tutte originally used for graphs [24, 25] and is called the *basis* or *activities* expansion. It enumerates each basis $B \subseteq E$ by a term $x^{i(B)}y^{e(B)}$, where the *numbers of internally and externally active elements* $i(B)$ and $e(B)$ are determined from a given linear order on the elements of E (see Definition 10). It was shown that even though $(i(B), e(B))$ for particular B might vary with the order, the resulting polynomial is independent of this order, and that it satisfies the Tutte equations. The second, called the *rank-nullity* generating function, is well-defined automatically because it enumerates each subset $A \subseteq E$ with the term $(x-1)^{(r(E)-r(A))}(y-1)^{(|A|-r(A))}$. This generating function is then shown to satisfy the Tutte equations. Zaslavsky noted that the activities expansion remains universal when parameters are included whereas the rank-nullity generating function expresses only the proper subset of Tutte functions which he called normal[26]. They are characterized by (CNF). See sec. 4.2.

Ellis-Monaghan and Traldi [14] remarked that the Tutte equation approach appears to give a shorter proof of the ZBR theorem than the activities expansion approach. Diao and Hetyei [11] proved specializations of Theorem 4 by means of the activities expansion approach. The inductive proofs on $|E|$ that we and [14] give demonstrate that every calculation of $T(M)$ from Tutte equations produces the same result when the conditions on the parameters and initial values are satisfied. It it then almost a tautology that the polynomial expression resulting from a particular calculation will equal the Tutte function value in the ring R. We show that every recursive calculation (see below) gives rise to an activities expansion, when the activities are defined in the more general way given by McMahon and Gordon [17]. We suggest a heuristic reason why the inductive Tutte equation approach is more succinct: The induction assures that *every* computation for a matroid with smaller $|E|$ gives the same result, not just those computations that are determined by linear orders on E.

To be precise, *recursive* means that the computation uses Tutte equations to find $T(M)$ in terms of the initial values for indecomposibles and/or of M' with $|E(M')| < |E(M)|$ using recursive computations. (Note the inductive definition.) All the recursive computations of $T(M)$ are expressible by "computation trees," formally defined by McMahon and Gordon [17]. Their motivation was to generalize activities expansions and the corresponding interval partitions of the subset lattice from matroids to greedoids. Unlike matroids, some greedoids do not have an activities expansion for their Tutte polynomial that derives from an element ordering.

Proofs of activities expansions for matroids, and their generalizations for P-ported matroids, seem more informative and certainly no harder when the expansions are derived from a general Tutte computation tree, than when the expansions are only those that result from an element order. From the retrospective that the Tutte equations specify a non-deterministic recursive computation [16], it seems artificial to start with element-ordered computations and then prove first that all linear orders give the same result and second that it satisfies the Tutte equations, in order prove that all recursions give the same result. We therefore take advantage of the Tutte computation tree formalism and the more general expansions it enables.

4.1. **Computation Tree Expansion.** We begin with the definition of what generalizes the matroid bases in the activities expansion when $P \neq \emptyset$. In the following, $\mathcal{B}(M)$ denotes the set of bases in M.

Definition 7. *Given P-ported matroid or oriented matroid M, a **P-subbasis** $F \in \mathcal{B}_P(M)$ is an independent set with $F \subseteq E(M)$ (so $F \cap P = \emptyset$) for which $F \cup P$ is a spanning set for M (in other words, F spans M/P).*

An equivalent definition was given in [22]. The following proposition shows our definition is equivalent to that given in [11]. C and D below are called "contracting and deleting sets" in that paper.

Proposition 8. *C is a P-subbasis if and only if $C \subseteq E((M))$ has no circuits and $D = E(M) \setminus C$ has no cocircuits.*

Proof. C has no circuits means C is an independent set in M. D has no cocircuits means D is independent in the dual of M, i.e., D is coindependent. D is coindependent if and only if $S(M) \setminus D = P \cup C$ is spans M. ☐

Proposition 9. *For every P-subbasis F there exists an independent set $Q \subseteq P$ that extends F to a basis $F \cup Q \in \mathcal{B}(M)$. Conversely, if $B \in \mathcal{B}(M)$ then $F = B \cap E = B \setminus P$ is a P-subbasis.*

Proof. Immediate. ☐

The next definition is also equivalent to one in [11]. It generalizes Tutte's definitions based on element orderings [25, 24] extended to matroids [10]. We will see that expansions based on computation trees generalize these further.

Definition 10 (Activities with respect to a P-subbasis and an element ordering O)**.** *Let ordering O have every $p \in P$ before every $e \in E$. Let F be a P-subbasis. Let B be any basis for M with $F \subseteq B$.*

- *Element $e \in F$ is internally active if e is the least element within its principal cocircuit with respect to B. Thus, this principal cocircuit contains no ports. The reader can verify this definition is independent of the B chosen to extend F. Elements $e \in F$ that are not internally active are called internally inactive.*
- *Dually, element $e \in E$ with $e \notin F$ is externally active if e is the least element within its principal circuit with respect to B. Thus, each externally active element is spanned by F. Elements $e \in E \setminus F$ that are not externally active are called externally inactive.*

Definition 11 (Computation Tree, following [17])**.** *A P-ported (Tutte) computation tree for M is a binary tree whose root is labeled by M and which satisfies:*

(1) *If M has non-separating elements not in P, then the root has two subtrees and there exists one such element e for which one subtree is a computation tree for M/e and the other subtree is a computation tree for $M \setminus e$.*

The branch to M/e is labeled with "e contracted" and the other branch is labeled "e deleted".

(2) *Otherwise (i.e., every element in $E(M)$ is separating) the root is a leaf.*

An immediate consequence is

Proposition 12. *Each leaf of a P-ported Tutte computation tree for M is labeled by the direct sum of some P-quotient (oriented if M is oriented) summed with loop and/or coloop matroids with ground sets $\{e\}$ for various distinct $e \in E$ (possibly none).*

It sometimes helps to revise Definition 11 to require that every leaf be labelled with an indecomposible; and then allow a single branch from M to M/e or to $M \setminus e$ labelled "e contracted as a coloop" or "e deleted as a coloop" depending on what kind of separator is $e \in E(M)$. We leave the corresponding revisions of further definitions to the reader.

Definition 13 (Activities with respect to a leaf)**.** *For a P-ported Tutte computation tree for M, a given leaf, and the path from the root to this leaf:*

- *Each $e \in E(M)$ labeled "contracted" along this path is called* ***internally passive****.*
- *Each coloop $e \in E(M)$ in the leaf's matroid is called* ***internally active****.*
- *Each $e \in E(M)$ labeled "deleted" along this path is called* ***externally passive****.*
- *Each loop $e \in E(M)$ in the leaf's matroid is called* ***externally active****.*

Proposition 14. *Given a leaf of a P-ported Tutte computation tree for M, the set of internally active or internally passive elements constitutes a P-subbasis of M which we say* ***belongs to the leaf****. Furthermore, every P-subbasis F of N belongs to a unique leaf.*

Proof. For the purpose of this proof, let us extend Definition 13 so that, given a computation tree with a given node i labeled by matroid M_i, $e \in E$ is called internally passive when e is labeled "contracted" along the path from root M to node i. Let IP_i denote the set of such internally passive elements.

It is easy to prove by induction on the length of the root to node i path that (1) $IP_i \cup S(M_i)$ spans M and (2) IP_i is an independent set in M. The proof of (1) uses the fact that elements labeled deleted are non-separators. The proof of (2) uses the fact that for each non-separator $f \in M/IP_i$, $f \cup IP_i$ is independent in M.

These properties applied to a leaf demonstrate the first conclusion, since each $e \in E$ in the leaf's matroid must be a separator by Definition 11.

Given a P-subbasis F, we can find the unique leaf with the algorithm below. Note that it also operates on arbitrary subsets of E.

Tree Search Algorithm: Beginning at the root, descend the tree according to the rule: At each branch node, descend along the edge labeled "e-contracted" if $e \in F$ and along the edge labeled "e-deleted" otherwise (when $e \notin F$). □

The above definitions and properties lead us to reproduce element order based activities:

Proposition 15. *Given element ordering O in which every $p \in P$ is ordered before each $e \notin P$, suppose we construct the unique P-ported computation tree $\mathcal{T}$ in which the greatest non-separator $e \in E$ is deleted and contracted in the matroid at each tree node.*

The activity of each $e \in E$ relative to ordering O and P-subbasis $F \subseteq E$ is the same as the activity of e defined with respect to the leaf belonging to F in $\mathcal{T}$.

Definition 16. *Given a computation tree for P-ported (oriented) matroid M, each P-subbasis $F \subseteq E$ is associated with the following subsets of non-port elements defined according to Definition 13 from the unique leaf determined by the algorithm given above.*

- $IA(F) \subseteq F$ *denotes the set of internally active elements,*
- $IP(F) \subseteq F$ *denotes the set of internally passive elements,*
- $EA(F) \subseteq E \setminus F$ *denotes the set of externally active elements, and*
- $EP(F) \subseteq E \setminus F$ *denotes the set of externally passive elements.*
- $A(F) = IA(F) \cup EA(F)$ *denotes the set of active elements.*

Proposition 17. *Given a P-ported Tutte computation tree for M, the boolean lattice of subsets of $E = E(M)$ is partitioned by the collection of intervals $[IP(F), F \cup EA(F)]$ (note $F \cup EA(F) = IP(F) \cup A(F)$) determined from the collection of P-subbases F, which correspond to the leaves.*

The boolean lattice of subsets of $E = E(M)$ is also partitioned by the collection of intervals $[EP(F), E \setminus F \cup IA(F)]$ (note $E \setminus F \cup IA(F) = EP(F) \cup A(F)$).

For a given $F \in \mathcal{B}_P(M)$, $A \subseteq E$ satisfies $A \in [IP(F), F \cup EA(F)]$ if and only if $(E \setminus A) \in [EP(F), E \setminus F \cup IA(F)]$.

Proof. Every subset $A \subseteq E = E(M) \setminus P$ belongs to the unique interval corresponding to the unique leaf found by the tree search algorithm given at the end of the previous proof.

The dual of that tree search algorithm, which descends along the edge labelled "e-deleted" if $e \in A'$, etc., will find the unique leaf whose interval $[EP(F), E \setminus F \cup IA(F)]$ contains A'.

When $A \in [IP(F), F \cup EA(F)]$, the dual algorithm applied to $A' = E \setminus A$ will find the same leaf. ☐

The following generalizes the activities expansion expression given in [26] to ported (oriented) matroids, as well as Theorem 8.1 of [22].

Proposition 18. *Given parameters x_e, y_e, X_e, Y_e, and P-ported matroid or oriented matroid M the Tutte polynomial expression determined by the sets in Definition 16 from a computation tree is given by*

$$\text{(PAE)} \qquad \sum_{F \in \mathcal{B}_P} [M/F|P]\, X_{IA(F)}\, x_{IP(F)}\, Y_{EA(F)}\, y_{EP(F)}.$$

Proof. (PAE) is an expression constructed by applying some of the Tutte equations. One monomial results from each leaf. It that leaf's matroid, each active element is a separator, and the active elements contribute $X_{IA(F)}Y_{EA(F)}$ to the monomial. The passive elements which contribute $x_{IP(F)}y_{EP(F)}$ are the tree edge labels in the path from the root to the leaf. Each $M/F|P$ denotes a P-quotient of M, so the expression is a polynomial in the parameters and in the initial values. Therefore, (PAE) expressions the result of the calculation when one substitutes $[M/F|P] = I(M/F|P)$. ☐

From Corollary 6 we conclude:

Theorem 19. *For every P-ported parametrized Tutte function T on $\mathcal{C}$ into ring R or an R-module, for every computation tree for $M \in \mathcal{C}$ (and so for every ordering of $E(M)$), the polynomial expression* (PAE) *equals $T^{\mathcal{C}}(M)$ of Corollary 6.*

4.2. **Expansions of Normal Tutte Functions.** .

After a notational translation, Zaslavsky's [26] definition of *normal* Tutte functions becomes those for which $T(\emptyset) = 1$, and for which there exist u, $v \in R$ so that for each $e \in E(M)$,

$$X_e = x_e + uy_e \text{ and } Y_e = y_e + vx_e. \tag{CNF}$$

Let us drop the $T(\emptyset) = 1$ constraint and then note that (CNF) applies immediately to P-ported Tutte functions. (Unfortunately, we use $(x_e, y_e, X_eT(\emptyset), Y_eT(\emptyset))$ for Zaslavsky's notations (b_e, a_e, x_e, y_e).) The normal Tutte functions include the classical two variable Tutte polynomial. Please observe that the equations of Theorem 4 are satisfied by (CNF) independently of the initial values. Hence all the expressions for normal Tutte functions will be in a ring freely generated by u, v, the x_e, y_e and the $[Q]$. We will therefore call them expansions for a Tutte polynomial. This Tutte polynomial is universal for P-ported separator-strong normal parametrized Tutte functions of matroids or oriented matroids. We can now generalize some known expansions.

4.2.1. *Boolean Interval Expansion.*

Corollary 20. *The following activities and boolean interval expansion formula is universal for normal Tutte functions and is obtained by substituting* (CNF) *into* $T^{\mathfrak{C}}(M)$*.*

$$T^{\mathfrak{C}}(M) = \sum_{F \in \mathcal{B}_P} [M/F|P] \Big(\sum_{\substack{IP(F) \subseteq K \subseteq F \\ EP(F) \subseteq L \subseteq E \setminus F}} x_{K \cup (E \setminus F \setminus L)}\, v^{|E \setminus F \setminus L|}\, y_{L \cup (F \setminus K)}\, u^{|F \setminus K|} \Big) \tag{1}$$

Proof. After substituting (CNF) we get

$$T^{\mathfrak{C}}(M) = \sum_{F \in \mathcal{B}_P} [M/F|P] \Big(x_{IP(F)} \prod_{e \in IA(F)} (x_e + y_e u) \;\; y_{EP(F)} \prod_{e \in EA(F)} (y_e + x_e v) \Big)$$

and then, by Definition 16, $IP(F) \uplus IA(F) = F$ and $EP(F) \uplus EA(F) = E \setminus F$. ☐

Lemma 21. *Given* $F \in \mathcal{B}_P$*,* $IP(F)$ *spans* $EA(F)$*.*

The pairs (K, L) *for which* $IP(F) \subseteq K \subseteq F$ *and* $EP(F) \subseteq L \subseteq E \setminus F$ *are in a one-to-one correspondance with the* A *satisfying* $IP(F) \subseteq A \subseteq F \uplus EA(F)$ *given by* $A = K \uplus (E \setminus F) \setminus L$*.*

For every such A*,*

$$|F \setminus K| = r(M) - r(M/F|P) - r(A) \tag{2}$$

and

$$|E \setminus F \setminus L| = |A| - r(A). \tag{3}$$

Proof. By our definition of activities, after all the elements of $IP(F)$ are contracted, all elements in $EA(F)$ are loops. (Note none of these elements are ports.)

Let $A = K \uplus (E \setminus F) \setminus L$. By our definition of activities, $IP(F) \uplus IA(F) = F$, so $IP(F) \subseteq A$. Similarly, $EP(F) \uplus EA(F) = E \setminus F$, so $A \cap (E \setminus F) \subseteq EA(F)$. Hence $A = K \uplus (E \setminus F) \setminus L$.

Since $IP(F)$ spans $EA(F)$, $K \subseteq IP(F)$ spans $EA(F)$. Since $A \subseteq K \uplus EA(F)$, K spans A. $K \subseteq F$ is a P-subbasis, so $|K| = r(K) = r(A)$ and $|F| = r(F)$. Therefore, $|F \setminus K| = r(F) - r(A)$.

Since F is a P-subbasis, $r(F \cup P) = r(M)$. By definition of contraction, $r(M/F|P) = r(F \cup P) - r(F)$, so $r(M/F|P) = r(M) - r(F)$. We conclude $|F \setminus K| = r(M) - r(M/F|P) - r(A)$.

$E \setminus F \setminus L = A \setminus K$, so $|E \setminus F \setminus L| = |A| - |K|$. As above, $|K| = r(A)$, so the last equation follows. ☐

Corollary 22.

$$T^{\mathcal{C}}(M) = \sum_{F \in \mathcal{B}_P} [M/F|P] \Big(\sum_{IP(F) \subseteq A \subseteq F \,\cup\, EA(F)} x_A y_{E \setminus A} u^{r(M)-r(M/F|P)-r(A)} v^{|A|-r(A)} \Big) \tag{4}$$

Proof. Apply Lemma 21 to the inner sum in Proposition 20. ☐

4.2.2. *Corank-nullity Expansion.*

Lemma 23. *Given $F \in \mathcal{B}_P(M)$, (K, L) and $A = K \cup E \setminus F \setminus L$ as in Lemma 21,*

$$M/F|P = M/A|P$$

(as matroids or oriented matroids).

Proof. Writing the contractions and deletions explicitly, $M/F|P = M/F \setminus (E \setminus F)$. By our definition of activities, all the $e \notin P$ in $M/IP(F) \setminus EP(F)$ are loops or coloops. Hence, when all these elements are removed from $M/IP(F) \setminus EP(F)$ whether by contraction or deletion, the result is the same matroid or oriented matroid. Since $IP(F) \subseteq A \subseteq F$ and $EP(F) \subseteq (E \setminus A) \subseteq (E \setminus F)$, we can construct $M/F|P$ or $M/A|P$ by forming $M/IP(F) \setminus EP(F)$ first, contracting the remaining elements of F or A, and last deleting all the remaining $e \notin P$. Therefore $M/A|P = M/F|P$. ☐

Theorem 24.

$$T^{\mathcal{C}}(M) = \sum_{A \subseteq E(M)} [M/A \mid P] x_A y_{E \setminus A} u^{r(M)-r(M/A|P)-r(A)} v^{|A|-r(A)}. \tag{PGF}$$

Proof. By Proposition 17, given any Tutte computation tree, the lattice of subsets of $E(M)$ is partitioned into intervals corresponding to P-subbases $\mathcal{B}_P$. However, given $F \in \mathcal{B}_P$, for every A satisfying Lemma 23, the P-quotient $M/F|P$ is equal to $M/A|P$ (as a matroid or oriented matroid). Hence we can interchange the summations in (4) and write (PGF). ☐

4.2.3. *Geometric Lattice Flat Expansion.* We generalize another formula from [6]:

Proposition 25. *Let N be an oriented or unoriented. Let $R_P(M)$ be given by* (PGF)*. In the formula below, F and G range over the geometric lattice of flats contained in M restricted to E.*

$$R_P(M)(u, v) = \sum_{Q_i} [Q_i] \sum_{\substack{F \leq E \\ [M/F|P]=[Q_i]}} u^{r(M)-r(Q_i)-r(F)} v^{-r(F)F} \sum_{G \leq F} \mu(G, F) \prod_{e \in G} (y_e + x_e v) \tag{5}$$

Proof. It follows the steps for theorem 8 in [6]. ☐

Part 2. **Objects, Graphs and Sums**

When $P = \emptyset$, the facts about separator-strong Tutte functions of matroid direct sums easily follow from the formula $T(M^1 \oplus M^2)T(\emptyset) = T(M^1)T(M^2)$. For example, T is strong if and only if $T(\emptyset) = T(\emptyset)^2$. The theory of separator-strong Tutte functions of graphs covered in [14] follows from the fact that any minor closed family of graphs (see below) $\mathcal{G}$ is partitioned into subfamilies $\mathcal{G}_k$, each with just one indecomposible, E_k, the edgeless graph with k unlabelled vertices, if $\mathcal{G}_k \neq \emptyset$. Tutte function formulas for disjoint and one-point graph unions, and the conditions for strongness (defined $T(G^1)T(G^2) = T(G)$ if matroids $M(G^1) \oplus M(G^2) = M(G)$) are then derived [14] in terms of the values $\alpha_k = I(E_k)$. Life is simple because matroid $M(E_k) = \emptyset$ for all k.

The corresponding facts become more complex when the definitions are naturally extended to P-ported matroids and graphs or to vertex labelled graphs. As with matroids, a P-ported graph G has some of the edges in P and the rest, $E(G)$, satisfy $E(G) \cap P = \emptyset$. Deletion, contraction, P-minors, P-families and P-quotients (i.e., irreducibles) are also defined as they are for P-ported matroids or oriented matroids. As in [14], deletion of an isthmus (i.e., coloop in the matroid) and contraction of a loop is forbidden.

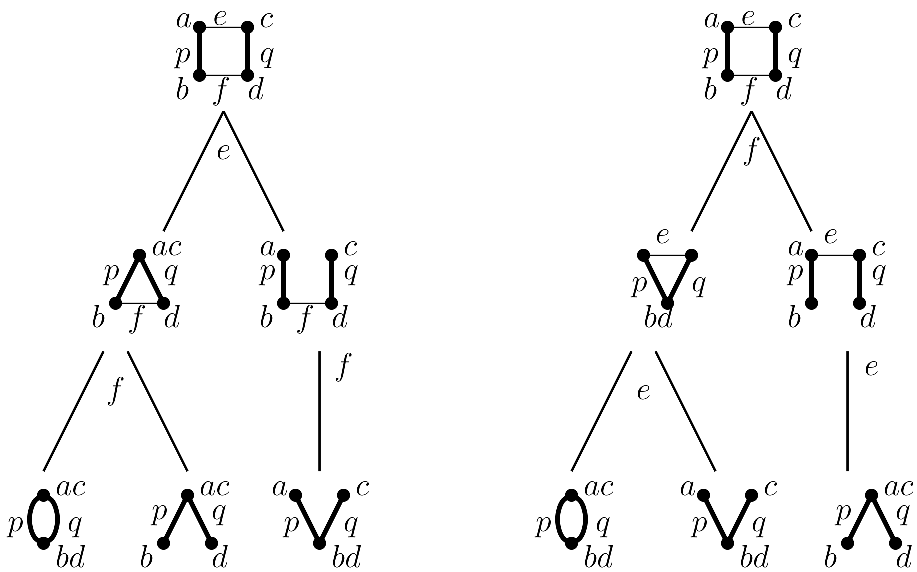
The main difficulty is illustrated by the following example. Let G be the circle graph of the five edges ordered (e, p, f, q, r) and take $P = \{p, q, r\}$. So, e and f are a series pair connected to P, but the P-quotient graphs $Q_1 = G/e \setminus f$ and $Q_2 = G/f \setminus e$ are different graphs, even though they have the same matroid. Q_1 is the path qrp and Q_2 is the path pqr. Function T might satisfy (TA) and (TSSM) even if $T(Q_1) = I(Q_1) \neq I(Q_2) = T(Q_2)$. So, if this $G \in \mathcal{G}$, a necessary condition for T to be a P-ported (separator-strong, as always) Tutte function would be

$$I(Q_1)(x_e y_f - y_f X_e) = I(Q_2)(x_f y_e - y_e X_f). \tag{6}$$

This equation does not have the form of those in the ZBR theorem for graphs [14] because the latter's equations, like the equations in Theorem 4, each has a single factor $I(Q_i)$ depending on one indecomposible.

The example relies on the elements of P being labelled. This leads us to formulate a extension of Tutte function theory for vertex labelled graphs. When the outcome, Theorem 37 is applied to graphs as in [14], we get Corollary 40 which demonstrates that the above example illustrates the *only* situation where the P-ported ZBR equations of Theorem 4 must be modified.

The next example illustrates the same phenonemon as the first, in a smaller graph, when the objects in $\mathcal{G}$ are graphs whose vertices are labelled by disjoint sets. Again, two different graphs have the same oriented matroid.



The expressions from these two Tutte computation trees are are

$$I(Q_1)x_ex_f + I(Q_2)x_ey_f + I(Q_3)y_eX_f$$

and

$$I(Q_1)x_ex_f + I(Q_3)x_fY_e + I(Q_2)y_fX_e = I(Q_1)x_ex_f + I(Q_2)y_fX_e + I(Q_3)x_fY_e,$$

which are equal if and only if

$$I(Q_2)(y_fX_e - x_ey_f) = I(Q_3)(y_eX_f - x_fY_e). \tag{7}$$

Q_2 and Q_3 are isomorphic as edge-labelled graphs but are different when the vertex labels are present.

There are two complications introduced into P-families of matroids when $P \neq \emptyset$. First, one matroid might have more than one P-quotient, i.e., indecomposible. The most simple example is a dyad matroid composed of one port and one non-port element; and its P-minors. Therefore, minimal P-minor closed families might have more than one indecomposible. Some will share matroids or oriented matroids and others will not. The second, which also occurs with the minor closed families of graphs [14] in the original $P = \emptyset$ form, is that the family is partitioned into disjoint P-minor closed subfamilies. Each subclass has its own indecomposibles, E_k in the case of graphs. Again, indecomposibles in different subclasses share matroids as do the E_k all of whose matroids are $\emptyset$. When $P \neq 0$, the indecomposibles of different subclasses might or might not share matroids or oriented matroids.

The ZBR theorem for graphs in [14] has conditions analogous to those in Theorem 2, except the factor α is replaced by $\alpha_k = I(E_k)$ depending on the subfamily.

P-ported matroids or oriented matroids can be combined by matroid direct sum $\oplus$. Graphs can be combined by disjoint union $\amalg$ or by a one-point union; then each such combination G of G^1 and G^2, if defined, satisfies $M(G) = M(G^1) \oplus M(G^2)$.

We following definitions are the immediate extensions of the corrsponding known definitions.

Definition 26. *A* strong *P-ported Tutte function T on a P-family $\mathcal{C}$ of matroids or oriented matroids satisfies $T(M^1)T(M^2) = T(M^1 \oplus M^2)$ when M^1, M^2 and $M^1 \oplus M^2$ are all in $\mathcal{C}$.*

Note that such a strong Tutte function is a separator-strong Tutte function with $X_e = T(U_1^e)$ and $Y_e = T(U_0^e)$ for all $e \in E(\mathcal{N})$.

We will give extensions of definitions of strong Tutte functions and of multiplicative Tutte functions of graphs below when we define P-families of objects with matroids or oriented matroids.

5. Objects with Matroids or Oriented Matroids

It is useful to think that a P-ported Tutte computation tree may have objects N for its node labels such as graphs. Each object N has an associated a P-ported matroid or oriented matroid $M(N)$. Elements are defined $S(N) = S(M(N))$, each $p \in S(N) \cap P$ is called a port, and $E(N) = S(N) \setminus P$. Loops, coloops and non-separators of N are characterized by their status in $M(N)$. So we say N is an *object with a matroid or an oriented matroid*. Often, but not always, N will be some matroid or oriented matroid representation.

Contraction N/e and deletion $N \setminus e$ of object N are defined when $e \in E(N)$, and e is not a coloop in $M(N)$ and e is not a loop in $M(N)$, respectively. Under those conditions, $M(N/e) = M(N)/e$ and $M(N \setminus e) = M(N) \setminus e$ (as matroids or oriented matroids). Thus P-minors are defined, and an indecomposible or P-quotient is a P-minor Q for which $S(Q) = S(M(Q)) \subseteq P$.

Definition 27. *An P-ported object N with a matroid or oriented matroid is described above together with $M(N)$, $E(N)$, $S(N)$, P-minors, etc.*

We say a P-family of objects $\mathcal{N}$ is a P-minor closed class of objects with matroid or oriented matroids.

Tutte computation trees are defined for such N. The matroid $M(N)$ of course constrains the structure of these trees. It is possible (as when the edgeless graphs G_k have different vertex sets but all $M(G_k) = \emptyset$) for different objects, even different indecomposibles, to have the same matroid or oriented matroid. It also natually occurs that $N/e \setminus f \neq N/f \setminus e$ (as objects) even though $M(N)/e \setminus f = M(N)/f \setminus e$. The latter equation when e, f are in parallel or in series (see Proposition 3) is critical to the above ZBR theorems. It is also conceivable that $N/e/f \neq N/f/e$ or $N \setminus e \setminus f \neq N \setminus f \setminus e$.

Since every P-minor N' of N has matroid or oriented matroid $M(N')$ the same as the corresponding minor of $M(N)$, we observe:

Lemma 28. *The Tutte computation trees for $M(N)$ are in a one-to-one correspondance with the Tutte computations tree for N where corresponding trees are isomorphic. In each isomorphism, corresponding branches have the same labels "e-contracted" or "e-deleted" with $e \in E(N) = E(M(N))$, and a node labelled N_i in the tree for N corresponds to a node labelled $M(N_i)$ in the tree for $M(N)$.*

Each computation tree value is given by the activities expansion (PAE) *reinterpreted for objects.*

We can still talk about Tutte decompositions and a Tutte computation tree for N even without a Tutte function. If we are given values $I(Q_i)$ for the indecomposibles, each Tutte computation tree for N yields a value in the R-module generated by the $I(Q_i)$. The Tutte decompositions, and the universal Tutte polynomial (if it exists!) of each $N \in \mathcal{N}$ are determined by $M(N)$ and the indecomposibles, i.e., P-quotients Q_i in N, which of course satisfy $Q_i \in \mathcal{N}$. This generalizes Zaslavsky's discussion[26].

Definition 29 (Separator-strong P-ported Tutte function on objects)**.** *Function T on $\mathcal{N}$ is a P-ported separator-strong Tutte function on $\mathcal{N}$ into the ring R containing parameters x_e, y_e, X_e, Y_e, or an R-module containing the initial values $T(Q_i) = I(Q_i)$ for indecomposibles, when for each $e \in E(N)$ for some $N \in \mathcal{N}$, if $T(N)$ satisfies* (TA) *and* (TSSM) *for all $N \in \mathcal{N}$.*

Therefore:

Proposition 30. *T is a P-ported separator-strong Tutte function on $\mathcal{N}$ if and only if for each $N \in \mathcal{N}$, all Tutte computation trees for $T(N)$ yield polynomial expressions that are equal in the range ring or R-module.*

We develop our first ZBR-type theorem for P-ported objects with matroids or oriented matroids. It is the generalization of the ZBR theorem for graphs as given by Ellis-Monaghan and Traldi[14]. It depends on a lemma similar to one of theirs.

Lemma 31. *Suppose P-family $\mathcal{N}$ is partitioned into disjoint P-minor closed subfamilies $\{\mathcal{N}_\pi\}$. Then T is a Tutte function on $\mathcal{N}$ if and only if T restricted to $\mathcal{N}_\pi$ is a Tutte function for each $\mathcal{N}_\pi$.*

Theorem 32. *Suppose P-family $\mathcal{N}$ is partitioned into disjoint P-minor closed subfamilies $\{\mathcal{N}_\pi\}$, and each initial value $I(Q_i)$ depends only on the matroid or oriented matroid $M(Q_i)$ and on the π for which $Q_i \in \mathcal{N}_\pi$,*

Then T is a Tutte function with given parameters (x, y, X, Y) and initial values $I(Q_i)$ if and only if it satisfies the equations of Theorem 4, interpreted for families of objects with matroids or oriented matroids.

Proof. As in [14], lemma 31 lets us prove the theorem for each π separately.

By Lemma 28, T is a P-ported Tutte function of family of objects $\mathcal{N}_\pi$ if and only if function $T'(M(N)) = T(N)$ on the P-family of the matroids or oriented matroids of $N \in \mathcal{N}_\pi$ is a P-ported Tutte function, since by hypothesis $I'(M(Q_i)) = I(Q_i) = I(M(Q_i))$ for corresponding indecomposibles $Q_i \in \mathcal{N}_\pi$ and $M(Q_i)$ in the matroid or oriented matroid P-family.

The conclusion follows from Theorem 4 applied to this matroid or oriented matroid P-family. □

Ellis-Monaghan and Traldi's ZBR theorem for graphs refers to one initial value $\alpha_k = I(E_k)$ for each non-empty subclass of graphs, with unlabelled vertices, that have k graph components. One natural ported generalization is to partition the P-ported graphs G according to (1) how many graph components k, (2) $P' = P \cap S(G)$ and (3) $\nu : P' \to \{1, \ldots, k\}$, where νp is which component contains edge p. Theorem 32 tells us:

Corollary 33. *Let a P-minor closed collection $\mathcal{G}$ of graphs with unlabelled vertices be partitioned into $\mathcal{G}_{k,P',\nu}$. Suppose initial values $I(G) = I_{k,P',\nu}(M(G))$ are given that depend only on the part and the matroid or oriented matroid of $G \in \mathcal{G}_{k,P',\nu}$. Then there is T, P-ported separator-strong parametrized Tutte function of graphs $\mathcal{G}$ satisfying $T(Q) = I(Q) = I_{k,P',\nu}(M(Q))$ whenever P-quotient $Q \in \mathcal{G}_{k,P',\nu}$ if and only if the identities of Theorem 4, interpreted for graphs, are satisfied with the given $I(Q)$.*

The next ZBR-type theorem addresses the problem illustrated by (7) requires that the P-family satisfy the following

Definition 34. *Object $N \in \mathcal{N}$ is* well-behaved *when for every independent set $C \subseteq E(N)$ and coindependent set $D \subseteq E(N)$ for which $C \cap D = \emptyset$, each of the $|C \cup D|!$ orders of contracting C and deleting D produces the same P-minor (which is an object) of N.*

Specifically, let $C = \{c_1, \ldots, c_j\}$, $D = \{d_{j+1}, \ldots, d_k\}$ and $R_i(N') = N'/c_i$ if $1 \le i \le j$ and $N' \setminus d_i$ if $j+1 \le i \le k$. The condition is $R_1 \circ \cdots \circ R_k(N) = R_{\sigma_1} \circ \cdots \circ R_{\sigma_k}(N)$ for every permutation σ of $\{1, \ldots, k\}$.

$\mathcal{N}$ *is* well-behaved *when each $N \in \mathcal{N}$ is well-behaved.*

By definition 27 all the minors are defined and $M(R_1 \circ \cdots \circ R_k(N)) = M(R_{\sigma_1} \circ \cdots \circ R_{\sigma_k}(N))$ independently of whether N is well-behaved or not. The point is that the objects themselves are the same.

We give two examples of well-behaved P-families.

Definition 35 (Graphs with set-labelled vertices)**.** *The elements of such a graph $S(G) = E(G) \cup (P \cap S(G))$ are edges. The vertices are labelled with non-empty finite sets so the two sets labelling distinct vertices in one graph are disjoint. Only non-loop edges $e \notin P$ can be contracted; when an edge is contracted, its two endpoints are replaced by one vertex whose label is the union of the labels of the two endpoints. Only non-isthmus edges $e \notin P$ can be deleted; deletion doesn't change labels. The graph has its graphic matroid if it is undirected and its oriented graphic matroid if it is directed.*

A graph with set-labelled vertices is well-behaved because the minor obtained by contracting forest C and deleting D is determined by merging all the vertex labels of each graph component of C and deleting edges $C \cup D$. Note that the deletions do not affect the vertex labels. Hence the set labels are not affected by the order of the operations.

Definition 36 (Graphs with set-labelled components)**.** *The elements of such a graph $S(G) = E(G) \cup (P \cap S(G))$ are edges. The path-connected components are labelled by non-empty finite sets so two components in the same graph always have disjoint labels. In other words, the set labels of the components are a partition π_V Only non-loop edges $e \notin P$ can be contracted and only non-isthmus edges $e \notin P$ can be deleted. The component labels are unchanged by these minor operations. Definition 35 specifies the matroids or oriented matroids.*

A non-well-behaved P-family $\mathcal{C}!$ can be constructed from any P-family of matroids $\mathcal{C}$ with some $M \in \mathcal{C}$ with $|E(M)| \ge 2$. Each member of $\mathcal{C}!$ is formed from some $M \in \mathcal{C}$ together with some history of deletions and contractions that can be applied to M. Let c_e and d_e be symbols for contracting and deleting $e \in E(\mathcal{C})$ respectively; a history h is a string of such symbols. Let $M|h$ be the P-minor obtained by performing history h on M, assuming each step is defined. The objects of $\mathcal{C}!$ are all pairs (M, h) for which P-minor $M|h \in \mathcal{C}$ is defined. The matroid of (M, h) is $M|h$, which determines the element set, loops and coloops. If $e \in E(M|h)$ is not a loop, then define $(M, h)/e = (M, hc_e)$. Similarly, if $e \in E(M|h)$ is not a coloop, $(M, h) \setminus e = (M, hd_e)$.

The point of this example is that even if the P-family is not well-behaved and so the indecomposibles do carry information about their history, Theorem 32 tells us that the Tutte function is still well defined if the initial values depend only on the matroid, or the oriented matroid, of the indecomposible.

The examples forced us to recognize that for N an object with a matroid or oriented matroid $M(N)$ with $e, f \in E(N)$ in series or in parallel, it might happen that $N/e \setminus f \ne N/f \setminus e$ even though, by Proposition 3, $M(N)/e \setminus f = M(N)/f \setminus e$. Note that Proposition 3 is about $\cdot/e \setminus f$ and $\cdot/f \setminus e$ which are not commutations of the same two operations.

Theorem 37 (ZBR Theorem for well-behaved P-families of objects with matroids or oriented matroids)**.** *Let $\mathcal{N}$ be a well-behaved P-family of objects with matroids or oriented matroids.*

The following two statements are equivalent.

(1) *T from $\mathcal{N}$ to R or an R-module is a P-ported separator-strong parametrized P-ported Tutte function with R-parameters (x, y, X, Y) whose values $T(Q_i)$ on P-quotients $Q_i \in \mathcal{N}$ are the initial values $I(Q_i)$.*

(2) *For every $N \in \mathcal{N}$:*

(a) *If $M(N) = U_1^{ef} \oplus M(Q_j) = U_1^{ef} \oplus M(Q'_j)$ with P-quotients $Q_j = N/e \setminus f$ and $Q'_j = N/f \setminus e$,*

$$I(Q_j)(x_e Y_f - y_f X_e) = I(Q'_j)(x_f Y_e - y_e X_f).$$

(b) *If $M(N) = U_2^{efg} \oplus M(Q_j) = U_2^{efg} \oplus M(Q'_j)$ with P-quotients $Q_j = N/e \setminus f/g$ and $Q'_j = N/f \setminus e/g$,*

$$I(Q_j)X_g(x_e y_f - y_f X_e) = I(Q'_j)X_g(x_f y_e - y_e X_f).$$

(c) *If $M(N) = U_1^{efg} \oplus M(Q_j) = U_1^{efg} \oplus M(Q'_j)$ with P-quotients $Q_j = N/e \setminus f \setminus g$ and $Q'_j = N/f \setminus e \setminus g$,*

$$I(Q_j)Y_g(x_e Y_f - y_f x_e) = I(Q'_j)Y_g(x_f Y_e - y_e x_f).$$

(d) *If $\{e, f\} = E(M(N))$ is a parallel pair connected to P,*

$$I(Q_j)(x_e Y_f - y_f x_e) = I(Q'_j)(x_f Y_e - y_e x_f)$$

where P-quotients $Q_j = N/e \setminus f$ and $Q'_j = N/f \setminus e$.

(e) *If $\{e, f\} = E(M(N))$ is a series pair connected to P,*

$$I(Q_j)(x_e y_f - y_f X_e) = I(Q'_j)(x_f y_e - y_e X_f)$$

where P-quotients $Q_j = N/e \setminus f$ and $Q'_j = N/f \setminus e$.

Proof. the Tutte function value for each tree depends only on the tree structure and the initial values. The fact the $\mathcal{N}$ is well-behaved allows us to conclude that ...

The rest is analogous to the proof we gave for Theorem 4. □

Corollary 38. *A P-family of objects with matroids satisfies a ZBR-type theorem with the identities given in Theorem 4 if, in addition $\mathcal{N}$ being well-behaved, the initial values I satisfy $I(N/e \setminus f) = T(N/f \setminus e)$ when $\{e, f\} = E(N)$ is a series or parallel pair, $I(N/e \setminus f/g) = I(N/f \setminus e/g$ when $\{e, f, g\} = E(N)$ is a triangle and $I(N/e \setminus f \setminus g) = I(N/f \setminus e \setminus g)$ when $\{e, f, g\} = E(N)$ is a triad.*

Corollary 39. *A P-family of objects with matroids satisfies a ZBR-type theorem with the identities given in Theorem 4 if, in addition to $\mathcal{N}$ being well-behaved, the object P-quotients $N/e \setminus f = N/f \setminus e$ when $\{e, f\} = E(N)$ is a series or parallel pair, $N/e \setminus f/g = N/f \setminus e/g$ when $\{e, f, g\} = E(N)$ is a triangle and $N/e \setminus f \setminus g = N/f \setminus e \setminus g$ when $\{e, f, g\} = E(N)$ is a triad.*

Proof. Clearly, if $N/e \setminus f = N/f \setminus e$ then $T(N/e \setminus f) = T(N/f \setminus e)$, etc. □

We conclude with second ported generalization of Ellis-Monaghan and Traldi's ZBR theorem for graphs, besides Corollary 33.

Corollary 40. *Let $\mathcal{G}$ be a ported P-family of graphs with unlabelled vertices, as in Corollary 33. Then there is T, P-ported separator-strong parametrized Tutte function of graphs $\mathcal{G}$ satisfying $T(Q) = I(Q)$ for all P-quotients $Q \in \mathcal{G}$ if and only if For every $G \in \mathcal{G}$:*

(1) *If $E(G)$ is dyad $\{e, f\}$ then*
$$I(Q)(x_e Y_f - y_f X_e) = I(Q)(x_f Y_e - y_e X_f).$$
where $Q = G/e \setminus f = G/f \setminus e$.

(2) *If $E(G)$ is triangle $\{e, f, g\}$ then*
$$I(Q)X_g(x_e y_f - y_f X_e) = I(Q)X_g(x_f y_e - y_e X_f).$$
where $Q = G/e \setminus f = G/f \setminus e$.

(3) *If $E(G)$ is triad $\{e, f, g\}$ then*
$$I(Q)Y_g(x_e Y_f - y_f x_e) = I(Q)Y_g(x_f Y_e - y_e x_f).$$
where $Q = G/e \setminus f = G/f \setminus e$.

(4) *If $E(G)\{e, f\}$ is a parallel pair connected to P,*
$$I(Q)(x_e Y_f - y_f x_e) = I(Q)(x_f Y_e - y_e x_f)$$
where $Q = G/e \setminus f = G/f \setminus e$.

(5) *If $E(G) = \{e, f\}$ is a series pair connected to P,*
$$I(Q)(x_e y_f - y_f X_e) = I(Q')(x_f y_e - y_e X_f)$$
where P-quotients $Q = G/e \setminus f$ and $Q' = G/f \setminus e$.

Note: The last is the *only* case where the equation is different from the one in the P-ported ZBR theorem for matroids or oriented matroids.

Proof. $\mathcal{G}$ is a well-behaved P-family of ported objects with matroids or oriented matroids, so Theorem 37 applies.

In all cases of Theorem 37 but the last, the two object minors are the same graph because the contracted edges are path-connected, so the equations of Theorem 37 are simplified. □

Both Corollaries 33 and 40 reduce to the ZBR theorem for graphs when $P = \emptyset$. The first uses the property that $\mathcal{G}$ is partitioned into minor-closed subclasses with indecomposibles E_k for which the initial values depend only on the matroid or oriented matroid to generalize the original ZBR equations. As with matroids, we find again that the Tutte functions can distinguish different orientations of the the same undirected graphs. The second relies on the commutivity of the graph minor operations and generalizes the fact that different initial values may be assigned to different indecomposibles, but then the conditions sufficient for the initial values to extend to a Tutte function must be stronger.

6. Direct and other Sums

It is a common situation that $\{N^1, N^2, N\} \subseteq \mathcal{N}$ and their matroids or oriented matroids $M(N^1) \oplus M(N^2) = M(N)$. Tutte computation trees help. The proposition below applies even to non-well-behaved N when the symbols $/B_i^j|P_j$ refer to sequences of deletions and contractions.

Definition 41. *If $\mathcal{T}_1$ and $\mathcal{T}_2$ are Tutte computation trees then $\mathcal{T}_1 \cdot \mathcal{T}_2$ is the tree obtained by appending a separate copy of $\mathcal{T}_2$ at each leaf of $\mathcal{T}_1$. The root is the root of the expanded $\mathcal{T}_1$.*

Proposition 42. *Suppose N, N^1 and N^2 are all in $\mathcal{N}$ and $M(N^1) \oplus M(N^2) = M(N)$. Then if $\mathcal{T}_1$ and $\mathcal{T}_2$ are Tutte computation trees for N^1 and N^2 respectively with values given by* (DS1) *and* (DS2)*, then there is a Tutte computation tree for N that yields the value given by* (DS)*.*

$$\text{(DS1)} \qquad \sum_{Q_i^1} I(Q_i^1)c_1(Q_i^1) \textit{ where } Q_i^1 = N/B_i^1|P_1.$$

$$\text{(DS2)} \qquad \sum_{Q_j^2} I(Q_i^2)c_2(Q_j^2) \textit{ where } Q_i^2 = N/B_i^2|P_2.$$

$$\text{(DS)} \qquad \sum_{Q_i^1, Q_j^2} I(Q_{i,j})c_1(Q_i^1)c_2(Q_j^2) \textit{ where } Q_{i,j} = N/B^1|P_1 \cup S(N^2)/B^2|P_2.$$

Furthermore, if T is a Tutte function on $\mathcal{N}$ and $T(N^1)$ and $T(N^2)$ equal the Tutte polynomials given by (DS1) *and* (DS2) *then $T(N)$ equals the polynomial given by* (DS)*.*

Proof. We show how to relabel $\mathcal{T}_1 \cdot \mathcal{T}_2$ to obtain a Tutte computation tree for N. $M(N^1) \oplus M(N^2) = M(N)$ is defined means $S(M(N^1)) \cap S(M(N^2)) = \emptyset$ and $S(M(N)) = S(M(N^1)) \cup S(M(N^2))$. Each node of $\mathcal{T}_1 \cdot \mathcal{T}_2$ is determined by by deleting and/or contracting some elements of $E(M(N^1)) \cup E(M(N^2))$. Relabel that node with the P-minor of N obtaining deleting and/or contracting the same elements respectively in the same order, those in N^1 preceding those in N^2. The result is a computation tree for N because $M(N^1) \oplus M(N^2) = M(N)$. Assume $P \subseteq S(M(N))$ (otherwise, take a smaller P) and let $P^1 = S(M(N^1)) \cap P$ and $P^2 = S(M(N^2)) \cap P$. At a leaf of the relabelled tree, there will be the P-quotient $N/B_1/B_2|P$ where B_1 is a P^1-subbasis of $M(N^1)$ and B_2 is a P^2-subbasis of $M(N^2)$. ∎

6.1. **Strong Tutte Functions.** Let us extend the definition of strong parametrized Tutte function to P-families $\mathcal{N}$ of objects with matroids and oriented matroids, in the way that abstracts the known notion of strong Tutte functions on minor closed families of graphs[14]. Of course, taking $\mathcal{N}$ to be a P-family $\mathcal{C}$ of matroids or oriented matroids gives us the extension to such $\mathcal{C}$. Then, there might still be indecomposibles besides or instead of $\emptyset$.

Definition 43. *A P-ported separator-strong Tutte function T on a P-family of objects $\mathcal{N}$ with matroids is called* strong *if whenever $\{N^1, N^2, N\} \subseteq \mathcal{N}$ and $M(N^1) \oplus M(N^2) = M(N)$, then $T(N^1)T(N^2) = T(N)$.*

We can use Proposition 42 to prove the generalization of the $T(\emptyset)T(\emptyset) = T(\emptyset)$ characterization of strong Tutte functions.

Theorem 44. *A P-ported separator-strong Tutte function T on a P-family of objects with matroids or oriented matroids $\mathcal{N}$ is* strong *if and only if T restricted to the indecomposibles of $\mathcal{N}$ is strong; i.e., whenever Q^1, Q^2 and Q are indecomposibles and $M(Q^1) \oplus M(Q^2) = M(Q)$ then $T(Q^1)T(Q^2) = T(Q)$.*

Proof. Every P-quotient is in $\mathcal{N}$, so clearly T restricted to the P-quotients is strong.

Conversely, suppose N^1, N^2 and N are in $\mathcal{N}$ and $M(N^1) \oplus M(N^2) = M(N)$, so Proposition 42 applies.

Since $M(N^1)\oplus M(N^2)=M(N)$, $M(N/(B_1\cup B_2)|P)=(M(N^1)/B_1|P)\oplus(M(N^2)/B_2|P)=M(N^1/B_1|P)\oplus M(N^2/B_2|P)$. We now use the fact that $Q_{ij}=N/B_1/B_2|P$, $Q_i^1=N^1/B_1|P$ and $Q_j^2=N^2/B_2|P$ are P-quotients and the hypothesis to write $T(Q_{ij})=T(Q_i^1)T(Q_j^2)$.

We therefore conclude $T(N)=T(N^1)T(N^2)$ from (DS1), (DS2) and (DS).

□

6.2. **Multiplicative Tutte Functions.** Often $\mathcal{N}$ comes equipped with one operation "$*$", or more, that satisfy the following definition. Examples for $P=\emptyset$ are disjoint union $\amalg$ and one-point unions of graphs [14]. These can be extended to P-ported graphs and disjoint union can be extended to P-families of objects.

Definition 45. *A partially defined binary operation "$*$" on a P-family of objects with matroids or oriented matroids $\mathcal{N}$ is* a matroidal direct sum *if whenever $N^1 * N^2\in\mathcal{N}$ is defined for $\{N^1,N^2\}\subseteq\mathcal{N}$, the matroids or oriented matroids satisfy $M(N^1)\oplus M(N^2)=M(N^1*N^2)$.*

Proposition 42 applies when $N^1*N^2=N$ is defined. It gives a general recipe for $T(N^1*N^2)$ which generalizes the identity [14] $T(M^1\oplus M^2)T(\emptyset)=T(M^1)T(M^2)$ for separator-strong Tutte functions of matroids. The P-ported generalization is more complicated and generally cannot be expressed by a product in the domain ring of T.

Proposition 46. *Suppose $*$ is a matroidal direct sum and N^1, N^2 and N^1*N^2 are each members of a P-family for which T is a Tutte function.*

If for P-quotients Q_i^j and R-coefficients $c_j(Q_i^j)$, $j=1$ and 2,

$$T(N^1)=\sum_{Q_i^1}T(Q_i^1)c_1(Q_i^1) \tag{MD1}$$

and

$$T(N^2)=\sum_{Q_j^2}T(Q_i^2)c_2(Q_j^2) \tag{MD2}$$

then

$$T(N^1*N^2)=\sum_{Q_i^1,Q_j^2}T(Q_i^1*Q_j^2)c_1(Q_i^1)c_2(Q_j^2). \tag{MD}$$

Proof. Substitute $Q_{i,j}=Q_i^1*Q_j^2$ in (DS) of Proposition 42. □

When $\mathcal{N}$ is a P-family of matroids or oriented matroids, direct matroid or oriented matroid sum is obviously a matroidal direct sum operation, and so Proposition 46 is applicable.

Corollary 47. [14] *Let $P=\emptyset$. $T(M^1\oplus M^2)T(\emptyset)=T(M^1)T(M^2)$ for Tutte function T of matroids.*

Proof. Our proof demonstrates how Proposition 46 generalizes this formula to P-families. The expansions DS1 and DS2 take the one-term form $T(M^j)=T(\emptyset)c_j(\emptyset)$, $j=1,2$, so $T(M^1)T(M^2)=T(\emptyset)^2c_1(\emptyset)c_2(\emptyset)$. Expansion DS is then $T(M^1\oplus T^2)=T(\emptyset)c_1(\emptyset)c_2(\emptyset)$. □

Following the definitions for graphs in [14], we write:

Definition 48. *Given a matroidal direct sum $*$ on $\mathcal{N}$, a Tutte function T on $\mathcal{N}$ is* multiplicative *(with respect to "$*$") if whenever N^1*N^2 is defined for $\{N^1,N^2\}\subseteq\mathcal{N}$, the Tutte function values satisfy $T(N^1)T(N^2)=T(N^1*T^2)$.*

A strong Tutte function is certainly multiplicative for any "$*$", but not conversely. A consequence of Proposition 42 is that, like strong Tutte functions, multiplicative Tutte functions are characterized by being that way on the indecomposibles.

Corollary 49. *A P-ported Tutte function T on P-family $\mathcal{N}$ is multiplicative with respect to matroidal direct product "$*$" if and only if for every pair of indecomposibles $\{Q_i, Q_j\} \in \mathcal{N}$ for which $Q_i * Q_j \in \mathcal{N}$ is defined, $T(Q_i)T(Q_j) = T(Q_i * Q_j)$.*

Proof. When $N^1 * N^2$ is defined, Proposition 42 applies because $M(N^1) \oplus M(N^2) = M(N^1 * N^2)$. $T(N^1 * N^2) = T(N^1)T(N^2)$ is then a consequence of $T(Q_i * Q_j) = T(Q_i)T(Q_j)$. $\square$

In the case when $P = \emptyset$ and the vertices are unlabelled, we can prove a strengthening of part of Ellis-Monaghan and Traldi's Corollary 3.13. It is stronger because it does not require any additional hypotheses on $\mathcal{G}$ to prove that all initial values that occur are the same idempotent.

Corollary 50. *Suppose T is parametrized Tutte function on a minor-closed class of graphs $\mathcal{G}$ (note $P = \emptyset$.) T is strong if and only if there is an idempotent $\alpha = \alpha^2 \in R$ and $T(E_k) = \alpha$ whenever $\mathcal{G}_k \neq \emptyset$.*

Proof. $M(E_k) = \emptyset$ for all $k \geq 1$ and $\emptyset \oplus \emptyset = \emptyset$, so $T(E_i)T(E_j) = T(E_k)$ whenever $\mathcal{G}_i$, $\mathcal{G}_j$ and $\mathcal{G}_k$ are all non-empty. Hence, if $\mathcal{G}_k \neq \emptyset$ then $T(E_k)T(E_k) = T(E_k) = \alpha$. Further, if $\mathcal{G}_j \neq \emptyset$ with $j \neq k$, $\alpha = T(E_k) = T(E_k)T(E_k) = T(E_j)$. $\square$

The other case of Corollary 3.13 [14] requires additional conditions for a Tutte function that is multiplicative on both disjoint union $\amalg$ and one-point unions to always be strong. Consider $\mathcal{N} = \{E_3, E_4, E_5, \ldots\}$, $T(E_k) = 1$ for $k \geq 3$, $k \neq 5$ and $T(E_5) = 0$. $T(E_3)T(E_4) = 1 \neq T(E_5)$, so T is not strong, but T is multiplicative on disjoint and one-point unions because E_5 cannot be expressed as either kind of union of graphs in $\mathcal{G}$. The other conditions are that $\mathcal{G}_k \neq \emptyset$ for all k and that $\mathcal{G}$ is closed under one-point unions and removal of isolated vertices.

Part 3. Additional Background

We presented in [8] a new kind of strong Tutte-like function on P-ported oriented graphic matroids (more generally, unimodular, i.e. regular oriented matroids) whose values vary with the orientation. Each function value $F(G)$ is in the exterior algebra over R^{2p}, where R is the reals extended by the x_e, y_e and $|P| = p$. The function obeys an *anti-commutative* variant of (TSM) below with exterior multiplication $\wedge$. (When $P = \emptyset$, it reduces to the reduced Laplacian determinant in the famous Matrix Tree Theorem [18, 5].) It is the first example we know of "the possibility of making use of a noncommutative generalization of the Tutte polynomial at some point in the future." mentioned by Bollobás and Riordan in [2]. We won't say more beyond that (1) each of the $\binom{2p}{p}$ Plücker coodinates of $F(G)$ is a P-ported Tutte function of the kind we cover here; and (2) that quadratic inequalities among some of them express negative correlation between edges in spanning trees, results also known as Rayleigh's inequality[**?**].

A second non-commutative possibility might be found in section 6 where we consider parametrized Tutte functions of graphs with ports. Different indecomposible graphs can have the same matroid[14]. To explore the issues, we define an abstraction called "P-ported objects with matroids or oriented matroids" to which we generalize graph results. The abstraction applies to situations where an object represents an initial matroid, graph, etc.

plus a history of deletions and contractions. The key feature is that the Tutte decompositions, represented by trees, of an object are identical to the Tutte decompositions of that object's matroid. The initial values on indecomposibles, used for a Tutte decomposition to determine a Tutte function value $T(N)$, might then depend on the order of the deletion and contraction reductions to obtain each indecomposible from N. This helps us understand the theory, but whether objects with minors that depend on reduction order have useful applications remains to be seen.

Our generalization is the kind of Tutte function determined by (TA) and (TSSM); the latter is the weakening of (TSM) so it applies only for cases where one of M_1 or M_2 is a *separator*, that is, a loop or coloop. We generalize by adding the restriction on port elements. The result, Theorem 4 (sec. 3), is a a straightforward generalization of Theorem 2 below paraphrased from [14] about the existance and universal form for the family of Tutte functions characterized by (TA) and (TSSM). This family was subsequently named *separator-strong Tutte functions* by [15] because it is wider than the strong Tutte functions characterized by (TA) and (TSM), which are the subject of [26]. We follow these authors' terminology when we extend the family to matroids, oriented matroids and graphs with distinguished port elements, and then to ported objects representing matroids or oriented matroids. The conclusions for P-ported parametrized strong Tutte functions easily follow from those for the separator-strong ones. See 44 in sec. 6.

We build upon [14] which reconciles the results of [26] and [2] with a common generalization. It generalizes the fields and strong Tutte functions of [26] to the commutative rings and separator-strong Tutte functions of [2], and the definedness on all matroids in [2] to definedness on a minor-closed class in [26]. Further, neither the matroid or graph Tutte functions need to be 0 or 1 on $\emptyset$.

6.3. **Background and Other Related Work.** Besides Brylawski's work, another early appearance of Tutte decomposition of a matroid or graph with a basepoint is [23]. Ellis-Monaghan and Traldi [14] explain that by leaving the reduction by e_0 to last so e_0 is always contracted as a coloop or deleted as a loop, the Tutte function value can be expressed by $T(M) = (rX_{e_o} + sY_{e_0})T(\emptyset)$ were r, s are not-necessarilly-unique elements in R. As one application, they give a formula for the parametrized Tutte polynomial for the parallel connection across e_0 which generalized Brylawski's. These r, s appear in the P-ported Tutte function expression $rT(U_1^{e_0}) + sT(U_0^{e_0})$ when $P = \{e_0\}$. They are parametrized generalizations of the coefficients of z' and x' in Brylawski's four variable Tutte polynomial.

Las Vergnas defined and gave basic properties of "set-pointed" Tutte polynomials (with no parameters) and used them to study matroid perspectives. The polynomial given in [20, 22] has a variable ξ_l for each subset in a collection of k subsets $P_l \subseteq P$, $l = 1, \ldots, k$. Each term in (PAE) had $\prod \xi_l^{r_i(P_l)}$ for $[Q_i]$ where r_i is the rank function of matroid Q_i. Therefore (TSM) was satisfied and the association of the term to (non-oriented) Q_i could be assured by taking all $2^{|P|}$ subsets for the P_l. The matroid perspective is the strong map $M \setminus E(M) \to M/E(M)$ given by the identity on P.

In [6], we reproduced Las Vergnes' theory with explicit P-quotient (matroid) variables (see the $[Q_i]$ symbols in Corollary 6 in sec. 3.3) in place of $\prod \xi_l^{r_i(P_l)}$. We then gave formulas for the P-ported Tutte polynomial for the union and its dual of matroids whose common elements are in P. These formulas work in a way similar to what appears in sec. 3.3. We extend to algebras the $\mathbb{Z}[u, w]$-module generated by the $[Q_i]$ by defining multiplications $\tilde{*}$ with the rules $[Q_i]\tilde{*}[Q_j] = r_{ij}[Q_{i,j}]$, with $r_{i,j} \in \mathbb{Z}[u,w]$ and $Q_{i,j} = Q_i * Q_j$ depending on (Q_i, Q_j) and whether $*$ represents union or its dual. It is not often recognized that series and parallel connection of matroids across basepoint p is equivalent to matroid union and its dual on matroids with only element p in common. We plan to investigate whether the formulas for parametrized Tutte polynomials of parallel connections in [14] can be generalized to the dual of union when $|P| > 1$, and to detail the relationship when $|P| = 1$.

Recent work on a different generalization, weak Tutte functions (see sec. 1.1), has been done by Ellis-Monaghan and Zaslavsky [27]. The distinction between weak Tutte functions (satisfying an additive identity only) and strong Tutte functions (which satisfy (TSM) and (TA)) seems first to have been made by Zaslavsky[26], for matroids. That paper also defined weak and strong Tutte functions of graphs. However, we use using Ellis-Monaghan and Traldi's definition of strong Tutte functions of graphs[14]. The latter restricts (TA) to non-separators (not just non-loops); and it requires $T(G^1)T(G^2) = T(G)$ whenever matroids $M(G^1) \oplus M(G_2) = M(G)$ (which we sometimes interpret as oriented), not just when G is the disjoint union of G^1 and G^2. The term separator-strong (I learned[15] after [14] appeared.) is used for Tutte functions of matroids and graphs as defined in [14]; recall that they satisfy (TA) and (TSSM). Normal is used in the same way as in[26]. We will repeat and extend these definitions with little further attribution.

We introduced P-ported parametrized Tutte polynomial for *normal* strong Tutte functions in [8], i.e., those with corank-nullity polynomial expressions. Most of the results in the current paper, when so restricted appeared in [8] or can be derived by adding parameters and oriented matriod considerations to material in [6]. These include computation tree[17] based activities expansions with terms corresponding to P-subbases (which are called "contracting sets" in [11]). In the normal case, the initial values can be assigned arbitrarilly. We used this to show the our extensor-valued Tutte-like function [6] is expressible by assigning extensors as the initial values. In this electrical network application, the indecomposibles are oriented graphic matroids and different values *are* assigned to different orientations of the same matroid.

We had left open questions about when non-normal P-ported parametrized Tutte functions are well-defined. They are whether arbitrary values can be assigned to the indecomposibles and what is the appropriate generalization of conditions on the parameters given by Zaslavsky [26], Bollobás and Riordan [2] and Ellis-Monaghan and Traldi [14]. Theorem 4 resolves these questions: The new conditions are obvious revisions of those for $P = \emptyset$. The indecomposibles can be assigned arbitrarilly so long as for each one separately, the conditions of Theorem 4 are satisfied. In addition, if the function is to be strong, Theorem 44 tells us that it is sufficient for the assignment to indecomposibles be strong.

Diao and Hetyei gave conditions on the parameters, similar to ours, for the Tutte polynomial to be well-defined for every assignment of values on the indecomposibles that obeys a symmetry condition. That condition is motivated by the graph specialization. They gave the very natural application to invariants of virtual knots calculated from their diagrams. Port edges, which they call *zero edges*, correspond to the virtual crossings, and the sign parameters of the other ± 1 edges derive from left-over or right-over sense of the regular crossings. Their preprint in fact motivated us to pursue the current topic[11]. This topic leads us to ask if the P-ported objects with matroids abstraction, and its related Tutte computation tree expansions for parametrized Tutte functions (sec. 6), can be usefully applied to objects besides graphs or directed graphs, such as various kinds of knot diagrams.

Another open project is to classify the solutions to the conditions of Theorem 4 (about separator-strong P-ported Tutte functions) and Theorem 44 (about the strong ones) for rings and for fields along the lines of [26] and [2].

In the next section we define particular expressions for $T^{\mathfrak{C}}(M)$ which will be called Tutte polynomials. Our purpose for allowing Tutte function values and polynomials to be in R-modules is that it facilitates giving formulas for Tutte functions of combinations such as direct sum in terms of multiplication rules that make the module into algebra.

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