DIRECT SUMS AND TUTTE FUNCTIONS OF GRAPHS AND OTHER OBJECTS WITH MATROIDS

SETH CHAIKEN

Abstract. The polynomials expressing the conditions that a parametrized P-ported Tutte function is well-defined for matroids each have one factor $I(Q)$, where Q is a P-quotient. Since the elements are labelled, the methodology applies to objects such as graphs with ports for which similar theorems can be proven. We abstract graphs to objects that have ported Tutte functions because they have matroids, but might have different Tutte function values on two objects with the same matroid. Two new theorems are given and are used to generalize known conditions for graphs without port edges to graphs with port edges. In some cases, the condition becomes $I(Q)r + I(Q')s = 0$, where r, s are polynomials in x, y, X, Y. The abstraction is then used to characterize ported Tutte functions of an object or a combination of two objects, whose matroid or oriented matroid is a direct matroid or oriented matroid sum. This extends with ports some known strong Tutte function and multiplicative Tutte function results.

1. Introduction

When $P = \emptyset$, the facts about separator-strong Tutte functions of matroid direct sums easily follow from the formula $T(M^1 \oplus M^2)T(\emptyset) = T(M^1)T(M^2)$. For example, T is strong if and only if $T(\emptyset) = T(\emptyset)^2$. The theory of separator-strong Tutte functions of graphs covered in [8] follows from the fact that any minor closed family of graphs (see below) $\mathcal{G}$ is partitioned into subfamilies $\mathcal{G}_k$, each with just one indecomposible, E_k, the edgeless graph with k unlabelled vertices, if $\mathcal{G}_k \neq \emptyset$. Tutte function formulas for disjoint and one-point graph unions, and the conditions for strongness (defined $T(G^1)T(G^2) = T(G)$ if matroids $M(G^1) \oplus M(G^2) = M(G)$) are then derived [8] in terms of the values $\alpha_k = I(E_k)$. Life is simple because matroid $M(E_k) = \emptyset$ for all k.

The corresponding facts become more complex when the definitions are naturally extended to P-ported matroids and graphs or to vertex labelled graphs. As with matroids, a P-ported graph G has some of the edges in P and the rest, $E(G)$, satisfy $E(G) \cap P = \emptyset$. Deletion, contraction, P-minors, P-families and P-quotients (i.e., indecomposibles) are also defined as they are for P-ported matroids or oriented matroids. As in [8], deletion of an isthmus (i.e., coloop in the matroid) and contraction of a loop is forbidden within Tutte equations.

The main difficulty is illustrated by the following example. Let G be the circle graph of the five edges ordered (e, p, f, q, r) and take $P = \{p, q, r\}$. So, e and f are a series pair connected to P, but the P-quotient graphs $Q_1 = G/e \setminus f$ and $Q_2 = G/f \setminus e$ are different graphs, even though they have the same matroid. Q_1 is the path qrp and Q_2 is the path pqr. Function T might satisfy (**??**) and (**??**) even if $T(Q_1) = I(Q_1) \neq I(Q_2) = T(Q_2)$. So, if this $G \in \mathcal{G}$, a necessary condition for T to be a P-ported (separator-strong, as always) Tutte

2000 *Mathematics Subject Classification.* Primary 05B35; Secondary 05C99, 05C15, 57M25, 94C05.
Key words and phrases. Tutte function, Tutte polynomial.
Version of April 13, 2009.

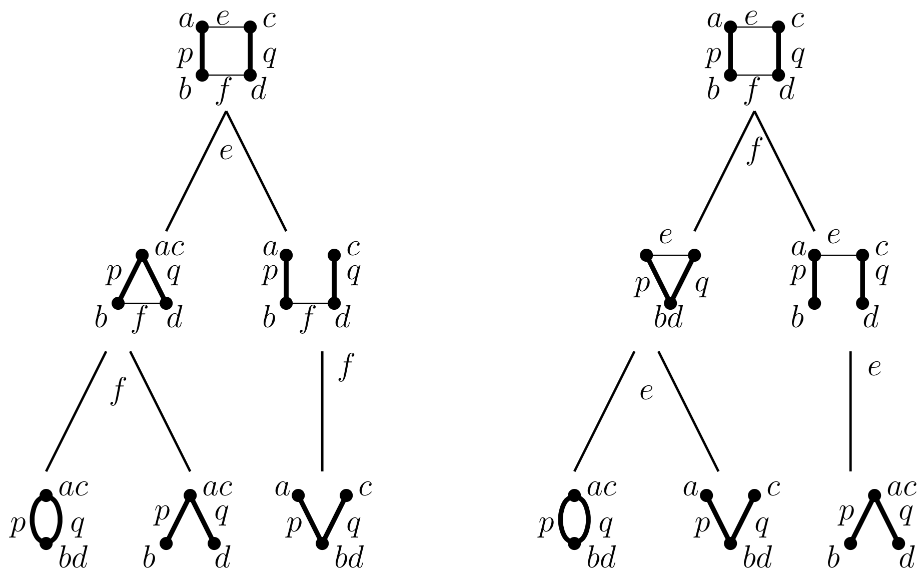

FIGURE 1. Two Tutte computation trees for the same graph with set labelled vertices. The indecomposibles at the bottom of the first tree are named Q_1, Q_2 and Q_3 from left to right. For non-separators, contraction edges are slanted left and deletion edges are slanted right. An edge representing a separator is drawn vertically.

function would be

$$I(Q_1)(x_e y_f - y_f X_e) = I(Q_2)(x_f y_e - y_e X_f).$$

This equation does not have the form of those in the ZBR theorem for graphs [8] because the latter's equations, like the equations in Theorem **??**, each has a single factor $I(Q)$ depending on one indecomposible.

The example relies on the elements of P being labelled. This leads us to formulate a extension of Tutte function theory for vertex labelled graphs. When the outcome, Theorem 12 is applied to graphs as in [8], we get Corollary 15 which demonstrates that the above example illustrates the *only* situation where the P-ported ZBR equations of Theorem **??** must be modified.

Figure 1 illustrates the same phenonemon as the first, in a smaller graph, when the objects in $\mathcal{G}$ are graphs whose vertices are labelled by disjoint sets. Again, two different graphs have the same oriented matroid.

The expressions from the two Tutte computation trees in Figure 1 are

$$I(Q_1)x_e x_f + I(Q_2)x_e y_f + I(Q_3)y_e X_f$$

and

$$I(Q_1)x_e x_f + I(Q_3)x_f Y_e + I(Q_2)y_f X_e = I(Q_1)x_e x_f + I(Q_2)y_f X_e + I(Q_3)x_f Y_e,$$

which are equal if and only if

$$I(Q_2)(y_f X_e - x_e y_f) = I(Q_3)(y_e X_f - x_f Y_e). \tag{1}$$

Q_2 and Q_3 are isomorphic as edge-labelled graphs but are different when the vertex labels are present.

There are two complications introduced into P-families of matroids when $P \neq \emptyset$. First, one matroid might have more than one P-quotient, i.e., indecomposible. The most simple example is a dyad matroid composed of one port and one non-port element; and its P-minors. Therefore, minimal P-minor closed families might have more than one indecomposible. Some will share matroids or oriented matroids and others will not. The second, which also occurs with the minor closed families of graphs [8] in the original $P = \emptyset$ form, is that the family is partitioned into disjoint P-minor closed subfamilies. Each subclass has its own indecomposibles, E_k in the case of graphs. Again, indecomposibles in different subclasses share matroids as do the E_k all of whose matroids are $\emptyset$. When $P \neq 0$, the indecomposibles of different subclasses might or might not share matroids or oriented matroids.

The ZBR theorem for graphs in [8] has conditions analogous to those in Corollary **??**, except the factor α is replaced by $\alpha_k = I(E_k)$ depending on the subfamily.

P-ported matroids or oriented matroids can be combined by matroid direct sum $\oplus$. Graphs can be combined by disjoint union $\amalg$ or by a one-point union; then each such combination G of G^1 and G^2, if defined, satisfies $M(G) = M(G^1) \oplus M(G^2)$. Zaslavsky's definition[16] for matroids is immediately extended:

Definition 1. *A **strong** P-ported Tutte function T on a P-family $\mathcal{C}$ of matroids or oriented matroids satisfies $T(M^1)T(M^2) = T(M^1 \oplus M^2)$ when M^1, M^2 and $M^1 \oplus M^2$ are all in $\mathcal{C}$.*

Note that such a strong Tutte function is a separator-strong Tutte function with $X_e = T(U_1^e)$ and $Y_e = T(U_0^e)$ for all $e \in E(\mathcal{N})$.

We will give extensions of definitions of strong Tutte functions and of multiplicative Tutte functions of graphs below when we define P-families of objects with matroids or oriented matroids.

2. Objects with Matroids or Oriented Matroids

It is useful to think that a P-ported Tutte computation tree may have objects N for its node labels such as graphs. Each object N has an associated a P-ported matroid or oriented matroid $M(N)$. Elements are defined $S(N) = S(M(N))$, each $p \in S(N) \cap P$ is called a port, and $E(N) = S(N) \setminus P$. Loops, coloops and non-separators of N are characterized by their status in $M(N)$. So we say N is an *object with a matroid or an oriented matroid*. Often, but not always, N will be some matroid or oriented matroid representation.

Contraction N/e and deletion $N \setminus e$ of object N are defined when $e \in E(N)$, and e is not a coloop in $M(N)$ and e is not a loop in $M(N)$, respectively. Under those conditions, $M(N/e) = M(N)/e$ and $M(N \setminus e) = M(N) \setminus e$ (as matroids or oriented matroids). Thus P-minors are defined, and an indecomposible or P-quotient is a P-minor Q for which $S(Q) = S(M(Q)) \subseteq P$.

Definition 2. *An P-ported object N with a matroid or oriented matroid is described above together with $M(N)$, $E(N)$, $S(N)$, P-minors, etc. A P**-family of objects** $\mathcal{N}$ is a P-minor closed class of objects with matroid or oriented matroids.*

Tutte computation trees are defined for such N. The matroid $M(N)$ of course constrains the structure of these trees. It is possible (as when the edgeless graphs G_k have different vertex sets but all $M(G_k) = \emptyset$) for different objects, even different indecomposibles, to have the same matroid or oriented matroid. It also natually occurs that $N/e \setminus f \neq N/f \setminus e$ (as objects) even though $M(N)/e \setminus f = M(N)/f \setminus e$. The latter equation when e, f are in parallel or in series (see Proposition **??**) is critical to the matroid ZBR theorems. It is also conceivable that $N/e/f \neq N/f/e$ or $N \setminus e \setminus f \neq N \setminus f \setminus e$.

Since every P-minor N' of N has matroid or oriented matroid $M(N')$ the same as the corresponding minor of $M(N)$, we observe:

Lemma 3. *The Tutte computation trees for $M(N)$ are in a one-to-one correspondance with the Tutte computation trees for N where corresponding trees are isomorphic. In each isomorphism, corresponding branches have the same labels "e-contracted" or "e-deleted" with $e \in E(N) = E(M(N))$, and a node labelled N_i in the tree for N corresponds to a node labelled $M(N_i)$ in the tree for $M(N)$.*

Each computation tree value is given by the activities expansion (**??**) *reinterpreted for objects.*

We can still talk about Tutte decompositions and a Tutte computation tree for N even without a Tutte function. If we are given values $I(Q)$ for the indecomposibles, each Tutte computation tree for N yields a value in the R-module generated by the $I(Q)$. The Tutte decompositions, and the universal Tutte polynomial (if it exists!) of each $N \in \mathcal{N}$ are determined by $M(N)$ and the indecomposibles, i.e., P-quotients Q in N, which of course satisfy $Q \in \mathcal{N}$. This generalizes Zaslavsky's discussion[16].

Definition 4 (Separator-strong P-ported Tutte function on objects)**.** *Function T on $\mathcal{N}$ is a P-ported separator-strong Tutte function on $\mathcal{N}$ into the ring R containing parameters x_e, y_e, X_e, Y_e, or an R-module containing the initial values $T(Q) = I(Q)$ for indecomposibles, when for all $N \in \mathcal{N}$,* (**??**) *and* (**??**) *are satisfied for each $e \in E(N)$.*

Therefore:

Proposition 5. *T is a P-ported separator-strong Tutte function on $\mathcal{N}$ if and only if for each $N \in \mathcal{N}$, all Tutte computation trees for $T(N)$ yield polynomial expressions that are equal in the range ring or R-module.*

We develop our first ZBR-type theorem for P-ported objects with matroids or oriented matroids. It is the generalization of the ZBR theorem for graphs as given by Ellis-Monaghan and Traldi[8]. It depends on a lemma similar to one of theirs.

Lemma 6. *Suppose P-family $\mathcal{N}$ is partitioned into disjoint P-minor closed subfamilies $\{\mathcal{N}_\pi\}$. Then T is a Tutte function on $\mathcal{N}$ if and only if T restricted to $\mathcal{N}_\pi$ is a Tutte function for each $\mathcal{N}_\pi$.*

Theorem 7. *Suppose P-family $\mathcal{N}$ is partitioned into disjoint P-minor closed subfamilies $\{\mathcal{N}_\pi\}$, and each initial value $I(Q)$ depends only on the matroid or oriented matroid $M(Q)$ and on the π for which $Q \in \mathcal{N}_\pi$,*

Then T is a Tutte function with given parameters (x, y, X, Y) and initial values $I(Q)$ if and only if it satisfies the equations of Theorem **??***, interpreted for families of objects with matroids or oriented matroids.*

Proof. As in [8], lemma 6 lets us prove the theorem for each π separately.

By Lemma 3, T is a P-ported Tutte function of family of objects $\mathcal{N}_\pi$ if and only if function $T'(M(N)) = T(N)$ on the P-family $\mathcal{C}^\pi = \{M(N) \mid N \in \mathcal{N}_\pi\}$ is a P-ported Tutte function, since by hypothesis $I'(M(Q)) = I(Q) = I(M(Q))$ for corresponding indecomposibles $Q \in \mathcal{N}_\pi$ and $M(Q) \in \mathcal{C}^\pi$. The conclusion follows from Theorem **??** applied to $\mathcal{C}^\pi$. ∎

Ellis-Monaghan and Traldi's ZBR theorem for graphs refers to one initial value $\alpha_k = I(E_k)$ for each non-empty subclass of graphs, with unlabelled vertices, that have k graph components. One natural ported generalization is to partition the P-ported graphs G according to (1) how many graph components k, (2) $P' = P \cap S(G)$ and (3) $\nu : P' \to \{1, \ldots, k\}$, where $\nu(p)$ is which component contains edge p. Theorem 7 tells us:

Corollary 8. *Let a P-minor closed collection $\mathcal{G}$ of graphs with unlabelled vertices be partitioned into $\mathcal{G}_{k,P',\nu}$. Suppose initial values $I(G) = I_{k,P',\nu}(M(G))$ are given that depend only on the part and the matroid or oriented matroid of $G \in \mathcal{G}_{k,P',\nu}$. Then there is T, a P-ported separator-strong parametrized Tutte function of graphs $\mathcal{G}$ satisfying $T(Q) = I(Q) = I_{k,P',\nu}(M(Q))$ whenever P-quotient $Q \in \mathcal{G}_{k,P',\nu}$ if and only if the identities of Theorem* **??**, *interpreted for graphs, are satisfied with the given $I(Q)$.*

The next ZBR-type theorem addresses the problem illustrated by equation (1). It requires that the P-family satisfy the following:

Definition 9. *Object $N \in \mathcal{N}$ is **well-behaved** when for every independent set $C \subseteq E(N)$ and coindependent set $D \subseteq E(N)$ for which $C \cap D = \emptyset$, each of the $|C \cup D|!$ orders of contracting C and deleting D produces the same P-minor (which is an object) of N.*

Specifically, let $C = \{c_1, \ldots, c_j\}$, $D = \{d_{j+1}, \ldots, d_k\}$ and $R_i(N') = N'/c_i$ if $1 \le i \le j$ and $N' \setminus d_i$ if $j+1 \le i \le k$. The condition is $R_1 \circ \cdots \circ R_k(N) = R_{\sigma_1} \circ \cdots \circ R_{\sigma_k}(N)$ for every permutation σ of $\{1, \ldots, k\}$.

*$\mathcal{N}$ is **well-behaved** when each $N \in \mathcal{N}$ is well-behaved.*

By definition 2 all the minors are defined and $M(R_1 \circ \cdots \circ R_k(N)) = M(R_{\sigma_1} \circ \cdots \circ R_{\sigma_k}(N))$ independently of whether N is well-behaved or not. The point is that the objects themselves are the same. We give two examples.

Definition 10 (Graphs with set-labelled vertices)**.** *The elements of such a graph $S(G) = E(G) \cup (P \cap S(G))$ are edges. The vertices are labelled with non-empty finite sets so the two sets labelling distinct vertices in one graph are disjoint. Only non-loop edges $e \notin P$ can be contracted; when an edge is contracted, its two endpoints are replaced by one vertex whose label is the union of the labels of the two endpoints. Only non-isthmus edges $e \notin P$ can be deleted; deletion doesn't change labels. The graph has its graphic matroid if it is undirected and its oriented graphic matroid if it is directed.*

A graph with set-labelled vertices is well-behaved because the minor obtained by contracting forest C and deleting D is determined by merging all the vertex labels of each graph component of C and removing edges $C \cup D$. The deletions do not affect the vertex labels. Hence the vertex labels are not affected by the order of the operations.

Definition 11 (Graphs with set-labelled components)**.** *The elements of such a graph $S(G) = E(G) \cup (P \cap S(G))$ are edges. The path-connected components are labelled by non-empty finite sets so two components in the same graph always have disjoint labels. In other words, the set labels of the components are a partition π_V. Only non-loop edges $e \notin P$ can be contracted and only non-isthmus edges $e \notin P$ can be deleted. The component labels are unchanged by these minor operations. Definition 10 specifies the matroids or oriented matroids.*

A non-well-behaved P-family $\mathcal{C}!$ can be constructed from any P-family of matroids $\mathcal{C}$ with some $M \in \mathcal{C}$ with $|E(M)| \ge 2$. Each member of $\mathcal{C}!$ is formed from some $M \in \mathcal{C}$ together with some history of deletions and contractions that can be applied to M. Let c_e and d_e

be symbols for contracting and deleting $e \in E(\mathcal{C})$ respectively; a history h is a string of such symbols. Let $M|h$ be the P-minor obtained by performing history h on M, assuming each step is defined. The objects of $\mathcal{C}!$ are all pairs (M, h) for which P-minor $M|h \in \mathcal{C}$ is defined. The matroid of (M, h) is $M|h$, which determines the element set, loops and coloops. If $e \in E(M|h)$ is not a loop, then define $(M, h)/e = (M, hc_e)$. Similarly, if $e \in E(M|h)$ is not a coloop, $(M, h) \setminus e = (M, hd_e)$.

The point of this example is that even if the P-family is not well-behaved and so the indecomposibles do carry information about their history, Theorem 7 tells us that the Tutte function is still well defined if the initial values depend only on the matroid, or the oriented matroid, of the indecomposible. (See Questions, 5.1.)

The examples forced us to recognize that for N an object with a matroid or oriented matroid $M(N)$ with $e, f \in E(N)$ in series or in parallel, it might happen that $N/e \setminus f \neq N/f \setminus e$ even though, by Proposition **??**, $M(N)/e \setminus f = M(N)/f \setminus e$. Note that Proposition **??** is about $\cdot/e \setminus f$ and $\cdot/f \setminus e$ which are not commutations of the same two operations.

Theorem 12 (ZBR Theorem for well-behaved P-families of objects with matroids or oriented matroids)**.** *Supposed $\mathcal{N}$ is well-behaved. The following two statements are equivalent.*

(1) *T from $\mathcal{N}$ to R or an R-module is a P-ported separator-strong parametrized P-ported Tutte function with R-parameters (x, y, X, Y) whose values $T(Q)$ on P-quotients $Q \in \mathcal{N}$ are the initial values $I(Q)$.*

(2) *For every $N \in \mathcal{N}$:*

   (a) *If $M(N) = U_1^{ef} \oplus M(Q) = U_1^{ef} \oplus M(Q')$ with P-quotients $Q = N/e \setminus f$ and $Q' = N/f \setminus e$,*
$$I(Q)(x_e Y_f - y_f X_e) = I(Q')(x_f Y_e - y_e X_f).$$

   (b) *If $M(N) = U_2^{efg} \oplus M(Q) = U_2^{efg} \oplus M(Q')$ with P-quotients $Q = N/e \setminus f/g$ and $Q' = N/f \setminus e/g$,*
$$I(Q)X_g(x_e y_f - y_f X_e) = I(Q')X_g(x_f y_e - y_e X_f).$$

   (c) *If $M(N) = U_1^{efg} \oplus M(Q) = U_1^{efg} \oplus M(Q')$ with P-quotients $Q = N/e \setminus f \setminus g$ and $Q' = N/f \setminus e \setminus g$,*
$$I(Q)Y_g(x_e Y_f - y_f x_e) = I(Q')Y_g(x_f Y_e - y_e x_f).$$

   (d) *If $\{e, f\} = E(M(N))$ is a parallel pair connected to P,*
$$I(Q)(x_e Y_f - y_f x_e) = I(Q')(x_f Y_e - y_e x_f)$$
where P-quotients $Q = N/e \setminus f$ and $Q' = N/f \setminus e$.

   (e) *If $\{e, f\} = E(M(N))$ is a series pair connected to P,*
$$I(Q)(x_e y_f - y_f X_e) = I(Q')(x_f y_e - y_e X_f)$$
where P-quotients $Q = N/e \setminus f$ and $Q' = N/f \setminus e$.

Proof. All the steps of the proof given for Theorem **??** can be adapted. The hypothesis the $\mathcal{N}$ is well-behaved means that $N'/e/f = N'/f/e$, $N'/e \setminus f = N' \setminus f/e$, etc., (equalities of objects) are true respectively whenever a step of the form $M'/e/f = M'/f/e$, $M'/e \setminus f = M' \setminus f/e$, etc., respectively occurs for matroids or oriented matroids.

The calculations done in all cases of Theorem **??** of where Proposition **??** is applied to write $Q = M'/e \setminus f = M'/f \setminus e$ when e, f are in series or parallel are replaced by $Q = N'/e \setminus f$

and $Q' = N'/f \setminus e$. Matroids or oriented matroids are replaced by objects with matroids or oriented matroids in the calculations. Notes to guide the reader appear in the proof. □

Corollary 13. *A P-family of objects with matroids satisfies a ZBR-type theorem with the identities given in Theorem* **??** *if, in addition $\mathcal{N}$ being well-behaved, the initial values I satisfy $I(N/e \setminus f) = I(N/f \setminus e)$ when $\{e, f\} = E(N)$ is a series or parallel pair, $I(N/e \setminus f/g) = I(N/f \setminus e/g)$ when $\{e, f, g\} = E(N)$ is a triangle and $I(N/e \setminus f \setminus g) = I(N/f \setminus e \setminus g)$ when $\{e, f, g\} = E(N)$ is a triad.*

Corollary 14. *A P-family of objects with matroids satisfies a ZBR-type theorem with the identities given in Theorem* **??** *if, in addition to $\mathcal{N}$ being well-behaved, the object P-quotients $N/e \setminus f = N/f \setminus e$ when $\{e, f\} = E(N)$ is a series or parallel pair, $N/e \setminus f/g = N/f \setminus e/g$ when $\{e, f, g\} = E(N)$ is a triangle and $N/e \setminus f \setminus g = N/f \setminus e \setminus g$ when $\{e, f, g\} = E(N)$ is a triad.*

Proof. Clearly, if $N/e \setminus f = N/f \setminus e$ then $I(N/e \setminus f) = I(N/f \setminus e)$, etc. □

We conclude with second ported generalization of Ellis-Monaghan and Traldi's ZBR theorem for graphs, besides Corollary 8.

Corollary 15. *Let $\mathcal{G}$ be a ported P-family of graphs with unlabelled vertices, as in Corollary 8. Then there is T, P-ported separator-strong parametrized Tutte function of graphs $\mathcal{G}$ satisfying $T(Q) = I(Q)$ for all P-quotients $Q \in \mathcal{G}$ if and only if for every $G \in \mathcal{G}$, each case applies:*

- a-d *Revise the corresponding case of Theorem 12 by writing $Q' = Q$, i.e., replace Q' by Q.*
- e *If $E(G) = \{e, f\}$ is a series pair connected to P,*

$$I(Q)(x_e y_f - y_f X_e) = I(Q')(x_f y_e - y_e X_f)$$

where P-quotients $Q = G/e \setminus f$ and $Q' = G/f \setminus e$. (This is case (e) of Theorem 12 verbatim.*)*

Proof. $\mathcal{G}$ is a well-behaved P-family of ported objects with matroids or oriented matroids, so Theorem 12 applies. In all cases of Theorem 12 but the last, the two object minors are the same graph because the contracted edges are path-connected, so the equations of Theorem 12 are simplified. □

Both Corollaries 8 and 15 reduce to the ZBR theorem for graphs when $P = \emptyset$. The first uses the property that $\mathcal{G}$ is partitioned into minor-closed subclasses with indecomposibles E_k for which the initial values depend only on the matroid or oriented matroid to generalize the original ZBR equations. As with matroids, we find again that the Tutte functions can distinguish different orientations of the the same undirected graphs. The second relies on the commutivity of the graph minor operations and generalizes the fact that different initial values may be assigned to different indecomposibles, but then the conditions sufficient for the initial values to extend to a Tutte function must be stronger.

3. Direct and Other Sums

It is a common situation that $\{N^1, N^2, N\} \subseteq \mathcal{N}$ and their matroids or oriented matroids satisfy $M(N^1) \oplus M(N^2) = M(N)$. Tutte computation trees help. The proposition below applies even to non-well-behaved N when the symbols $/B_i^j|P_j$ refer to sequences of deletions and contractions.

Definition 16. *If $\mathcal{T}_1$ and $\mathcal{T}_2$ are Tutte computation trees then $\mathcal{T}_1 \cdot \mathcal{T}_2$ is the tree obtained by appending a separate copy of $\mathcal{T}_2$ at each leaf of $\mathcal{T}_1$. The root is the root of the expanded $\mathcal{T}_1$.*

Proposition 17. *Suppose N, N^1 and N^2 are all in $\mathcal{N}$ and $M(N^1) \oplus M(N^2) = M(N)$. Then if $\mathcal{T}_1$ and $\mathcal{T}_2$ are Tutte computation trees for N^1 and N^2 respectively with values given by* (DS1) *and* (DS2)*, then there is a Tutte computation tree for N that yields the value given by* (DS)*.*

$$\sum_{Q_i^1} I(Q_i^1)c_1(Q_i^1) \textit{ where } Q_i^1 = N/B_i^1|P_1. \tag{DS1}$$

$$\sum_{Q_j^2} I(Q_j^2)c_2(Q_j^2) \textit{ where } Q_j^2 = N/B_j^2|P_2. \tag{DS2}$$

$$\sum_{Q_i^1,Q_j^2} I(Q_{i,j})c_1(Q_i^1)c_2(Q_j^2) \textit{ where } Q_{i,j} = N/B_i^1|(P_1 \cup S(N^2))/B_j^2|P_2. \tag{DS}$$

Furthermore, if T is a Tutte function on $\mathcal{N}$ and $T(N^1)$ and $T(N^2)$ equal the Tutte polynomials given by (DS1) *and* (DS2) *then $T(N)$ equals the polynomial given by* (DS)*.*

Proof. We show how to relabel $\mathcal{T}_1 \cdot \mathcal{T}_2$ to obtain a Tutte computation tree for N. $M(N^1) \oplus M(N^2) = M(N)$ is defined means $S(M(N^1)) \cap S(M(N^2)) = \emptyset$ and $S(M(N)) = S(M(N^1)) \cup S(M(N^2))$. Each node of $\mathcal{T}_1 \cdot \mathcal{T}_2$ is determined by by deleting and/or contracting some elements of $E(M(N^1)) \cup E(M(N^2))$. Relabel that node with the P-minor of N obtaining deleting and/or contracting the same elements respectively in the same order, those in N^1 preceding those in N^2. The result is a computation tree for N because $M(N^1) \oplus M(N^2) = M(N)$. Assume $P \subseteq S(M(N))$ (otherwise, take a smaller P) and let $P^1 = S(M(N^1)) \cap P$ and $P^2 = S(M(N^2)) \cap P$. At a leaf of the relabelled tree, there will be the P-quotient $N/B_1/B_2|P$ where B_1 is a P^1-subbasis of $M(N^1)$ and B_2 is a P^2-subbasis of $M(N^2)$. □

3.1. **Strong Tutte Functions.**

Let us extend the definition of strong parametrized Tutte function to P-families $\mathcal{N}$ of objects with matroids and oriented matroids, in the way that abstracts the known notion of strong Tutte functions on minor closed families of graphs[8]. We can then specialize $\mathcal{N}$ to $\mathcal{C}$, a P-family of matroids or oriented matroids. There might still be indecomposibles besides or instead of $\emptyset$.

Definition 18. *A P-ported separator-strong Tutte function T on a P-family of objects $\mathcal{N}$ with matroids is called **strong** if whenever $\{N^1, N^2, N\} \subseteq \mathcal{N}$ and $M(N^1) \oplus M(N^2) = M(N)$, then $T(N^1)T(N^2) = T(N)$.*

We now generalize to $P \neq \emptyset$ the $T(\emptyset)T(\emptyset) = T(\emptyset)$ characterization of strong Tutte functions.

Theorem 19. *A P-ported separator-strong Tutte function T on a P-family of objects with matroids or oriented matroids $\mathcal{N}$ is strong if and only if T restricted to the indecomposibles of $\mathcal{N}$ is strong; i.e., whenever Q^1, Q^2 and Q are indecomposibles and $M(Q^1) \oplus M(Q^2) = M(Q)$ then $T(Q^1)T(Q^2) = T(Q)$.*

Proof. Every P-quotient is in $\mathcal{N}$, so clearly T restricted to the P-quotients is strong.

Conversely, suppose N^1, N^2 and N are in $\mathcal{N}$ and $M(N^1) \oplus M(N^2) = M(N)$, so Proposition 17 applies.

Since $M(N^1) \oplus M(N^2) = M(N)$, $M(N/(B_1 \cup B_2)|P) = (M(N^1)/B_1|P) \oplus (M(N^2)/B_2|P) = M(N^1/B_1|P) \oplus M(N^2/B_2|P)$. We now use the fact that $Q_{ij} = N/B_1/B_2|P$, $Q_i^1 = N^1/B_1|P$ and $Q_j^2 = N^2/B_2|P$ are P-quotients and the hypothesis to write $I(Q_{ij}) = I(Q_i^1)I(Q_j^2)$.

We therefore conclude $T(N) = T(N^1)T(N^2)$ from (DS1), (DS2) and (DS). □

3.2. **Multiplicative Tutte Functions.**

Graphs can be combined in several ways all so the matroid of the combination is the direct sum of the matroids of the parts. This motivates:

Definition 20. *A partially defined binary operation "$*$" on a P-family of objects with matroids or oriented matroids $\mathcal{N}$ is* ***a matroidal direct sum*** *if whenever $N^1 * N^2 \in \mathcal{N}$ is defined for $\{N^1, N^2\} \subseteq \mathcal{N}$, the matroids or oriented matroids satisfy $M(N^1) \oplus M(N^2) = M(N^1 * N^2)$.*

Proposition 17 applies when $N^1 * N^2 = N$ is defined. It gives a general recipe for $T(N^1 * N^2)$ which generalizes the identity [8] $T(M^1 \oplus M^2)T(\emptyset) = T(M^1)T(M^2)$ for separator-strong Tutte functions of matroids. The P-ported generalization is more complicated and generally cannot be expressed by a product in the domain ring of T.

Proposition 21. *Suppose $*$ is a matroidal direct sum and N^1, N^2 and $N^1 * N^2$ are each members of a P-family for which T is a Tutte function.*

If for P-quotients Q_i^j and R-coefficients $c_j(Q_i^j)$, $j = 1$ and 2,

$$T(N^1) = \sum_{Q_i^1} T(Q_i^1)c_1(Q_i^1)$$

and

$$T(N^2) = \sum_{Q_j^2} T(Q_i^2)c_2(Q_j^2)$$

then

$$T(N^1 * N^2) = \sum_{Q_i^1, Q_j^2} T(Q_i^1 * Q_j^2)c_1(Q_i^1)c_2(Q_j^2).$$

Proof. Substitute $Q_{i,j} = Q_i^1 * Q_j^2$ in (DS) of Proposition 17. □

When $\mathcal{N}$ is a P-family of matroids or oriented matroids, direct matroid or oriented matroid sum is obviously a matroidal direct sum operation, and so Proposition 21 is applicable.

Corollary 22. [8] *Let $P = \emptyset$. $T(M^1 \oplus M^2)T(\emptyset) = T(M^1)T(M^2)$ for Tutte function T of matroids.*

Proof. Our proof demonstrates how Proposition 21 generalizes this formula to P-families. The expansions DS1 and DS2 take the one-term form $T(M^j) = T(\emptyset)c_j(\emptyset)$, $j = 1, 2$, so $T(M^1)T(M^2) = T(\emptyset)^2 c_1(\emptyset)c_2(\emptyset)$. Expansion DS is then $T(M^1 \oplus T^2) = T(\emptyset)c_1(\emptyset)c_2(\emptyset)$. □

Following the definitions for graphs in [8], we write:

Definition 23. *Given a matroidal direct sum $*$ on $\mathcal{N}$, a Tutte function T on $\mathcal{N}$ is* ***multiplicative*** *(with respect to "$*$") if whenever $N^1 * N^2$ is defined for $\{N^1, N^2\} \subseteq \mathcal{N}$, the Tutte function values satisfy $T(N^1)T(N^2) = T(N^1 * T^2)$.*

A strong Tutte function is certainly multiplicative for any "$*$", but not conversely.

Corollary 24. *A P-ported Tutte function T on P-family $\mathcal{N}$ is multiplicative with respect to matroidal direct product "$*$" if and only if for every pair of indecomposibles $\{Q_i, Q_j\} \in \mathcal{N}$ for which $Q_i * Q_j \in \mathcal{N}$ is defined, $T(Q_i)T(Q_j) = T(Q_i * Q_j)$.*

Proof. When $N^1 * N^2$ is defined, Proposition 17 applies because $M(N^1) \oplus M(N^2) = M(N^1 * N^2)$. $T(N^1 * N^2) = T(N^1)T(N^2)$ is then a consequence of $T(Q_i * Q_j) = T(Q_i)T(Q_j)$. □

Corollary 25. *Suppose T is parametrized Tutte function on a minor-closed class of graphs $\mathcal{G}$ with unlabelled vertices and $P = \emptyset$. T is strong if and only if there is an idempotent $\alpha = \alpha^2 \in R$ and $T(E_k) = \alpha$ whenever $\mathcal{G}_k \neq \emptyset$.*

Proof. $M(E_k) = \emptyset$ for all $k \geq 1$ and $\emptyset \oplus \emptyset = \emptyset$, so $T(E_i)T(E_j) = T(E_k)$ whenever $\mathcal{G}_i$, $\mathcal{G}_j$ and $\mathcal{G}_k$ are all non-empty. Hence, if $\mathcal{G}_k \neq \emptyset$ then $T(E_k)T(E_k) = T(E_k) = \alpha$. Further, if $\mathcal{G}_j \neq \emptyset$ with $j \neq k$, $\alpha = T(E_k) = T(E_k)T(E_k) = T(E_j)$. □

This strengthens part of Corollary 3.13 in [8]. The other part does require its additional conditions to establish strongness of a Tutte function on graphs given as multiplicative on both disjoint union $\amalg$ and one-point unions. Consider $\mathcal{N} = \{E_3, E_4, E_5, \ldots\}$, $T(E_k) = 1$ for $k \geq 3$, $k \neq 5$ and $T(E_5) = 0$. $T(E_3)T(E_4) = 1 \neq T(E_5)$, so T is not strong, but T is multiplicative on disjoint and one-point unions because E_5 cannot be expressed as either kind of union of graphs in $\mathcal{G}$. The other conditions are that $\mathcal{G}_k \neq \emptyset$ for all k and that $\mathcal{G}$ is closed under one-point unions and removal of isolated vertices.

Part 1. Remarks, Background, Problems, and References

4. Non-commutativity

We presented in [4] a new kind of strong Tutte-like function on P-ported oriented graphic matroids (more generally, unimodular, i.e. regular oriented matroids) whose values vary with the orientation. Each function value $F(G)$ is in the exterior algebra over R^{2p}, where R is the reals extended by the x_e, y_e and $|P| = p$. The function obeys an *anti-commutative* variant of (**??**) with exterior multiplication $\wedge$. (When $P = \emptyset$, F is the reduced Laplacian determinant in the famous Matrix Tree Theorem [11, 2].) It is the first example we know of "the possibility of making use of a noncommutative generalization of the Tutte polynomial at some point in the future." mentioned by Bollobás and Riordan in [1]. We won't say more beyond that (1) each of the $\binom{2p}{p}$ Plücker coodinates of $F(G)$ is a P-ported Tutte function of the kind we covered; and (2) that quadratic inequalities among some of them express negative correlation between edges in spanning trees, results also known as Rayleigh's inequality[5, 14, 6].

A second non-commutative possibility might be found in section 3. The abstraction applies to situations where an object represents an initial matroid, graph, etc. plus a history of deletions and contractions. The key feature is that the Tutte decompositions of an object are identical to the Tutte decompositions of that object's matroid. The initial value on indecomposible object Q might then depend on the order of the deletion and contractions reductions to to obtain it from N. This helps us understand the theory, but whether objects with minors that depend on reduction order have useful applications remains to be seen.

5. Background and Other Related Work

Besides Brylawski's work, another early appearance of Tutte decomposition of a matroid or graph with a basepoint is [15]. Ellis-Monaghan and Traldi [8] explain that by leaving the reduction by e_0 to last so e_0 is always contracted as a coloop or deleted as a loop, the Tutte function value can be expressed by $T(M) = (rX_{e_o} + sY_{e_0})T(\emptyset)$ were r, s are not-necessarilly-unique elements in R. As one application, they give a formula for the parametrized Tutte polynomial for the parallel connection across e_0 which generalized Brylawski's. These r, s appear in the P-ported Tutte function expression $rT(U_1^{e_0}) + sT(U_0^{e_0})$ when $P = \{e_0\}$. They are parametrized generalizations of the coefficients of z' and x' in Brylawski's four variable Tutte polynomial.

Las Vergnas defined and gave basic properties of "set-pointed" Tutte polynomials (with no parameters) and used them to study matroid perspectives. The polynomial given in [12, 13] has a variable ξ_l for each subset in a collection of k subsets $P_l \subseteq P$, $l = 1, \ldots, k$. Each term in (**??**) had $\prod \xi_l^{r_i(P_l)}$ for $[Q_i]$ where r_i is the rank function of matroid Q_i. Therefore (**??**) was satisfied and the association of the term to (non-oriented) Q_i could be assured by taking all $2^{|P|}$ subsets for the P_l. The matroid perspective is the strong map $M \setminus E(M) \to M/E(M)$ given by the identity on P.

In [3], we reproduced Las Vergnes' theory with explicit P-quotient (matroid) variables (see the $[Q_i]$ symbols in Corollary **??** in sec. **??**) in place of $\prod \xi_l^{r_i(P_l)}$. We then gave formulas for the P-ported Tutte polynomial for the union and its dual of matroids whose common elements are in P. These formulas work in a way similar to what appears in sec. **??**. We extend to algebras the $\mathbb{Z}[u, w]$-module generated by the $[Q_i]$ by defining multiplications $\tilde{*}$ with the rules $[Q_i]\tilde{*}[Q_j] = r_{ij}[Q_{i,j}]$, with $r_{i,j} \in \mathbb{Z}[u, w]$ and $Q_{i,j} = Q_i * Q_j$ depending on (Q_i, Q_j) and whether $*$ represents union or its dual. It is not often recognized that series and parallel connection of matroids across basepoint p is equivalent to matroid union and its dual on matroids with only element p in common. We plan to investigate whether the formulas for parametrized Tutte polynomials of parallel connections in [8] can be generalized to the dual of union when $|P| > 1$, and to detail the relationship when $|P| = 1$.

Recent work on a different generalization, weak Tutte functions (see sec. **??**), has been done by Ellis-Monaghan and Zaslavsky [17]. The distinction between weak Tutte functions (satisfying an additive identity only) and strong Tutte functions (which satisfy (**??**) and (**??**)) seems first to have been made by Zaslavsky[16], for matroids. That paper also defined weak and strong Tutte functions of graphs. However, we use using Ellis-Monaghan and Traldi's definition of strong Tutte functions of graphs[8]. The latter restricts (**??**) to non-separators (not just non-loops); and it requires $T(G^1)T(G^2) = T(G)$ whenever matroids $M(G^1) \oplus M(G_2) = M(G)$ (which we sometimes interpret as oriented), not just when G is the disjoint union of G^1 and G^2. The term separator-strong (I learned[9] after [8] appeared.) is used for Tutte functions of matroids and graphs as defined in [8]; recall that they satisfy (**??**) and (**??**). Normal is used in the same way as in[16].

We introduced P-ported parametrized Tutte polynomials for normal strong Tutte functions in [4], i.e., those with corank-nullity polynomial expressions. Most of the results in the current paper, when so restricted appeared in [4] or can be derived by adding parameters and oriented matriod considerations to material in [3]. These include computation tree[10] based activities expansions with terms corresponding to P-subbases (which are called "contracting sets" in [7]). In the normal case, the initial values can be assigned arbitrarilly. We used this to show the our extensor-valued Tutte-like function [3] is expressible by assigning extensors as the initial values. In this electrical network application, the indecomposibles are oriented graphic matroids and different values *are* assigned to different orientations of the same matroid.

5.1. **Questions.** Diao and Hetyei's work [7] is part of a larger literature on knot invariants which leads us to ask if the P-ported objects with matroids abstraction, and its related Tutte computation tree expansions for parametrized Tutte functions (sec. 3), can be usefully applied to objects besides graphs or directed graphs, such as various kinds of knot diagrams.

Another open project is to classify the solutions to the conditions of Theorem **??** (about separator-strong P-ported Tutte functions) and Theorem 19 (about the strong ones) for rings and for fields along the lines of [16] and [1].

Finally, the theory of objects with matroids or oriented matroids compared with P-ported Tutte functions of oriented matroids raises the question of what role orientation plays in P-ported Tutte function theory. An oriented matroid can be considered to be an object with a non-oriented matroid. In other words, the orientation is a property of the object that is constrained by, but not uniquely determined by the object's non-oriented matroid. However, our results show that when the initial values depend only on the orientations in addition to depending on the matroids, the Tutte function is characterized by the equations of Theorem **??**, not Theorem 12. The reason is that when interchanging the elements when deleting and contracting a series or parallel pair, both the resulting matroids and their orientations are the same. The question we raise is whether there are interesting objects with matroids and certain other properties besides matroid orientation such that Theorem **??** characterizes the Tutte functions, when the initial values depend only on the matroids and those other properties.

6. Acknowledgements

I thank the Newton Institute for Mathematical Sciences of Cambridge University for hospitality and support of my participation in the Combinatorics and Statistical Mechanics Programme, January to July 2008, during which some of this work and many related subjects were reviewed and discussed.

I thank the organizers of the Thomas H. Brylawsky Memorial Conference, Mathematics Department of The University of North Carolina, Chapel Hill, October 2008, Henry Crapo, Gary Gordon and James Oxley for fostering collaboration beween people who carry on the memory of Prof. Brylawsky and editing this journal issue.

I thank Lorenzo Traldi for bringing to my attention and discussing Diao and Hetyei's work, as well as Joanne Ellis-Monaghan, Gary Gordon, Elizabeth McMahon and Thomas Zaslavsky for helpful conversations and communications at the Newton Institute and elsewhere, and my University at Albany colleague Eliot Rich for mutual assistance with writing.

This work is also supported by a Sabbatical leave granted by the University at Albany, Sept. 2008 to Sept. 2009.

References

[1] B. Bollobás and O. Riordan. A Tutte polynomial for colored graphs. *Combin. Probab. Computat.*, 8(1–2):45–93, 1999.

[2] S. Chaiken. A combinatorial proof of the all minors matrix tree theorem. *SIAM J. Algebraic Discrete Methods*, 3(3):319–329, 1982.

[3] S. Chaiken. The Tutte polynomial of a P-ported matroid. *J. Combin. Theory Ser. B*, 46:96–117, 1989.
[4] S. Chaiken. Ported tutte functions of extensors and oriented matroids. `http://arXiv.org/abs/0605707`, August 2006.
[5] Y.-B. Choe. A combinatorial proof of the rayleigh formula for graphs. *Discrete Math.*, 2003. in press.
[6] Y.-B. Choe, J. G. Oxley, A. D. Sokal, and D. G. Wagner. Homogeneous multivariate polynomials with the half-plane property, Dec. 2002. `http://arxiv.org/abs/math/0202034 [math.CO]`.
[7] Y. Diao and G. Hetyei. Relative Tutte polynomials for colored graphs and virtual knot theory. preprint, May 2008. `http://www.math.uncc.edu/preprint/2008/2008_06.pdf`.
[8] J. A. Ellis-Monaghan and L. Traldi. Parametrized Tutte polynomials of graphs and matroids. *Combinatorics, Probability and Computing*, 15:835–854, 2006.
[9] J. A. Ellis-Monaghan and T. Zaslavsky. Personal Communication.
[10] G. Gordon and E. McMahon. Interval partitions and activities for the greedoid Tutte polynomial. *Adv. Appl. Math.*, 18(1):33–49, Jan. 1997.
[11] F. Harary. *Graph theory.* Addison-Wesley, Reading, Mass., 1969.
[12] M. Las Vergnas. Extensions normales d'un matroïde, polynôme de Tutte d'un morphisme. *C. R. Acad. Sci. Paris Sér. A-B*, 280(22):Ai, A1479–A1482, 1975.
[13] M. Las Vergnas. The Tutte polynomial of a morphism of matroids. I. Set-pointed matroids and matroid perspectives. *Ann. Inst. Fourier (Grenoble)*, 49(3):973–1015, 1999. Symposium à la Mémoire de François Jaeger (Grenoble, 1998).
[14] C. Semple and D. Welsh. Negative correlation in graphs and matroids. *Combin. Probab. Comput.*, 17:423–435, 2008.
[15] C. A. B. Smith. Patroids. *J. Comb. Theory ser. B*, 16:64–76, 1974.
[16] T. Zaslavsky. Strong Tutte functions of matroids and graphs. *Trans. Amer. Math. Soc.*, 334(1):317–347, 1992.
[17] T. Zaslavsky. Tutte functions. Presentation at the Thomas H. Brylawsky Memorial Conference, Chapel Hill, October 18 2008. joint work in progress with J. A. Ellis-Monaghan.

Computer Science Department, The University at Albany (SUNY), Albany, NY 12222, U.S.A.

E-mail address: `sdc@cs.albany.edu`