# Restricted or Ported Tutte Decomposion and Analogs of All-Minors Laplacian Expansions

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## What is a Strong Tutte function?

Some history. Zaslavsky (1992) in "Strong Tutte Functions of Matroids and Graphs" showed what happens with Tutte equations (on a field), with 4 parameters (or weights) (Different notation!) $g_e$, $r_e$, $i_{\mathbf{loop(e)}}$ and $i_{\mathbf{coloop(e)}}$ for element $e$:

1. For all $\mathbf{N}$ with separator (neither loop nor coloop) $e \in S(\mathbf{N})$,

$$F(\mathbf{N}) = g_e F(\mathbf{N}/e) + r_e(\mathbf{N} \backslash e)$$

2. When $\mathbf{N} = \mathbf{N_1} \oplus \mathbf{N_2}$,

$$F(\mathbf{N}) = F(\mathbf{N_1})F(\mathbf{N_2})$$

3. When $\mathbf{N}$ is a loop or coloop on $e$, an initial value is given:

$$F(\mathbf{N}) = i_{\mathbf{N}}$$

(Z.'s term: Point values.) This means there are two parameters (besides $g_e$, $r_e$) for each $e$, so

$$F(\mathbf{loop(e)}) = i_{\mathbf{loop(e)}} \text{ and } F(\mathbf{coloop(e)}) = i_{\mathbf{coloop(e)}}$$

# What happens?

## Those equations might not have a solution!

For (typically *lots of*) equations involving a common *function* $F$, for them "to have a solution" MEANS there exists a function $F$ on some domain of matroids so all the equations are satisfied with that $F$.
This MEANS $F(\mathbf{N})$ is what is computed by applying Tutte equations *in any order they are applicable*.

## Z.'s result

Strong Tutte functions are classified into seven types, each given by conditions on the weights and the initial values.
Amazingly, the conditions for there to be a solution are all derived by requiring all Tutte decompositions of **2 or 3 point matroids** in the domain to compute *the same value*.

# Maybe more history

All things matroids and Tutte polynomial were around Zaslavsky and the rest of the 1970's MIT gang.
After a couple of years, I tried my hand at drawing algorithms for planar graphs, and was led to Tutte's "How to draw a graph", and Brook, Smith, Stone and Tutte's "Dissecting a square into squares." Both inverted submatrices of a graph's Laplacian; both had the Matrix Tree Theorem to prove this was possible. Harmonic functions on vertices were used to place vertices (after fixing places of some) and to find sizes of squares so they tiled a square in a given combinatorial pattern.

# Solving electrical problems by counting trees

## Very shocking fact-Maxwell's or Kirchhoff's rule

The equivalent resistance $R_{uv}$ between nodes $u$ and $v$ of a resistor network $N$ with edge conductances $g_e$ $(= r_e^{-1})$ is

$$R_{uv} = \frac{\sum_{F \text{ a spanning tree in } N/(uv) \text{ with } u,v \text{ identified}} \prod_{e \in F} g_e}{\sum_{F \text{ a spanning tree in } N} \prod_{e \in F} g_e}$$

So weighted tree enumeration don't just tell us some matrices are invertable.

Thinking matroids, $N/(uv)$ is $N$ with a different kind of edge, an interface edge $p = uv$ added. Then,

Numerator is $\sum g_F$ over $F$ bases in $N/p$.
Denominator is $\sum g_F$ over $F$ bases in $N \backslash p$.
BOTH of these sums are weighted Tutte functions.

# Why call $(uv)$ a port?

from "The Tutte Polynomial of a Ported Matroid" sdc 1989

We have been motivated by electrical network considerations where the branches used to connect the network to other networks are distinguished from the branches or variables associated with devices such as resistors or capacitors..

# Ported/Set Pointed/Relative Tutte Functions

## Definition (sdc 1989)

(easily updated with weights and oriented matroids) Let $M(E, P)$ be a $P$-ported oriented matroid with rank function $\rho$. The $P$-ported weighted rank generating function $r_P(M)$ is

$$r_P(M) = \sum_{A \subseteq E} [M/A|P] g_A r_{\overline{A}} x^{\rho(M)-\rho(M/A|P)-\rho(A)} y^{|A|-\rho(A)}$$

Here $S(M) = P(M) \coprod E(M)$, and $r_e, g_e$ are weights for each $e \in E$.
For any oriented matroid $M$ for which $E(M) = \emptyset$, $[\emptyset] = 1$ and

$$[M] = [M_1][M_2] \cdots [M_k]$$

where $M = M_1 \oplus M_2 \cdots M_k$ and each $M_i$ is connected. These bracket *oriented* matroid symbols comprise, with (the well-known) Tutte Polynomial variables $x$ and $y$, the variables in $r_P(M)$.

# $P$-ported parametrized Tutte Equations

They are the usual, except deletion/contraction of $e$ is forbidden when $e \in P$.

### Definition

An oriented matroid $M(P, E)$ is $P$-ported when its ground set $S(M) = P \coprod E$. A function $F$ on oriented matroids is a $P$-ported weighted Tutte function if

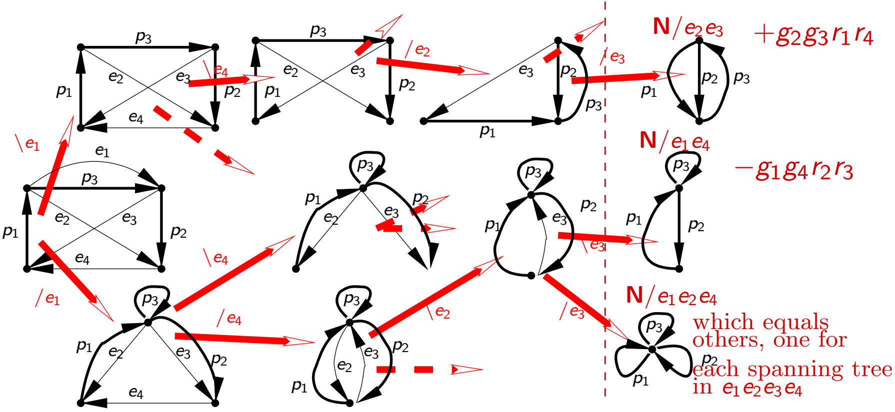
- Whenever $e \in E(M)$, and $e$ is a non-separator in $M$, $F(M) = g_e F(M/e) + r_e F(M \backslash e)$.
- Whenever $M = M_1 \oplus M_2$, $F(M) = F(M_1)F(M_2)$.

### Theorem

*$r_P$ defined above is (such) a Tutte function.*

# Ported Tutte Decomposition (not all of it)

The decomposition ends with minors $\mathbf{N}/F\overline{F}$.
We show component $\mathbf{L}_E(\mathbf{N}\ \mathbf{N})[p_{\alpha 1}p_{\beta 1}p_{\alpha 3}]$:



# Zaslavsky 1992: It doesn't all work. $r_P$ is not universal.

$$r_P(\mathbf{loop(e)}) = g_e y + r_e$$
$$r_P(\mathbf{coloop(e)}) = r_e x + g_e$$

The two Tutte decompositions of the circuit $U^1_{ef}$ on $e, f$ to compute a prospective Tutte function $F$ give

$$e \text{ first } F(U^1) = g_e F(loop(f)) + r_e F(coloop(f))$$
$$f \text{ first } F(U^1) = g_f F(loop(e)) + r_f F(coloop(e))$$

We still need those 4 point values, and they can't be chosen independently of the 4 weights. Zaslavsky called the class where there are arbitrary $x$, $y$ values and the point values are $g_e y + r_e$ and $r_e x + g_e$ *normal Tutte functions*.

# Related work.

- $|P| = 1$ and series/parallel connections on pointed matroids (Brylawsky (1971)), extended to unions and dual-unions over $P$ (also sdc 1989).
- Matroids called set-pointed on $P$ encoded by products of many variables (Las Vergnas 1975).
- Weighs/colors/parameters (Zaslavsky 1992, Bollobás and Riordan 1999, Ellis-Monaghan and Traldi 2006. Much motivation from maps on surfaces and knot theory.
- Dao and Hetyei (2012) named carried out BRZs classfication program, called the matroids relative. Motivated by knots with ports for virtual crossings. Easy to see this extends to oriented matroids.

# What this talk is about.

Some ways determinants make Tutte functions.
How the graph and other Laplacians ACTUALLY ARE Tutte functions, not just a partcular determinant.
Ported Tutte functions are needed to tell this story.
The only Tutte function I know valued in a **non-commutative ring** (the signed commutative exterior algebra).
Only normal Tutte functions are relevant, and we only need them with $x = y = 0$ (which does $P$-subbasis enumeration).

# Tutte Functions using determinants: Our setup

- Matrices $N_\alpha$, $N_\beta^\perp$; full row rank, columns indexed by $P \coprod E$.
  $\mathrm{rank}(N_\alpha) + \mathrm{rank}(N_\beta^\perp) = |E| + |P|$.
  $P_\alpha, P_\beta \leftrightarrow P$, $P_\alpha \cap P_\beta = \emptyset$.
- Weight (parameter) matrices
  $G = \mathrm{diag}\{g_e\}_{e \in E}$, $R = \mathrm{diag}\{r_e\}_{e \in E}$.
- Matrix with columns $P_\alpha \coprod P_\beta \coprod E$

$$L = L\begin{pmatrix} N_\alpha \\ N_\beta^\perp \end{pmatrix} = \left[\begin{array}{c|c|c} N_\alpha(P) & 0 & N_\alpha(E)G \\ \hline 0 & N_\beta^\perp(P) & N_\beta^\perp(E)R \end{array}\right]$$

Define

$$F(L) = (( \binom{2p}{p} )) - \text{tuple of determinants } L[Q_\alpha \overline{Q_\beta} E(\text{all of } E)])$$

indexed by length $p = |P|$ sequences $Q_\alpha \overline{Q_\beta} \subseteq P_\alpha P_\beta$ where $Q_\alpha \subseteq P_\alpha$ and $\overline{Q_\beta} \subseteq P_\beta$.

Column $e$ of $L$ when $e \notin P$ is

$$\begin{bmatrix} N_{\alpha,1,e}g_e \\ N_{\alpha,2,e}g_e \\ \dots \\ N_{\alpha,r_1,e}g_e \\ N^{\perp}_{\beta,1,e}r_e \\ N^{\perp}_{\beta,2,e}r_e \\ \dots \\ N^{\perp}_{\beta,r_2,e}r_e \end{bmatrix} = \begin{bmatrix} N_{\alpha,1,e} \\ N_{\alpha,2,e} \\ \dots \\ N_{\alpha,r_1,e} \\ 0 \\ 0 \\ \dots \\ 0 \end{bmatrix} g_e + \begin{bmatrix} 0 \\ 0 \\ \dots \\ 0 \\ N^{\perp}_{\beta,1,e} \\ N^{\perp}_{\beta,2,e} \\ \dots \\ N^{\perp}_{\beta,r_2,e} \end{bmatrix} r_e$$

So, for all $e \in E$, that is $e \notin P$:

$$F(L)_{Q_\alpha \overline{Q_\beta}} = L[Q_\alpha \overline{Q_\beta} E] =$$

$$g_e L \begin{pmatrix} N_\alpha / e \\ N^{\perp}_\beta \backslash e \end{pmatrix} [Q_\alpha \overline{Q_\beta} E] + r_e L \begin{pmatrix} N_\alpha \backslash e \\ N^{\perp}_\beta / e \end{pmatrix} [Q_\alpha \overline{Q_\beta} E].$$

## Since deletion and contraction are done only for $e \notin P$

we get a **Ported** (sdc) or **Set-pointed** (Las Vergnas) or **relative** (Dao and Hetyei) Tutte Function.

$|Q_\alpha \overline{Q_\beta}| = p$, so $\binom{2p}{p}$ determinants $L[Q_\alpha \overline{Q_\beta} E]$ make the tuple:

$$F(L) = g_e FL \begin{pmatrix} N_\alpha / e \\ N_\beta^\perp \backslash e \end{pmatrix} + r_e FL \begin{pmatrix} N_\alpha \backslash e \\ N_\beta^\perp / e \end{pmatrix}$$

where

$N/e$ means remove the $g_e$ or $r_e$ but otherwise keep column $e$

$N \backslash e$ means replace column $e$ by 0.

## Plücker coordinates

These determinants can be considered an *affine* version of the (projective) Plücker coordinates for the row space of $L$ projected into $K^{P_\alpha \coprod P_\beta}$. We need affine so Tutte's + identity makes sense.

$$FL\begin{pmatrix} N_\alpha \\ N_\beta^\perp \end{pmatrix} = g_e FL\begin{pmatrix} N_\alpha/e \\ N_\beta^\perp\backslash e \end{pmatrix} + r_e FL\begin{pmatrix} N_\alpha\backslash e \\ N_\beta^\perp/e \end{pmatrix} \qquad (*)$$

Real deletion/contraction removes $e$ from the ground set of the matroid or other object, but $N/e, N\backslash e$ still have column $e$. But (*) holds for all $e \in E$, so Laplace's expansion is a basis expansion:

$$L[Q_\alpha \overline{Q_\beta} E] = \sum_{A \subseteq E} g_A r_{\overline{A}} N_\alpha[Q_\alpha A] N_\beta^\perp[\overline{Q_\beta A}] \epsilon(Q_\alpha A, \overline{Q_\beta A})$$

The $A$ term is $\neq 0$ iff $Q_\alpha A$ is a column basis for $N_\alpha$ and $\overline{Q_\beta A}$ is a column basis for $N_\beta^\perp$. So, for each $Q_\alpha \overline{Q_\beta}$

$$L[Q_\alpha \overline{Q_\beta} E] = \pm \sum_{A \subseteq E} g_A r_{\overline{A}} N_\alpha[Q_\alpha A] N_\beta^\perp[\overline{Q_\beta A}] \epsilon(A, \overline{A})$$

(The non-zero terms all have $|A| = \text{rank}(N_\alpha) - |Q_\alpha|$.)

# Quick and dirty fix

1. Drag column $e$ to the far right.
   Changes sign of $F(L)$ by $\epsilon(E'e)$.
2. Left multiply by a determinant 1 matrix that sends the last column to $(0, ..., 1g_e, 0, ..., 1r_e)^{\mathbf{t}}$ (if the top or bottom submatrix has just 1 row, do the hack: $\mathbf{N}/e$ is number $\mathbf{N}_{1,e}$ that acts like a matrix with columns $E'$ and no rows.)
3. Drag the row with the $1g_e$ to the bottom.
   Changes sign of $F(L)$ by $(-1)^{\mathrm{r}\mathbf{N}_\beta^\perp}$
4. With $e$ deleted/contracted from the $\mathbf{N}$s defining $L$, define $F$ by $FL_{Q_\alpha \overline{Q_\beta}} = L[Q_\alpha \overline{Q_\beta} E']$

## Result

$$FL\begin{pmatrix} N_\alpha \\ N_\beta^\perp \end{pmatrix} = \epsilon(E'e)\left(g_e(-1)^{\mathrm{r}(N_\beta^\perp)} FL\begin{pmatrix} N_\alpha/e \\ N_\beta^\perp \backslash e \end{pmatrix} + r_e FL\begin{pmatrix} N_\alpha \backslash e \\ N_\beta^\perp / e \end{pmatrix}\right)$$

# Simplify calculations /w minors via Exterior Algebra

Full $r$-row minors of matrix $N$ with columns indexed by $S$:

$$\begin{array}{ccc} (e_1) & (e_2) & (e_3) \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ \hline \end{array}$$
$$\mathbf{N}[e_1 e_3] = (a_1 b_3 - a_3 b_1)$$

Coefficients when the exterior product of $N$'s row vectors $\mathbf{N}$ are expressed in basis $\{\mathbf{e}_{i_1} \wedge \mathbf{e}_{i_2} \cdots \mathbf{e}_{i_r} | i_1 < i_2 \cdots < i_r\}$:

$$\begin{array}{cc} & (a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 + a_3 \mathbf{e}_3) \\ \wedge & (b_1 \mathbf{e}_1 + b_2 \mathbf{e}_2 + b_3 \mathbf{e}_3) \\ \hline \multicolumn{2}{c}{((a_1 b_3 - a_3 b_1)\mathbf{e}_1 \mathbf{e}_3 + \cdots)} \end{array}$$

We sometimes omit the $\wedge$ and we can always write:

$$(\text{Exterior product})\mathbf{N} = \sum_{A \subseteq S; |E| = r} \mathbf{N}[A]\mathbf{A}$$

Each subset $A$ is ordered $a_1 a_2 \ldots a_r$ **arbitrarilly** but $\mathbf{A}$ denotes the exterior product of (row coordinate vectors) **in the same order**

$$\mathbf{A} = \mathbf{a_1 a_2 \ldots a_r}$$

# Catalogs of Oriented Matroid operations on OM($N$) of matrix $N$ and on $\mathbf{N} = \wedge(\text{rows}(N))$

| Op is on: | chirotopes | exterior products |
|---|---|---|
| which are: | $\chi : B \to \{0, \pm\}$ | decomposibles in $\wedge$ |
| case we use: | $\chi : B \mapsto \text{sign}(N[B])$ | $\mathbf{N} : B \mapsto \mathbf{N}[B]$ |
| OPERATION | | |
| deletion $\bullet\backslash A$ | restriction | restriction |
| contraction $\bullet/A$ | $\pm\chi' : B \mapsto \chi(BA)$ | $\mathbf{N}/A : B \mapsto \mathbf{N}[BA]$ |
| duality $\bullet^{\perp}$ | $\pm\chi^{\perp} : B \mapsto \chi(\overline{B})\epsilon(\overline{B}B)$ | $\mathbf{N}^{\perp} : B \mapsto \mathbf{N}[\overline{B}]\epsilon(\overline{B}B)$ |

We must choose some global orientation $\epsilon$ in order to define duality as an exterior alg. operation!
$\epsilon$ is an alternating sign function on all finite sequences of elements.

This implies commutations

$$(\mathbf{N}\backslash X)^{\perp} = \epsilon(S')\epsilon(S'X)(\mathbf{N}^{\perp}/X)$$

$$(\mathbf{N}/X)^{\perp} = \epsilon(S')\epsilon(S'X)(-1)^{|X|r\mathbf{N}^{\perp}}(\mathbf{N}^{\perp}\backslash X)$$

# Our setup - again

- Matrices $N_\alpha$, $N_\beta^\perp$; full row rank, columns indexed by $P \coprod E$.
  $\mathrm{rank}(N_\alpha) + \mathrm{rank}(N_\beta^\perp) = |E| + |P|$.
  $P_\alpha, P_\beta \leftrightarrow P$, $P_\alpha \cap P_\beta = \emptyset$.
- Weight (parameter) matrices
  $G = \mathrm{diag}\{g_e\}_{e\in E}$, $R = \mathrm{diag}\{r_e\}_{e\in E}$.
- Matrix with columns $P_\alpha \coprod P_2 \coprod E$

$$L\begin{pmatrix} N_\alpha \\ N_\beta^\perp \end{pmatrix} = \left[\begin{array}{c|c|c} N_\alpha(P) & 0 & N_\alpha(E)G \\ \hline 0 & N_\beta^\perp(P) & N_\beta^\perp(E)R \end{array}\right]$$

Define

$$F(L) = ((\binom{2p}{p})) - \text{tuple of determinants } L[Q_\alpha \overline{Q_\beta} E])$$

indexed by sequences $Q_\alpha \overline{Q_\beta} \subseteq P_\alpha P_\beta$ where $Q_\alpha \subseteq P_\alpha$, $\overline{Q_\beta} \subseteq P_\beta, |Q_\alpha \overline{Q_\beta}| = p = |P|$.

$$L\begin{pmatrix} N_\alpha \\ N_\beta^\perp \end{pmatrix} = \left[\begin{array}{c|c|c} N_\alpha(P) & 0 & N_\alpha(E)G \\ \hline 0 & N_\beta^\perp(P) & N_\beta^\perp(E)R \end{array}\right] \qquad F(L) = \text{tuple } (L[Q_\alpha \overline{Q_\beta} E])$$

Translate into exterior algebra definitions:

$$\mathbf{L}\begin{pmatrix} \mathbf{N}_\alpha \\ \mathbf{N}_\beta^\perp \end{pmatrix} := (\iota(\mathbf{N}_\alpha)(P_\alpha) + \iota_G(\mathbf{N}_\alpha(E))) \wedge (\upsilon(\mathbf{N}_\beta^\perp)(P_\beta) + \upsilon_R(\mathbf{N}_\beta^\perp)(E))$$
$$= (\iota_G(\mathbf{N}_\alpha) \wedge \upsilon_R(\mathbf{N}_\beta^\perp))$$

$$\mathbf{F}_E(\mathbf{L}) := \mathbf{L}/E = \sum_{Q_\alpha, \overline{Q_\beta}} \mathbf{L}[Q_\alpha \overline{Q_\beta} E]\mathbf{Q}_\alpha \overline{\mathbf{Q}_\beta}$$
$$= ((\iota(\mathbf{N}_\alpha)\backslash e(\text{no } \mathbf{e}) + g_e(\iota(\mathbf{N}_\alpha)/e) \wedge \mathbf{e})$$
$$\wedge (\upsilon(\mathbf{N}_\beta^\perp)\backslash e(\text{no } \mathbf{e}) + r_e(\upsilon(\mathbf{N}_\beta^\perp)/e) \wedge \mathbf{e}))/E$$
$$\text{2 of 4 terms} = \Big(r_e \qquad \iota(\mathbf{N}_\alpha)\backslash e \wedge (\upsilon(\mathbf{N}_\beta^\perp)/e) \wedge \mathbf{e}$$
$$\text{vanish} \quad + g_e(-1)^{r(\mathbf{N}_\beta^\perp)}(\iota(\mathbf{N}_\alpha)/e) \wedge (\upsilon(\mathbf{N}_\beta^\perp)\backslash e) \wedge \mathbf{e}\Big)/E$$

$$L\begin{pmatrix} N_\alpha \\ N_\beta^\perp \end{pmatrix} = \left[\begin{array}{c|c|c} N_\alpha(P) & 0 & N_\alpha(E)G \\ \hline 0 & N_\beta^\perp(P) & N_\beta^\perp(E)R \end{array}\right] \qquad F(L) = \text{tuple } (L[Q_\alpha \overline{Q_\beta} E])$$

$$\mathbf{F}_E(\mathbf{L}) = \mathbf{L}/E = \Big(r_e \qquad \iota(\mathbf{N}_\alpha \backslash e) \wedge (\upsilon(\mathbf{N}_\beta^\perp / e)) \wedge \mathbf{e}$$

$$+g_e(-1)^{r(\mathbf{N}_\beta^\perp)}(\iota(\mathbf{N}_\alpha / e)) \wedge (\upsilon(\mathbf{N}_\beta^\perp \backslash e)) \wedge \mathbf{e}\Big)/E$$

$$= r_e\left(\mathbf{L}\begin{pmatrix} \mathbf{N}_\alpha \backslash e \\ \mathbf{N}_\beta^\perp / e \end{pmatrix} \wedge \mathbf{e}/E\right) + g_e(-1)^{r(\mathbf{N}_\beta^\perp)}\left(\mathbf{L}\begin{pmatrix} \mathbf{N}_\alpha / e \\ \mathbf{N}_\beta^\perp \backslash e \end{pmatrix} \wedge \mathbf{e}/E\right)$$

$$(\mathbf{N}\backslash e)^\perp = \epsilon(S')\epsilon(S'e)(\mathbf{N}^\perp / e) \; ; \; (\mathbf{N}/e)^\perp = \epsilon(S')\epsilon(S'e)(-1)^{|\{e\}| r\mathbf{N}^\perp}(\mathbf{N}^\perp \backslash e)$$

Result

$$= \epsilon(S)\epsilon(S'e)(r_e\left(\mathbf{L}\begin{pmatrix} \mathbf{N}_\alpha \backslash e \\ (\mathbf{N}_\beta \backslash e)^\perp \end{pmatrix} \wedge \mathbf{e}/E\right) + g_e\left(\mathbf{L}\begin{pmatrix} \mathbf{N}_\alpha / e \\ (\mathbf{N}_\beta / e)^\perp \end{pmatrix} \wedge \mathbf{e}/E\right))$$

With $\mathbf{L}(\mathbf{N}_\alpha \;\; \mathbf{N}_\beta) = \mathbf{L}\begin{pmatrix} \mathbf{N}_\alpha \\ \mathbf{N}_\beta^\perp \end{pmatrix}$, and more sign calculations:

## Definition

For $E$, $P$ sets written as ordered sequences,

$$\mathbf{F}_E(\mathbf{N}_\alpha \;\; \mathbf{N}_\beta) = \mathbf{L}(\mathbf{N}_\alpha \;\; \mathbf{N}_\beta)/E$$

## Theorem

$$\epsilon(PE)\mathbf{F}_E(\mathbf{N}_\alpha \;\; \mathbf{N}_\beta) =$$
$$\epsilon(PE')\left(g_e\mathbf{F}_{E'}(\mathbf{N}_\alpha/e \;\; \mathbf{N}_\beta/e) + r_e\mathbf{F}_{E'}(\mathbf{N}_\alpha\backslash e \;\; \mathbf{N}_\beta\backslash e)\right)$$

## Corollary

$\mathbf{F} = \mathbf{F}_E(\mathbf{N}_\alpha \;\; \mathbf{N}_\beta) = \pm \sum_{H \subset E} g_H r_{\overline{H}} \mathbf{F}_\emptyset(\mathbf{N}_\alpha / H \backslash \overline{H} \;\; \mathbf{N}_\beta / H \backslash \overline{H})$

## Applying the Tutte Polynomial

- THEREFORE: We can obtain $\mathbf{F}$ by doing a ported Tutte decomposition, keeping track of the contraction and deletion order $H$,$\overline{H}$. Then, when we get nodes with no more $e \in E$, substitute the exterior product $\mathbf{F}_\emptyset(\mathbf{N}_\alpha / H \backslash \overline{H} \mathbf{N}_\beta / H \backslash \overline{H})$ which is $\mathbf{N}_\alpha / H | P \wedge \mathbf{N}_\beta / H | P$
- When the $\mathbf{N}_\alpha = \mathbf{N}_\beta$ represent regular matroids by unimodular matrices, we can do the familiar substition of the Tutte function value(s) $F([M/H|P])$ for the matroid variable (product) $[M/H|P]$. (More research needed to develop how to make sure proper $\pm$ signs are maintained everywhere.)

## Corollary

1. *Componentwise,* $\sum_{Q_\alpha, Q_\beta} \mathbf{F}_E[Q_\alpha \overline{Q}_\beta] \mathbf{Q}_\alpha \overline{\mathbf{Q}_\beta} =$

$$= \pm \sum_{Q_\alpha, Q_\beta} \sum_{H \in E} g_H r_{\overline{H}} \mathbf{N}_\alpha[Q_\alpha H] \mathbf{N}_\beta^\perp[\overline{Q_\beta H}]$$

$$= \pm \sum_{Q_\alpha, Q_\beta} \sum_{H \in E} g_H r_{\overline{H}} \mathbf{N}_\alpha[Q_\alpha H] \mathbf{N}_\beta[Q_\beta H]$$

2. *Two expr. for products of numbers* $\mathbf{N}_\alpha[Q_\alpha H] \mathbf{N}_\beta[Q_\beta H]$*:*

$$(\mathbf{N}_\alpha/Q_\alpha)[H] \cdot (\mathbf{N}_\beta/Q_\beta)[H] = (\mathbf{N}_\alpha/H)[Q_\alpha] \cdot (\mathbf{N}_\beta/H)[Q_\beta]$$

3. *It's non-zero iff* $H$ *is a common basis (in the matroids of)* $\mathbf{N}_\alpha/Q_\alpha$ *and* $\mathbf{N}_\beta/Q_\beta$
   *iff* $Q_\alpha$ *is a basis in* $\mathbf{N}_\alpha/H$ *and* $Q_\beta$ *is a basis in* $\mathbf{N}_\beta/H$

# Weighted Laplacian-like matrices

Generalize a graph's incidence matrix: Make $P$ label the rows, $E$ the columns of any matrices $A_\alpha, A_\beta$. Take all $r_e \neq 0$. Then, $N_\alpha = (I(P) \;\; A_\alpha(E))$ and $N_\beta = (I(P) \;\; A_\beta(E))$, and

$$L\begin{pmatrix} N_\alpha \\ N_\beta^\perp \end{pmatrix} = \left[\begin{array}{c|c|c} I & 0 & A_\alpha G \\ \hline 0 & -A_\beta^t & IR \end{array}\right] = L\left(N_\alpha \quad N_\beta\right). \text{ Do row ops:}$$

$$\begin{pmatrix} I & -A_\alpha G R^{-1} \\ 0 & R^{-1} \end{pmatrix} L = \begin{pmatrix} I & A_\alpha G R^{-1} A_\beta^t & 0 \\ 0 & -R^{-1}A_\beta^t & I \end{pmatrix}, \text{ and therefore}$$

$$\epsilon(Q_\alpha \overline{Q_\alpha})\mathbf{F}_E(\mathbf{L})[Q_\alpha \overline{Q_\beta}] = \frac{1}{r_E}\sum_{B\in E} g_B r_{\overline{B}} A_\alpha[\overline{Q_\alpha} B] A_\beta[\overline{Q_\beta} B]$$

is the Cauchy-Binet expansion of any minor $(\overline{Q_\alpha}, \overline{Q_\beta})$ of the weighted graph Laplacian-like matrix $A_\alpha G R^{-1} A_\beta^t$.
(Note $\frac{1}{r_E} r_{\overline{B}} = (r^{-1})_B$.)

# Examples

$N_\alpha = N_\beta = N; A =$ graph's incidence matrix w/ columns $(0, ..0, 1, 0, .., -1, 0, .., 0)^t$ for each edge; reps. graphic matroid.

The all-minors Matrix Tree Theorem for weighted undirected graphs

$A_\alpha, N_\alpha$ as above. $A_\beta =$ only the $+1$ entries of $A$ for a directed graph, so $+1$ is for an edge head on a vertex.

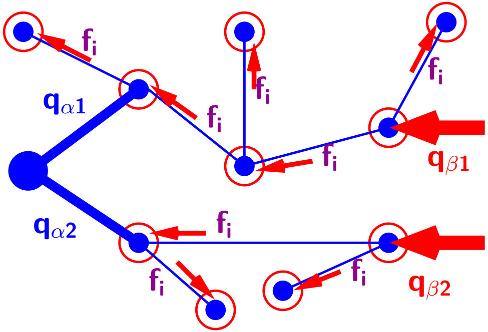
The all-minors Matrix Tree Theorem for weighted directed graphs

$N_\alpha = N_\beta = N; A =$ gain graph's incidence matrix w/ columns $(0, .., 0, 1, 0, .., -\gamma_e, 0, .., 0)^t$ for $e$ with gain $\gamma_e \in \mathbf{C}$.

All-minors expansions of a signed, genr. gain graph's Laplacian

NB: Edge Gains $\gamma_e$ are DIFFERENT ATTRIBUTES from weights/parameters $g_e, r_e$

# All-Minors Digraph Matrix Tree Theorem Example

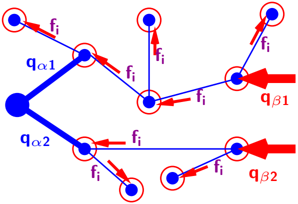



This contributes the term

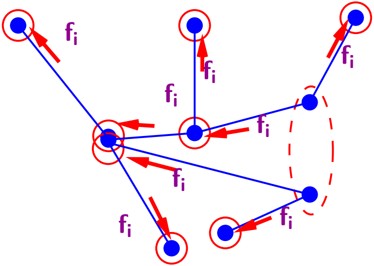
$g_F r_{\overline{F}} \mathbf{N}_\alpha[Q_\alpha F] \mathbf{N}_\beta[Q_\beta F]$.

The $\mathbf{q}_{\alpha 1}, \mathbf{q}_{\alpha 2}$ port edges $\cup$ the $f_i$ elements as edges in the graphic matroid comprise a spanning tree.

The $\mathbf{q}_{\beta 1}, \mathbf{q}_{\beta 2}$ port arrows $\cup$ the $f_i$ elements as arrows in a partition matroid comprise a basis. Each part (a red cirle) of the partition is the set of arrows incident to a vertex, except the star vertex.

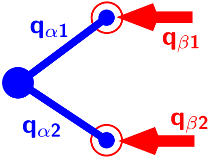



Contract the ports.




Count the bases in common.
$g_F r_{\overline{F}} \mathbf{N}_\alpha / Q_\alpha[F] \mathbf{N}_\beta / Q_\beta[F]$.

Contract the non-ports.




$\alpha$ and $\beta$ ports are bases in the contracted $N_\alpha$ and $N_\beta$ matroids.
$g_F r_{\overline{F}} \mathbf{N}_\alpha / F[Q_\alpha] \mathbf{N}_\beta / F[Q_\beta]$.

# Resistive Network style problems Solved by Tutte Functions

With the $\binom{2p}{p}$ tuple of $(p+n) \times (p+n)$ minors of $\mathbf{L}(\mathbf{N} \;\; \mathbf{N})$ all including columns $E$, every electrical style problem can be analyzed.

## Input

Choose $1 \leq k \leq p$, and choose from among the set of $2p$ variables $\{v_1, \ldots, v_p; i_1, \ldots, i_p\}$ these 4 subsets:

- $k$ "source" variables $S = \{s_1, \ldots, s_k\}$.
- $p - k$ "zero" variables $Z = \{z_1, \ldots, z_{p-k}\}$ so $S \cap Z = \emptyset$, ie. $|S \cup Z| = p$.
- $k'$ "response" variables $R = \{r_1, \ldots, r_{k'}\}$
- $p - k'$ "don't care" variables $D = \{d_1, \ldots, d_{p-k}\}$

# Question and Answer

Does there exist a $k' \times k$ matrix $\Xi$ for all source values $s_i$,

$$\Xi(s_1, ..., s_k)^t = (r_1, ..., r_{k'})^t \text{ is the unique solution in}$$

$$\{(r_1, ..., r_{k'}) | L(\mathbf{N} \ \ \mathbf{N})(v_1, ..., v_p; i_1, ..., i_p)^t = 0,$$

$$s_i \text{ are given }, z_i = 0,$$

$$\text{and there exist } d_1, ..., d_{p-k'}\}??$$

## Answer for $S, Z, R, D$

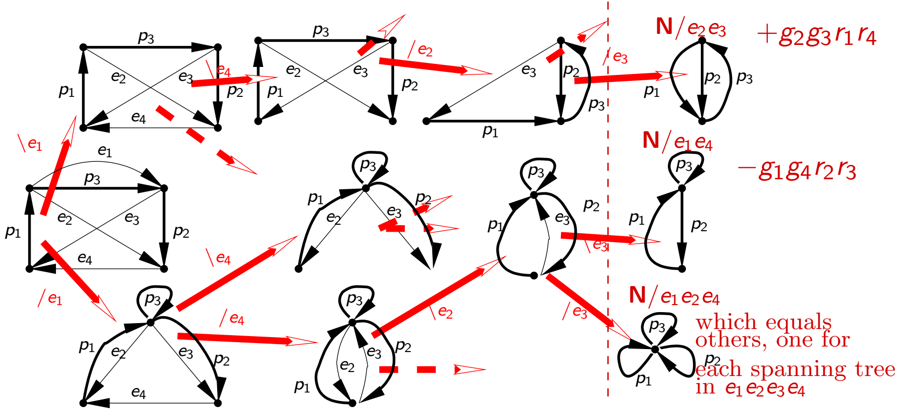
- If $\mathbf{F}_E(\mathbf{N} \ \ \mathbf{N})[SZ] \neq 0$, then $\Xi$ exists.
- If so, every minor of $\Xi$ is, for some $Q_{\text{Num1}}, Q_{\text{Num2}}, Q_{\text{Den1}}, Q_{\text{Den2}} \subset P_\alpha P_\beta$

$$\frac{\sum_{F \subseteq E} \mathbf{N}[Q_{\text{Num1}} F]\mathbf{N}[Q_{\text{Num2}} F] g_F r_{\overline{F}}}{\sum_{F \subseteq E} \mathbf{N}[Q_{\text{Den1}} F]\mathbf{N}[Q_{\text{Den2}} F] g_F r_{\overline{F}}}$$

Remember, each (field valued) sum, being a component of $\mathbf{F}_E(\mathbf{N} \ \ \mathbf{N})$, IS A TUTTE FUNCTION.

# Ported Tutte Decomposition (incomplete)

The decomposition ends with minors $\mathbf{N}/F\overline{F}$.
We show component $\mathbf{L}_E(\mathbf{N}\ \mathbf{N})[p_{\alpha 1}p_{\beta 1}p_{\alpha 3}]$:



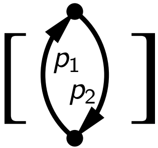
# Conditions (what sets $F$ are enumerated by one det. $C_i$)

The **conditions ...** are on the rank, nullity of $F$ and, WHAT ORIENTED MINOR is $G/F \setminus (E \setminus F)$, the minor with ONLY PORT EDGES from contracting $F$ and deleting the other resistor edges, leaving the ports.

The conditions for a given $C_k$ *sometimes* make all the signs the same (eg: $C_i$ and $C_j$ in 1-port equivalent resistance $R = C_i/C_j$)

*Othertimes*, the oriented **P-minors** in the completed Tutte decomposition of $C_k$ determine some $+$ and some - signs.

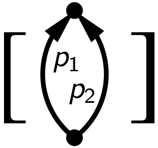
When $[G/F|P]$ is




the term is

$+g_F r_{E\setminus F}$

When $[G/F|P]$ is




the term is

$-g_F r_{E\setminus F}$

# Known to EEs: Linear electrical networks with IDEAL AMPLIFIERS

$N_\alpha i(P,E) = 0$ expresses Kirchhoff's current law on currents $i_e$ in the network edges (along edge direction) and currents $i_p$ into vertices from external connections.

$N_\beta^\perp v(P,E) = 0$ expresses Kirchoff's voltage law: The voltage rise along a network edge $v_e = v_h - v_t$ is the difference of the head and tail vertex potentials. (Sometimes the vertex potentials are imposed by external connections.)

$N_\alpha = N_\beta$ in ordinary resistor networks.

## Different Graphs for $N_\alpha$ and $N_\beta$

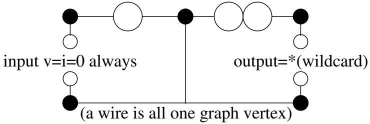
W. K. Chen models networks with ideal amplifiers by $N_\alpha$ by one graph on $(P,E)$ called the **Current Graph** and another graph also on $(P,E)$ called the **Voltage Graph**.

# Voltage and Current graphs $G_V$, $G_I$

"Voltage graph" $G_V$ (EEs Hasler and Neirynck 1986, not Gross, et. al.) $\mathbf{v} \in$ Coboudaries W/ SOME $v_e \equiv 0$

"Current graph" $G_I$ represents KCL $\mathbf{i} \in$ Cycles WITH SOME FLOWS $\equiv 0$

- They are EQUAL GRAPHS for resistor networks.
- For networks with idealized amplifiers, they are not equal.




The output voltage and current are whatever makes the input voltage and current BOTH BE zero.

- (More) realistic amp. model = idealized amp. + resistors.

**open**

$G_v = G \backslash e$

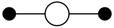
$G_I = G \backslash e$

**short**

$G_v = G / e$

$G_I = G / e$

**nullator**



$G_v = G / e$

$G_I = G \backslash e$

**norator**

$G_v = G \backslash e$

$G_I = G / e$

# Chain Complexes View (Alg. Topology, Homological Alg.)

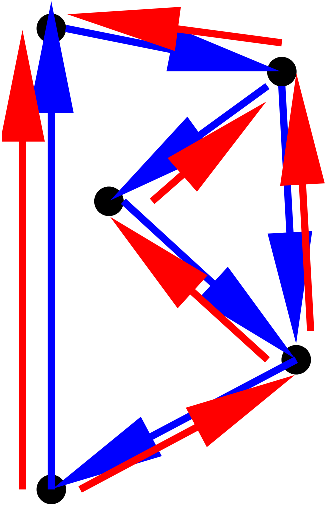
A graph is a $k$-dim simplicial complex $X$ with $k = 1$.
In general, for us, the $k$-chains $C_k = Z[P \coprod E] = \{\sum_{x \in P \coprod E} c_e e\}$ are the free abelian group with basis $P \coprod E$.
The $k$-cochains $C^k = \mathrm{Hom}(C_k, \mathbb{R})$ is the $\mathbb{R}$-module of linear maps from $C_k$ to a coefficient ring $\mathbb{R}$.
The $k$-complex $X = \coprod_{j=0}^{k} X_j$ ($X_j$ is the set of $j$-simplices) determines, (or the chain complex might just be subspaces given with) **boundary maps** $\partial_j : C_j \to C_{j-1}$ for $j = 0, ..., k$ that satisfy $\partial_{j-1} \circ \partial_j = 0$ for each $j$.
The dual $\delta^j : C^{j-1} \to C^j$ is defined by $(\delta^j(u^*))(v) = u^*(\partial_j(v))$.

In the case $N_\alpha = N_\beta$, generalizing:

- $\mathbf{N}$ ($\wedge$ of the rows on $N_\alpha$) represents the $k$-coboundary group $B^k = \mathrm{img}(\delta_k)$.
- The equation $N_\alpha \begin{pmatrix} I \\ G \end{pmatrix} (J_P \;\; X_E)^t = 0$ says $\begin{pmatrix} I \\ G \end{pmatrix} (J_P \;\; X_E) \in Z_1$, is a $k$-cycle. (Electrically, a flow of currents in edges.)
- $\mathbf{N}^\perp$ ($\wedge$ of the rows of $N^\perp$) represents the $k$-cycle group $Z_k = \ker(\partial_k)$.
- The equation $N^\perp \begin{pmatrix} I \\ R \end{pmatrix} (V_P \;\; X_E)^t = 0$ says $\begin{pmatrix} I \\ R \end{pmatrix} (V_P \;\; X_E) \in Z_1$, is a $k$-coboundary $\delta_k\psi$. (Electrically, $\delta_1\psi$ maps each edge (1-simplex) to the difference of electrical potential assigned to vertices (a 1-cochain) $\delta_1(\psi)(v_0 v_1) = \psi(v_1) - \psi(v_0)$.

# Electribraic Topology–Happy Birthday, have fun Tom.

The left edge is a port containing an electric source.
Red:1-coboundary Diffs of a potential $\psi$ (0-cobdy). Coeffs $v_e$ are $\geq 0$ for the edge $e = u_0 u_1$ dirs indicated, so $v_e = \psi(u_1) - \psi(u_0)$.
Blue:1-cycle Current (charge flow), $\geq 0$ with arrow. In resistor edge w/ conductance $g_e$, $c_e = g_e(-(\psi(u_1) - \psi(u_0)))$.
In the port, either the pot. diff. or the current is set by the source. Other coefficents are determined by Kirchhoff's and Ohm's laws.



# Research Notes Accumulated in Frames

# Ideas in Describing the Setup

- The expression $\mathbf{M}[A]$ for a "component of the Plücker coodinate representative" means application of an element of an exterior algebra to an element of the algebra's dual: Let $A$ signify $\wedge_{\mathbf{e} \in A} \mathbf{e}$.
- This abstracts and might simplify the presentation because to identify sequence $A$ with the $\wedge$ of its elements will automatically make permutations of $A$ be equal to the sign of the permutation times $A$.

# Codomain of Our Tutte Function–Equivalent Definitions

- Rank 1 Step $p$ antisymmetric tensors.
- Non-zero exterior products of $p$ vectors. Maybe this is why mathematicians did not give it a particular name, like they did for the Grassmannian.
- Dim. $p \times 2p$ full rank $p$ matrices modulo left action of $\mathrm{SL}_p$.
- All length $\binom{p}{2p}$ tuples representing (projective space) points in Grassmannian $G(p, 2p)$.
- All length $\binom{p}{2p}$ tuples satisfying the Grassmann-Plucker relations.

It is a subset of the grade $p$ subspace of the exterior algebra. It is closed under linear combinations of the form $gF(N/e) + rF(N \setminus e)$.

# Lines in the Grassmannian

The Grassmannian is a submanifold of $Pr_{\binom{n}{r}}$. Each instance of the deletion/contraction identity specifies a pair of points on a (projective space) line that lies in the Grassmannian. What is known about lines that lie in the Grassmannian?

# Unused, maybe wrong slides

# Tutte equations, functions and Good Questions

1. For all $\mathbf{N}$ with separator $e \in S(\mathbf{N})$,

$$F(\mathbf{N}) = g_e F(\mathbf{N}/e) + r_e(\mathbf{N} \backslash e)$$

2. When $\mathbf{N} = \mathbf{N_1} \oplus \mathbf{N_2}$,

$$F(\mathbf{N}) = F(\mathbf{N_1}) F(\mathbf{N_2})$$

3. When $\mathbf{N}$ is indecomposible,

$$F(\mathbf{N}) = i_{\mathbf{N}}$$

$F$ is Tutte function when all the Tutte equations are satisfied.
Good Questions: When does $\mathcal{N}$ and parameters ACTUALLY HAVE a Tutte function? If so, what is a *universal* Tutte function?

# Some answers–for Graphs and Matroids

### Only loops and coloops need initial values

The only $\mathbf{N}$ with no separators and no $\mathbf{N} = \mathbf{N}_1 \oplus \mathbf{N_2}$ for $\mathbf{N}_i \neq \emptyset$ are $\mathbf{loop}(e)$ and $\mathbf{coloop}(e)$.

### The famous Tutte Polynomial

Adding all $g_e = r_e = 1$, the Tutte polynomial $F(\mathbf{N})(x, y)$ obtained from $i_{\mathbf{loop}(e)} = x$, $i_{\mathbf{coloop}(e)} = y$ and $i_\emptyset = 1.$ is a universal Tutte function.

### Normal Tutte Functions for Matroids

(Zaslavsky, Bollobás/Riordan) With arbitrary $g_e, r_e$, and $x$,$y$, the *normal* Tutte functions for matroids are obtained with $i_{\mathbf{coloop(e)}} = g_e y + x$, $i_{\mathbf{loop(e)}} = r_e x + y$ and $i_\emptyset = 1.$ They are exactly the ones with a weighted rank-nullity generating function.
There's a big story about what relationships among the $g_e, r_e, i_{\mathbf{coloop(e)}}, i_{\mathbf{loop(e)}}, i_\emptyset$ give others.

# Hopf Alg. from Minor Systems (Krajewski, Moffatt, Tanasa 2017)

## Definition (Minor System)

- Finite combinatorial objects $\{N\}$ w/ ground sets $E(N)$, graded by $|E(N)|$; unique 1 with $E(1) = \emptyset$; $E(N)$ consists of objects at level $|E(N)|$.
- For distinct $e, f \in E(N)$, deletion & contraction ops so both ($\backslash\backslash e$ or $//e$) commute with both ($\backslash\backslash f$ or $//f$).

.

# Some generalization..

Tutte equations are satisfied in a very general setup:

1. Elements $\{e\}$ each with parameters $g_e, r_e$.
2. A category $\mathcal{N}$ of objects $\mathbf{N}$ each with ground set $S = S(\mathbf{N})$ of elements.
3. For some *decomposible* $\mathbf{N}$, for one or more *separators* $e \in S(\mathbf{N})$, the *contraction* and *deletion* operations are defined with results $\mathbf{N}/e$ and $\mathbf{N}\backslash e$ in $\mathcal{N}$, with ground sets $S(\mathbf{N})\backslash\{e\}$
4. Some $\mathbf{N} = \mathbf{N}_1 \oplus \mathbf{N_2}$ are direct sums, where $S(\mathbf{N}_1) \cap S(\mathbf{N}_2) = \emptyset$.
5. For each indecomposible $\mathbf{N}$ with no separators there is an additional parameter $i_{\mathbf{N}}$ called the *initial value*.