Ported or Relative Oriented Matroids and Electric Circuits

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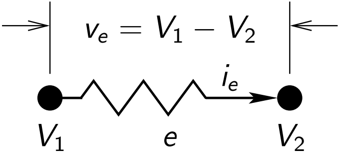
Why Electricity, EE?

- Scholarly topic suggested by G.-C. Rota $\approx$ 1980?.
- $\approx$ 100 yrs. geometry-like intuition of circuit configurations known by engineers, EE books: "Intuitive Analog Circuit Design (2013)" [11]; "Non-linear Circuits" [5] translates to our Oriented Matroid pair model.
- Geometry of linear spaces and oriented matroids; Tutte decomp. w/ techniques from Barnabi, Brini and Rota's Exterior Calculus [1])
- Real behavior $\approx$ ideal plus perturbations, ideal constraints predict intended real behavior,
- Interesting, accessible, intuitively understandable intentential designs, applicable, easy to both simulate and build physically, dimension $\approx$ 12 or 24, depending on formulation
- Analogs to chemical (and real algebraic geometry [8]), biological, elastic/tensegrity strs. etc., random walks ...
- Merely one scalar non-linearity can cause chaos.

Kirchhoff (1847) [6] Maxwell (1891) [7] The equivalent resistance PROBLEM IS SOLVED by the Matrix Tree Theorem. (1) POSE! the VARIABLES or COORDINATES

$\boxed{v_e, i_e : e \in E}$

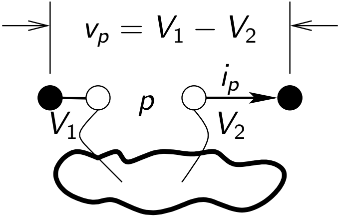


(Use **voltage drops along the flow**, not potentials V_1, V_2.)

ordinary, resistor edge $e \in E$

(IF linear Ohm's law use $|E|$ variables ($g_e v_e = r_e i_e$) ELSE use $2|E|$ variables.)

$\boxed{v_p, i_p : p \in P}$



DISTINGUISHED, PORT edge $p \in P$

The interface to an environment is modelled with $2|P|$ variables.

(math, not EE sign convention)

(2) POSE: EQUATIONS. Preview the consequences.

- (KCL) $(i_e)_{e \in S}$ is a cycle (a flow).
- (KVL) $(v_e)_{e \in S}$ is a cocycle
- (constituitive Law) $i_e = g_e(v_e)$ non-linear, usually monotonic increasing $R \to R$.
  (Sometimes use Ohm's approximation $i_e = g_e v_e$)

Combinatorics!

The signs $\{+, -, 0\}$ have a DUAL-PAIR ORIENTED MATROID structure (combinatorial, geometric, topological).

Engineering with amplifiers!

There's good unique solvablility due to STRUCTURE, when the NON-DUAL PAIR (for voltages and currents) is ALMOST DUAL: No common covectors.

SOLUTION: Equiv. Resistance $:\equiv -(v_p/i_p)$ observed at a port p by the environment EQUALS a Ratio of Spanning Tree Enumerators! (Port edge p locates the *2 terminals*.)

$$-\left(\frac{v_p}{i_p}\right) = \frac{\mathrm{WTS}(G/p)}{\mathrm{WTS}(G\backslash p)} = \frac{\text{Matrix-Tree Det}(G/p)}{\text{Matrix-Tree Det}(G\backslash p)}$$

- "Maxwell's rule" uses MatrTreeT on 2 DIFFERENT GRAPHS

  (G/p and $G\backslash p$) (Sorry, amplifiers come later.)

- Weighted Tree Sum (WTS) is a colored Tutte function:

$$\mathrm{WTS}(G') = g_e \mathrm{WTS}(G'/e) + r_e \mathrm{WTS}(G\backslash e) \text{ for all } e \notin P$$

$$\mathrm{WTS}(\mathrm{coloop}(e)) = g_e$$

$$\mathrm{WTS}(\mathrm{loop}(e)) = r_e$$

Multiple Ports. (your stereo: 3=power plug & 2 speakers)

- One formula expresses $\binom{2|P|}{|P|}$ different Matrix Tree Theorems...
- ... long vertex-based proofs are shortened; Rayleigh inequalities too.
- Interesting **non-commutative ranges** of new ORIENTED MATROID Tutte invariants with pattern:

  $$\mathrm{TF}(N(P \cup E)) = F(N(P \cup E)/E)$$

  (They distinguish DIFFERENT ORIENTATIONS of the SAME MATROID.)
- Ported/Relative OM Tutte Poly. terms embed SPECIFIC MINORS as variables, making proofs just with $\partial T/\partial x_e$ easier.
- Formalize composition of systems [9], Tutte poly. splitting formulas.
- Model practical devices (transistors, op amps); Label variables to observe.
- Align EE applications with knots [3] (Ported = "Relative") and combinatorial geometry [13] (Ported = "Set Pointed").

Constraint/Generator Duals and 2 Results.

- (Part 1) Technique:

$$\text{Solution Space} = \bigcap \text{Constraint Subspaces}$$

- Result: An exterior algebraic algebraic Tutte function: Each of its $\binom{2|P|}{|P|}$ Plücker coordinates satisfies a Matrix Tree Theorem.
  This and det. formulas easily prove Rayleigh inequalities.

- (Part 2) Combine with:

$$\text{Solution Space} = \text{Closure(Set of Generators)}$$

- To apply: An oriented matroid's COVECTOR SET encodes ALL POSSIBLE $(+,-,0)$ coordinate behaviors or δs.
- Result: An oriented matroid pair model for some non-linear problem (AMPLIFIER!) well-posedness. (How? Sign contradictions $\Rightarrow$ a KERNEL$=\{(0)\}$.)

Part 1) Use Matrix M in CONSTRAINTS $MX = 0$ to get...

The Tutte-like function $\mathbf{M}_E()$: Extensor $\mathbf{N} \rightarrow$ Extensor $\mathbf{M_E}(\mathbf{N})$.
(STUDENT NOTE: An EXTENSOR represents the row-space of an $r \times s$ $r-$rank matrix M by the $\binom{s}{r}$-TUPLE of the DETERMINANTS of M's $r \times r$ submatrices. Plücker coords.)
Given N (matrix), construct $N^{\perp}$ with orthog. comp. row space.
Construct: $(G = \mathrm{diag}(g_e),\ R = \mathrm{diag}(r_e))$

$$M = \left[\begin{array}{c|c|c} N(P) & 0 & N(E)G \\ \hline 0 & N^{\perp}(P) & N^{\perp}(E)R \end{array}\right]$$

with columns labelled by $P_I \cup P_V \cup E$.
Extensor $\mathbf{M}$ over $k[g_e, r_e](P_V \cup P_I \cup E)$ is the $\wedge$-product of M's **row vectors**. The contraction result $\mathbf{M}_E(\mathbf{N}) = \mathbf{M}/E$ appears:

$$\mathbf{M} = \mathbf{M}_E(\mathbf{N})\mathbf{e_1}\mathbf{e_2}\cdots\mathbf{e}_{|E|} + (\cdots)$$

$\mathbf{M}_E(\mathbf{N})$ is our Tutte function $\mathbf{N} \rightarrow$ Ext. Alg.

Contracting means "Eliminate variables"

ELIMINATE the variables indexed by E, leaving $2|P|$ variables labelled by P_I and P_V. ie, CONTRACT E. **Answer $\mathbf{M_E}$ IS:**

$$\mathbf{M}_E = \bigwedge_{\text{JOIN over rows}}^{\text{Exterior}} \left[\begin{array}{c|c} A_{I,I} & A_{I,V} \\ \hline A_{V_I} & A_{V,V} \end{array}\right] [\mathbf{p_{I_1}}, \cdots, \mathbf{p_{I_p}}; \mathbf{p_{V_1}}, \cdots, \mathbf{p_{V_p}}]^{\mathbf{t}}$$

$$= \ldots + C_i\mathbf{XXX} + \ldots ; \text{Equiv. Resistance} = \text{certain} \;\; C_i/C_j$$

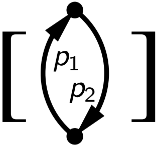
All the other C_k's have similar interpretations.
$\binom{2|P|}{|P|}$ **Matr. Tree Theorems:** Each $C_k(N)$ (a PRINCIPAL MINOR of MATRIX A ABOVE!) $= g_e C_k(N/e) + r_e C_k(N\backslash e)$ ($e \notin P$, e not (co)loop).
Each C_k is a signed weighted enumerator of forests satisfying **conditions ...**

Conditions (what sets F are enumerated by one det. C_i)

The **conditions ...** are on the rank, nullity of F and, WHAT ORIENTED MINOR is $G/F \setminus (E \setminus F)$, the minor with ONLY PORT EDGES from contracting F and deleting the other resistor edges, leaving the ports.

The conditions for a given C_k *sometimes* make all the signs the same (eg: C_i and C_j in 1-port equivalent resistance $R = C_i/C_j$)
Othertimes, the oriented **P-minors** in the completed Tutte decomposition of C_k determine some $+$ and some - signs.

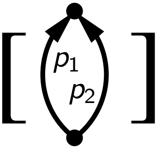
When $[G/F|P]$ is



the term is

$+g_F r_{E \setminus F}$

When $[G/F|P]$ is



the term is

$-g_F r_{E \setminus F}$

Application: Rayleigh Identity, "Neg. Spanning Tree Correlation"

$\Gamma_e(G)$ is equivalent conductance across e. Rayleigh: $0 \leq \frac{\partial \Gamma_p}{\partial g_f} = \frac{\partial \frac{T_G}{T_{G/e}}}{\partial g_f}$

is equivalent to

$$0 \leq \frac{\partial T_G}{\partial g_f} T_{G/e} - T_G \frac{\partial T_{G/e}}{\partial g_f} = T_{G/f} T_{G/e} - T_G T_{G/e/f}$$

In fact,

$$T_{G/f} T_{G/e} - T_G T_{G/e/f} = \left(T^{+}_{G/e \;\&\; G/f} - T^{-}_{G/e \;\&\; G/f} \right)^2$$

$T^{\pm}_{G/e \;\&\; G/f}$ enumerate the $\pm$ common spanning trees.

Known Partial and Full Combinatorial Proofs

$$T_{G/f}T_{G/e} - T_G T_{G/e/f} = \left(T^{+}_{G/e \,\&\, G/f} - T^{-}_{G/e \,\&\, G/f}\right)^2$$

$T^{\pm}_{G/e \,\&\, G/f}$ enumerate the $\pm$ common spanning trees.

Choe (2004) proved essentially this using the vertex-based all-minors matrix tree theorem, combinatorial cases and Jacobi's theorem relating the minors of a matrix to the minors of its inverse..

Cibulka, Hladky, Lacroix and Wagner (2008) gave a completely bijective proof that utilizes some natural 2:2 and 2:1 correspondances.

Difficulty: Some terms on the left cancel and some reduce to terms with coefficients ± 2.

Linear Alg./Oriented Matroid Proof of Rayleigh's Identity

Let R be the transfer resistance matrix for 2 ports across e and f. Our result implies that

$$\det R = \begin{vmatrix} R_{ee} & R_{ef} \\ R_{fe} & R_{ff} \end{vmatrix} = +\frac{T_{G/e/f}}{T_G}$$

It and better-known results tell us

$$R_{ee} = \frac{T_{G/e}}{T_G}; \quad R_{ff} = \frac{T_{G/f}}{T_G}; \quad R_{ef} = R_{fe} = \frac{T^{+}_{G/e\ \&\ G/f} - T^{-}_{G/e\ \&\ G/f}}{T_G}$$

$T_{G/f}T_{G/e} - T_G T_{G/e/f} = \left(T^{+}_{G/e\ \&\ G/f} - T^{-}_{G/e\ \&\ G/f}\right)^2$ is immediate after substituting these into

$$\det R = R_{ee}R_{ff} - (R_{ef})^2$$

The $+$ follows from physical grounds if the $g_e, r_e \geq 0$. Our characterization and proof are combinatorial.

New Rayleigh's Identities!

The same method generates identities and inequalities from

$$\begin{vmatrix} R_{ee} & R_{ef} & R_{eg} \\ R_{fe} & R_{ff} & R_{fg} \\ R_{ge} & R_{gf} & R_{gg} \end{vmatrix} = +\frac{T_{G/e/f/g}}{T_G} \geq 0$$

when all $r_{..}, g_{..} \geq 0$, ETC...

(Applications???)

Might the same methods address a much harder problem: The same inequality for forests instead of spanning trees?

(Part 2) Common Covector Model

The cycle space of G_I GENERATES the covectors of an **oriented matroid** over $(E \cup P_I)$.

0

(signs indexed by E) (by P_I)

Non-linear monotone resistors CONSTRAIN SIGNS of voltage drops (from ↓) and flows (from ↑) TO BE EQUAL

(signs indexed by E) (by P_V)

The cocycle space of G_V GENERATES the covectors of an oriented matroid over $E \cup P_V$.

G_V

SOMETIMES EQUALS

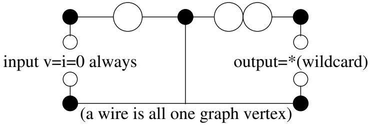
G_I

0

Voltage and Current graphs G_V, G_I

"Voltage graph" G_V (EE [5, 10], NOT Gross, ...) represents KVL $\mathbf{v} \in$ Cocycles W/ SOME $v_e \equiv 0$	"Current graph" G_I represents KCL $\mathbf{i} \in$ Cycles WITH SOME FLOWS $\equiv 0$

- They are EQUAL GRAPHS for resistor networks.
- For networks with idealized amplifiers, they are not equal.



The output voltage and current are whatever makes the input voltage and current BOTH BE zero.

- (More) realistic amp. model = idealized amp. + resistors.

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		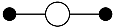	
$G_v = G \backslash e$	$G_v = G / e$	$G_v = G / e$	$G_v = G \backslash e$
$G_I = G \backslash e$	$G_I = G / e$	$G_I = G \backslash e$	$G_I = G / e$

“Colors” are parameters on every Tutte decomposition step

The Bollobos/Riordan/Zaslavsky [2, 14], Traldi-Ellis-Monaghan [12, 4], (sdc unpub) BRZ theory for well-definedness of “Relative Tutte Polynomials for Colored Graphs” ALL GOES THROUGH (Diao and Hetyei [3]): The 3 BRZ conditions on (colors,initial values) GENERALIZE TO 5; activity theory WORKS TOO, when based on linear orders on the non-port-elements.

In a nutshell

The 5 conditions $\Longrightarrow$ activities define an unambiguous Tutte function from the deletion/contraction and initial value formulas. Additional conditions $\Longrightarrow$ the Tutte function has a rank-nullity expansion.

(The rank-nullity conditions are satisfied in our application.)

To specify the activity/deletion-contraction linear order GLOBALLY is UNNECESSARY.

The Gordon/McMahon computation-tree-based activity theory also generalizes. (sdc).

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