Tutte Activities Based on More General Element Orders (draft)

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In 1997 Gordon and McMahon [1] extended from matroids to greedoids the internal and external element activities and the resulting subset-interval partition analysis for the Tutte polynomial. They found it necessary to abandon the traditional definition of activities based on a single (but arbitrary) linear element order. Instead, the internal and external, active versus inactive element classification was based on properties of each individual root-to-leaf path down any computation tree for the Tutte polynomial. Independence of element order was generalized to independence of the computation tree.

We report that these results (for matroids) naturally generalize when both (1) the deletion/contraction and loop/coloop removal operations, for each element, are **parametrized** or **colored**, and (2) a subset of **port** elements P is given and the resulting **restricted** or **ported** Tutte functions satisfy $T(M) = g_e T(M/e) + r_e T(M\backslash e)$ **only for** $e \notin P$.

After they are extended to involve the initial values $T(M')$ for the P**-minors** M', the Bollobas/Riordan/Zaslavsky conditions on the parameters are quite easily shown to be necessary and sufficient for the Tutte polynomial to be well-defined.

1 Brief History of Tutte Functions

1. Well-defined invariants like the chromatic (poly.) function and the count of spanning trees were shown to satisfy
$$T(G) = T(G/e) \pm T(G \setminus e).$$
2. Alternative expansions for some Tutte functions, like Whitney's
$$\chi(G, \lambda) = \sum_{A \subseteq E(G)} (-1)^{|A|} \lambda^{k(G|A)} \quad 2^{|E|} \text{ terms}$$
$\sum$ could be SHORTENED to "broken-circut-free" subsets A, based on an *arbitrary linear ordering of* $E(G)$
3. Tutte and Brylawski's universal 2-variable solution to
$$T(G) = T(G/e) \pm T(G \setminus e).$$
expressed by
$$T(G) = \sum_{\text{Bases} B \subseteq E} x^{|IA(B)|} y^{|EA(B)|}$$
internally (IA) and externally (EA) active elements based on a arbitrary linear ordering of E

Tutte's and others proved **every computation tree** computes every Tutte function value **correctly BECAUSE** the Tutte eq. have a unique solution.

But *independence of the linear ordering defining the activities* HAD TO BE PROVED FIRST.

2 GM's Computation Tree Approach

Sets IA and EA are assoc. to **leaves** of an **arbitrary computation tree** $\mathcal{T}_j$
$IA(\text{leaf})$ = isthmuses at leaf's graph or matroid.
$EA(\text{leaf})$ = loops at leaf's graph or matroid.
Routine induction ("Let $\mathcal{T}_1$, $\mathcal{T}_2$ be two trees of a smallest counterexample. ... Contradiction![1]") proves:

$$\sum_{\text{leaf of } \mathcal{T}_1} x^{IA_1(\text{leaf})} y^{EA_1(\text{leaf})} = \sum_{\text{leaf of } \mathcal{T}_2} x^{IA_2(\text{leaf})} y^{EA_2(\text{leaf})}$$

[1] Technical lemmas about series and parallel elements are applied.

3 What generalizes to GM computation tree approach

Greedoids (GM 1997)
Sometimes the **only computation trees that exist** are NOT BASED ON A LINEAR element ORDER!

2 Interval partitions of the boolean lattice 2^E defined using **one part for each computation tree leaf**.

Zaslavsky-Riordan-Bollobas, Traldi on weight (or color) pairs (x_e, y_e) on labelled edges and initial values (X_e, Y_e) on loops and insthuses – necessary and sufficient for the weighted or colored Tutte equations to have a solution.

P**-ported or restricted or set-pointed** Tutte functions T with weights or colors. (Diao-Hetyei, computation tree proofs by sdc) TWO more ZBR-type conditions re. **initial values of T on the P-quotients** are needed besides ZBR's original three.

ORIENTED matroids INCLUDING DISTINCT P-quotients.(sdc)

4 What DOES NOT Go Through

Given a linear element order, $B \subset E$ is a broken-circuit if there exists $e \notin E$ so $B \cup e$ is a circuit and e is the (FOR US!) GREATEST element in $B \cup e$.

Given a computation tree (unless it's from a linear order) "$B \subseteq E$ is a broken-circuit" **is not well-defined**.

5 What does go through

What's a broken circuit B, COMPUTATIONALLY? ANSWER: There is a root-to-leaf path down which which each element of B is contracted, and a **LOOP IS PRODUCED** when contracting the last element of B.

When the tree is from a linear order, "$A \subseteq E$ is broken-circuit free" **if and only if** "the leaf of $\mathcal{T}$ gotten to by searching for A has no loops (ie., $EA(\text{leaf}) = \emptyset$).

AND Whitney's expansion generalizes to those sets A leading to loop-free leaves in an arbitrary computation tree.

References

[1] Gary Gordon and Elizabeth McMahon. Interval partitions and activities for the greedoid Tutte polynomial. *Adv. Appl. Math.*, 18(1):33–49, Jan. 1997.