PORTED SEPARATOR-STRONG PARAMETRIZED TUTTE FUNCTIONS

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Abstract. We discuss Tutte functions of parametrized matroids in which certain "port" elements have been distinguished. The port elements are held back from deletion and contraction during Tutte decompositions. We prove conditions, specified on the parameters and other relevant values, that are necessary and sufficient for a ported Tutte function to be well-defined on ported minor-closed classes. They generalize together known conditions for definedness on minor-closed sets but with no ports, and for definedness everywhere but with the port restriction.

The ported Tutte functions might have different values on different orientations of the same matroid. The application to graphs where these differences do occur is motivated by expressions of solutions to electrical resistive network and other discrete Laplacian-based problems by new kinds of Tutte functions.

An activities based expression and the corresponding interval partition of the boolean lattice of port-free subsets are given for any recursive computation of a Tutte function value, not just one based on an element order. These are then specialized to a corank-nullity and a geometric lattice expansion for the ported generalization of normal Tutte functions.

1. Introduction

Two natural and useful generalizations of the Tutte equations and the resulting Tutte polynomials T and decompositions are known wherein:

(1) A set P of distinguished elements, which we call "ports", is specified. The operations of deleting and contracting non-separator e, and of removing a separator e so that $T(G \oplus e) = XT(G)$ or $YT(G)$ depending on whether e is a coloop or loop, are then restricted only to $e \notin P$. See [15, 14] where the Tutte polynomials are called "set-pointed", [5], and Brylawsky's early work [4] on the $|P| = 1$ case.

(2) Four parameters x_e, y_e, X_e, Y_e are given for each element e. x_e and y_e are coefficients in the additive Tutte equation for contracting and deleting e when e is a non-separator. X_e and Y_e generalize the common X and Y in (1) to different parameters for different e. The resulting additive Tutte equation is $T(G) = x_e T(G/e) + y_e T(G \setminus e)$. When a set P of port elements (1) is specified, parameters are given only for $e \notin P$.

We use the separator multiplicative Tutte equation shown in (1) because the general multiplicative equation $T(G_1 \oplus G_2) = T(G_1)T(G_2)$ can be derived from it given additional assumptions, in particular $T(\emptyset) = T(\emptyset)T(\emptyset)$ when $P = \emptyset$ [9, Corollary 2.3]. For this note, we only address consequences of generalizations (1) and (2). We adopt Zaslavsky's terminology[19] to call the resulting functions **separator strong Tutte functions**. Ported generalizations of the formula $T(\emptyset)T(G_1 \oplus G_2) = T(G_1)T(G_2))$ [9] and related results about strong Tutte functions[19], direct sums and graphs (based on [19, 2, 9]) are left for future publication.

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The underlying problem with the parametrized generalizion (2) is that different computations of $T(G)$ from the separator multiplicative and additive Tutte equations and the value of T on $\emptyset$ give different results. Only for special parameter values (such as $X_e = X$, $Y_e = Y$ and $x_e = y_e = 1$ for all e, which define classical Tutte polynomials) do the Tutte equations have a solution. This problem has been resolved by Zaslavsky who published the so-called ZBR conditions[9] on the parameter values for collections of Tutte equations to have a solution [19, 9]. Those conditions were applied to classify the Tutte functions into fields[19] and rings[2]. Two equivalent algebraic formulations of the ZBR conditions have been given. One can specify a set of polynomial equations on the parameters necessary and sufficient for $T(G)$ to be defined from the Tutte equations for all G in a given domain. Alternatively, given the domain, the range of T can be specified by the quotient module or ring $R[\{x_e, y_e, X_e, Y_e\}]/I$ where I is the ideal generated by the polynomials f, for each equation $f = 0$ in the set.

In either formulation, a solution to the Tutte equations on some domain is called a Tutte function. A polynomial expression for a Tutte function that is universal for all those Tutte functions is called a Tutte polynomial. When the parameters (for us, the x_e, y_e, X_e, Y_e) and values ($T(Q)$ for indecomposibles Q) are considered to be the variables, the Tutte polynomial is non-unique[20]. We can then say that the ZBR conditions (on the values of the parameters and the $T(Q)$) are necessary and sufficient for all the Tutte polynomials for the same graph or matroid to express the same value.

The subject of this note is extending the ZBR conditions to the combination of generalizations of (1) and (2) slightly beyond the work of Diao and Hetyei, together with ported generalizations of various expansions of Tutte functions, and the emergance of paramatrized Tutte functions depending on matroid orientation when the port constraint (1) is imposed. The development suggests that it is more natural to consider a Tutte function to be determined by the separator multiplicative equations with parameters (X_e and Y_e) plus the **initial value** $T(\emptyset)$ rather than by considering the loop and coloop graphs or matroids to be the indecomposible objects. With the port constraint, our indecomposibles are (or in the case of graphs, have) matroids, including $\emptyset$, on zero or more port elements only.

1.1. **ZBR conditions and indecomposibles.** The first step towards characterizing Tutte functions over various choices of field or ring values and function domain was taken by Zaslavsky[19]. He used induction on the number of matroid elements to show that the conditions are exactly those for the existance of Tutte functions on only three very simple kinds of matroids and all their minors: The “dyad” or uniform rank 1 matroid on two elements, the “triad” or uniform rank 1 matroid on three elements and the “triangle” or uniform rank 2 matroid on three elements. See the top of figure 1. Diao and Hetyei found that the three ZBR conditions on the parameters are necessary and sufficient for ported (which they called “relative”) Tutte functions to be defined for all graphs and matroids. Our generalization consists of ZBR-type conditions that apply to five simple kinds of configurations within an appropriately restricted minor closed class. These conditions are necessary and sufficient for ported Tutte functions to be defined on the given class. They are illustrated in the bottom of figure 1. Each condition applies to only certain matroids in the class with only two or three non-port elements. The minor Q mentioned in the condition is an **indecomposible**, which means that none of the Tutte equations apply to $T(Q)$.

When $T(G)$ is well-defined, it can then be expressed by a not necessarilly unique polynomial in the parameters and $T(Q)$ on indecomposibles Q. Generalizing $T(\emptyset)$, we call such

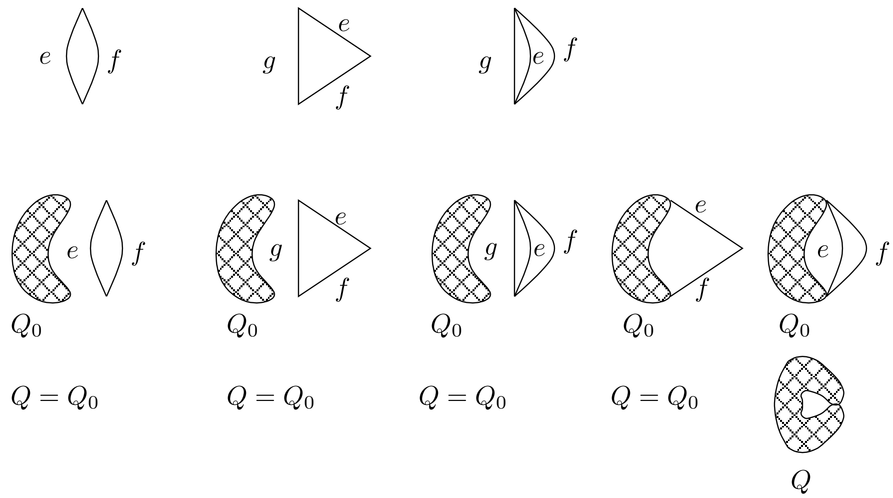

FIGURE 1. Dyad, triangle and triad [19] represented by graphs. Below them are the five kinds of matroids for which our conditions are derived by using that two computations from Tutte equations must give the same value. Each condition has the form $T(Q)f = 0$ where f is a polynomial in the parameters for e, f, and, in two of the five cases, g. In the last case, $f = (x_e Y_f + y_e x_f) - (x_f Y_e + y_f x_e)$.

$T(Q)$ **initial values**. Generally speaking, a function T, not necessarilly a classical polynomial, is called a **separator-strong** [10] **ported** [5] **Tutte function** when it satisfies the parametrized additive and separator multiplicative Tutte equations. Such functions are sometimes only defined on a particular domain of matroids, graphs, and certain matroid representations[6].

Theorem 3 demonstrates how the characterizations[19, 2, 9] of Tutte functions defined on minor-closed subsets $\mathcal{G}$ generalize the conditions given by Diao and Hetyei for relative Tutte functions[1] to be defined everywhere. Thus the programs of Zaslavsky [19], and of Bollobás and Riordan [2] of classifying all Tutte functions may be carried out in the ported or relative case.

Diao and Hetyei's conditions consist of Zaslavsky's conditions on all dyads, triads and triangles[2] of non-port elements together with a "block invariant" property of the Tutte function restricted to graphs or matroids with port elements only.

Our's consist of one equation $T(Q)f = 0$ for each instance of of one of five types of $G \in \mathcal{G}$, where all but two or three of G's elements are ports and Q is a minor of G obtained by removing the non-port elements. In the first three, $f = 0$ expresses Zaslavsky's conditions on dyads, triangles and triads on non-ports. The last two resemble the triangle and triad conditions in that one of the three elements is replaced with a quotient Q of Q_0, where Q_0 is connected to the other two elements in either a series or parallel pair. See figure 1.

[1]It is unfortunate that the term "Tutte polynomial" is often used in the sense of our usage of "Tutte function", even in situations where the difference matters.

[2]When the conditions apply to *all* dyads and triads over a ground set, it is immediate that they apply to all the triangles also.

We defer our proof to the end because the result is such a mild generalization of Diao and Hetyei's work. Our proof avoids introducing an activities-based expression based on a linear ordering of the elements.

It should be noted that if Tutte functions over proper subclasses of matroids or graphs are considered, then each condition derives from a particular subclass member. The three independent kinds of conditions found by Zaslavsky must be replaced by five independent kinds when we generalize with a non-empty set of ports. We were surprised that although each condition is $F = 0$ for some a polynomial F that includes parameters for 2 or 3 elements of E, each such $F = T(Q)f$ includes *one and only one* $T(Q)$ which occurs as a factor.

1.2. **Tutte functions depending on orientations.** The resulting ported Tutte functions might depend on the orientation of an oriented matroid M, not just on the underlying matroid. Here is the simplest case. Let Q^+ be the oriented matroid on port elements $\{p, q\}$ consisting of a 2-circuit with signature $\pm(+,+)$ and Q^- be the one with signature $\pm(+,-)$. An example is any ported Tutte function T with $T(Q^+) = 1$ and $T(Q^-) = -1$. Note that the distinct oriented matroids $Q^\pm$ have identical underlying matroids.

An electrical resistor network is one of many analogs modeled by the discrete Laplacian matrix[8]. It has been known from Kirchhoff[13] that the transfer resistance (called transpedance in [3, 18]), is the ratio of two polynomials whose monomials are $\pm$ products of conductances g_e over sets of edges, where each set is a kind of forest. Transfer resistance is usually defined given two pairs of nodes. It is the potential difference across one pair due to unit current flowing in and out at the other pair. Suppose two new edges $\{p, q\}$ are introduced solely to demark the two given pairs of nodes. We take p and q to be ports and we do not supply parameters or conductances g for them. The denominator is the ported Tutte function that enumerates the spanning trees containing neither p nor q. Although the number of terms in the numerator is also ported Tutte function of the underlying graphic, non-oriented matroid[5], the numerator itself cannot be because the signs of the terms vary.

It is an immediate result of our explicit extension of Diao and Hetyei's theory to oriented matroids that the above numerator is *also* a ported Tutte function. It is the extension of T above with $T(Q') = 0$ for all decomposibles except for U^r_{pq} and all $x_e = X_e = g_e$, $y_e = Y_e = 1$. The positive terms (multiples of $T(U^+_{ef}) = 1$) correspond to spanning forests F with exactly two trees such that $F \cup \{p, q\}$ contains a unique circuit and, when oriented, that signed circuit contains p and q in the same direction. The negative terms include $T(U^-_{ef}) = -1$.

We had developed this subject in [6] where the Tutte equations had to be verified directly. We had demonstrated and applied a variation of Tutte functions of linear matroids whose values lie in an exterior algebra, so this is not subsumed by the present work.

1.3. **Tree-based activity expressions.** When the conditions are satisfied, all recursive computations based on the ported Tutte equations give the same result. The Tutte computation tree based formulation of activities [12] enables us to define an activities-type expansion for every such computation, not just those based on a linear ordering of E. The known theory generalizes to the ported case, including for example, interval partitions of the Boolean lattice 2^E. There is one term and boolean interval for each independent subset $F \subseteq E$ for which $F \cup P$ is spanning; such F specialize to bases when $P = \emptyset$. The blocks of such interval partitions are then partitioned according to the common $Q = M/A|P$ for A in each block.

Our inductive, non-activities proof facilitates the generalization Diao and Hetyei's activities based expression of $T(G)$ (needed in their proof) so that the activities are not based on a fixed ordering of elements. Instead, the activities are based on an arbitrary recursive calculation of $T(G)$ expressed as a tree as in [16]. This result applies the idea of the "computation tree" based activities expression for Tutte polynomials of greedoids given by Gordon and MacMahon to ported or restricted Tutte polynomials of matroids or graphs[12].

1.4. **Ported normal Tutte functions.** We conclude the note by addressing the ported generalization of **normal** Tutte functions, named by Zaslavsky as those that have corank-nullity expressions, in contrast to the activity-based expressions that exist for every Tutte function.

2. Matroid Tutte Function Characterization

Definition 1. *Let P be a set of **ports** and $\mathcal{C}$ be a class of matroids or oriented matroids closed under taking minors or oriented minors by deleting or contracting only elements $e \notin P$. Such a $\mathcal{C}$ is called a P-**family**. Suppose $\mathcal{C}$ is given with four parameters (x_e, y_e, X_e, Y_e), each in a commutative ring R, for each $e \notin P$ that is an element in some $M \in \mathcal{C}$.*

The ground set of $M \in \mathcal{C}$ is denoted by $S(M)$ and $E(M) = S(M) \setminus P$, the ground set elements of M that are not ports.

*A **separator-strong parametrized** P**-ported Tutte function** T maps $\mathcal{C}$ to R or to an R-module and satisfies conditions* (TA) *and* (TSSM) *below for all $M \in \mathcal{C}$ and all e in $E(M)$.*

(TA)
$$T(M) = x_e T(M/e) + y_e T(M \setminus e)$$
if $e \notin P$ and e is a non-separator, i.e., neither a loop nor a coloop.

(TSSM)
If $e \notin P$ is a coloop in M then $T(M) = X_e T(M/e)$.
If $e \notin P$ is a loop in M then $T(M) = Y_e T(M \setminus e)$.

The $Q \in \mathcal{C}$ for which neither (TA) nor (TSSM) apply, equivalently, $E(Q) = \emptyset$, are called the **indecomposibles** or P**-quotients** in $\mathcal{C}$. $U_r^{ef\cdots}$ denotes the uniform rank r matroid on $\{e, f, \ldots\} \subseteq E$.

2.1. **Paramaters and Initial Values.** Classifications of parametrized Tutte functions T for given rings (or fields as rings) and given function domain are partly based on the value $T(\emptyset)$. For example, Tutte functions traditionally satisfy $T(G_1 \oplus G_2) = T(G_1)T(G_2)$ for all direct summands, while in [19] separator-strong Tutte functions are the generalization where G_1 or G_2 is required to be a separator. Hence $T(\emptyset) \neq T(\emptyset)T(\emptyset)$ is possible for separator-strong Tutte functions.

Here we take a different approach which seems to be more natural for separator-strong ported Tutte functions with $P \neq \emptyset$. Instead of specifying a Tutte function by parameters denoted above by x_e, y_e together with "initial values" $T(U_e^r)$ for each rank $r = 0$ or $r = 1$ matroid on a singleton ground set $\{e\}$, we provide four parameters x_e, y_e, X_e, Y_e for each $e \in E$, and $T(U_e^r)$ is determined by a Tutte equation $T(U_e^0) = Y_e T(\emptyset)$ or $T(U_e^0) = Y_e T(\emptyset)$. We consider $\emptyset$ (assuming $\emptyset \in \mathcal{G}$) to be an indecomposible. The separator multiplicative Tutte equations $T(U_e^0 \oplus G) = Y_e T(G)$ and $T(U_e^1 \oplus G) = X_e T(G)$ therefore subsume the special case of an "initial value" needed for a loop or coloop e when $e \notin P$.

We use the term **initial value** for $T(Q)$ whenever Q is an indecomposible. (In [19], the $T(U_e^r)$ are called initial values, whereas for us, for example, $T(U_e^1) = X_e T(\emptyset)$ where X_e is called a parameter and $T(\emptyset)$ is called an initial value.)

2.2. **Ported Matroid ZBR Theorem.** Proposition 2 helps simplify the statement of Theorem 3

Proposition 2. *Suppose e, f are in series, or are in parallel, in matroid or oriented matroid M. (1) The minors $M/e \setminus f = M/f \setminus e$ are equal as matroids. (2) If M is oriented, the oriented minors $M/e \setminus f = M/f \setminus e$ are equal as oriented matroids.*

Proof. If e, f are in series, note that $M/e \setminus f = M \setminus f/e$. e is a coloop in $M \setminus f$, so $M \setminus f/e = M \setminus \{e, f\}$, which is clearly the same matroid or oriented matroid if e, f are interchanged. The relevant theory of minors of oriented matroids can be found in [1].

If e, f are in parallel, e, f are in series in the matroid or oriented matroid dual M^* of M. By the first case, $M^* \setminus e/f = M^* \setminus f/e$ as matroids or as oriented matroids. Thus $M/e \setminus f = (M^* \setminus e/f)^* = (M^* \setminus f/e)^* = M/f \setminus e$ as matroids or as oriented matroids. ☐

Theorem 3. *The following two statements are equivalent.*

(1) *Given an initial value $T(Q)$ in an R-module for each P-quotient $Q \in \mathcal{C}$, the Tutte equations* (TA) *and* (TSSM) *extend T to a unique P-ported separator-strong parametrized Tutte function with with R-parameters (x, y, X, Y).*

(2) (a) *For every $M = U_1^{ef} \oplus Q \in \mathcal{C}$ with P-quotient Q (U_1^{ef} is a dyad),*
$$T(Q)(x_e Y_f + y_e X_f) = T(Q)(x_f Y_e + y_f X_e).$$
(b) *For every $M = U_2^{efg} \oplus Q \in \mathcal{C}$ with P-quotient Q (U_2^{efg} is a triangle),*
$$T(Q)X_g(x_e y_f + y_e X_f) = T(Q)X_g(x_f y_e + y_f X_e).$$
(c) *For every $M = U_1^{efg} \oplus Q \in \mathcal{C}$ with P-quotient Q (U_1^{efg} is a triad),*
$$T(Q)Y_g(x_e Y_f + y_e x_f) = T(Q)Y_g(x_f Y_e + y_f x_e).$$
(d) *If $\{e, f\} = E(M)$ is a parallel pair connected to P,*
$$T(Q)(x_e Y_f + y_e x_f) = T(Q)(x_f Y_e + y_f x_e)$$
where P-quotient $Q = M/e \setminus f = M/f \setminus e$.
(e) *If $\{e, f\} = E(M)$ is a series pair connected to P,*
$$T(Q)(x_e y_f + y_e X_f) = T(Q)(x_f y_e + y_f X_e)$$
where P-quotient $Q = M/e \setminus f = M/f \setminus e$.

When $P = \emptyset$, this reduces to the Zaslavsky-Bollobás-Riordan theorem for matroids[9]: In that case, only $Q = \emptyset$ occurs and the last two conditions become vacuous.

2.3. **Universal Tutte Polynomial.** It is easy to follow [2, 9] to define a universal, i.e., most general P-ported parametrized Tutte function $T^{\mathcal{C}}$ for $\mathcal{C}$. We take indeterminates x_e, y_e, X_e, Y_e for each $e \in E(M), M \in \mathcal{C}$ and an indeterminate $[Q]$ for each P-quotient $Q \in \mathcal{C}$. Let $\mathbb{Z}[x, y, X, Y]$ denote the integer polynomial ring generated by the x_e, y_e, X_e, Y_e indeterminates, and define $\widetilde{\mathbb{Z}}$ to be the $\mathbb{Z}[x, y, X, Y]$-module generated by the $[Q]$. Let $I^{\mathcal{C}}$ denote the ideal of $\widetilde{\mathbb{Z}}$ generated by the identities of Theorem 3, comprising for example $[Q](x_e Y_f + y_e X_f - x_f Y_e - y_f X_e)$ for each instance of case (a), etc. The universal Tutte function has values in the quotient module $\widetilde{\mathbb{Z}}/I^{\mathcal{C}}$. Finally, observe that the range of Tutte function T can be considered to be the R-module generated by the values $T(Q)$ where ring R contains the x, y, X, Y parameters. If the $T(Q) \in R$, consider the ring R to be the R-module generated by R. We follow [9] to write the corresponding consequence of Theorem 3:

Corollary 4. *Let $\mathcal{C}$ be a P-minor closed class of matroids or oriented matroids. Then there is a $\widetilde{\mathbb{Z}}/I^{\mathcal{C}}$-valued function $T^{\mathcal{C}}$ on $\mathcal{C}$ with $T^{\mathcal{C}}(Q) = [Q]$ for each P-quotient $Q \in \mathcal{C}$ that is a P-ported parametrized Tutte function on $\mathcal{C}$ where the parameters are the $x, y, X, Y \in \widetilde{\mathbb{Z}}$. Moreover, if T is any R-parametrized Tutte function with parameters x'_e, y'_e, X'_e, Y'_e, then T is the composition of $T^{\mathcal{C}}$ with the homomorphism determined by $[Q] \to T(Q)$ for P-quotients Q and $x_e \to x'_e$, etc., for each $e \in E(\mathcal{C})$.*

3. Tutte Computation Trees and Activities

Zaslavsky[19] noted that the *basis* or *activities* expansion, introduced by Tutte [17, 18] for graphs, applies to all well-defined parametrized Tutte functions whereas the parametrized corank-nullity generating function expresses only a proper subset of them, which he called normal. We show how to give activities-type expansions comprised of one expansion for every recursive computation of a ported Tutte function value, not just those determined by a linear element ordering. We apply to ported matroids the expansions based on activities defined with Tutte computation trees, which Gordon and McMahon [12] used to demonstrate that a definition of Tutte polynomials of greedoids is consistant with Tutte equations. Unlike matroids, some greedoids do not have activities expansions for their Tutte polynomials that derive from element orderings[12]. When we have $P \neq \emptyset$, each tree leaf is generally one of many indecomposibles. The use of trees whose leaves are indecomposibles for analyzing a variation of Tutte decompositions for graph appears in [16].

Ellis-Monaghan and Traldi [9] remarked that the Tutte equation approach appears to give a shorter proof of the ZBR theorem than the activities expansion approach. The proofs by induction on $|E|$ demonstrate that every calculation of $T(M)$ from Tutte equations produces the same result when the conditions on the parameters and initial values are satisfied. We then know immediately that the polynomial expression resulting from a particular calculation equals the Tutte function value. We suggest a heuristic reason why the Tutte equation approach is more succinct: The induction assures that *every* computation with smaller $|E|$ gives the same result, not just those computations that are determined by linear orders on E. Proofs of activities expansions for matroids, and their generalizations for P-ported matroids, seem more informative and certainly no harder when the expansions are derived from a general Tutte computation tree, than when the expansions are only those that result from an element order. From the retrospective that the Tutte equations specify a non-deterministic recursive computation [11], it seems artificial to start with element-ordered computations and then prove first that all linear orders give the same result and second that it satisfies the Tutte equations, in order prove that all recursions give the same result. We therefore take advantage of the Tutte computation tree formalism and the more general expansions it enables.

3.1. **Computation Tree Expansion.**

Definition 5. *Given P-ported matroid or oriented matroid M, a P-**subbasis** F is an independent set with $F \subseteq E(M)$ (so $F \cap P = \emptyset$) for which $F \cup P$ is a spanning set for M (in other words, F spans M/P). $\mathcal{B}_P(M)$ denotes the set of P-subbases.*

Equivalent definitions were given in [15] and in [7]. In the latter, F is called a "contracting set." The following is immediate and useful:

Proposition 6. *For every P-subbasis F there exists an independent set $Q \subseteq P$ that extends F to a basis $F \cup Q \in \mathcal{B}(M)$ of M. Conversely, if $B \in \mathcal{B}(M)$ then $F = B \cap E = B \setminus P$ is a P-subbasis.*

Definition 7 (Computation Tree, following [12])**.** *A P-ported (Tutte) computation tree for M is a binary tree whose root is labeled by M and which satisfies:*

(1) *If M has non-separating elements not in P, then the root has two subtrees and there exists one such element e for which one subtree is a computation tree for M/e and the other subtree is a computation tree for $M \setminus e$.*

The branch to M/e is labeled with "e contracted" and the other branch is labeled "e deleted".

(2) *Otherwise (i.e., every element in $E(M)$ is separating) the root is a leaf.*

Each leaf is labelled with a P-quotient Q plus a direct sum of loop or coloop matroids on elements respectively e or f in E, expressed as $[Q] \prod X_e \prod Y_f$.

Definition 8 (Activities with respect to a leaf)**.** *For a P-ported Tutte computation tree for M, a given leaf, and the path from the root to this leaf:*

- *Each $e \in E(M)$ labeled "contracted" along this path is called* ***internally passive****.*
- *Each coloop $e \in E(M)$ in the leaf's matroid is called* ***internally active****.*
- *Each $e \in E(M)$ labeled "deleted" along this path is called* ***externally passive****.*
- *Each loop $e \in E(M)$ in the leaf's matroid is called* ***externally active****.*

Proposition 9. *Given a leaf of a P-ported Tutte computation tree for M, the set of internally active or internally passive elements constitutes a P-subbasis of M which we say* ***belongs to the leaf****. Furthermore, every P-subbasis F of N belongs to a unique leaf.*

Proof. For the purpose of this proof, let us extend Definition 8 so that, given a computation tree with a given node i labeled by matroid M_i, $e \in E$ is called internally passive when e is labeled "contracted" along the path from root M to node i. Let IP_i denote the set of such internally passive elements.

It is easy to prove by induction on the length of the root to node i path that (1) $IP_i \cup S(M_i)$ spans M and (2) IP_i is an independent set in M. The proof of (1) uses the fact that elements labeled deleted are non-separators. The proof of (2) uses the fact that for each non-separator $f \in M/IP_i$, $f \cup IP_i$ is independent in M.

These properties applied to a leaf demonstrate the first conclusion, since each $e \in E$ in the leaf's matroid must be a separator by Definition 7.

Given a P-subbasis F, we can find the unique leaf with the algorithm below. Note that it also operates on arbitrary subsets of E.

Tree Search Algorithm: Beginning at the root, descend the tree according to the rule: At each branch node, descend along the edge labeled "e-contracted" if $e \in F$ and along the edge labeled "e-deleted" otherwise (when $e \notin F$). □

We leave the reader to check that the classsical element-order based activities expansion, as extended with ports explicitly in[7], is reproduced with the unique P-ported computation tree in which the greatest non-separator $e \in E$ is deleted and contracted in the matroid at each tree node, when the elements are ordered so $p \in P$ is before each $e \notin P$.

Definition 10. *Given a computation tree for P-ported (oriented) matroid M, each P-subbasis $F \subseteq E$ is associated with the following subsets of non-port elements defined according to Definition 8 from the unique leaf determined by the algorithm given above.*

- $IA(F) \subseteq F$ *denotes the set of internally active elements,*
- $IP(F) \subseteq F$ *denotes the set of internally passive elements,*
- $EA(F) \subseteq E \setminus F$ *denotes the set of externally active elements, and*
- $EP(F) \subseteq E \setminus F$ *denotes the set of externally passive elements.*
- $A(F) = IA(F) \uplus EA(F)$ *denotes the set of active elements.*

Proposition 11. *Given a P-ported Tutte computation tree for M, the boolean lattice of subsets of $E = E(M)$ is partitioned by the collection of intervals $[IP(F), F \uplus EA(F)]$ determined from the collection of P-subbases F, which correspond to the leaves. (Note $F \uplus EA(F) = IP(F) \uplus A(F)$.)*

Each $A \in [IP(F), F \uplus EA(F)]$ in one interval determines the same matroid or oriented matroid P-quotient by $M/A|P = M/IP(F)|P$.

The boolean lattice of subsets of $E = E(M)$ is also partitioned by the collection of intervals $[EP(F), (E \setminus F) \uplus IA(F)]$. (Note $(E \setminus F) \uplus IA(F) = EP(F) \uplus A(F)$.)

Each $A' \in [EP(F), (E\setminus F) \uplus IA(F)]$ of one of these intervals determines the same matroid or oriented matroid P-quotient by $M \setminus B'/(E \setminus B') = M/F|P$.

For a given $F \in \mathcal{B}_P(M)$, $A \subseteq E$ satisfies $A \in [IP(F), F \uplus EA(F)]$ if and only if $(E\setminus A) \in [EP(F), (E \setminus F) \uplus IA(F)]$.

Proof. Every subset $A \subseteq E = E(M) \setminus P$ belongs to the unique interval corresponding to the unique leaf found by the tree search algorithm given at the end of the previous proof.

P-quotient $M/A|P$ is independent of the order of the deletions and contractions. So let $IP(F) \subseteq A$ be contracted and $(E \setminus EA(F)) \subseteq (E \setminus A)$ be deleted first. The remaining elements of E are loops or coloops, so the P-quotient is independent of whether they are deleted or contracted.

The dual of that tree search algorithm, which descends along the edge labelled "e-deleted" if $e \in A'$, etc., will find the unique leaf whose interval $[EP(F), E \setminus F \cup IA(F)]$ contains A'.

When $A \in [IP(F), F \cup EA(F)]$, the dual algorithm applied to $A' = E \setminus A$ will find the same leaf. The P-quotient is determined by $IP(F)$. □

The following generalizes the activities expansion expression given in [19] to ported (oriented) matroids, as well as Theorem 8.1 of [15].

Proposition 12. *Given parameters x_e, y_e, X_e, Y_e, and P-ported matroid or oriented matroid M the Tutte polynomial expression determined by the sets in Definition 10 from a computation tree is given by*

$$\text{(PAE)} \qquad \sum_{F \in \mathcal{B}_P} [M/F|P]\, X_{IA(F)}\, x_{IP(F)}\, Y_{EA(F)}\, y_{EP(F)}.$$

Proof. (PAE) is an expression constructed by applying some of the Tutte equations. One monomial results from each leaf. In that leaf's matroid, each active element is a separator, and the active elements contribute $X_{IA(F)}Y_{EA(F)}$ to the monomial. The passive elements which contribute $x_{IP(F)}y_{EP(F)}$ are the tree edge labels in the path from the root to the leaf. Each $M/F|P$ denotes a P-quotient of M, so the expression is a polynomial in the parameters and in the initial values. Therefore, (PAE) expresses the result of the calculation when one substitutes $[M/F|P] = T(M/F|P)$. □

From Corollary 4 we conclude:

Theorem 13. *For every P-ported parametrized Tutte function T on $\mathcal{C}$ into ring R or an R-module, for every computation tree for $M \in \mathcal{C}$, (and so for every ordering of $E(M)$), the polynomial expression* (PAE) *equals $T^{\mathcal{C}}(M)$ of Corollary 4.*

3.2. **Expansions of Normal Tutte Functions.** After a notational translation, Zaslavsky's [19] definition of **normal** Tutte functions discriminates those for which $T(\emptyset) = 1$, and those for which there exist u, $v \in R$ so that for each $e \in E(M)$,

$$X_e = x_e + uy_e \text{ and } Y_e = y_e + vx_e. \tag{CNF}$$

The equations of Theorem 3 are readily verified and so we naturally extend this definition, with the $T(\emptyset) = 1$ condition dropped, to ported separator-strong parametrized Tutte functions. All the expressions for normal Tutte functions are therefore in the ring freely generated by u, v, the x_e, y_e and the $[Q]$. We can now generalize some known expansions for the Tutte polynomial $T^{\mathcal{C}}$ after the (CNF) substitution. The rank function for M is denoted by r.

Corollary 14 (Boolean Interval Expansion)**.** *The following activities and boolean interval expansion formula is universal for normal Tutte functions:*

$$T^{\mathcal{C}}(M) = \sum_{F \in \mathcal{B}_P} [M/F|P] \Big(\sum_{\substack{IP(F) \subseteq K \subseteq F \\ EP(F) \subseteq L \subseteq E \setminus F}} x_{K \cup (E \setminus F \setminus L)}\, v^{|E \setminus F \setminus L|}\, y_{L \cup (F \setminus K)}\, u^{|F \setminus K|} \Big)$$

Proof. Substitute (CNF) into $T^{\mathcal{C}}(M)$ and use $IP(F) \uplus IA(F) = F$ and $EP(F) \uplus EA(F) = E \setminus F$ from Definition 10. ☐

Lemma 15. *Given $F \in \mathcal{B}_P$, $IP(F)$ spans $EA(F)$.*

The pairs (K, L) for which $IP(F) \subseteq K \subseteq F$ and $EP(F) \subseteq L \subseteq E \setminus F$ are in a one-to-one correspondance with the A satisfying $IP(F) \subseteq A \subseteq F \uplus EA(F)$ given by $A = K \uplus (E \setminus F) \setminus L$.

For every such A,

$$|F \setminus K| = r(M) - r(M/F|P) - r(A)$$

and

$$|E \setminus F \setminus L| = |A| - r(A).$$

Proof. (See Figure 2 in Appendix.) By our definition of activities, after all the elements of $IP(F)$ are contracted, all elements in $EA(F)$ are loops. (Note none of these elements are ports.)

Let $A = K \uplus (E \setminus F \setminus L)$. By our definition of activities, $IP(F) \uplus IA(F) = F$, so $IP(F) \subseteq A$. Similarly, $EP(F) \uplus EA(F) = E \setminus F$, so $A \cap (E \setminus F) \subseteq EA(F)$. Hence $IP(F) \subseteq A \subseteq F \uplus EA(F)$. Conversely, given such an A, take $K = A \cap F$ and $L = (E \setminus F) \setminus A$.

Since $IP(F)$ spans $EA(F)$ and $K \supseteq IP(F)$, K spans $EA(F)$. Since $A \subseteq K \uplus EA(F)$, K spans A. $K \subseteq F$, F is a P-subbasis, so K and F are independent, hence $|K| = r(K) = r(A)$ and $|F| = r(F)$. Therefore, $|F \setminus K| = r(F) - r(A)$.

Since F is a P-subbasis, $r(F \cup P) = r(M)$. By definition of contraction, $r(M/F|P) = r(F \cup P) - r(F)$, so $r(M/F|P) = r(M) - r(F)$. We conclude $|F \setminus K| = r(M) - r(M/F|P) - r(A)$.

$E \setminus F \setminus L = A \setminus K$, so $|E \setminus F \setminus L| = |A| - |K|$. As above, $|K| = r(A)$, so the last equation follows. ☐

Corollary 16.

$$T^{\mathcal{C}}(M) = \sum_{F \in \mathcal{B}_P} [M/F|P] \Big(\sum_{IP(F) \subseteq A \subseteq (F \cup EA(F))} x_A y_{E \setminus A} u^{r(M)-r(M/F|P)-r(A)} v^{|A|-r(A)} \Big) \tag{1}$$

Proof. Apply Lemma 15 to the inner sum in Proposition 14. ☐

Theorem 17 (Corank-nullity expansion)**.**

$$T^{\mathcal{C}}(M) = \sum_{A \subseteq E(M)} [M/A \mid P] x_A y_{E \setminus A} u^{r(M)-r(M/A|P)-r(A)} v^{|A|-r(A)}. \tag{PGF}$$

Remark: This extends to separator-strong ported Tutte functions with parameters on possibly oriented matroids an expression from [14] reproduced in [5]. It can also be proved by corank-nullity generating function methods.

Proof. By Proposition 11, given any Tutte computation tree, the lattice of subsets of $E(M)$ is partitioned into intervals corresponding to P-subbases $\mathcal{B}_P$. In each interval, P-quotient $M/F|P$ is equal to $M/A|P$ (as a matroid or oriented matroid) for $A \in [IP(F), F \cup EA(F)]$. Hence we can interchange the summations in (1) and write (PGF). ☐

Proposition 18 (Geometric Lattice Flat Expansion)**.** *Let M be an oriented or unoriented. In the formula below, F and G range over the geometric lattice of flats $\mathcal{L} = \mathcal{L}(M|E)$ contained in M restricted to $E = E(M)$. (E is the top of $\mathcal{L}$ and $\leq$ is its partial order.)*

$$T^{\mathcal{C}}(M) = \sum_Q [Q] \sum_{\substack{F \leq E \\ [M/F|P]=[Q]}} u^{r(M)-r(Q)-r(F)} v^{-r(F)} \sum_{G \leq F} \mu(G,F) \prod_{e \in G} (y_e + x_e v)$$

Proof. This generalizes and follows the steps for theorem 8 in [5]. (See Appendix.) ☐

4. Proof of the Ported ZBR Theorem

The proof is independent of Diao and Hetyei's. It is different in that it is not based on element activities determined by a linear element order. It results from adding considerations of ports to Zaslavsky's proofs in [19]. We base our sketch below on the "straightforward adaption" presented by Ellis-Monaghan and Traldi [9], giving a few details and pointing out differences.

Proof. As in [9], the necessary relations are easy to deduce by applying (TA) and (TSSM) in two different orders to a general configuration in each of the five families.

Now on to the converse. As in [9], the strategy is to first verify that the conditions imply $T(M)$ is well-defined for $n = |E(M)| = 0, 1$ and 2. Note that for us, the count n does not include $|P \cap S(M)|$. Second, we rely on the hypothesis the $\mathcal{C}$ is closed under P-minors in order to verify that in a larger minimum n counterexample, the elements of $E(M)$ are either all in series or all in parallel, and then the conditions imply that all calculation orders give the same result. All the cases involve two different combinations of deleting and contracting of several elements in $E(M)$ where both combinations produce the same P-quotients.

Let $M \in \mathcal{C}$ be a counterexample with minimum $n = |E(M)|$. Therefore, whenever M' is a proper P-minor of M, $T(M')$ is well-defined. (TA) and (TSSM) have the property that given M and $e \in E(M)$, exactly one equation applies. Therefore, calculations that yield different values for $T(M)$ must start with reducing by different elements of $E(M)$. Since $T(M)$ is an already given initial value when $n = 0$, we can assume $n \geq 2$.

M cannot contain a separator $e \in E(M)$, because this e is a separator in every P-minor of M containing e. Therefore, as observed in [9], every computation has the same result $X_e T(M/e)$ or $Y_e T(M \setminus e)$ depending on whether e is a coloop or a loop.

Let e be one element in $E(M)$. Since no element in $E(M)$ is a separator, $V = x_e T(M/e) + y_e T(M \setminus e)$ is well-defined, and so is $x_{e'} T(M/e') + y_{e'} T(M \setminus e')$ for each other $e' \in E$. We follow [9] and define $D = \{e' \in E(M) \mid V = x_{e'} T(M/e') + y_{e'} T(M \setminus e')\}$. The induction hypothesis then tells us that there is at least one $f \in E(M) \setminus D$. (Recall $e, e', f \notin P$.)

Suppose that e is a separator in both $M \setminus f$ and M/f and f is a separator in both $M \setminus e$ and M/e. Then, T would be well-defined for all four of these P-minors and so we can write

$$T(M) = x_e x_f T(M/\{e,f\}) + x_e y_f T(M/e \setminus f) + y_e x_f T(M \setminus e/f) + y_e y_f T(M \setminus \{e,f\}).$$

Both computations give the same value because in this situation the reductions by e and f commute. So, for M to be a counterexample, there must be $e \in D$ and $f \notin D$ ($e, f \notin P$) to which one case of the following lemma applies:

Lemma 19. [19] *Let e, f be nonseparators in a matroid M. Within each column the statements are equivalent:*

(1) *e is a separator in $M \setminus f$.*	(1) *e is a separator in M/f.*
(2) *e is a coloop in $M \setminus f$.*	(2) *e is a loop in M/f.*
(3) *e and f are in series in M.*	(3) *e and f are in parallel in M.*
(4) *f is a separator in $M \setminus e$.*	(4) *f is a separator in M/e.*

We claim that one of the following five cases must be satisfied:

(1) $n = 2$ and $E(M) = \{e, f\}$ is a dyad.
(2) $n \geq 3$ and $E(M)$ is a circuit not connected to P.
(3) $n \geq 3$ and $E(M)$ is a cocircut not connected to P.
(4) $n \geq 2$ and for some $\emptyset \neq P' \subseteq P$, $P' \cup E(M)$ is a cocircuit.
(5) $n \geq 2$ and for some $\emptyset \neq P' \subseteq P$, $P' \cup E(M)$ is a circuit.

As in [9], we draw the conclusion that if $e \in D$ and $f \notin D$ then e, f are either series or parallel. It was further proven that a series pair and a parallel pair cannot have exactly one element in common. Therefore, the pairs e, f satisfying the conditions are either all series pairs or all parallel pairs. By minimality of n, $E(M)$ is either an n-element parallel class or an n-element series class. The last two cases are distinguished from the first three according to whether or not $E(M)$ is disconnected or not from elements of P in matroid M. We now use (TA) and (TSSM) to show that, in each case, the calculations that start with e and those that start with f have the same result, which contradicts $e \in D$ and $f \notin D$.

We give the details for case (d). By hypothesis, each of $T(M/e)$, $T(M \setminus e)$, $T(M/f)$, $T(M \setminus f)$, $T(M/f/e) = T(M/e/f)$, $T(M/e \setminus f)$ and $T(M/f \setminus e)$ is well-defined. (Remark: By Proposition 2, $M \setminus f/e = M \setminus e/f$.

Starting with e and with f, (TA) gives the two expressions:

$$V = x_e T(M/e) + y_e x_f T(M \setminus e/f) + y_e y_f T(M \setminus e \setminus f)$$

$$V \neq x_f T(M/f) + y_f x_e T(M \setminus f/e) + y_f y_e T(M \setminus f \setminus e)$$

Let M' be the P-minor obtained by deleting each element in $E(M)$ except for e and f ($M' = M$ if $n = 2$.) Since $E(M') = \{e, f\}$ and e, f are in parallel connected to P within M', (d) of the hypotheses tells us that

$$T(Q)(x_e Y_f - y_f x_e) = T(Q')(x_f Y_e - y_e x_f),$$

where $Q = M'/e\setminus f$, $Q' = M'/f\setminus e$. Since e, f are in parallel within M', matroids or oriented matroids $Q = Q'$ by Proposition 2. Since $A = E(M) \setminus \{e, f\}$ is a set of loops ($\emptyset$ if $n = 2$) in $M/e \setminus f$ ($N/e \setminus f$) and in $M/f \setminus e$ ($N/f \setminus e$), we write $Y_A = \prod_{a\in A} Y_a$ (1 if $A = \emptyset$) and use (TSSM) to write

$$T(M/e \setminus f) = Y_A T(Q),\;\; T(M/f \setminus e) = Y_A T(Q'),$$
$$T(M/e) = Y_f Y_A T(Q), \quad \text{and} \quad T(M/f) = Y_e Y_A T(Q').$$

So, since $M \setminus f \setminus e = M \setminus e \setminus f$,

$$x_e T(M/e) + y_e x_f T(M \setminus e/f) = x_f T(M/f) + y_f x_e T(M \setminus f/e)$$

contradicts $V \neq x_f T(M/f) + y_f x_e T(M \setminus f/e) + y_f y_e T(M \setminus f \setminus e)$ The remaining cases can be completed analogously. It might be noted that our proof differs slightly from [9] in that the cases of $n = 3$ and $n \geq 4$ are not distinguished. ☐

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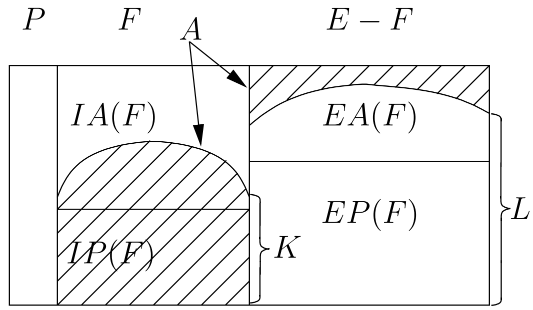

Figure 2. Illustration for proof of Lemma 15.

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Appendix A. Proof of Proposition 18

For each subset $A \subseteq E$, $[Q] = [M/A|P] = [M/F|P]$ is determined by the unique flat F in $L(E)$ spanned by $A \subseteq E$. So, we write (PGF) by

$$T^{\mathfrak{C}}(M) = \sum_Q [Q] \sum_{\substack{F\in\mathcal{L} \\ [M/F|P]=[Q]}} \sum_{\substack{A\subseteq F \\ A \text{ spans } F}} x_A y_{E\setminus A} u^{r(M)-r(M/A|P)-r(A)} v^{|A|-r(A)}.$$

Factoring, we get

$$T^{\mathfrak{C}}(M) = \sum_Q [Q] \sum_{\substack{F\in\mathcal{L} \\ [M/F|P]=[Q]}} u^{r(M)-r(Q)-r(F)} v^{-r(F)} \sum_{\substack{A\subseteq F \\ A \text{ spans } F}} x_A y_{E\setminus A} v^{|A|}.$$

A is summed over the spanning sets of F. Let $Z(F)$ denote this last sum. Since every subset of F spans some flat in $\mathcal{L}$,

$$\sum_{0\le G\le F} Z(G) = \sum_{e\in F} (y_e + x_e v).$$

Möbius inversion gives

$$Z(F) = \sum_{0\le G\le F} \mu(G, F)(y_e + x_e v).$$

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