An Exterior Algebra Valued Tutte Function on Linear Matroid Pairs

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Abstract

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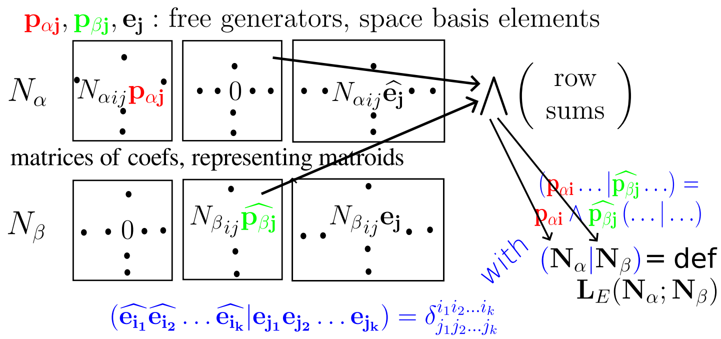
Let N be a linear representation (i.e, matrix) of a matroid whose ground set S includes a finite, distinguished subset P. We give function $L(N)$ that, unlike what we know of other Tutte functions and work like the Hopf algebra variants of Krajewski, Moffatt and Tanasa, has values in an *anti-commutative* algebra. Let deletion and contraction be limited to $e \notin P$. Then, the values are in the exterior algebra generated by $P_\alpha \coprod P_\beta$. The construction relies on concrete minor operations to establish consistent signs of the constituent terms so that, with suitable accounting for sequential orderings of set elements, $L(N) = L(N \setminus e) + L(N/e)$ in the exterior algebra. Our construction is derived from the structure of the equilibrium equations for linear electrical networks, and of their generalization to multi-dimensional elastic frameworks. Further, the construction does not require orthogonality for the spaces that generalize spaces of feasible currents and voltages, or of forces and displacements. Hence L will be defined on equal rank pairs (N_α, N_β) (where originally, $N_\alpha = N_\beta = N$). We take the Tutte identities those for Welsh and Kayibi's linking polynomial of matroid pairs. With $N_\alpha \neq N_\beta$, we can derive the digraph all-minors matrix tree theorem by taking P to be the set of vertices. We so get ratio of common basis expansion solutions for linear electrical and other linear systems with multi-terminal amplifiers (where a voltage or force at one place is a multiple of current or displacement at a different place). To incorporate resistance (r_e), conductance (g_e), elasticity coefficient, etc. parameters, we use parametrized Tutte function theory for which $L(N) = r_e L(N \setminus e) + g_e L(N/e)$; the term for common basis B includes $\prod_{e \in B} g_e \prod_{e \notin B \cup P} r_e$.



$$\text{1} = \mathbf{L}_{E\setminus e}(\mathbf{N}_\alpha \setminus \mathbf{e};\mathbf{N}_\beta \setminus \mathbf{e}) + \mathbf{L}_{E\setminus e}(\mathbf{N}_\alpha/\mathbf{e};\mathbf{N}_\beta/\mathbf{e})$$
$$\in \bigwedge P_\alpha \cup \widehat{P_\beta} : \text{ Exterior Algebra (anti-comm!)}$$

$$\text{2} = \sum_{F\subseteq E}([\mathbf{N}_\alpha/F|P_\alpha] \quad | \quad [\mathbf{N}_\beta/F|\widehat{P_\beta}])$$
$$= \sum_{F\subseteq E}[\mathbf{N}_\alpha/F|P_\alpha] \wedge [\mathbf{N}_\beta/F|\widehat{P_\beta}]$$

$$\text{Like } |\text{Graph Laplacian}| = |I\ I^t| = \sum_{F\subseteq E}|I(F)|^2$$
$$= \sum_{T:\text{span. tree}} 1 \text{ (Cauchy-Binet expansion)}$$

We start with two matrices, N_α and N_β. The matroids they represent have the **p** and **e** symbols, each exclusively hatted or not hatted, for their ground set elements. ... For now, think of the row spaces.

We make two exterior algebra elements, boldface $\mathbf{N}_\alpha$ and $\mathbf{N}_\beta$ to represent the row spaces.

We construct function L of those $\mathbf{N}_\alpha$ and $\mathbf{N}_\beta$, ... by a bilinear pairing that performs composition ... $\mathbf{N}_\alpha$ bar $\mathbf{N}_\beta$.

This bilinear operation has exterior algebra, not field or commutative ring values.

Result 1 is that L obeys Tutte's deletion and contraction identity: BUT only for e type elements, not the p's.

There is also a direct sum identity ... L of a direct sum is an exterior product, not a commutative ring product.

We have TO CAREFULLY DEFINE the exterior algebra operations for deletion & contraction & direct sum so the signs in the L–s we combine are consistent.

As pure, ... or indecomposable anti-symmetric tensors, ... that is, products of vectors, the boldface $\mathbf{N}$–s represent linear subspaces of a big space generated by matroid ground set elements and their hatted versions.

So with these designated bases, related to ground sets, we get duals of basis vectors. The choice of column labels makes hatting consistent with dualizing. We'll use duals later.

$\mathbf{N}_\alpha$ and $\mathbf{N}_\beta$ represent points in the Grassmannian, and have Plucker coordinates. The Plucker coordinates are the maximal minors of the matrices. So, the matroid bases are encoded by which Plucker coordinates are non-zero.

To construct an extensor from a matrix, I multiply each column's boldface symbol with its entries. Boldface $\mathbf{N}$ is the exterior product of the row sums.

Finally L equals the bilinear pairing which expresses an linear endomorphism of the exterior algebra generated by the p–s.

I find it interesting that, ... in order to get a Tutte function out of this, it seems require two special things.

We need to make the Tutte function relative.

That means we to hold some elements back from deletion or contraction, ... so they do not all disappear from the algebra where the function value will live.

The distinguished elements we don't delete or contract I like to call PORTS.

In applications, ports relate to variables used to specify inputs or parameters, ... like how much you force or electric current you put into nodes, ... and responses, ... like how the nodes move, or change their voltages.

But when you use L-s value, you can disregard the duality status of p elements, ... so you can interchange inputs and responses ANY WAY that the matroid of L tells you is feasible and well-posed. It encodes which combinations of variables are independent. Electrical engineers like to solve for matrix forms of L and them MULTIPORT LINEAR DEVICE models.

Two, it seems this exterior algebra Tutte function needs to be constructed on two arguments, labelled α and β.

We recover the basis enumerator when the two are equal and P is empty. It is the sum of squared determinants though, not always ones.

Now for the final step.

To define the final bilinear pairing function we distinguish four kinds of generators:
vector $\mathbf{e}$'s, vector $\mathbf{p}$'s and dual vector $\widehat{\mathbf{e}}$ hats and $\widehat{\mathbf{p}}$ hats.

It's defined here *WITH* these rules for algebra basis monomials. Dual vectors in the left evaluate on vectors in the right, but reverse that ... and they behave like anticommutative coefficients.

The $\mathbf{N}$–s mix vectors and dual vectors ... they represent linear mappings. The hatted and unhatted status of p–s and e–s are interchanged. $\mathbf{N}_\beta$ is like the adjoint of $\mathbf{N}_\alpha$. So ... the bilinear pairing is composition of mappings ... like matrix multiplication.

Therefore ... I call this the Cauchy-Binet form. We get result 2. The common independent set expansion here hides many common basis expansions. One of them is the famous matrix tree theorem.

I finish with some take-homes and morals, and my name, Seth Chaiken of Albany, NY. Three punchlines: One, ports are IM-PORT-ANT. Two, let's do matroid recursion on matrices in exterior algebra. Three, we find a Tutte function there. Thank you!

1. Distinguish matroid element "ports" associated with electric or elastic system parameter and solution variables of interest. (All vars are paired: (voltage, current), (force, displacement), etc. One gets a pair of submodels with dual matroids in elementary situations; not duals otherwise.)

2. Exterior algebra forms of deletion and contraction of a non-port yield a pair of simpler systems.

3. Cancelling non-port elements with a kind of bilinear pairing yields the parameter/solution variables of interest relation, in the form of an **exterior algebra valued function** of systems, that **is a Tutte function** (when the minor and direct sum operations are sign-consistent).

With the suitable incidence matrix form, we get the all-minors matrix tree theorem; but all the minors are packed into **one exterior algebra object** that is a Tutte function of graphs.

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