

$\mathbf{p}_{\alpha j}, \mathbf{p}_{\beta j}, \mathbf{e}_j$: free generators, space basis elements
 $\begin{bmatrix} N_{\alpha ij} \mathbf{p}_{\alpha j} \\ \vdots \\ N_{\alpha} \end{bmatrix} \quad \begin{bmatrix} \cdots 0 \cdots \\ \vdots \end{bmatrix} \quad \begin{bmatrix} \cdots N_{\alpha ij} \widehat{\mathbf{e}}_j \cdots \\ \vdots \end{bmatrix}$
 matrices of coefs, representing matroids
 $\begin{bmatrix} N_{\beta} \vdots \\ \cdot 0 \cdot \\ \vdots \end{bmatrix} \quad \begin{bmatrix} N_{\beta ij} \widehat{\mathbf{p}}_{\beta j} \\ \vdots \end{bmatrix} \quad \begin{bmatrix} \cdot \cdot N_{\beta ij} \mathbf{e}_j \cdot \cdot \\ \vdots \end{bmatrix}$
 $(\widehat{\mathbf{e}}_{i_1} \widehat{\mathbf{e}}_{i_2} \cdots \widehat{\mathbf{e}}_{i_k} | \mathbf{e}_{j_1} \mathbf{e}_{j_2} \cdots \mathbf{e}_{j_k}) = \delta_{j_1 j_2 \cdots j_k}^{i_1 i_2 \cdots i_k}$
 $(\mathbf{p}_{\alpha i} \cdots | \widehat{\mathbf{p}}_{\beta j} \cdots) = \mathbf{p}_{\alpha i} \wedge \widehat{\mathbf{p}}_{\beta j} (\cdots | \cdots)$
 $(\mathbf{N}_{\alpha} | \mathbf{N}_{\beta}) = \text{def } \mathbf{L}_E(\mathbf{N}_{\alpha}; \mathbf{N}_{\beta})$

① $= \mathbf{L}_{E \setminus e}(\mathbf{N}_{\alpha} \setminus \mathbf{e}; \mathbf{N}_{\beta} \setminus \mathbf{e}) + \mathbf{L}_{E \setminus e}(\mathbf{N}_{\alpha} / \mathbf{e}; \mathbf{N}_{\beta} / \mathbf{e})$
 $\in \wedge \mathbf{P}_{\alpha} \cup \widehat{\mathbf{P}}_{\beta}$: Exterior Algebra (anti-comm!)

② $= \sum_{F \subseteq E} ([\mathbf{N}_{\alpha} / F | \mathbf{P}_{\alpha}] \quad | \quad [\mathbf{N}_{\beta} / F | \widehat{\mathbf{P}}_{\beta}])$
 $= \sum_{F \subseteq E} [\mathbf{N}_{\alpha} / F | \mathbf{P}_{\alpha}] \wedge [\mathbf{N}_{\beta} / F | \widehat{\mathbf{P}}_{\beta}]$

Like $|\text{Graph Laplacian}| = |I \ I^t| = \sum_{F \subseteq E} |I(F)|^2$
 $= \sum_{T: \text{span. tree}} 1$ (Cauchy-Binet expansion)

1. Distinguish matroid element “ports” associated with electric or elastic system parameter and solution variables of interest. (All vars are paired: (voltage, current), (force, displacement), etc. One gets a pair of submodels with dual matroids in elementary situations; not duals otherwise.)
2. Exterior algebra forms of deletion and contraction of a non-port yield a pair of simpler systems.
3. Cancelling non-port elements with a kind of bilinear pairing yields the parameter/solution variables of interest relation, in the form of an **exterior algebra valued function** of systems, that **is a Tutte function** (when the minor and direct sum operations are sign-consistent).

With the suitable incidence matrix form, we get the all-minors matrix tree theorem; but all the minors are packed into **one exterior algebra object** that is a Tutte function of graphs.
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