

1

a. b. c. d. e.

2

a. b. c. d. e. f.

3

a. $\delta_1(t) = \lim_{a \rightarrow \infty} \frac{a}{2} e^{-a|t|}$

Property #1

$$\delta_1(t - t_0) = \lim_{a \rightarrow \infty} \frac{a}{2} e^{-a|t - t_0|}$$

Let's explore two cases $|t - t_0| = 0$ and $|t - t_0| = m \neq 0$, in case 1, the limit takes the value of $\lim_{a \rightarrow \infty} \frac{a}{2} e^0 = \lim_{a \rightarrow \infty} \frac{a}{2}$ which is clearly non-zero.

When $|t - t_0| = m \neq 0$ the limit is:

$$\lim_{a \rightarrow \infty} \frac{a}{2} e^{-am} = \lim_{a \rightarrow \infty} \frac{\frac{a}{2}}{e^{am}}$$

using L'Hopital's rule we get that:

$$= \lim_{a \rightarrow \infty} \frac{\frac{1}{2}}{me^{am}} = 0$$

Property #2

$$\begin{aligned} \lim_{a \rightarrow \infty} \int_{-\infty}^{\infty} \frac{a}{2} e^{-a|t|} dt &= \lim_{a \rightarrow \infty} \frac{a}{2} 2 \int_0^{\infty} e^{-at} dt \\ &= \lim_{a \rightarrow \infty} \frac{1}{a} e^{-at} \Big|_0^{\infty} = \lim_{a \rightarrow \infty} \frac{a}{a} \\ &= 1 \end{aligned}$$

b. $\delta_2(t) = \lim_{a \rightarrow 0} \frac{1}{2a} \Pi\left(\frac{t-a}{2a}\right)$

Property #1

$$\delta_2(t - t_0) = \lim_{a \rightarrow 0} \frac{1}{2a} \Pi\left(\frac{(t - t_0) - a}{2a}\right)$$

By using standard shifting and scaling we can write $\Pi\left(\frac{(t-t_0)-a}{2a}\right)$ piecewise:

$$\Pi\left(\frac{(t - t_0) - a}{2a}\right) = \begin{cases} 1 & : t \in [t_0, t_0 + 2a] \\ 0 & : t \notin [t_0, t_0 + 2a] \end{cases}$$

As the $a \rightarrow 0$ the only value that stays consistently non-0 is $t = t_0$ all the other values become zero.

Property #2

$$\begin{aligned} \lim_{a \rightarrow 0} \int_{-\infty}^{\infty} \frac{1}{2a} \Pi\left(\frac{t-a}{2a}\right) dt &= \lim_{a \rightarrow 0} \frac{1}{2a} \int_0^{2a} dt = \lim_{a \rightarrow 0} \frac{2a}{2a} \\ &= 1 \end{aligned}$$

4

a.

$$z \times z^* = (re^{j\theta})(re^{-j\theta}) = r^2 e^{j(\theta-\theta)} = r^2$$

b.

$$\frac{z}{z^*} = \frac{re^{j\theta}}{re^{-j\theta}} = \frac{r}{r} e^{j(\theta+\theta)} = e^{2j\theta}$$

c.

$$\begin{aligned} (z_1 z_2)^* &= (r_1 e^{j\theta_1} r_2 e^{j\theta_2})^* = (r_1 r_2 e^{-j(\theta_1+\theta_2)})^* = r_1 r_2 e^{-j(\theta_1+\theta_2)} \\ z_1^* z_2^* &= (r_1 e^{j\theta_1})^* (r_2 e^{j\theta_2})^* = (r_1 e^{-j\theta_1}) (r_2 e^{-j\theta_2}) = r_1 r_2 e^{-j(\theta_1+\theta_2)} \\ &\rightarrow z_1^* z_2^* = (z_1 z_2)^* \end{aligned}$$

d.

$$\begin{aligned} \left(\frac{z_1}{z_2}\right)^* &= \left(\frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}}\right)^* = \left(\frac{r_1}{r_2} e^{j(\theta_1-\theta_2)}\right)^* = \frac{r_1}{r_2} e^{j(\theta_2-\theta_1)} \\ \frac{z_1^*}{z_2^*} &= \frac{r_1 e^{-j\theta_1}}{r_2 e^{-j\theta_2}} = \frac{r_1}{r_2} e^{j(\theta_2-\theta_1)} \\ &\rightarrow \frac{z_1^*}{z_2^*} = \left(\frac{z_1}{z_2}\right)^* \end{aligned}$$

5

a. $u(t)$

$$\begin{aligned} Odd &= \frac{u(t) - u(-t)}{2} \\ Even &= \frac{u(t) + u(-t)}{2} = \frac{1}{2} \end{aligned}$$

b. $\cos(2\pi t)u(t)$

$$\begin{aligned} Odd &= \frac{\cos(2\pi t)u(t) - \cos(-2\pi t)u(-t)}{2} = \frac{\cos(2\pi t)}{2}(u(t) - u(-t)) \\ Even &= \frac{\cos(2\pi t)u(t) + \cos(-2\pi t)u(-t)}{2} = \frac{\cos(2\pi t)}{2}(u(t) + u(-t)) \\ &= \frac{\cos(2\pi t)}{2} \end{aligned}$$

6

a. $h(t) = e^t u(t)$ is not *BIBO* stable system, an example of a bounded input resulting in an unbound output is $x(t) = u(t)$ which results in the output

$$\begin{aligned} y(t) &= h(t) * x(t) = \int_{-\infty}^{\infty} u(\tau) e^{t-\tau} u(t-\tau) d\tau \\ &= \int_0^t e^{t-\tau} d\tau = e^t \int_0^t e^{-\tau} d\tau = \\ &= \boxed{e^t(1 - e^{-t})} \end{aligned}$$

which is an unbound function.

b. $h(t) = (t-1)^2 e^{1-t} u(t)$ The system is *BIBO* stable since:

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} |u(t) e^{1-t} (t-1)^2 dt|$$

$$= e$$

which is bound, hence the integral is bound.

c. $h[n] = u[n-4]$ is not *BIBO* stable system, an example of a bounded input resulting in an unbound output is $x(t) = u(t)$ which results in the output:

$$y[t] = h[t] * x[t] = \sum_{k=-\infty}^{\infty} u[k] u[n-k-4] = \sum_{k=0}^{n-4} 1$$

$$= n-4+1 = \boxed{n-3}$$

which is an unbound function. Therefore, the system is not *BIBO* stable

d. $\cos[2\pi n] u[n]$ is not *BIBO* stable system, an example of a bounded input resulting in an unbound output is $x[n] = u[n]$ which results in the output

$$y[t] = h[t] * x[t] = \sum_{k=-\infty}^{\infty} u[k] u[n-k] = \sum_{k=0}^n 1 = \boxed{n+1}$$

which is clearly unbound.

e. $\sum_{n=-\infty}^{\infty} \delta(t-2n)$ is not *BIBO* stable system, an example of a bounded input resulting in an unbound output is $x(t) = u(t)$ which results in the output

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} u(t-\tau) \sum_{n=-\infty}^{\infty} \delta(\tau-2n) d\tau$$

$$= \int_{-\infty}^t \sum_{n=-\infty}^{\infty} \delta(\tau-2n) d\tau = \sum_{n=-\infty}^{\infty} \int_{-\infty}^t \delta(\tau-2n) d\tau$$

$$= \sum_{n=-\infty}^{\lfloor \frac{t}{2} \rfloor} 1 = \boxed{\infty}$$

which is clearly unbound

7

a. $x(t) = \Pi(t-1); h(t) = r(t)$

$$\begin{aligned}
 y(t) &= x(t) * h(t) = \int_{-\infty}^{\infty} r(\tau) \Pi(t-\tau-1) d\tau \\
 &= \int_{(t-1)-\frac{1}{2}}^{(t-1)+\frac{1}{2}} r(\tau) d\tau = \int_{t-\frac{3}{2}}^{t-\frac{1}{2}} r(\tau) d\tau \\
 &= \begin{cases} \int_{t-\frac{3}{2}}^{t-\frac{1}{2}} \tau d\tau & : t \geq \frac{3}{2} \\ \int_0^{t-\frac{1}{2}} \tau d\tau & : t \in [\frac{1}{2}, \frac{3}{2}) \\ 0 & : t < \frac{1}{2} \end{cases} \\
 &= \boxed{\begin{cases} \frac{(t-\frac{1}{2})^2 - (t-\frac{3}{2})^2}{2} & : t \geq \frac{3}{2} \\ \frac{(t-\frac{1}{2})^2}{2} & : t \in [\frac{1}{2}, \frac{3}{2}) \\ 0 & : t < \frac{1}{2} \end{cases}}
 \end{aligned}$$

b. $x(t) = e^{-t}u(t); h(t) = \Pi(t - \frac{1}{2})$

$$\begin{aligned}
 y(t) &= x(t) * h(t) = \int_{-\infty}^{\infty} e^{-\tau} u(\tau) \Pi(t-\tau-\frac{1}{2}) d\tau \\
 &= \int_{(t-\frac{1}{2})-\frac{1}{2}}^{(t-\frac{1}{2})+\frac{1}{2}} e^{\tau} u(\tau) d\tau = \int_{t-1}^t e^{\tau} u(\tau) d\tau \\
 &= \begin{cases} \int_t^{t-1} e^{\tau} d\tau & : t \geq 1 \\ \int_0^t e^{\tau} d\tau & : t \in [0, 1) \\ 0 & : t < 0 \end{cases} \\
 &= \boxed{\begin{cases} e^t - e^{t-1} & : t \geq 1 \\ e^t & : t \in [0, 1) \\ 0 & : t < 0 \end{cases}}
 \end{aligned}$$

c. $x(t) = \sum_{n=-\infty}^{\infty} \delta(t - \frac{1}{2} - n); h(t) = \Pi(t) \sin(2\pi t)$

$$\begin{aligned}
 y(t) &= x(t) * h(t) = \sum_{n=-\infty}^{\infty} \delta(t - \frac{1}{2} - n) * h(t) = \sum_{n=-\infty}^{\infty} h(t - \frac{1}{2} - n) = \\
 &\sum_{n=-\infty}^{\infty} \Pi(t - \frac{1}{2} - n) \sin(2\pi(t - \frac{1}{2} - n)) = \sum_{n=-\infty}^{\infty} \Pi(t - \frac{1}{2} - n) \sin(2\pi t - \pi - 2n\pi) = \\
 &\quad - \sum_{n=-\infty}^{\infty} \Pi(t - \frac{1}{2} - n) \sin(2\pi t) = \\
 &\quad \boxed{\sin(2\pi t)}
 \end{aligned}$$