AP Calculus CD: An approach via intuition.

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Created in preparation for S'26.

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0 Building Intution for the Complex Plane.

0.1 Part 1: Basics and Motifs.

In the wondrous land of $\mathbb{C}$, one may witness miracles even beyond that of the naïve high school student's imagination; for a few examples, a function "differentiable once" is actually "differentiable infinitely many times over", there is a direct formula between said derivatives via integrals, and functions can be identified simply by looking at a few (in)equalities and modified forms. Real analysis should have (if taken properly) all but shattered these hopes, but this course will gaslight, gatekeep, and girlboss you into these (bad) habits again, so how about we solulu by delulu?

Let us now review some basic properties of complex numbers; first, any number has a rectangular form $a + bi$ and a polar form $r\angle\theta$, and given an interval of length 2π of real numbers, the fundamental equivalence

$$a + bi = r\angle\theta = r(c_\theta + is_\theta)$$

is a bijection, so a radius and argument of a complex number are uniquely determined. Curves and regions are defined (per usual) as vanishing loci of some expression of z and $\bar{z}$; in particular, the notations $\mathbb{D}_{c,r}$ and $\partial_{c,r}$ are used to represent an open disk of radius r centred at c and its boundary respectively.
Since this is ANALYSIS, one expects some differentiation to happen; welp, here it is.

New operators.

Given ∂_x and ∂_y, one defines the operators

$$\partial_z \overset{\text{def}}{=} \frac{1}{2}(\partial_x - i\partial_y) \text{ and } \partial_{\bar{z}} \overset{\text{def}}{=} \frac{1}{2}(\partial_x + i\partial_y),$$

and functions termed "holomorphic on $U \overset{\text{open}}{\subset} \mathbb{C}$" are those that satisfy the following condition.

$$\forall w \in U, [\partial_{\bar{z}} f]_w = 0.$$

Further again, now choosing to consider second-order $\mathbb{C}$-derivatives, we can define a new class of functions.

har_U.

First, for $\mathbb{R}$-valued $f \in C^2_U$ for $U \overset{\text{op.}}{\subset} \mathbb{C}$,

$$f \in \mathsf{har}_U \overset{\text{def}}{\Leftrightarrow} (\partial_x^2 + \partial_y^2) f = 0.$$

A complex-valued $h = \mathsf{re}\, h + i\, \mathsf{im}\, h$ is har_U precisely if both $\mathsf{re}\, h$ and $\mathsf{im}\, h$ are har_U as in the above definition.

Using our new operators as well, one can state the harmonic condition solely using ∂_z and $\partial_{\bar{z}}$ alone.

$$\partial_x^2 + \partial_y^2 = \partial_x^2 - (i\partial_y)^2 = 4 \times \frac{\partial_x + i\partial_y}{2} \cdot \frac{\partial_x - i\partial_y}{2i} = 4\partial_z\partial_{\bar{z}},$$

noting that ∂_z and $\partial_{\bar{z}}$ (thankfully) commute.

Next, in order to treat $\mathbb{C}$ as a closed surface-type object, one may attach a so-named point at infinity $[\infty]$ and denote this compactification by $\mathbb{C}_\infty$. Only one point at infinity is attached due to intuiting $\mathbb{C}$ via polar coordinates; trying to shove in $-\infty$ into the same surface does NOT bode well with limits (since one can also see ∞ as $\lim_{|\alpha|\to\infty} \alpha$ for α complex).
Continuing on with a more "flair" to $\mathbb{C}$, we note the following fact concerning $\mathsf{op}_{\mathbb{C}}$.

Open sets of $\mathbb{C}$.

Any open set $U \overset{\text{open}}{\subset} \mathbb{C}_\infty$ is within the set $\left\langle \mathbb{D}_{w,r}, \mathbb{H}\angle\theta, {}^{\complement}\overline{\mathbb{D}_{w,r}} \middle| \cup_I, \cap_{<\infty} \right\rangle$.

For the minimalists, the object $\mathbb{H}$ turns out to be superfluous via the relation

$$\mathbb{H} = \bigcup_{n\in\mathbb{N}} \mathbb{D}_{ni,n}$$

by noting that all points above the real axis (and NONE at or below) lie in AT LEAST one of the open disks. Of course, corollary-wise, open polygons are possible with finite intersections of $\mathbb{H}\angle\theta_1$. Recalling that openness is PRESERVED under arbitrary union (while simple/path-connectedness may not be), this side quest suddenly doesn't seem like a side quest anymore, and it isn't; topological properties like these are central to your further study, and $\mathbb{C}$'s structures serve as a wonderful jumping point.

0.2 Part 2: Special maps and their effects on planar regions.

One may recall from BC that many well-behaved functions have appropriate series expansions of the form

$$\sum_{n\geqq N} a_n(x-c)^n$$

for c the real center of expansion and a_n real coefficients. Swapping the "real" for "complex" still results in valid c-centred expansions; the demonstrations of this are via the ratio and root tests (also vestiges from your study of BC).
Again we pull from pre-collegiate maths for the next topic: linear-fractional transforms (LFTs). They are functions of the form

$$z \overset{f}{\mapsto} \frac{az+b}{cz+d} (a,b,c,d \in \mathbb{C}),$$

and the table below summarizes some special families within this large zoo.

Type	img.	Note.
Reciprocal.	$z^{-1} = \frac{1}{z}$	Brings outside $\leftrightarrow$ inside.
Cayley (K)	$\frac{z-i}{z+i}$	$\mathbb{R} \mapsto \partial_{0,1}$.
Rotation	$z \mapsto \rho z$	$\lvert\rho\rvert = 1$; rotation counterclockwise by $\arg\rho$.
Blaschke (0-forcing)	$z \overset{\mathrm{bla}_u}{\mapsto} \frac{z-u}{1-\bar{u}z}$	For $u \in \mathbb{C}$, maps a value to/from zero.

It is swiftly noted that LFTs as a set are closed under composition (awful algebra). After some other sorts of algebraic manipulation,it is easily shown that LFTs break down <u>CONSISTENTLY</u> into the following conformal sequence:

$$z \overset{\text{in}}{\to} \left([z - w_1], \left[\frac{1}{z}\right], [z \times w_2], [z + w_3] \right) \overset{\text{out}}{\to} \text{your desired LFT.}$$

The second transform in particular represents an algebraic "flip" or inversion ACROSS $\partial_{0,1}$, and this becomes again a central topic in later sections (cof_U, for example). This completes the treatise on fractional functions (at least, for LFTs) for the moment.
Now, we aim to look at (possibly) transcendental maps; that is, any map that doesn't fall into meromorphic territory. We again summon a table to our visage.

Type	img.	Note.
Exponential.	$\exp z$	Preserves radial symmetry and linearly stretches angles (so $\theta \mapsto 2\theta$is possible with careful composition).
Logarithmic.	$\lg z$	Acts on $\mathbb{C}^\times$;
Sinusoidal.	$z \mapsto s_z$ OR $z \mapsto c_z$.	Arises from specific linear combos involving $\exp$.
Power.	$z \mapsto z^r$	Works for real r given a branch of $\lg$.

Note that due to $\mathbb{C}$'s geometry, these maps end up being more fundamental figurewise than the LFTs; it is also additionally noted that these maps do not necessarily preserve the number of finite borders (so maybe not arc-arc).In fact, we note that the following chain of implications stands for valid functions on some domain.

Valid operations on a region.

Let $U \subset \mathbb{C}$. The following chain of implications holds.

$$\text{Simply connected} \Rightarrow \text{I} \Leftrightarrow \text{P} \Rightarrow \text{L} \Rightarrow \text{S}.$$

One may generalize this procedure to create arbitrary arc-arc maps. We do so (again) in the cof_U section.

0.3 Part 3: More Laurent series review: Residues and singularities.

As compared to Taylor expansions that are infinite extensions (exponent-wise) of polynomials, Laurent expansions can be thought of as infinite extensions of rational functions. This solidifies their form as

$$\sum_{n\in\mathbb{Z}} a_n(z-w)^n \overset{\text{def}}{=} \sum_{j\in\mathbb{N}_1} \frac{a_j}{(z-w)^j} + \sum_{k\in\mathbb{N}_0} a_k(z-w)^k,$$

where the first series is also known as the <u>principal part</u> and the second is known as the <u>analytical part</u>.
We below take the singular point to be at $c \in \mathbb{C}$.

Type	coef.	lim.
Removable	$\forall j < 0, a_j = 0.$	$\exists L \in \mathbb{C}$ s.t. $\lvert f\rvert \xrightarrow{\mathbb{D}^{\times}_{c,\varepsilon}} L$ as $\varepsilon \searrow 0$
Pole	$\exists j < 0$ s.t. $\forall k < j$, $a_k = 0.$	$\exists N \in \mathbb{N}_1$ minimal s.t. $\lvert(z-c)^N f\rvert \xrightarrow{\mathbb{D}^{\times}_{c,\varepsilon}} r \in \mathbb{R}_{>0}$ ditto.
Essential	$\forall j \in \mathbb{Z}$, $\exists k < j$ s.t. $a_k \neq 0.$	$\exists \theta_1, \theta_2 \in [0, 2\pi)$ s.t. $\theta_1 \neq \theta_2$ and $\lim_{r\searrow 0} f_{r\angle\theta_1} \neq \lim_{r\searrow 0} f_{r\angle\theta_2}$.

In words, removable singularities are "holes" like in BC, poles are rational expressions over some linear factor (so CLEARING DENOMINATORS fixes the finiteness issue), and essential singularities are wacky (AT LEAST 2 directions do not give the same limit value).
We next move on to another derived quantity in the realm of complex functions;

Residues at a point.

For a function $f : \mathbb{C} \setminus (\alpha_n)_{\mathbb{N}}$, its "residue at w" for w finite is defined to be

$$\operatorname{res}_w f \overset{\text{def}}{=} \frac{1}{2\pi i}\int_\gamma \frac{f_z}{(z-w)^{n+1}}dz.$$

Stated differently, it is the coefficient a_{-1} in the a-centred expansion of f.
In the case of $w = \infty$, one instead has that

$$\operatorname{res}_\infty f = -\operatorname{res}_0 \frac{f_{z^{-1}}}{z^2}.$$

We note that the <u>CLOSED</u> contour γ must avoid the punched-out holes $(\alpha_n)_{\mathbb{N}}$; additionally we also note that the number of holes punched out is AT MOST countable.
For a swift demonstration as to the soundness of $\operatorname{res}_\infty$, we compute as below.

$$-\frac{f_{z^{-1}}}{z^2} = z^{-2}\sum_{\mathbb{Z}} a_n \left(\frac{1}{z}\right)^n = \sum_{\mathbb{Z}} a_n z^{-n-2} \xrightarrow{\therefore} n = -n-2, \text{ so } n = -1,$$

where the implication is looking for the fixed coefficients after the transform. As the fixed coefficient is s.t. $n = -1$, the residue definition is reasonable.
One may also view the minus sign appended to the quantity as a "reveral of orientation"; after all, $\mathbb{C}_\infty$ is really the Riemann sphere, and all "finite counterclockwise" contours appear clockwise to ∞ (so adjusting this is necessary).

1 Computing Integrals: Fitting to types.

1.1 Complex Integration

Integration is introduced early in Math 220 with the path integral of a function, which looks like

$$\int_\gamma f(z)dz,$$

where $\gamma : [a, b] \to \mathbb{C}$ is a piecewise continuously differentiable curve. If f has an antiderivative in a neighborhood of the region contained in γ, then $f(z, \bar{z}) = f(z)$, then we can easily evaluate integrals through the Fundamental Theorem of Calculus:

$$F(\gamma(b)) - F(\gamma(a)) = \int_\gamma f(z)dz.$$

However, the class and qualifying exam will not ask you to find antiderivatives to solve integrals (of which we can utilize results in $\mathbb{R}$ to assist). Rather, it is more interesting to compute integrals of functions which do not have an easily determined antiderivative. The two techniques that accomplish this are curve parameterizations and the residue theorem.

1.1.1 Antiderivatives

1.1.2 Parameterization

If our function is not holomorphic, then the straightforward means of determining a path integral is to introduce a parameterization. These will often either be a circle of radius R:

$$\gamma_{\text{circle}} = Re^{i\theta}, \quad 0 \le \theta \le 2\pi,$$

or line segment on $[a, b]$:

$$\gamma_{\text{segment}} = a(1-t) + bt, \quad 0 \le t \le 1.$$

Examples

$\int_{\partial D(0,1)} \bar{z}\, dz$	simplest nontrivial example; no antiderivative exists

1.1.3 Residue theorem

For more sophisticated contours, explicitly determining the integral can become difficult. In the case that a function is analytic except at finitely many points, we can fortunately appeal to the following powerful theorem:

Residue Theorem

Suppose that $U \subseteq \mathbb{C}$ is a simply connected open set in $\mathbb{C}$, and that $P_1, \ldots, P_n$ are distinct points of U. Suppose that $f : U \setminus \{P_1, \ldots, P_n\} \to \mathbb{C}$ is a holomorphic function and γ is a closed, piecewise C^1 curve in $U \setminus \{P_1, \ldots, P_n\}$. Then

$$\oint_\gamma f dz = 2\pi i \sum_{i=1}^{n} \operatorname{Res}_f(z_i) \cdot \operatorname{Ind}_\gamma(z_i),$$

where $\operatorname{Ind}_\gamma z_i = (z_i)_\gamma \stackrel{\text{def}}{=} (2\pi i)^{-1} \int_\gamma \frac{dw}{w - z_i}$.

In other words, the integral is the weighted sum of the residues of f by the index of γ around the singularities.

Examples

$\oint_{\partial D(0,1)} \bar{z}\ dz$	Example 1
$\int_{\partial D(0,1)} \bar{z}\ dz$	abc...

1.2 Real integrals with complex technology

The most interesting and frequent applications of the residue theorem compute *real integrals*. How could this work? Observe that some complex integrals have contours which align with the real axis, and this parameterized region can be thought of as a real integral. Since the residue theorem is a convenient way to evaluate contour integrals, it may be possible to evaluate (a subset of) real integrals if the remaining complex integral is sufficiently simple.

As a matter of order, the candidate real integral is given first instead of a general complex integral. Thus we have a kind of inverse problem: how do we turn a real integral into a sufficiently simple complex integral? The task breaks down into a few steps:

1. *Complexify the integral*: Turn the real integrand $f(x)$ into a complex integrand $g(z)$ through some mapping. The default "complexification" is the assignment $x \mapsto z$ (that is, $g(z) = f(z)$).

2. *Construct a contour*: Make a closed curve via the union of line segments and circular arcs. Some line segments will necessarily lie on the real axis to correspond with the real integral.

3. *Evaluate the contour integral*: The line segments on the real axis correspond to a real integral, and the other contours should either be a) multiples of the real integral, b) simple to compute, or c) zero by the Jordan Lemma. This is your Left-Hand Side (LHS)

4. *Calculate residues*: The constructed contour is closed, so evaluate the residues (and indices) of the singularities contained in the contour. This is your Right-Hand Side (RHS)

5. *Apply Residue Theorem*: By the residue theorem, LHS = RHS. Algebra should reveal a solution to the original, real integral.

The steps above are good initial guide, but there are some edge cases to identify. First, the naive map $x \mapsto z$ may not be a viable complexification if the resulting contour integral cannot use Jordan's Lemma. This can occur with quotients where $\sin z$ or $\cos z$ are in the numerator. Second, the complexification can result in a holomorphic function which has a branch cut, such as $\ln(z)$ and $\sqrt{z}$. This necessarily influences the kinds of contours you can construct.

Examples

$\int_0^\infty \frac{1}{1+x^4}dx$	simplest nontrivial example
$\int_0^\infty \frac{\cos x}{1+x^4}dx$	naive complexification does not work; use $\Re(e^{iz})$ instead.
$\int_0^\infty \frac{\ln x}{1+x^4}dx$	complexification requires a branch cut

1.3 Series with complex technology

Some series can also be evaluated using the residue theorem (though these questions are less frequently asked for the exam).

1.4 Part 2: Meromorphic.

We make use of the notation f_0^{-1} to denote the MULTISET (which includes multiplicity) of zeros of a function f, and we extend this definition to other images as well (k-preimages and such).
Next, we address an extension of "rational" functions from previous classes; a brief definition is given below.

mer_U.

Let $f, g \in \mathsf{hol}_U$ for a common region $U \subset \mathbb{C}$. For $g \neq 0$, any quotient $\frac{f}{g}$ defines a <u>meromorphic</u> function.

Of note is the fact that holomorphic functions cannot possess limit points for their zeros (if so, the function collapses to identically zero). This means that, outside of sufficiently small disks, the holomorphic region of a function is somewhat characteristic. In fact, thanks to the residue theorem, we have the following "structure theorem" for meromorphic functions.

Structure Theorem.

Let $f \in \mathsf{mer}_U$ for $U \subset \mathbb{C}$. There exists $(\alpha_k)_{\mathbb{N}_1}$ and $(\beta_l)_{\mathbb{N}_1}$ (and g) s.t.

$$f = g + \sum_{j \in \mathbb{N}_1} \mathrm{ord}_{\alpha_j} f + \sum_{k \in \mathbb{N}_1} \frac{\mathrm{res}_{\beta_j} f}{z - \beta_k}.$$

In other words, all meromorphic functions are characterized by their zeros and singularities (up to a scalar factor?)

偏角の原理や凡ゆる積分の奴。Let us now consider a "formula" sure to show up in later sections; while it may take until the "zeros" chapter to see its true powers, there is no shame in working through it as an initial exposure here.

Argument principle for mer_U.

Let $f \in \mathsf{mer}_U$ and $\gamma \overline{\in} U$ be a (reasonable?) cycle disjoint to $f_0^{-1} \cup f_\infty^{-1}$. Then

$$(2\pi i)^{-1} \int_\gamma \frac{f'}{f} = \sum_{\alpha \in f_0^{-1}} \alpha_\gamma \mathrm{ord}_\alpha f - \sum_{\beta \in f_\infty^{-1}} \beta_\gamma \mathrm{ord}_\beta \frac{1}{f}.$$

One can simplify this case to obtain useful corollaries for later application; actually, there's a wonderful (more complicated) corollary that multiplies the interior logarithmic derivative by a holomorphic function. This version is listed here.

Perturbed Argument.

Under the same conditions as above, for $h \in \mathsf{hol}_U$,

$$(2\pi i)^{-1} \int_\gamma \frac{hf'}{f} = \sum_{\alpha \in f_0^{-1}} h_\alpha \alpha_\gamma - \sum_{\beta \in f_\infty 1} h_\beta \beta_\gamma.$$

In fact, when h is precisely a monomial z^k for $k \in \mathbb{Z}$, one obtains the relation

$$(2\pi i)^{-1} \int_\gamma z^k \frac{z^k f'}{f} = z_1^k + z_2^k + \cdots + z_n^k$$

, which can be viewed as the "Viete's analogue" in CD. Let us now use this in a small practice.

Lebl(5.4.5).

Let f cubic with $f_0 = 1$ satisfy

$$(2\pi i)^{-1}\int_{\partial_{2i,2}} \frac{zf'}{f} = 1+2i, (2\pi i)^{-1}\int_{\partial_{-2i,2}} \frac{zf'}{f} = -i, (2\pi i)^{-1}\int_{\partial_{5,1}} \frac{zf'}{f} = 5.$$

Determine f completely, and assume that f_0^{-1} does not intersect the contours.

2 The treasures of cof_U.

Your basic (holy grail) is the following principle:

The number of arcs bounding a region is an invariant under LFTs.

We also note that we have the following.

Power functions $z \mapsto z^q$ for rational q expand angles linearly and blow radii polynomially.

The exponent being relatively nice is due to existence of n^{th} roots (so "branch of lg" hijinks); keep it simple, stupid.

2.1 A callback.

Please recall section 0.2; there, we gave the basic concept of a conformal map (biholomorphic and non-collapsing). Our standard reference will be that of

$$\mathsf{cof}_{\mathbb{D}} = \left\{\rho \cdot \mathrm{bla}_u z \middle| |\rho| = 1, |u| < 1\right\} = \partial_{0,1}\mathrm{bla}_{\mathbb{D}}$$

viewed as a set (proof left to reader). Note that this translates to, up to a final rotation, just Blaschke factors applied in sequence, which hopefully allows their importance to resonate. However, an even simpler (possibly more ubiquitous) set of conditions exists.

Schwartz Lemma

If $f \in \mathsf{hol}_{\mathbb{D}\to\mathbb{D}}$ fixes the origin, then $|f_0'| < 1$ and $|f_z| \leqq |z|$, and a rotation is the unavoidable conclusion when some $w \in \mathbb{D}^\times$ is s.t. $|f_w| = |w|$.

In other words, point-rotation forces diskwise rotation.
We now give a table compiling the conformal families of common regions. Note that $\mathsf{cof}_{\mathbb{D}}$ was already given above.

cof_U for standard regions.

$\mathsf{cof}_{\mathbb{C}} = \{az + b | a \neq 0, b \in \mathbb{C}\}$
$\mathsf{cof}_{\mathbb{C}_\infty} = \left\{\frac{az+b}{cz+d} | ad \neq bc\right\}$

2.2 Zipping up cuts.

Many examples will feature regions slit by arc-type cuts; many a problem will then be to "get rid of" said cuts. There are two wildly different approaches to this.

[1] Straighten out the "cut" into an arc.
[2] Erase the arc.

The first can usually be done via an appropriate LFT; map a point to ∞ and another to 0 if you wish to obtain a ray (at minimum) through the origin. The third point (for unique arc determination) gives its direction. The latter, on the other hand, is demonstrable via the mapping $z \mapsto \sqrt{z^2+\alpha}$ for α the endpoint for the cut $\overline{0\alpha}$.
Noting that arcs are uniquely determined by three points (including ∞), calculations can be made. One may also utilize preservation of the cross ratio

$$[[w_1, w_2, w_3, w_4]] = \frac{w_4 - w_3}{w_1 - w_4} \cdot \frac{w_2 - w_1}{w_3 - w_2}$$

for LFTS to directly calculate (which is advised against, since this tends to time crunch). Let us tackle a quick example for the sake of it.Calculation.Give the LFT

2.3 Sometimes, you just can't.

Sometimes, there are regions that cannot be conformally mapped to each other; those that can be mapped to the open unit disk in particular have with them a special piece of data guaranteed by the following statement.

Riemann's mapping theorem

Any simply connected proper NONEMPTY subregion of $\mathbb{C}$ (called U here) has with it a UNIQUE associated map $\mathrm{rie}_U \in \mathsf{cof}_{U\to\mathbb{D}}$.

The power here, then, is that of case-reduction; instead of developing separate theory for special cases of h.s.c. U, we can map to a familiar region, apply results there, and transfer them back to U via the conjugation

$$\mathrm{rie}_U^{-1} \circ (\text{ whatever }) \circ \mathrm{rie}_U;$$

most of the work is then churning through calculations.
As is a common theme in maths, many an excellent strategy is to bring the situation to a home base; in $\mathbb{C}$, a common choice is $\mathbb{D}$, and this theorem is exactly that choice.

3 Identifying Functions: A Detective's Work.

3.1 Part 1: Things best intuited.

The following collection of facts are frequently used in identifying classes of functions.
(1) All $p \in \langle x, y, xy \rangle_{\mathbb{R}}$ are $\mathsf{har}_{\mathbb{C}}$.
(2) The logarithmic derivative $\frac{f'}{f}$ is closely related to locating zeros and poles of a meromorphic function f.
(3) Limit points for a holomorphic in an open region tend to force something by identity principle.
(4) Maximum and minimum moduli tend to pop out more by using either reciprocals or exp to force non-negativity or another (half-infinite interval) condition.
(5) Bounding the modulus of the difference between a function and its guess/approximations by 0 (so identity principle) is a good instinct.
(6) INEQUALITIES ARE YOUR CITY.
Inhaling these guidelines like fish in water allows for your thinking time to be spent writing; many of these principles are simple enough to develop live, but this distracts from the point collection. As usual, practice is key here.

3.2 Part 2: Forcing via mappings.

As with many techniques in maths, some of the "best" options are those that force one into known situations (or hope anyway). Said aforementioned maps are commonly conformal (hence the section placement), and AGAIN, specific maps to be used will be hinted at via peculiarities with U. Many times, this will be done via sweeping out singularities. Below are compiled results for map determinations from an input region to an output region.

cof_U for standard regions.

in	out	condition	result
$\partial_{0,1}$	$\partial_{0,1}$	hol	rational.
w	h	a	t

Somewhat related to this concept of "map-forcing" is the extension of a holomorphic function across a line; you likely know of this as (Schwarz's) reflection lemma. Let us review it below.

Reflection.

Let $V \overset{\text{op.}}{\subset} \mathbb{C}$ open be such that $V \cap (\text{real axis}) = (a,b)$, an open interval. Define $U \overset{\text{def}}{=} V \cap \mathbb{H}/$ and let $f \in \mathsf{hol}_U$ be s.t. $\mathsf{im}\, f_z \overset{U \ni z \to x}{\longrightarrow} 0$. The function f possesses a holomorphic extension g to $U \cup \bar{U}$ given below.

$$g_z = f_z \mathbf{1}_U + \overline{f_{\bar{z}}} \mathbf{1}_{\bar{U}} + (\lim_{U \ni z \to x} \mathsf{re}\, f_z) \mathbf{1}_{(a,b)}.$$

This chunk of a theorem, as gargantuan as it is, is rather slick in application; shoving the details to when necessary, we can (as the name entails) "reflect a hol. domain with spine ($\mathbb{R}$-portion)" to obtain a valid extension. When to use it is usually obvious, and we recommend the UCI qual problems (**HERE**) for immediate use.

3.3 Part 3: Bruh.

Ain't this section a real mood? This is the "little stinkers" part (also known as "no pingas"). Let's now get to the real stuff (although this is for a complex course). For example, take this question that can be deftly slain by Picard's theorems.

Frequent visitor.

Let $f : \mathbb{D} \overset{\mathsf{hol}}{\longrightarrow} \mathbb{D}$ be such that $a \mapsto a$ and $b \mapsto b$. Prove that f is the identity function.

Yes, most of this question is concerning a conformal sequence drawn from $\mathsf{cof}_{\mathbb{D}}$; however, the associated diagram (zushiki) and its navigation is the crux of this. Draw from both chapters 0 and 2 for this sake.

When it comes to identifying functions, a key insight is to note that "the difference between a function and its approximations <u>SHOULD</u> approach 0" ideally; that is, we make use of the <u>IDENTITY PRINCIPLE</u> yet again.

We present yet another useful upper bound for function identification here.

Cauchy-Riemann Estimates.

Let $f \in \mathsf{hol}_U$ for $p \in U \subset \mathsf{hol} f$. The following upper bound holds.

$$\left| \left[\partial_z^k f \right]_P \right| \leqq \frac{k}{r^k} \cdot |f|^{\infty}_{L_{\mathbb{D}_{p,r}}}.$$

Here now is a key problem directly taken from UCI quals that (evidently) makes use of this.

Template.

Determine the set

$$\left\{ f : \mathbb{H} \overset{\mathsf{hol}}{\longrightarrow} \mathbb{C} | i\sqrt{n} \mapsto n, |[\partial_z^n f]_i| \leqq 3 \right\}.$$

The set notation here straight up has a deriative-bound; in fact, a good guess to note here is that the most this function can "wiggle by" (AKA away from zero modulus) is e^{3z} (series expansion); therefore, by order arguments (seen next chapter), your required search need only happen over polynomials and exponentials with linear arguments.
If "pole-like" behaviours start to arise instead, note that some can multiply by a monomial product of the form

$$\prod_{\text{finite}} (z - w_j)$$

to work with a function that conceivably has fewer hiccups. If this causes raucous behaviours, Blaschke factors are more ideal, and said treatise is treated in previous sections.

3.4 Part 4: $\mathsf{hol}_{\mathbb{C}}$.

Entire functions have wonderful properties w.r.t. many aspects of $\mathbb{C}$-analysis; many of these, per nicety, are inequality-centred. There then is a tag-team effort between these and limits; usually,... (see UCI something something).
Let us observe one example below that encapsulates this theory.

Model example.

Give all entire functions f such that, for $n \in \mathbb{N}_1$, the relation below holds.

$$f \circ \frac{1}{\ell_{n+1}} \leqq \frac{1}{n}$$

Here,the key is to note that as $n \to \infty$,the point $(0, 0)$ becomes a sort of "limit point" for f; in fact, this then motivates proving that f is identically zero by identity principle.
Taking a sudden turn, we note that entire functions (being entire) possess a "growth factor" (called *order*) that serves as another differentiator between functions. A rough (and INCORRECT) way to summarize it is the inf of exponents s.t. growth of e^{order} outpaces the function given SUFFICIENTLY LARGE RADII.

3.5 Part 5: $\mathsf{har}_{\mathbb{C}}$.

Another (painfully) evident theme in 220 is that of the harmonic function, and some basic identifiers are given here.

Forcing har_U.

The following conditions specify a harmonic function.

A core function to study for har_U is the Poisson kernel; its definition is given below.

Poisson kernel.

Let $\rho \in \partial_{0,1}$ and $a \in \mathbb{D}$. The Poisson kernel parameterized by this is the function

$$z \overset{\mathrm{pkr}_3 z}{\mapsto} .$$

A warining to the reader is that this quantity may be defined in other texts with a scalar $(2\pi)^{-1}$ tacked on as well.
As is the theme of this course, classes of functions that pop out with particularly nice hidden properties are treasured; the harmonic functions are no different, as shown next.

Central values from edge values.

For $u \in \mathsf{har}_{\mathrm{nei}_{\overline{\mathbb{D}}}}$, for $a \in \mathbb{D}$, we have that

$$u_a = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{pkr}_\theta z u_{e^{i\theta}}.$$

In fact, the famous Harnack inequality (G&K, pg. 224) also assists in inequalities for har_U functions.

Harnack.

For $u : \mathrm{nei}_{\overline{\mathbb{D}}_{0,R}} \overset{\mathsf{har}}{\to} \mathbb{R}_{>0}$,the following holds for $|z| < R$.

$$\frac{R - |z|}{R + |z|} u_0 \leqq u_z \leqq \frac{R + |z|}{R - |z|} u_0.$$

One notes that the bounds are reciprocal (so ranges such as $[\frac{1}{7}, 7]$ and $[\frac{2}{3}, \frac{3}{2}]$ arise),and approximations (sensibly) become tighter given z closer to (???). In theme with this section, we also possess results concerning a boundary problem $\mathrm{dir}_{\partial_{0,1}} f$.

Dirichlet Problem for the Disc.

Let $f \in C^0_{\partial_{0,1}}$. The following function u is $\mathsf{har}_{\mathbb{D}}$ and $C^0_{\overline{\mathbb{D}}}$.

$$z \overset{u}{\mapsto} \frac{\mathbf{1}_{\mathbb{D}}}{2\pi} \int_0^{2\pi} f_{e^{it}} \cdot \frac{1 - |z|^2}{|z - e^{it}|^2} dt + f \mathbf{1}_{\partial_{0,1}}.$$

Another notation for this function will be $\mathrm{dir}_{\mathbb{D}}\, f$.

As the function's structure shows, one can "color in" a $\mathsf{har}\mathbb{D}$ function given minimal conditions on $\partial_{0,1}$, so a HARMONIC EXTENSION is possible. As per usual, the Poisson kernel worms its way in again; please note this for the sake of easy memorization.

4 Zeros

A central motivation for complex analysis comes through the Fundamental Theorem of Algebra, which states that the zeros of polynomials live in $\mathbb{C}$. Therefore, it should not be surprising that the qualifying exam frequently asks you to identify the zeros of holomorphic functions (which, as Greene and Krantz observe, can be viewed as generalized complex polynomials). The tools employed for this task include the Argument Principle, Rouché's theorem, and Jensen's Formula.

4.1 Rouché's Theorem

In general, finding zeros of a function is difficult. After all, this is why computational solvers exist (see bisection method, Newton's method, etc. in applied mathematics). So Rouché's theorem does not provide a way to explicitly find the zeros of an arbitrary holomorphic function, but instead relates the *existence* of zeros for a function f with another function g. This is interesting in a problem-solving context, where if we can pick a sufficiently nice comparison function g whose zeros are easily determined, then we can conclude information about the zeros for the "difficult" function f. The way f and g must be related in order to equate their zeros is given rigorously by Rouché's Theorem.

Rouché's Theorem

Let $U \subset \mathbb{C}$ be open, $\bar{D}(p, r) \subset U$ with $f, g : U \to \mathbb{C}$ and f, g holomorphic. If $\forall w \in \partial D(p, r)$ we have

$$|f(w) - g(w)| < |f(w)| + |g(w)|,$$

then the number of zeros for f and g in $D(p, r)$ agree (counting multiplicity).

That is, we only need to check the behavior of f and g at the boundary to verify if they have the same number of zeros in some disk $D(p, r)$. It is worthwhile to note that Rouché's theorem generalizes to other regions beyond disks (for instance, half circles, rectangles, etc.) and the weaker condition $|f(w) - g(w)| < |g(w)|$ is frequently sufficient to complete most problems.

4.1.1 Problem solving with Rouché

Here are some general tips:

(1) In practice, one does not (and cannot!) check every point $w \in D(p, r)$ satisfies the strict inequality $|f(w) - g(w)| < |f(w)| + |g(w)|$. Instead, the following sufficient condition is checked:

$$\max_{w \in \partial D(p,w)} |f(w) - g(w)| < \min_{w \in \partial D(p,w)} \Big[|f(w)| + |g(w)|\Big].$$

(2) The quantities in (1)

$$\max_{w \in \partial D(p,w)} |f(w) - g(w)| \quad \text{and} \quad \min_{w \in \partial D(p,w)} \Big[|f(w)| + |g(w)|\Big]$$

are usually estimated by the triangle inequality.

(3) Frequently the weaker inequality

$$\max_{w\in\partial D(p,w)} |f(w) - g(w)| < \min_{w\in\partial D(p,w)} |g(w)|.$$

can be satisfied to complete the zeros argument.

(4) If the inequality in (1) does not hold, it is often violated at only finitely many points $\{z_1, \ldots, z_n\}$. Identifying these points (often through the equality condition of the triangle-inequality), inserting them back into f and g, and verifying the inequality $|f(z_i)-g(z_i)| < |f(z_i)| + |g(z_i)|$ for each $i \in \{1, \ldots n\}$ is sufficient to conclude the number of zeros for f and g in $D(p, r)$ match.

(5) There are cases where f and g are zero on the boundary, so the hypothesis for Rouché will not be satisfied. There are two main responses:

   (a) Determine if for all ϵ sufficiently small (say, $\epsilon < \epsilon_0$) Rouché's Theorem holds for any proper subdisk $D(p, r - \epsilon)$. If true, then the zeros of f and g match on $D(P, r)$ when taking $\epsilon \to 0$.

   (b) Identify the common zeros of f and g and factor them from the original function f. Check to see if new functions satisfy Rouché's Theorem.

(6) Occasionally no viable comparison function g is known. If this is true and the original function f is a polynomial, it can be helpful to recall one of Vieta's formulae.

Vieta's Formula

Let P be a polynomial of degree n

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

and $\{r_i\}_{i=1}^n$ be the roots of P. Then

$$\prod_{i=1}^{n} r_i = (-1)^n \frac{a_0}{a_n}.$$

The formula's main utility comes by determining if all the roots of a polynomial are outside a given disk.

(7) If no other result is fruitful, it is technically possible to parameterize the boundary of the disk and find maxima and minima through calculus to verify the inequality in (1). This is a computationally intensive exercise, though, and is not recommended. At this point, you should probably just work on a different qual problem.

Examples

Evaluate the number of zeros in the given region for the following functions:

How many zeros of $3z^5 + 2z + 1$ on $D(0, 1)$?	simplest nontrivial example
How many zeros of on $\mathbb{H}$?	Zeros on a non-disk domain
Example 3.5	Case where $\lvert g(z)\rvert$ inequality is not sufficient
Example 3	sufficient max-min inequality not satisfied; check particular points.
Example 4	Problematic boundary (infinitely many violations?) Use ϵ-argument.
Example 5	Use parameterization
Example 6	Use Vieta's formula.

4.2 Jensen's Formula

There is another way that we can characterize zeros of holomorphic functions using results from investigating harmonic functions. The essential idea is just to exploit the mean value property, and the result is Jensen's Formula:

Jensen's Formula

Let $f : U \supset D(0, R) \to \mathbb{C}$ be holomorphic with $f(0) \neq 0$ and take $a_1, \ldots, a_n$ as the zeros of f in $\overline{D}(0, R)$. Then

$$\log|f(0)| + \sum_{k=1}^{n} \log\left|\frac{R}{a_k}\right| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|d\theta.$$

While this equation can be directly used, it is usually more helpful to use a corollary:

Jensen Formula Corollary

Let f be entire with $f(0) \neq 0$ and $n(R)$ be the number of zeros of f in $D(0, R)$. Then

$$n(R) \leq \frac{\log \max_{|z|=\alpha R} |f(z)| - \log|f(0)|}{\log \alpha}$$

with $\alpha > 1$.

The main utility of this expression is that it relates the number of zeros of a holomorphic function with its behavior at a boundary. Note the similarity and difference with Rouché. Rouché also gives information about zeros using information at the boundary, but this expression requires knowledge about the zeros of a comparison function to have any utility. Meanwhile, Jensen's Formula exchanges the equality conclusion from Rouché with an inequality conclusion that only depends on the function f itself (no comparison function g needed).

Given the differences between Jensen and Rouché, there are different kinds of qual questions which use Jensen instead of Rouché. These frequently give information about the *order* of a holomorphic function (definition provided below for convenience), and the task is to use this information to deduce facts about the function's zeros or identify the function itself.

Order of an Entire Function

An entire function f is of finite order λ for $r > R$ if

$$|f(z)| \leq e^{|z|^\lambda}, \quad |z| > R,$$

where

$$\lambda = \inf\{a : |f(z)| \leq e^{|z|^a}, a > 0\}.$$

Examples

Example 1	simplest nontrivial example
Example 2	a
Example 3	a

Argument Principle and Gauss-Lucas Theorem

Finally, we briefly mention the Argument Principle and the Gauss-Lucas Theorem. The former actually implies Rouché's Theorem, though Rouché is used more frequently. The latter is a nice but niche result that has come up a couple times on previous exams, which for thoroughness we present here.

Argument Principle (Weighted)

Let f, g be holomorphic on an open set $U \subseteq \mathbb{C}$ such that $\bar{D}(P, r) \subseteq U$. Assume that f has zeros at $z_1, z_2, \ldots, z_p \in D(P, r)$, with multiplicities $n_1, n_2, \ldots, n_p$, and no zeros lie on the boundary $\partial D(P, r)$. Then

$$\frac{1}{2\pi i} \oint_{\partial D(P,r)} g(z) \frac{f'(z)}{f(z)} dz = \sum_{j=1}^{p} n_j \cdot g(z_j).$$

Gauss-Lucas Theorem

If P is a polynomial with complex coefficients, then all zeros of P' belong to the convex hull of the set of zeros of P. That is, the zeros of P', denoted w_i, may be written as

$$w_i = \sum_{i=0}^{n} a_{ij} z_j,$$

where $\sum_{j=1}^{n} a_j = 1$.

Example

Summary

Here is a summary about the different zeros tools available and when it may be more/less applicable to use them:

Method	Question	Data
Rouché's Theorem	ABC	ABC
Jensen's Formula	ABC	ABC

Table 1: **Summary of methods for characterizing zeros**

4.3 Zeros, zeros, zeros

Yeah; this is the section that will stand out most, as shifting from function comparison to solution set comparison seems again to be a pit of quicksand. However, Rouche's theorems (both weak and strong) both require the creation of a "comparison function" to, well, compare your original one with, and the authors believe this to be sufficiently strong connections for the sake of this section.
We first list both Rouche's and defer the reader to appropriate practice problems after showing a few easy templates below.

Rouche.

(Weak) On a contour $\gamma \subset U \overset{\text{open}}{\subset} \mathbb{C}$, let $f, g \in \mathsf{hol}_U$ be such that $|f - g|_{L^\infty_\gamma} < |g|_{L^\infty_\gamma}$. Then $^{\#}\mathrm{sol}^{\times}_{\gamma^\circ} f = {}^{\#}\mathrm{sol}^{\times}_{\gamma^\circ} g$.

(Strong) Replace the inequality above with $|f - g|_{L^\infty_\gamma} < |f|_{L^\infty_\gamma} + |g|_{L^\infty_\gamma}$ and obtain the same conclusion.

Both versions are provided here for the same purpose as (say) both root and ratio tests; the stronger version works for more situations than the weak version, but the weak version

is easier to compute for (and visualize).
Key notes for using this would be to use a comparison function hinted at from the problem statement; for example, a polynomial whose coefficients are perturbed from tends to be a great choice (Sp'22), and far from the origin, exponential growth outpaces that of even the most aggressive polynomial.
A more powerful relation and an even more powerful bound arises when one starts to contemplate orders of entire functions; we list said allies now.

Jensen and sol-disk expansion.

Let $f \in \mathsf{hol}_{\mathrm{nei}_{\overline{\mathbb{D}}_{0,r}}}$ and $0 \not\mapsto 0$. If $\mathrm{sol}^{\times}_{\overline{\mathbb{D}}_{0,r}} f = \{a_1, a_2, \cdots, a_N\}$,the following relation holds.

$$\ell_{|f_0|} + \sum_{j=1}^{N} \ell_{|r/a_j|} = \frac{1}{2\pi} \int_0^{2\pi} \lg |f_{re^{i\theta}}|.$$

If $n : \mathsf{hol}_{\mathrm{nei}_{\overline{\mathbb{D}}_{0,r}}} \longrightarrow \mathbb{N}_0$ denotes the size of the solution set with respect to r, then one has the following bound for $a > 1$.

$$n_f \leqq \frac{\lg |f|_{L^{\infty}_{\partial_{0,ar}}} - \lg |f_0|}{\ell_a}$$

This second result becomes especially useful for questions that ask one to bound holomorphic growth based on order, since it allows for algebraic manipulation to (usually) an end contradiction. A hot example of this is presented now.

Said hot example.

Give all $f \in \mathsf{hol}_{\mathbb{C}}$ such that $k^{1/4} \mapsto k$ for k natural and $|f_z| \leqq e^{|z|}$ for $z \gg 0$.

Note that the entire function here has order condition via the exponential and zero behaviours via the other condition. As per usual, the key is to note that an easy "slow-growing" example exists in $z \overset{f}{\mapsto} z^4$, and identity principle combined with n_f applied to $g_z \overset{\text{def}}{=} \frac{f_z - z^4}{z^m}$ for m the order of 0 in $f - z^4$ gives uniqueness via forcing a contradiction outside the case where $g \overset{\mathsf{id}}{=} 0$.

5 Bird's eye view: Many tricks in sequence.

5.1 It's not as simple as direct application.

As the name of this section states, rarely is it that exclusively one tool is tested in a question. The weeb in you should..,

5.2 巧妙な手続き。

5.3 Normal Families (aptly named).

In this class, the concept of the normal family seems almost arbitrary and NOT normal; however, as you may metagame, the properties are splendid. As per usual, the QE rarely has one verifying "by definition" that something is normal, as easier theorems exist that turn the unwieldy condition to one more apt for computations.
Let us first review the definition of a normal family.

hol-Normal Family.

Let $\mathbb{F}$ be a collection of hol_U functions, U here being a common domain. We say that $\mathbb{F}$ is U-normal (notation nor_U if any $\mathbb{F}$-sequence $(f_n)_{n\in\mathbb{N}}$ admits a subsequence $(f_{n_k}) \subset (f_n)$ and $g \in \mathsf{hol}_U$ such that

$$f_{n_k} \xrightarrow{\forall U_{\text{sub}} \Subset U, n_k \to \infty} g \text{ uniformly.}$$

As predicted, the last limit condition defines "U-normal convergence" for said region, and as previously noted, the next theorem turns this checking to a computational game.

Montel.

A family $\mathbb{F}$ of hol_U-functions is normal precisely when $\mathbb{F}$ is U-locally bounded; that is, $\forall f \in \mathbb{F}$ can be locally bounded on any compact subset of U by a constant C_{sub} that may rely on said compact subset.

At UCI, this magic theorem is the one permissible key to cutting down computations. In reality, the choice of compact subset is historically a closed disk; as a warning, this means that borders ARE factored for, and "wiggle room via" ε is naught but forced.
Let us move to small examples that staircase problems in normal convergence.

Staircase.

(1) The $\mathsf{hol}_{\mathbb{D}}$-family $(nz)_{n\in\mathbb{N}}$ fails to be normal, since
$[f_n]_0 \overset{n\to\infty}{\longrightarrow} 0$ while $[f_n]_{1/2} = \frac{n}{2} \overset{n\to\infty}{\longrightarrow} \infty \neq 0$.

(2) Let $\mathbb{F}_2 = \left\{ f \in \mathsf{hol}_U \middle| \int_{\mathbb{D}} |f^2| dS < \infty \right\}$. $\mathbb{F}_2$ is normal as $|\mathrm{f}_z| \leqq \frac{|f|_{L^2_{\mathbb{D}}}}{|z, \mathbb{D}_{0,1}|_2 \pi^{1/2}}$.
Here, L^2-norm and distance $|\cdot,\cdot|_2$ are both locally bounded.

(3) Let $\mathbb{F}_3 = \left\{ f \in \mathsf{hol}_{\mathbb{D}} \middle| \sup_{r\in(0,1)} \left(\int_0^{2\pi} |f_{re^{it}}|^2 dt \right)^{1/2} < \infty \right\}$. We have $\mathbb{F}_3 \in \mathsf{nor}_{\mathbb{D}}$ by the following calculation.
$|\mathrm{f}_z|^2 \overset{\text{CIT!}}{=} \lim_{r\nearrow 1} \left| \int_{\partial 0,r} \frac{f_w^2}{w-z} dw \right| \leqq \frac{1}{2\pi} \lim_{r\nearrow 1} \int_0^{2\pi} \frac{|f_{re^{it}}|^2}{r-|z|} \cdot r dt \leqq \frac{|f|^2_{L_{\mathbb{D}}}}{2\pi(1-|z|)}$.

(4) Dirichlet space on Disk.

(5) Block space on Disk.

(6) (Whatever here.)

The tools that are used here are similar to analysis; namely, there is FTC for integrals ($f_{\text{end}} = f_{\text{start}} + \int_{\text{start}}^{\text{end}} f'$), the triangle inequality, and (other inequalities that come into play when considering...)

6 The real stuff.

6.1 Test Types 1: Inequalities.

[QUEST.] F'21(3): For $f \in \mathsf{hol}_{\mathbb{C}}$ with $|f_z| \leqq \exp|z|$, show that (n'th coef. of series) $\leqq \left(\frac{e}{n}\right)^n$ for $n \geqq 1$.
[THINK.] The solid statement one has for bounds on coefficients of series expansions comes from Cauchy's extimates; using those should be sufficient.
[SOLVE.] By Cauchy's estimates, one has the following on $D_{0,r}$.

$$|[\partial_z^n f]_0| \leqq \frac{n!}{r^n} \max_{\partial_{0,r}} |f_z| \leqq \frac{n!}{r^n} \exp r.$$

After rearrangement, $\underbrace{\frac{|[\partial_z^n f]_0|}{n!}}_{\text{need this.}} \leqq \frac{e^r}{r^n} \overset{r \ll n}{\leqq} \frac{e^n}{n^n}$, which completes the proof.$_{/\!/}$
[CHEAT.] There isn't really any here.

[QUEST.] F'21(8): For $f \in \mathsf{hol}_{D_{0,1}}$ with $|f_z| \leqq 1$ here, prove that $z \in D_{0,1}$ implies

$$\frac{|f_0| - |z|}{1 + |f_0||z|} \leqq |f_z| \leqq \frac{|f_0| + |z|}{1 - |f_0||z|}.$$

[THINK.] Noting that f is a hol. self-map of the unit disk, the best one can do is Schwartz. We must adjust for such. Additionally, one must note the symmetry of the required statement and use a similarly symmetric relation on $|\bullet|$.
[SOLVE.] Since $0 \overset{f}{\mapsto} 0$ is not guaranteed, we first define $\mathrm{bla}_{f_0} : z \mapsto \frac{z - f_0}{1 - \overline{f_0} z}$ and note that $\mathrm{bla} f_0 \circ f$ satisfies all conditions for Schwartz. Therefore, the composite function has an uupper bound; this is expressed as

$$\left| \frac{f_z - f_0}{1 - \overline{f_0} f_z} \right| \leqq |z|,$$

and algebraic manipulations after pertinent use of the inequalities

$$|f_z - f_0| \geqq |f_z| - |f_0|, |1 - \overline{f_0} f_z| \leqq 1 + |f_0||f_z|$$

then lead to the right-most $\leqq$ relation.
Now,for the other required $\leqq$ relation,one makes use of the logical equivalence

$$|a - b| \leqq |c| \Leftrightarrow |b| - |c| \leqq |a| \leqq |b| + |c|$$

to replace the - and + above with ...$_{/\!/}$
[CHEAT.] There isn't really any here.

[QUEST.]F'24(4): For $P(z) \in \mathbb{C}[z]$ of degree d s.t. $x \in [-1, 1]$ implies $|P_x| \leqq 1$, show that $|P_5| \leqq 10^d$. As necessary, feel free to consider the function $z \overset{f}{\mapsto} z^d P_{(z+z^{-1})/2}$ on the unit disk $D_{0,1}$.
[THINK.]From a simple side calculation, it is evident that the LFT in the inside of P maps $\mathbb{D} \to [\pm 1]$, so one can then use boundedness properties that are Schwartz-flavoured.
[SOLVE.]
[CHEAT.]The hint is our cheat.

6.2 Test Types 2: Normal Families.

Normal families of <u>meromorphic</u> functions are all about subsequences converging nicely. Thanks to Montel's theorem, this is logically equivalent to UBC (uniform boundedness w.r.t. compact subsets of the domain). I still make use of previously established notation in the discussions below.
[QUEST.] F'17(7): Let $\mathcal{F} \subset \mathsf{hol}_{\mathbb{D}}$ s.t. ${}^{\forall}f \in \mathcal{F}$, $|f_0|^2 + \int_{\mathbb{D}} |f'_z|^2 \leqq 1$. WTS: $\mathcal{F} \overset{\text{nom}}{\subset} \mathsf{hol}_{\mathbb{D}}$.
[THINK.] Note that holomorphic functions are closed under differentiation, have the center sub-integrand relation, and that FTC exists. Additionally note that all quantities in the given equality are non-negative and so must be less than one.
[SOLVE.]From the sub-integral inequality for hol, for $\alpha \in \mathbb{D}$,

$$|f'_\alpha|^2 \leqq \frac{1}{\pi\left(\frac{1-|\alpha|}{2}\right)^2} \int_{D_{\alpha, \frac{1-|\alpha|}{2}}} |f'_z|^2 \leqq \frac{1}{\pi\left(\frac{1-|\alpha|}{2}\right)^2} \int_{\mathbb{D}} |f'_z|^2 \leqq \frac{4}{\pi(1-|\alpha|)^2},$$

so we are successful in a point-dependent (so local) bound for our derivative. By FTC, $\triangle$, and ML, we note further that

$$|f_\alpha| = \left| f_0 + \int_{\overline{0\alpha}} f'_z \right| \leqq |f_0| + \int_{\overline{0\alpha}} |f'_z| \leqq 1 + |\alpha|\sqrt{2} \times \sqrt{\frac{4}{\pi(1-|\alpha|)^2}} = 1 + \frac{2|\alpha|\sqrt{2}}{\pi(1-|\alpha|)},$$

a finite bound on the closed arbitrary (compact) disk. By Montel's theorem, we have sufficient conditions to conclude $\mathcal{F} \overset{\text{nom}}{\subset} \mathsf{hol}_{\mathbb{D}}$.$_{/\!/}$
[CHEAT.]

[QUEST] F'23(5): For f entire and $f^{\circ n}$ the n^{th} composition of f (so 2 means $f \circ f$ and so on), verify/deny that $(f^{\circ n}|_{\mathbb{D}})_{n \in \mathbb{N}_0}$ is normal.
[THINK.] The problem is asking about boundedness w.r.t. $\mathbb{D}$ as $n \to \infty$; this eliminates all polynomial functions of degree $\geqq 2$ as COUNTERexamples (norms go to 0). Of course, $\lg_e$ functions are also out as $0 \notin$ domain; however, this does inspire a simple counterexample.
[SOLVE.] For our denial, we invoke the function $z \overset{f}{\mapsto} z + 1$; by easy calculations, f_n must send z to $z + n$. No matter which $z \in \mathbb{D}$ is chosen, we note that as $n \to \infty$, we have that $|f_n \circ z| \to \infty$, meaning that there is no UBC for all subregions of $\mathbb{D}$. By Montel, this is log. eq. to non-normality, so the denial is finished.$_{/\!/}$

6.3 Test Types 3: Zero-centric.

[QUEST.] F'21(2): $\mathrm{sol}_{\mathbb{C}} P \subset D_{0,1} \overset{(\exists)}{\Rightarrow} \mathrm{sol}_{\mathbb{C}}(\partial_z P) \subset D_{0,1}$.
SOLVE. Gauss-Lucas.

6.4 Test Types 4: Equalities.

The weak heading here is to accommodate the residue and integral calculations as well.
[QUEST.] F'17(6):

$$\int_{\mathbb{R}_{>0}} \frac{\ell_x^2}{1 + x^2} \overset{(\exists)}{=} \frac{\pi^3}{8}.$$

[THINK.] Note that lg always requires branch cuts emanating from 0; where one takes them is up to the solver, and this author chooses to use $\arg z \in [0, 2\pi)$ for the problem. Additionally, note that the singularities of the denominator are simple (derivative-integral conversion deals with all non-essential sing.'s derivatives, anyways). Keep the principle of choosing contours alive throughout the problem.
[SOLVE.] With $R \gg 1$ and $0 < \varepsilon, r \ll 1$, choose the contour

$$\gamma \overset{\text{def}}{=} (R\angle 0 \to \pi) \cup [-R, -r] \cup [r\angle\pi \to 0] \cup [r, R],$$

where all arrows indicate directions of traversal. The interior of this keyhole contour contains the order 1 singularities $\pm i$ of f (the integrand), and by swift calculation, $\text{res}_i\, f = \left[\frac{\ell_z^2}{z+i}\right]_i = \frac{(\ell_{|i|} + \frac{\pi}{2} i)^2}{2i}$ and $\text{res}_{-i}\, f._{/\!/}$
[CHEAT.]

[QUEST.]F'23(1). Along a semicircle with endpoints ± 1,compute $\int_\gamma z s_z$.
[THINK.] Computing the integral directly is ghoulish (nested exp and such); however, the integrand is $\mathsf{hol}_{\mathbb{C}}$. This makes contour deformation valid, so one may use a simpler STRAIGHT contour to evaluate.
[SOLVE.] By contour deformation, naming our integrand f, $\int_\gamma f = \int_{-1}^1 f$; this turns our integral into a real BC-type integral. By integrating by parts, we obtain that

$$\int_\gamma f = [s_z - z c_z]_{-1}^1 = 2s_1 - 2c_1,$$

so the integral itself has numerical value $2(s_1 - c_1)._{/\!/}$

[QUEST.]F'23(2) Let the domain $\Omega \subset \mathbb{C}$ be h.s.c. if for $\forall(f \in \mathsf{hol}_\Omega, \gamma \subset \Omega$ simple closed $\tilde{\mathrm{C}}^1_\Omega)$, $\int_\gamma f = 0$. Prove that $\{x + iy \in \mathbb{C} | y > x\}$ is h.s.c. while $\mathbb{D}^\times$ is not.
[THINK.]Note the lack of puctures for the former region and the one hole for $\mathbb{D}^\times$ at 0; the strategy (as usual) is to prove that all points admit a contour in the first part and that at LEAST one contour admits a non-zero integration for the second part.
[SOLVE.]For ease, we note that $\gamma \overset{\text{def}}{=} \partial_{0,\frac{1}{2}}$ and $z \overset{f}{\mapsto} \frac{1}{z}$ gives an integration of $2\pi i \neq 0$.
For the first part, for any appropriate contour as specified, there is a point interior to said contour along with a non-touching disk of radius r since the region here is open (tilted half-plane). By contour deformation, we must have that the integrals of all f hol. along either countours do not differ, and by CIF, the latter integral swiftly evaluates to 0. As all choices made are independent of choice of point, this region is indeed h.s.c. as required.$_{/\!/}$
[CHEAT.]The half-plane region here is conformal to $\mathbb{D}$ via the conformal sequence $\left([\times \frac{1-i}{\sqrt{2}}] = [\angle \frac{-\pi}{4}], [\frac{z-i}{z+i}]\right)$. Since h.s.c. is preserved through conformal transforms, this is a reduction proof.

[QUEST.] F'23(6): Let $f, g : \mathbb{C} \overset{\mathsf{har}}{\to} \mathbb{R}$ s.t. $f|_\mathbb{R} = 0$ and $g|_{\partial_{0,1}} = 0$. State whether either f or g is necessarily the zero f'n.
[THINK.] For f, we are given a restriction on values within the space $\mathbb{C}$; in contrast, for g, we have values on a CIRCULAR boundary. This makes Poisson integration available for g

in contrast to f.
[SOLVE.] For f, we conjure a counterexample via the function $z \mapsto xy$ for $x =$(real part of z) and $y =$(imag part of z). Since $\mathbb{R}$ has $y = 0$, the restriction condition holds, and since $[\partial_x^2 + \partial_y^2] \circ f = 0$, this function is also harmonic. Our counterexample is thus born.

However, for g,we make use of a kernel formula. Guided by the "boundary harmonic determines central value" principle for har f'ns, we have explicitly that

$$\mathbb{D} \ni z \overset{g}{\mapsto} (2\pi)^{-1} \int_0^{2\pi} \frac{1 - |z|^2}{|z - e^{it}|^2} dt,$$

and as formed, $a \overset{g}{\mapsto} 0$. By identification, ${}^{\exists}f \in \mathsf{hol}_{\mathbb{C}}$ s.t. $f = u + iv$, but on $\partial_{0,1}$, $f = iv$ results. By the relation $\partial_{\bar{z}} f = 0$ (expand into partials), we must have that v is identically Ki for real constant K. However,since this relation holds for all complex values and the real part of f is u, we obtain that $u \overset{\text{id.}}{=} 0$.$_{/\!/}$

S1 Worked Examples

S1.1 Building Intuition for the Complex Plane

S1.2 Computing Integrals: Fitting to Types

- Evaluate the integral
$$\int_0^\infty \frac{1}{1+x^4}dx.$$
Solution: Abcdefg...

S1.3 Zeros

S1.3.1 Rouché's Theorem

- Problem: How many zeros of $3z^5 - 2z + 1$ on $D(0,1)$?
Let $f(z) = 3z^5 - 2z + 1$ and $g(z) = 3z^5$. Then
$$\begin{aligned} |f(z) - g(z)|_{\partial D(0,1)} &= |2z - 1|_{\partial D(0,1)} \\ &\leq |2z|_{\partial D(0,1)} + 1, \\ &= 3 \end{aligned}$$

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