Exploration of Quotient Vector Spaces

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1 Review of Vector Spaces

Definition 1.1. A vector space over any arbitrary field $\mathbb{F}$ is a set V, with the following operations:

(a) Vector Addition, which obeys the following,

$$\forall x, y \in V,\ x + y \in V$$

(b) Scalar Multiplication, which obeys the following,

$$\forall x \in V,\ \forall \lambda \in \mathbb{F},\ \lambda x \in V$$

Moreover, the two operations must obey the following properties:

(1) $\forall x, y, z \in V,\ (x + y) + z = x + (y + z)$

(2) $\forall x, y \in V,\ x + y = y + x$

(3) $\exists 0_V \in V$ such that $\forall x \in V, x + 0_V = x$

(4) $\forall x \in V, \exists$ "-x" such that $x + (-x) = 0_V$

(5) $\forall x, y \in V$ and $\forall \lambda \in \mathbb{F},\ \lambda(x + y) = \lambda x + \lambda y$

(6) $\forall x \in V$ and $\forall \lambda, \mu \in \mathbb{F},\ (\lambda + \mu)x = \lambda x + \mu x$

(7) $\forall x \in V$ and $\forall \lambda, \mu \in \mathbb{F},\ (\lambda\mu)x = \lambda(\mu x)$

(8) $\forall x \in V\ 1x = x$

With these 8 axioms, we say that V is a vector space over field $\mathbb{F}$.

Example 1.1. *The set of n-tuples $\mathbb{R}^n$ with addition and multiplications defined coordinate wise, is a vector space. (Below we will prove some of the above axioms)*

Proof. .
(4) Let $x \in \mathbb{R}^n$. Then $x = (x_1, x_2, \cdots, x_n)$. Then we claim that the tuple $(-x_1, -x_2, \cdots, -x_n)$ is the desired vector $-x \in \mathbb{R}^n$.

$$(x_1, x_2, \cdots, x_n) + (-x_1, -x_2, \cdots, -x_n) = (x_1 + -x_1, x_2 + -x_2, \cdots x_n + -x_n) = (0, 0, \cdots, 0) \equiv 0_V$$

as needed.
The proofs for other axioms are also trivial and follow a similar structure as above. □

Note: Example comes from [3]

Remark. There are few extra properties of V that we will state and not prove.

- $\forall x \in V,\ 0_V x = 0_V$
- The zero vector 0_V is unique
- $\forall x \in V$, the additive inverse $-x \in V$ is unique

Now that we have a vector space, we will discuss about vector subspaces.

Definition 1.2. Let V be a vector space, and let $W \subset V$ be a non-empty subset. Then W is a vector subspace of V, if W is a vector space itself with the same operations as V.

Intuitively, a subspace is just a smaller set with the same properties as the original set.

2 Intro and Motivation for Quotient Vector Spaces

In linear algebra we often talk about different vector spaces and their subspaces such as the vector space of all functions, and one of its subspace, the space of all continuous functions. In all such examples, the vector space and the subspace is a collection of vectors with properties of addition and multiplication. These vectors are individual elements and are all unique.

In the section below, we will construct a vector space V/S where S is a subspace of vector space V. However, in this vector space, each "vector" is not an individual element, rather a large family of vectors in V. This is similar to equivalence classes in modular arithmetic (for example, $[1]_7 \equiv [8]_7$.)

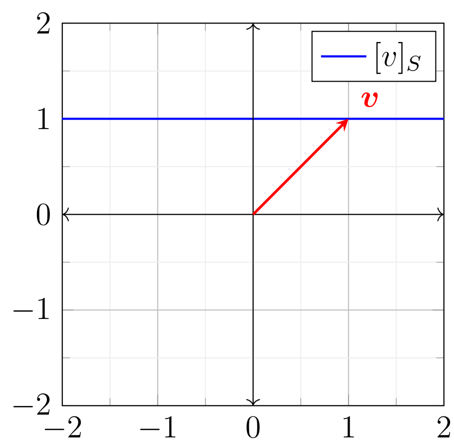
3 Quotient Vector Spaces

Definition 3.1. Let S be a subspace of a finitely generated vector space V over field $\mathbb{F}$ and let $v \in V$. Then we define a coset, denoted by $[v]_S$ by,

$$[v]_S = \{v + s \mid \text{for some } s \in S\}$$

Geometrically, $[v]_S$ is a hyperplane in the directions of S through vector v, i.e. $[v]_s$ is a set of vectors in V that can be reached by adding v to all vectors in W [4]. Lets consider the following example to demonstrate this.

Example 3.1. *Consider the vector space* $V = \mathbb{R}^2$ *and subspace* $S = \mathbb{R}$. *And take* $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in V$. *Then,* $[v]_S$ *is a horizontal line passing through the coordinate* $(1, 1)$.



Naturally, we also want to define what it means for two cosets to equal each other. Thus,

Proposition 3.1. *Let S be a subspace of a finitely generated vector space V over field $\mathbb{F}$ and let $v_1, v_2 \in V$. Then $[v_1]_S = [v_2]_S$ iff $v_1 - v_2 \in S$.*

Proof.
$\implies$ Suppose $[v_1]_S = [v_2]_S$. Then since $v_1 \in [v_1]_s$ ($v_1 = v_1 + 0$ and $0 \in S$ as S is a subspace), then by assumption $v_1 \in [v_2]_S$. Thus there exists a $s \in S$ such that $v_1 = v_2 + s$. Hence $v_1 - v_2 = s \in S$ as needed.

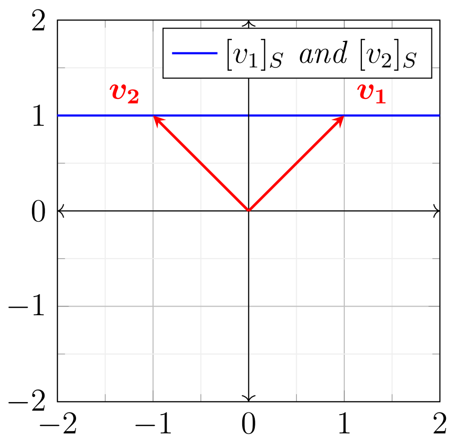
$\impliedby$ Suppose $v_1 - v_2 \in S$. Then let $s = v_1 - v_2$. And so,

$$v_1 = v_2 + s, \text{ and } v_2 = v_1 + (-s)$$

Hence $v_1 \in [v_2]_S$ and $v_2 \in [v_1]_S$. Thus, $[v_1]_S = [v_2]_S$ as needed. □

Note: Proposition and proof comes from [1] Lets consider the following related example to demonstrate the proposition.

Example 3.2. *Consider the vector space $V = \mathbb{R}^2$ and subspace $S = \mathbb{R}$. Take $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in V$ and $v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \in V$ Then, $[v_1]_S$ is a horizontal line passing through the coordinate $(1, 1)$, and similarly $[v_2]_S$ is a horizontal line passing through $(-1, 1)$.*



Now that we have defined cosets, we will define a collection of cosets and define addition and multiplication on the collection to turn the collection into a vector space.

Definition 3.2. Let S be a subspace of a finitely generated vector space V over field $\mathbb{F}$. Then the set V/S is defined by,

$$V/S = \{[v]_S \mid v \in V\}$$

i.e. V/S is the set of cosets of S in V.

With this, we define the following operations. Let $[v_1]_S, [v_2]_S \in V/S$ and let $\lambda \in \mathbb{F}$. Then **vector addition** is defined as,

$$[v_1]_S + [v_2]_S = [v_1 + v_2]_S$$

and, **scalar multiplication** is defines as,

$$\lambda[v_1]_S = [\lambda v_1]_S$$

With Definition 3.2 and vector addition and scalar multiplication defined above we get the following proposition.

Proposition 3.2. *Let S be a subspace of a finitely generated vector space V over field $\mathbb{F}$. Then with the above given operations, V/S is a vector space over $\mathbb{F}$.*

Before we give a proof of the theorem, we need to show that addition and multiplication as defined above is well defined (given unique inputs, there has to be unique outputs). However, since elements in V/S are not single vectors, rather hyper-planes with many representations, we need to show that, if $[v_1]_S = [v_2]_S$ then,

$$[v]_S + [v_1]_S = [v]_S + [v_2]_S \tag{1}$$

i.e. addition should be independent of the vectors we pick to represent the hyper-plane.

Proof. (Addition is well defined)
Let S be a subspace of a finitely generated vector apace V over field $\mathbb{F}$. And let $v, v' \in V$ and $s_1, s_2 \in S$. Then by Proposition 3.1,

$$[v]_S = [v + s_1]_S$$
$$[v']_S = [v' + s_2]_S$$

Then by definition of addition,

$$[v]_S + [v']_S = [v + s_1]_S + [v' + s_2]_S = [v + v' + s_1 + s_2]_S = [v + v']_S + [s_1 + s_2]_S = [v + v']_S$$

We were able to make the simplification in the last step, since $s_1, s_2 \in S$ so $s_1 + s_2 \in S$ and hence $[s_1 + s_2]_S = [0]_S$. And by axiom (3) of vector spaces, addition by the additive identity, $[v]_S + [0]_S = [v]_S$. (We haven't yet verified the 8 axioms, which we will do below)

(Multiplication is well defined)
Let S be a subspace of a finitely generated vector apace V over field $\mathbb{F}$. And let $v \in V$, $s_1 \in S$ and $\lambda \in \mathbb{F}$. Then by Proposition 3.1,

$$[v]_S = [v + s_1]_S$$

Then by definition of multiplication,

$$\lambda[v]_S = \lambda[v + s_1]_S = [\lambda(v + s_1)]_s = [\lambda v] + \lambda s_1]_S = [\lambda v]_S$$

We were able to make the last simplification since $\lambda s_1 \in S$ and S is a subspace. Thus we have shown that addition and multiplication as defined above is well defined. Now to show that V/S is a vector space we need to verify the 8 axioms mentioned in the previous section. □

Note: Proof is based on [4]

Proof. Here we will only verify one of the axioms, as others are trivial and/or follow the same proof structure.

1. Existence of additive identity: Consider the subset $[0_V]_S$. Then we claim that $[0_V]_S := 0_{V/S}$ is the additive inverse. Let $[a]_S \in V/S$ then,

$$[0_V]_S + [a]_S = [0_V + a]_S = [a]_S$$

as needed.

□

With this introduction to Quotient Vector Spaces, we can look at an isomorphism theorem.

4 Isomorphism Theorem

To give a brief idea of this theorem, lets look at the following example (based on [1]).

Example 4.1. *Let $V = \mathbb{R}^2$ be a vector space over $\mathbb{R}$. And let $S \subset V$ be the x-axis then, S is a subspace. Now given $(x, y), (x', y') \in \mathbb{R}^2$ then, by Proposition 3.1,*

$$[(x, y)]_S = [(x', y')]_S \textit{ iff } (x, y) - (x', y') \in S$$

i.e. $y - y' = 0$ since S is the x-axis. Thus, a vector in V/S is entirely labeled by its y coordinate. Restated, there is a linear map from V/S to the y-axis that sends,

$$[x, y]_S \longrightarrow (0, y)$$

Moreover this map is an isomorphism. With this, lets state the entire theorem.

Theorem 4.1. *Let V be a finite dimensional vector space over field $\mathbb{F}$ and let S be a subspace of V. Let $T : V \longrightarrow S$ be a linear transformation. And define a linear transformation, $T^* : V/\ker T \longrightarrow \mathrm{range}(T)$ by,*

$$T^*([v]_S) = Tv$$

Then, T^* is a well-defined transformation and moreover, it is an isomorphism.

Proof.
1) First we will show that T^* is well-defined, i.e. for each unique input we get a unique output. Let $[v]_{\ker T}, [v']_{\ker T} \in V/\ker T$. Assume that $[v]_{\ker T} = [v']_{\ker T}$. Then, $v - v' \in \ker T$ and so $T(v - v') = 0$. Hence $Tv = Tv'$. And thus,

$$T^*([v]_{\ker T}) = T^*([v']_{\ker T})$$

shows that T^* is well defined.

2) Now we show that T^* is linear. From basic linear algebra we know that a transformation A is linear when,

$$A(\alpha u + \beta v) = \alpha Au + \beta Av$$

Thus, let $\alpha, \beta \in \mathbb{F}$, and let $[v]_{\ker T}, [v']_{\ker T} \in V/\ker T$. Then,

$$T^*(\alpha[v]_{\ker T} + \beta[v']_{\ker T}) = T^*([\alpha v + \beta v']_{\ker T}) = T(\alpha v + \beta v)$$

Now since T is linear by assumption,

$$T(\alpha v + \beta v) = \alpha T v + \beta T v' = \alpha T^*([v]_{\ker T}) + \beta T^*([v']_{\ker T})$$

Hence T^* is linear.

3) Now we show that T^* is surjective. Let $b \in \mathit{range}(T) \implies b \in S$ since $S \subseteq \mathit{range}(T)$. Then, there exists $v \in V$ such that $Tv = b \implies T^*([v]_{\ker T}) = b$ and thus we have shown that T^* is surjective.

4) And finally we show that T^* is injective. Assume that $[v]_{\ker T} \in \ker T^*$, i.e.,

$$T^*([v]_{\ker T}) = T([v]_{\ker T}) = 0 \implies v \in \ker T$$

and so, $[v]_{\ker T}$ is the zero vector of $V/\ker T$, thus we have shown that $\ker T^* = \{0\}$ and so T^* is injective and hence T^* is an isomorphism.
Proof based on [1] $\square$

The theorem stated above is often called the "First Isomorphism Theore". And now lets look at an example to demonstrate this theorem.

Example 4.2. *Lets consider the following linear transformation,*

$$\begin{aligned} T : \mathbb{R}^{n\times n} &\to \mathbb{R}^{n\times n} \\ A &\to A - A^T \end{aligned}$$

Lets consider the kernal of T.

$$\ker T = \{A \mid A - A^T = 0\} = \{A \mid A^T = A\}$$

this is the set of all symmetric matrices. Now lets consider the range of T,

$$T(\mathbb{R}^{n\times n}) = \{A - A^T \mid A \in \mathbb{R}^{n\times n}\}$$

Lets call $A - A^T := B$, then consider $B^T = A^T - A = -(A - A^T) = -B$. Thus the range of T is the set of all anti-symmetric matrices. And so, Theorem 4.1 tells us that,

$$\mathbb{R}^{n\times n}/n \times n \text{ symmetric matrices} \cong n \times n \text{ anti-symmetric matrices}$$

where the isomorphism is given by T^*

Example based on [2]

References

[1] Santiago Canez. *Notes of Quotient Spaces*. URL: https://www3.nd.edu/~jdiller/teaching/archive/spring12_20820/quotient-spaces.pdf. (accessed: 07.06.2022).

[2] James Cook. *Linear Algebra: Lecture 23 part 2: quotient space examples,1st isomorphism theorem*. URL: https://www.youtube.com/watch?v=2Pjm8EqMa-8. (accessed: 07.06.2022).

[3] David B Damiano. *A Course in Linear Algebra*. Cambridge University Press, 1988. ISBN: 0486469085.

[4] *Quotient Spaces*. URL: https://math.okstate.edu/people/binegar/4063-5023/4063-5023-l13.pdf. (accessed: 07.06.2022).