PHY324 - Pendulum Project

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Abstract

In this report, we will verify/refute various claims about a simple pendulum moving in a plane. By constructing a pendulum with variable amplitude θ_0, length L and mass m, we conclude that the time period is dependent on θ_0 and m and had inconclusive results for the dependence of T on L. Further, we conclude although the pendulum swing appears as a decaying exponential, the provided equation is not a valid fit equation explaining the amplitude decay. We also quantify the asymmetry of the pendulum constructed via the vertical shift in the fit parameters. Finally, from the various calculated fit parameters, we notice that the decay constant τ has a linear relation with the quantity $L + D$.

1 Introduction

A simple pendulum consists of a mass attached to a pivot via a string, moving under the influence of gravity. To model a realistic pendulum, one may consider the effects of drag and set up the appropriate differential equation to arrive at the following,

$$\theta(t) = \theta_0 e^{-t/\tau} \cos\left(\frac{2\pi}{T}t + \phi_0\right) \tag{1}$$

where θ_0 is the initial angle from which the pendulum was released from[1]. The decay constant, τ that depends on the drag properties, T is the time period of one swing and ϕ_0 is the phase shift.

We also assume that the pendulum is symmetric and thus Equation 1 does not have a vertical shift term. Moreover, in this report we will verify the validity of τ, T being independent of t and,

$$T = 2(L + D)^{1/2} \tag{2}$$

where L is the length of the string and D is the distance from the center of mass of the object to the point attached to the string, assuming that the center of mass lies along the line of the string[2].

2 Procedure

2.1 Materials & Apparatus

The experiment required a stand to hold the pendulum and another to hold the camera which records the entire pendulum swing. Masses of 100 ± 1g, 200 ± 1g, 500 ± 1g were used. A protractor was attached to the stand as seen in Figure 1. The length of rope in this experiment varied from 10.0 ± 0.1cm to 60.0 ± 0.1cm, however due to how the string was attached, double the length of rope is required. After recording each swing, Tracker[3] software was used to find the relative horizontal and vertical position of the CoM with respect to the pivot. From this setup, it can be seen that m, L, θ_0 can be varied. Thus in this experiment, we will hold two variables constant and vary the third for all three combinations.

To ensure the pendulum moved in a 2D plane, the pendulum was attached to the rod via two strings in a "V" shape, to dampen out any motion taking place outside the plane of motion. To ensure the pendulum was as symmetric as possible, the knot was tied directly at the bottom of the rod.

[1] *When solving for Equation 1 we make the assumption that θ_0 is small.*
[2] *Under the implicit assumption that $D^2 << L^2$.*
[3] Downloaded form: https://physlets.org/tracker/

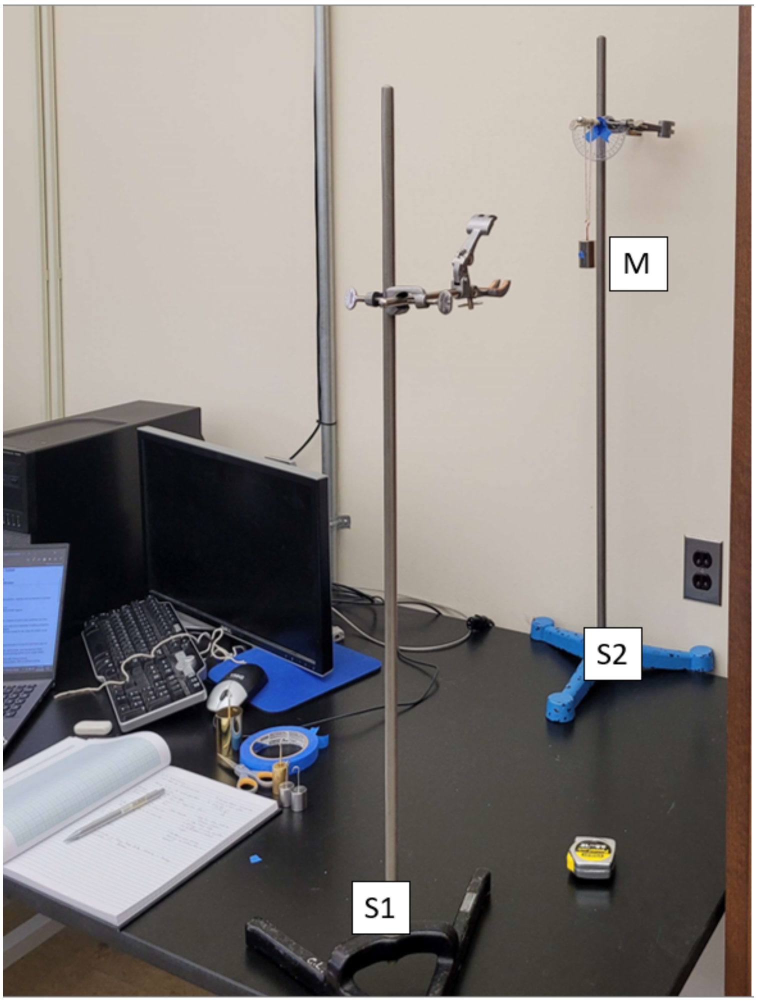

Figure 1: Setup of experiment showing both stands S1 (for camera) and S2 (for pendulum) and mass M. Protractor is also attached to the stand and is centered. Notice that M has a bright blue tape marking the center of mass. This is done to allow the auto-tracker to find the CoM easily. *Note: When working with larger string lengths, the distance between the stands was increased, and a larger protractor was used.*

2.2 Various Procedures

Constant Mass & Length, Varying Initial Angle:
We setup the pendulum as seen in Figure 1 and used a mass of $m = 0.200 \pm 0.001$kg (with the distance to the center of mass $D = 2.15 \pm 0.05$cm). Attaching the string in the "V" shape and measuring the vertical distance between the mass and the pivot, the length was found to be $l = 14.5 \pm 0.1$cm (note that the measurement were made after the mass was hung on the string, as that increases the length of the rope slightly). To ensure that the string on the pivot did not slide as the pendulum moves, we taped the top of the string to the pivot. Using the attached protractor, we vary the angle between 25° and 55°, for a a total of 5 trials. We then started the recording (under a well illuminated area, with uniform background color and making sure that the stands were firmly in place and did not vibrate)[4] and released the mass ensuring no slack was in the string, and no external force was exerted on the mass. The recording were stopped when the swing had decayed to less than 10°.

Constant Initial Angle & Length, Varying Mass:
Similar to the previous section, We setup the pendulum as seen in Figure 1 and measure the length to be $l = 54.5 \pm 0.1$cm. Due to a larger length, the camera was taken further back and adjusted such that it was parallel to the plane of motion, ensuring that parallax effect was a minimum. We varied the mass of the object from $m_1 = 100 \pm 1$g, $m_2 = 200 \pm 1$g and $m_3 = 500 \pm 1$g with the distance to the center of mass being $D_1 = 1.75 \pm 0.05$cm, $D_2 = 2.15$cm and $D_3 = 2.80 \pm 0.05$cm respectively.

It was also noticed that for this longer string size, the rope would twist as it was swinging. To

[4]For the Tracker application to function correctly, the video must be in a MP4 format with codex H-264. Moreover, cropping the video to just include the swing and removing audio, will aid the import time to the software.

minimize this rope was allowed to twist beforehand, and then taped to prevent unravelling. Then the length between the pivot and hook on the mass was measured. For each trial, we released the mass from $45° \pm 5°$ as viewed from the recordings. However, since the videos was taken from a much further distance, the angle readings were difficult to make and thus, we take an uncertainty of $5°$. (In a later attempt, one can consider a mechanized method to release the mass, thus giving very precise angle measurements.)

Constant Initial Angle & Mass, Varying Length:
Finally, for this combination, we fixed the mass at $m = 200 \pm 1$g with $D = 2.15 \pm 0.05$cm and an initial release angle of $\theta_0 = 40° \pm 5°$ (for the same reason as the last section). Then lengths were varied from 34.2 ± 0.1cm, 42.5 ± 0.1cm and 51.7 ± 0.1cm, where we measured after the rope had twisted and was taped.

3 Results & Discussion

Constant m, l and Varying θ_0:
For this analysis the $23° \pm 1°$ data-set was used. Remaining 4 data-sets were analyzed in the same manner.

Using the Tracker software, we traced through every-other frame (Camera was recording at 60FPS) and used the in-built auto-tracker to go through each frame and identify the center of mass, via the colored tape on the object. We calibrate the software by providing that length of the rope and the location of the pivot which acts as the origin. Due to the occasional motion blur, we had to manually place certain the CoM in some frames. The Tracker then provides a data-set of time, horizontal and vertical position of the center of mass, and we take time uncertainty to be $1/30 \approx 0.03$s as we are skipping every other frame. We also the position uncertainty of 0.005cm due to the occasional manual CoM placement. Using, $\theta = \arctan(x/y)$, we calculate the angle between the string and the vertical, and propagate the uncertainty accordingly. Figure 2 shows the complete decaying pendulum swing. As mentioned in the Figure caption, the plot is not centered about the horizontal axis leading us to conclude that the pendulum design used in the experiment was not symmetric. This could be caused, as the pivot was a cylindrical rod, around which the rope was tied and thus the rope would have wrapped around the pivot as the mass moved. As such, we will use the vertical shift as a quantifier of the "asymmetry" of the pendulum.

Moreover, notice the region around 100s where the negative angle data is not a smooth decay. This leads us to conclude that the Tracker application did not always mark the same position on the object as the CoM. This may have also been caused by significant motion blur or vibrations in the stands holding the pendulum and camera.

Now under the assumption that T is constant throughout the pendulum motion, we fit a plane cosine wave to the first 3 oscillations of the data to calculate the time period and phase shift. Figure 3 shows this fit, and the calculated the fitting parameters were found to be,

$$\theta_0 = 0.41 \pm 0.01, \quad T = 0.921 \pm 0.001, \quad \phi_0 = -0.62 \pm 0.01$$

Now to fit the exponential decay, we fit both the upper decay and lower decay separately, due to the asymmetry of the pendulum. Using `scipy.signal.find_peaks` we fit the peaks and valleys of the data-set against, $y(t) = \theta_0 e^{-t/\tau} + c$, where θ_0 is the initial angle calculated before, τ is the decay constant and c is a linear shift to account for the asymmetry. Figure 4 shows the upper and lower decay fits, and the decay constants and average shift were found to be,

$$\tau_{\text{upper}} = 242 \pm 3\text{s}, \quad \tau_{\text{lower}} = 226 \pm 6\text{s}, \quad c_{\text{avg}} = 0.063 \pm 0.001 \text{ rad}$$

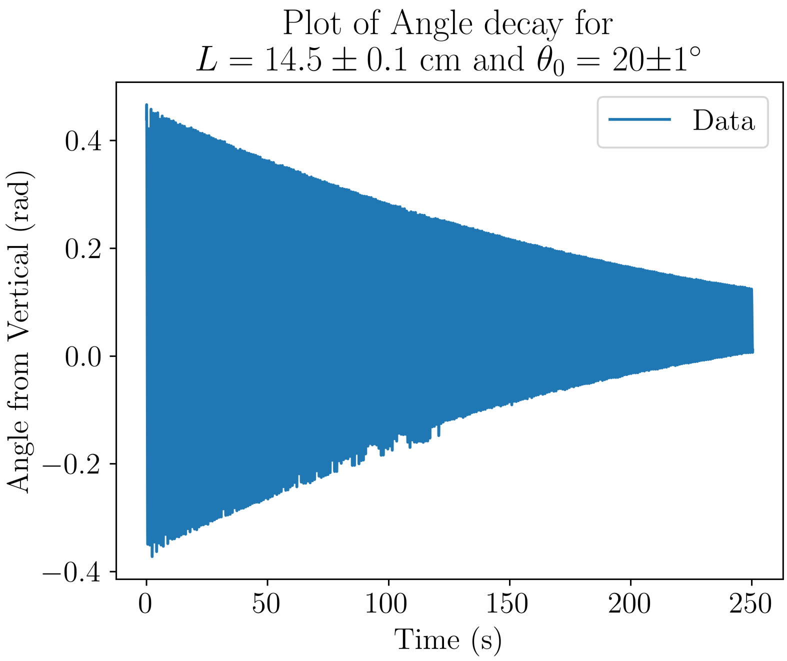

Figure 2: Complete decaying pendulum swing of a 200 ± 1g mass on a string of length $l = 14.5 \pm 0.1$cm with an initial angle of $20° \pm 1°$. Angle and time uncertainty are not plotted, to increase ease of readability. Notice that the plot is not centered at the horizontal and the discrepancy in negative angle data around the 100s region.

Following the same procedure for the remaining data-sets, we calculate the required fitting parameters (complete table in Table 1). In Figure 5 we see the relationship between the time period and initial release angle. Notice that the relations is clearly quadratic in nature, which leads us to conclude that the T is not independent the initial angle of release. This can be explained mathematically, as the time period equation $T = 2\pi\sqrt{l/g}$ comes from a differential equation which assume the initial angle is small, and thus taking a Taylor expansion up to θ. However for larger angles of release, we must instead Taylor expand up to a θ^2 term to get a better approximation.

Since the time period is dependent on the angle, we cannot assume that the time period is constant for pendulums with large angles and thus we predict that Equation 1 is not a valid fit equation. In Figure 9 (in Appendix) we plot Equation 1 with the calculated fitting parameters along with the pendulum swing and we also plot the residual in Figure 10 (in Appendix). Notice that the calculated fit does not decay as quickly as the actual pendulum swing. This suggests that the decay constant is evolving in time and is not constant. This makes sense as the decay constant acts as a drag term which depends on the geometry of the object and more importantly the speed of the object. Thus as the pendulum slows down during its swing, the decay constant τ also changes.

Moreover, from the residual plot, we can see a beat-like behavior.[5] This suggests that pendulum swing is changing its frequency and hence the time period as it evolves in time, confirming the previously calculated fact that the time period is dependent of the angle. Thus we conclude that the pendulum swing is decaying, however it does not obey Equation 1.

Constant m, θ_0 and Varying l:

For this analysis the $l = 42.5 \pm 0.1cm$ data-set was used. Remaining 2 data-sets were analyzed in the same manner.

[5] Occurs in the physics of music, when two sine waves of slightly different frequencies are added together, creating a quickly oscillating inner sine wave and a slower envelope frequency.

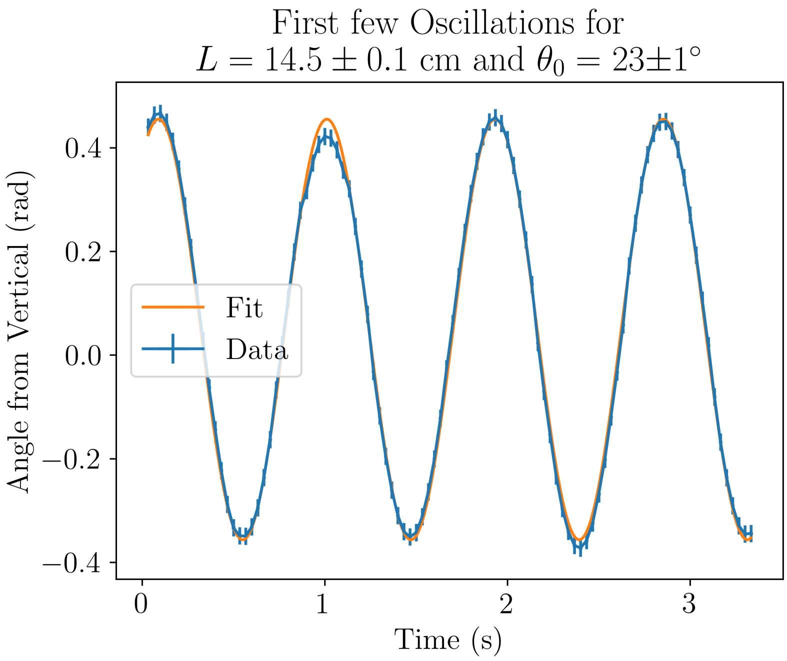

Figure 3: First 3 Oscillation of the full data-set in Figure 2 fitted with a plane cosine wave scaled by the initial angle and shifted by a phase-shift. Error bars included but not significant enough plot a separate residual plot.

Following the same process as above, we plot the first few oscillations to calculate the time period T. We then fit the upper and lower exponential decay separately, since the pendulum is asymmetric. Figure 11 and residual Figure 12 (in Appendix) shows a similar behavior as seen before. Although notice that the fit decays faster than the data. Moreover, in the residual we notice that the beat phenomena is more pronounced and is also decaying. The three data-sets fitting parameters are given in Table 2 and in Figure 6 we plot the relation between the length of the string and the time period of the swing and a linear and power fit. It was found that the reduced chi-square for both fits were 0, however the power fit had a chi-squared probability of 1.00. This suggests an over-fit of the data, however as there are only 3 data-points, it is difficult to conclude which fit is better. (Further experiments should consider more trials and use a rigid rod instead of a string to ensure lower uncertainty in the length and more confidence in the relationship between l and T.) The power fit has parameters given by, $y(t) = at^n + b$, where $a = 0.46 \pm 0.15$ m, $n = 1.2 \pm 0.3$, $b = -0.3 \pm 0.2$ m. Notice that we plot the relation as length vs. time period, to get the exact relationship. Inverting the power n, we get that $T \propto L^{0.83}$ which is different than the proposed relation in Equation 2, which relates $T \propto L^{0.5}$. Given the large uncertainty in n, it is difficult to conclude whether the proposed relationship given by Equation 2 is valid.

Constant L, θ_0 and Varying m:

For this analysis the $m = 500 \pm 1g$ data-set was used. Remaining 2 data-sets were analyzed in the same manner.

Following the same process as before, we fit each portion accordingly and record the parameters in Table 3.[6] The upper and lower exponential fit again had reduced chi-square less than 0.5 for all three trials, with a chi-squared probability greater than 0.6. Similar to before, we plot the relationship between the mass on the pendulum and the time period in Figure 7 and notice that decreasing linear fit. The reduced chi-square was calculated to be 0 since the data-set only contained 3 data-points and the chi-squared probability was found to be 1.00, which is expected for a linear fit.

[6] Due to the similarity to the previous full data fit, the full fit and residual for this trial is not provided.

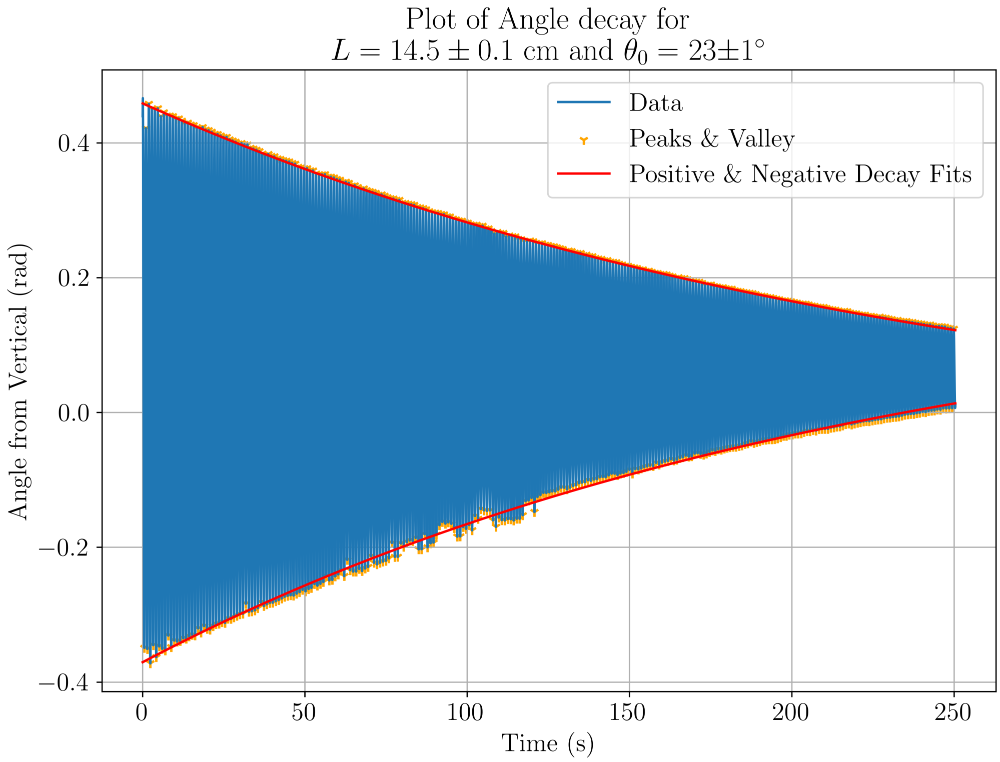

Figure 4: Upper and lower exponential decay fit of the peaks and valleys of the data-set. The calculated upper and lower reduced chi-squared were both less than 0.1 and the calculated chi-squared probability for both fits were $p = 1.00$

Thus the figure suggests that the time period of a pendulum is linearly dependent of the mass of the object. However, we know that the pendulum is independent of mass. We predict that this linear relationship is due to the fact that the time period itself is not constant during the full pendulum motion. We know that the time period is dependent of the angle and is evolving in time. Moreover, increasing the mass, changes the geometry of the object which may impact the decay constant and thus change the time period.

Extra:
Moreover, from the three data-tables in the Appendix, when plotting $L + D$ against τ_{avg} we notice a linear trend as seen in Figure 8 and no other trend were noticed between m and τ or θ_0 and τ. This suggests that along with the geometry of the object, τ also depends on the length of the pendulum. The calculated linear fit parameters were found to be, $y(x) = mx + b$, where $m = -3.9 \pm 0.2$cm/s and $b = 381 \pm 11$s. That is, the decay constant decreases roughly one unit for every 4cm increase in the length of the pendulum.

4 Conclusion

In this experiment we construct a pendulum where l, m, θ_0 can each vary independently, to refute/verify various claims such as the time period T being independent of θ_0, m, the decay being exponential, the dependence of L on T given by $T = (L + D)^{1/2}$.

By fixing m, L and varying θ_0, we fit Equation 1 to the data-set and found the various fitting parameters. Plotting θ_0 and T we found a quadratic relationship between the two quantities which is explained when considering higher order Taylor expansions. Moreover, from the residual plot of

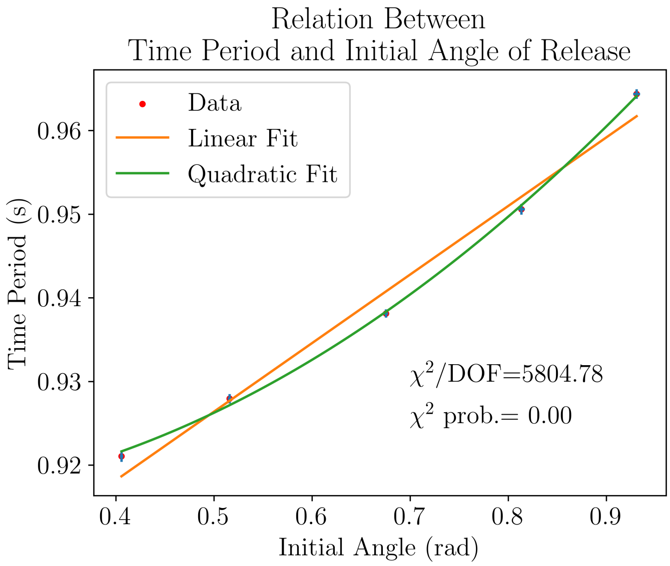

Figure 5: Relations between pendulum time period and initial angle of release for a pendulum of mass $m = 200 \pm 1$g and length $l = 14.5 \pm 0.1$cm. A linear and quadratic fit are shown along with the reduced chi-square and chi-squared probability of the quadratic fit.

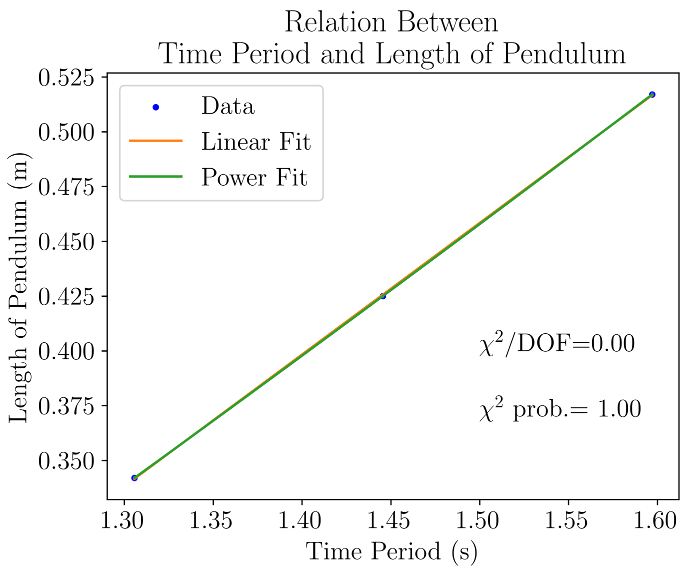

Figure 6: Relation between the length of the pendulum and the time period for a mass $m = 200 \pm 1$g and initial release angle of $\theta_0 = 45° \pm 5°$. A linear and power fir are shown along with the reduced chi-squared and chi-squared probability of the power fit.

the full fit, we noticed a beat-like behavior which suggested that the time period T was evolving in time. The full fit in Figure 9 also shows that the fit does not decay as quickly as the data-set, which leads us to conclude that the pendulum decay may not be exponential as given by Equation 1. When fitting the full data-set, we also included a vertical shift term to account for the asymmetry in the pendulum, caused as a result of how the string was attached to the pendulum. The shift term given in the Tables in the Appendix quantify the asymmetry of each trial.

By fixing m, θ_0 and varying L we again calculate the fitting parameters, as summarized in the Table in the Appendix. When plotting the relation between the length and the calculated time period, we fit both a linear and power-law fit and found that the power-law resulted in a higher chi-squared probability. However, due to both fits being very likely, it is difficult to refute/verify whether $T = (L + D)^{1/2}$. Having more trials for this combination of the pendulum would further aid

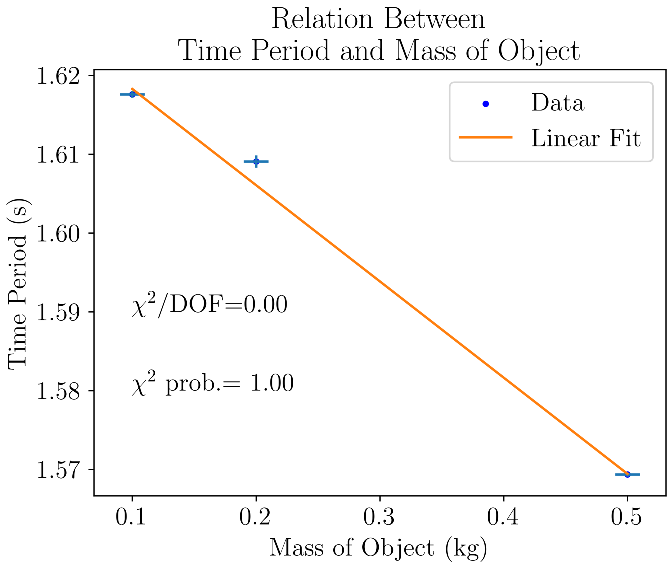

Figure 7: Relation between the mass on the pendulum and the time period for a pendulum of length $l = 54.4 \pm 0.1$cm and an initial angle of $\theta_0 = 45° \pm 5°$. The fitting parameters are $m = -0.122 \pm 0.001$s/kg and $b = 1.63 \pm 0.01$s. Notice that the reduced chi-squared probability is 1.00 despite it missing one data-point. This is due to the reduced chi-square being 0.00 which has a much larger impact on the probability.

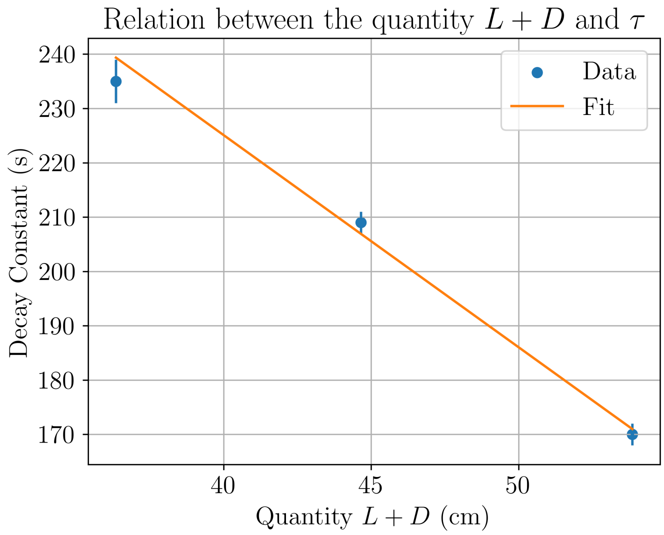

Figure 8: Relation between τ and the quantity $L + D$ coming from Table 2, along with a linear fit. The calculated reduced chi-square and chi-squared probability were found to be 0.00 and 1.00 which is expected given that the fit is linear.

this claim. Similarly by fixing θ_0, l and varying m, we plot the relation between the calculated time period and m and found a decreasing linear relationship. Finally, we also noticed that the calculated decay constant for the fixed m, θ_0 trials had a negative linear relationship with the quantity $L + D$.

To improve the results of this experiment, we can consider a mechanized rotor connected to the pendulum at the pivot which allows for a higher precision of initial angle measurement. Using a rigid rod instead of a sting will also reduce the second largest source of error, as it results in far accurate length measurements, which no mid-swing rotation and stretching.

5 Appendix

5.1 Data Tables

Initial Angle ($\pm 1°$)	Time Period (± 0.01s)	Phase Shift (± 0.01)	τ_{upper} (s)	τ_{lower} (s)	Average Linear Shift (± 0.001 rad)
23	0.92	-0.62	242 ± 3	226 ± 6	0.063
30	0.93	-0.06	302 ± 9	253 ± 5	0.023
39	0.94	-0.33	212 ± 1	238 ± 1	0.052
47	0.95	0.28	205 ± 1	218 ± 1	0.040
53	0.96	0.06	213 ± 3	207 ± 1	0.033

Table 1: Curve fit parameters for pendulum of mass 200 ± 1g and length $l = 14.5 \pm 0.1$cm. Where we fit two decaying exponential to the entire data-set and a plain cosine wave to the initial few oscillations.

Length (± 0.1cm)	Time Period (± 0.01s)	Phase Shift (± 0.02)	τ_{upper} (s)	τ_{lower} (s)	Average Linear Shift (± 0.002 rad)
34.2	1.31	3.35	215 ± 2	255 ± 3	0.028
42.5	1.45	3.26	210 ± 2	208 ± 1	0.008
51.7	0.60	3.53	164 ± 1	176 ± 2	0.003

Table 2: Curve fit parameters for pendulum of mass 200 ± 1g and initial release angle of $\theta = 45° \pm 5°$. Where we fit two decaying exponential to the entire data-set and a plain cosine wave to the initial few oscillations.

Mass (± 1g)	Time Period (± 0.01s)	Phase Shift (± 0.02)	τ_{upper} (s)	τ_{lower} (s)	Average Linear Shift (± 0.002 rad)
100	1.62	3.39	326 ± 2	331 ± 2	-0.002
200	1.61	2.63	203 ± 2	189 ± 1	-0.006
500	1.57	3.73	214 ± 1	212 ± 2	0.004

Table 3: Curve fit parameters for pendulum with length $l = 54.5 \pm 0.1$cm and initial release angle of $\theta = 40° \pm 5°$. Where we fit two decaying exponential to the entire data-set and a plain cosine wave to the initial few oscillations.

5.2 Figures

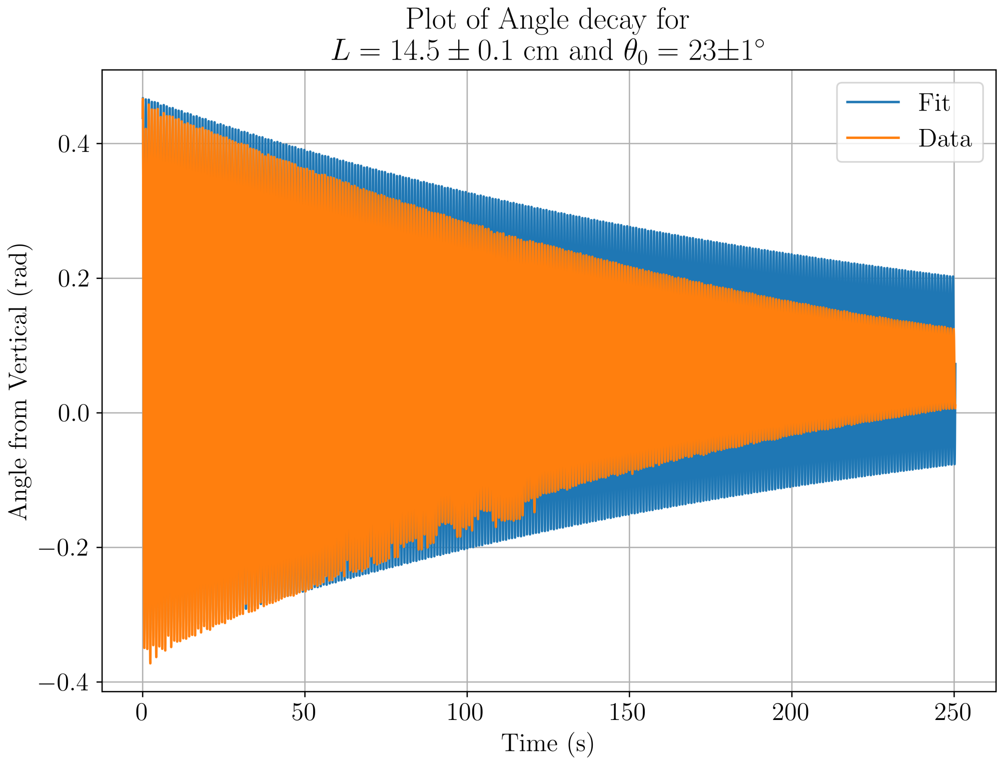

Figure 9: Proposed Equation 1 with the calculated fit parameters along with the original pendulum data. Notice that the fit does not decay as quickly as the data-set. Moreover in the following Figure we notice that the two decaying cosines are not in-phase.

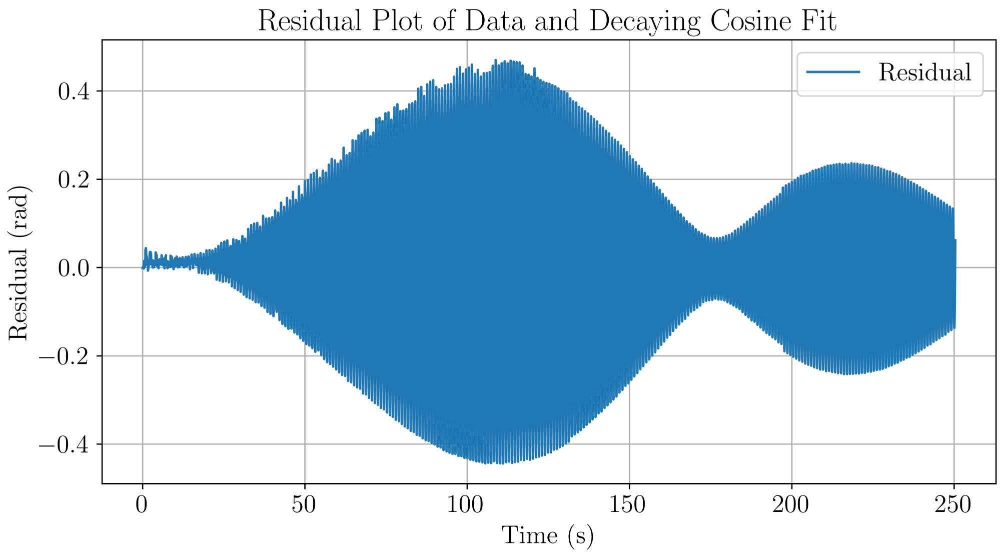

Figure 10: The Residual plot of the proposed fit equation and the pendulum data. Notice the beat like behavior, which one sees in the physics of music, and occurs when two waves of slightly different frequencies are added. A similar pattern is observed here, which leads us to conclude that the assume constant time period T should not be constant.

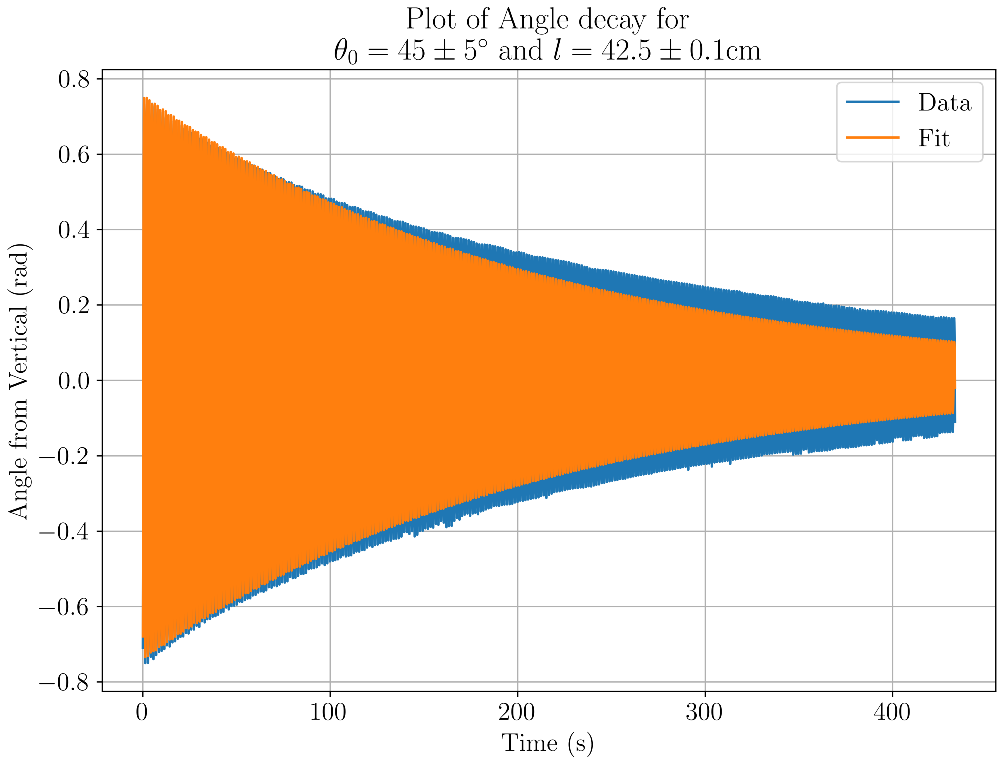

Figure 11: Proposed Equation 1 with the calculated fit parameters along with the original pendulum data. Notice that the fit decays faster then the data-set. Moreover in the following Figure we notice that the two decaying cosines are not in-phase.

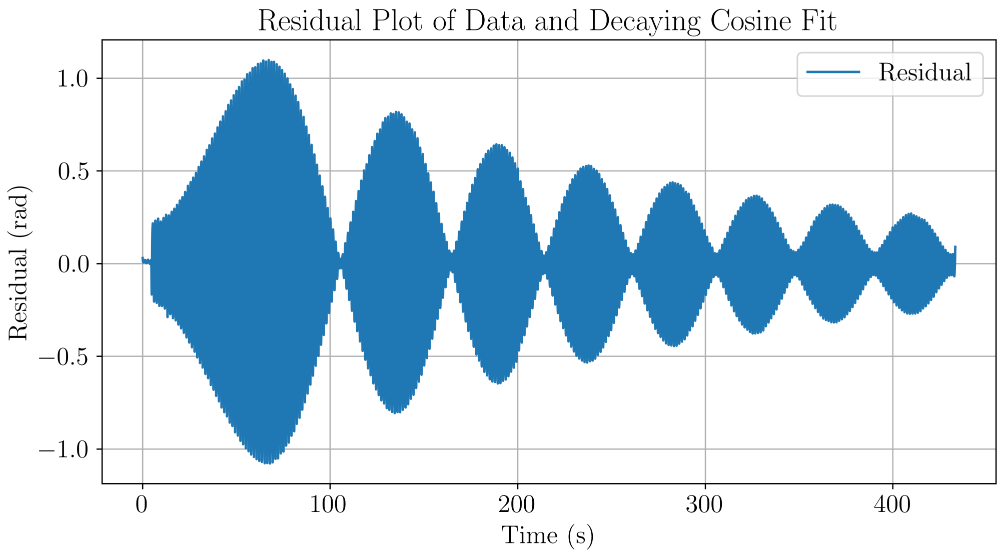

Figure 12: The Residual plot of the proposed fit equation and the pendulum data. Notice the beat like behavior, which one sees in the physics of music, and occurs when two waves of slightly different frequencies are added. A similar pattern is observed here, which leads us to conclude that the assume constant time period T should not be constant.