Image Graph Spectrum

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June, 2017

Outline

Introduction to Image Graph

- For image segmentation, clustering, etc. `Graph Spectral Analysis` has been extensively used.
- Image is represented as a weighted complete graph. Each pixel represents a vertex.
- Edge weights are assigned as per affinity between pixels. Affinity can be defined on the basis of various properties :
  - Intensity difference between two vertices
  - Gradient difference between two vertices
  - Difference between some other feature calculated at each pixel e.g. sift

Image Graph

- Image Graph is represented as $G(V, E, W)$
- V contains all image pixels as vertices. If there are total n pixels in the image then $|V| = n$
- E contains all pairwise relationship between every pair of vertices(pixels) thus making G a complete graph. $|E| = \binom{n}{2}$ for undirected graph
- The weight $w_{ij} \geq 0$ associated with an edge $(v_i, v_j) \in E$ encodes the affinity between the pixels represented by vertices v_i and v_j. We can collect these weights into an $n \times n$ affinity matrix $W = (w_{ij})_{i,j=1,\ldots,n}$

Function p

- We want to define a function $p : V \rightarrow \mathbb{R}$ such that it is a continuous function i.e. difference between $p(v_i)$ and $p(v_j)$ inversely follows w_{ij}
- It is equivalent to say that we want to minimize

$$\lambda = \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij}(p(v_i) - p(v_j))^2$$

- Let Matrix P be defined as $\begin{bmatrix} p(v_1) \\ p(v_2) \\ \vdots \\ p(v_{|V|}) \end{bmatrix}$

Incident Matrix

- For any directed graph $G(V, E)$, consider

$$V = \{v_1, v_2, \ldots\ v_{|V|}\}$$

$$E = \{e_1, e_2, \ldots, e_{|E|}\}$$

- Incident Matrix ∇ is $|E| \times |V|$ matrix such that if k^{th} edge is from v_i to v_j with weight w_{ij} then
  - $\nabla_{ki} = +w_{ij}$
  - $\nabla_{kj} = -w_{ij}$
  - $\nabla_{km} = 0, \forall m \neq i, j$

Laplacian Matrix

- $L = \nabla^T \nabla$ is called laplacian of graph
- L is $|V| \times |V|$ matrix where

$$L_{ii} = \sum_{j=1}^{|V|} w_{ij}$$

$$\underset{i \neq j}{L_{ij}} = -w_{ij}$$

- $L = D - W$ where D is degree matrix and W is adjacency matrix

Laplacian's relation to function p

- We can show that $P^T LP = \frac{\lambda}{2}$

$$\begin{aligned} P^T LP &= P^T(D - W)P \\ &= P^T DP - P^T WP \end{aligned}$$

- Take $d_{ii} = (i^{th}$ diagonal entry in $D)$ and $p_i = p(v_i)$
- First term is

$$P^T DP = \sum_{i=1}^{|V|} d_{ii} p_i^2$$

- second term is

$$P^T WP = \sum_{i=1}^{|V|} \sum_{j=1}^{|V|} p_i p_j w_{ij}$$

Laplacian's relation to function p (cont.)

$$
\begin{aligned}
P^T L P &= \sum_{i=1}^{|V|} d_{ii} p_i^2 - \sum_{i=1}^{|V|} \sum_{j=1}^{|V|} p_i p_j w_{ij} \\
&= \sum_{i=1}^{|V|} \sum_{j=1}^{|V|} w_{ij} p_i^2 - \sum_{i=1}^{|V|} \sum_{j=1}^{|V|} p_i p_j w_{ij} \\
&= \frac{1}{2} \left(\sum_{i=1}^{|V|} \sum_{j=1}^{|V|} w_{ij} p_i^2 + \sum_{i=1}^{|V|} \sum_{j=1}^{|V|} w_{ij} p_j^2 - 2 \sum_{i=1}^{|V|} \sum_{j=1}^{|V|} p_i p_j w_{ij} \right) \\
&= \frac{1}{2} \sum_{i=1}^{|V|} \sum_{j=1}^{|V|} w_{ij} (p_i - p_j)^2 \\
&= \frac{\lambda}{2}
\end{aligned}
$$

Courant-Fischer Formula

- Courant-Fischer Formula for any $n \times n$ symmetric matrix A

$$\lambda_1 = \min_{||x||=1}(x^T Ax)$$
$$\lambda_2 = \min_{\substack{||x||=1 \\ x \perp v_1}}(x^T Ax)$$
$$\vdots$$
$$\lambda_{max} = \max_{||x||=1}(x^T Ax)$$

  Here $v_1, v_2, v_3, \cdots$ are eigenvectors corresponding to eigenvalues $\lambda_1, \lambda_2, \lambda_3, \cdots$ where $\lambda_1 \leq \lambda_2 \leq \lambda_3 \cdots$
- We know $P^T LP = \frac{\lambda}{2}$ and L is symmetric. Thus eigenvectors of L corresponding to smallest eigenvalues, represent such possible P's that minimizes λ

Conclusion

- For any image, a weighted graph is constructed considering each pixel a vertex.
- Edge weights are assigned according to affinity of vertices.
- Laplacian is obtained from adjacency matrix using formula $L = D - W$
- Normalized laplacian can be obtained by formula $L = I - D^{-\frac{1}{2}} W D^{-\frac{1}{2}}$
- Eigen Decomposition of L gives v_1 as a trivial solution and $v_2, v_3, \cdots$ as desired solutions