Brachistochrone Curve

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Abstract

This work investigates the brachistochrone problem, posed by Johann Bernoulli in 1696—the curve of fastest descent under gravity. We derive the classical solution under uniform gravity, showing that the cycloid satisfies the Euler–Lagrange equation and all sufficiency conditions for a strong minimum. The problem is then extended to a linearly varying gravitational field, where the optimal curve is governed by a modified differential equation, solvable numerically. The study illustrates both analytical and computational aspects of time-optimal paths under different gravity profiles.

1 Introduction

The motion of particles under gravity has fascinated mathematicians and physicists for centuries. One of the most famous problems in this context is the *brachistochrone problem*, which asks: given two points at different heights in a vertical plane, what is the curve along which a particle will slide from the higher to the lower point in the shortest possible time, assuming no friction?

Unlike the shortest path in distance, the path of quickest descent is influenced by both the slope and the particle's acceleration due to gravity. This makes the problem nontrivial and historically significant, as its solution led to the development of the calculus of variations, a branch of mathematics concerned with finding extrema of functionals.

We derive the brachistochrone in a plane under uniform gravity, derive the Euler-Lagrange equation for the time functional, and obtain the classical cycloidal solution. We then explore extensions to non-uniform gravitational fields, highlighting how variations in gravity modify the optimal path. This study provides a theoretical framework and a basis for computational exploration of time-optimal trajectories.

2 Brachistochrone Curve under Uniform Gravity (Newtonian Gravity)

Let the particle move in the vertical xy-plane, with the x-axis horizontal and the y-axis directed vertically downward. The particle starts from rest at the origin.

By conservation of mechanical energy, the speed of the particle at a vertical depth y is

$$\frac{1}{2}mv^2 = mgy \quad \Rightarrow \quad v(y) = \sqrt{2gy}.$$

For an infinitesimal arc length ds,

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx,$$

the corresponding time taken is

$$dt = \frac{ds}{v} = \frac{\sqrt{1 + (y')^2}}{\sqrt{2gy}}\, dx.$$

Hence, the total time of descent is

$$T[y] = \int dt = \int \frac{\sqrt{1 + (y')^2}}{\sqrt{2gy}}\, dx.$$

The problem therefore reduces to finding the function $y(x)$ that minimizes the functional $T[y]$.

Extremization of a Functional-Euler-Lagrange Equations

In the calculus of variations, one seeks the function $y(x)$ that extremizes a functional of the form

$$J[y] = \int_{x_1}^{x_2} L(y, y', x)\, dx,$$

where L is the Lagrangian density depending on $y(x)$, its derivative $y'(x)$, and possibly the independent variable x.

A necessary condition for $y(x)$ to extremize $J[y]$ is

$$\delta J = 0.$$

Evaluating the variation and integrating by parts leads to the Euler-Lagrange equation,

$$\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) - \frac{\partial L}{\partial y} = 0,$$

which determines the extremizing function $y(x)$.

Extremization of a Time Functional

Ignoring constants, we get,

$$L = \frac{\sqrt{1+(y')^2}}{\sqrt{y}}$$

$$\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) - \frac{\partial L}{\partial y} = 0$$

$$\frac{\partial L}{\partial y'} = \frac{1}{\sqrt{y}} \cdot \frac{y'}{\sqrt{1+(y')^2}} \qquad \frac{\partial L}{\partial y} = -\frac{\sqrt{1+(y')^2}}{2y\sqrt{y}}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{\sqrt{y}}\frac{y'}{\sqrt{1+(y')^2}}\right) + \frac{\sqrt{1+(y')^2}}{2y\sqrt{y}} = 0$$

Using the product rule $u\frac{dv}{dx} + v\frac{du}{dx}$:

$$\frac{-y'}{2y\sqrt{y}}\frac{y'}{\sqrt{1+(y')^2}} + \frac{1}{\sqrt{y}}\frac{d}{dx}\left(\frac{y'}{\sqrt{1+(y')^2}}\right) + \frac{\sqrt{1+(y')^2}}{2y\sqrt{y}} = 0$$

Now,

$$\begin{aligned}
\frac{d}{dx}\left(\frac{y'}{\sqrt{1+(y')^2}}\right) &= \frac{\sqrt{1+(y')^2}y'' - y'\frac{y'y''}{\sqrt{1+(y')^2}}}{1+(y')^2} \\
&= \frac{y''(1+(y')^2) - y''(y')^2}{[1+(y')^2]\sqrt{1+(y')^2}} \\
&= \frac{y''}{[1+(y')^2]\sqrt{1+(y')^2}}
\end{aligned}$$

$$\Rightarrow \frac{-(y')^2}{2y\sqrt{y}\sqrt{1+(y')^2}} + \frac{1}{\sqrt{y}}\frac{y''}{[1+(y')^2]\sqrt{1+(y')^2}} + \frac{\sqrt{1+(y')^2}}{2y\sqrt{y}} = 0$$

Multiply with $[2y\sqrt{y}][1+(y')^2][\sqrt{1+(y')^2}]$,

$$-(y')^2\big(1+(y')^2\big) + 2yy'' + \big[1+(y')^2\big]^2 = 0$$

$$\Rightarrow (y')^2 + 2yy'' + 1 = 0$$

Let

$$P = \frac{dy}{dx} \;\Rightarrow\; P = y'$$

$$\Rightarrow \frac{dP}{dx} = \frac{d^2y}{dx^2}$$

Using the chain rule,

$$y'' = \frac{dP}{dx} = \frac{dP}{dy}\frac{dy}{dx} = P\frac{dP}{dy}$$

Substituting,

$$P^2 + 2yP\frac{dP}{dy} + 1 = 0$$

$$P^2 + 1 = -2yP\frac{dP}{dy}$$

$$\frac{1}{y}\,dy = \frac{-2P}{1+P^2}\,dP$$

Integrating,

$$\ln y = -\ln\big(1+P^2\big) + \ln\big(c^2\big)$$

$$\ln y = \ln\left(\frac{c^2}{1+P^2}\right)$$

$$y = \frac{c^2}{1+P^2}$$

$$\Rightarrow 1 + (y')^2 = \frac{c^2}{y} \tag{1}$$

$$\frac{dy}{dx} = \sqrt{\frac{c^2}{y} - 1}$$

If you redefine $c^2 \to \frac{1}{k}$, you get,

$$\frac{dy}{dx} = \sqrt{\frac{1}{ky} - 1}$$

$$dx = \sqrt{\frac{Ky}{1-Ky}}\,dy$$

Let $Ky = \sin^2\alpha \implies K\,dy = 2\sin\alpha\cos\alpha\,d\alpha$

$$\begin{aligned}
dx &= \sqrt{\frac{\sin^2\alpha}{\cos^2\alpha}}\cdot\frac{2\sin\alpha\cos\alpha}{K}\,d\alpha \\
K\,dx &= \frac{\sin\alpha}{\cos\alpha}[2\sin\alpha\cos\alpha]\,d\alpha \\
K\,dx &= 2\sin^2\alpha\,d\alpha \\
K\,dx &= (1-\cos 2\alpha)\,d\alpha \\
Kx &= \alpha - \frac{\sin 2\alpha}{2} + \frac{C_0}{2} \\
2Kx &= 2\alpha - \sin 2\alpha + C_0 \\
\implies x &= \frac{1}{2K}(2\alpha - \sin 2\alpha + C_0)
\end{aligned}$$

And

$$\begin{aligned}
y &= \frac{\sin^2\alpha}{K} \\
y &= \frac{1-\cos 2\alpha}{2K} \\
\implies y &= \frac{1}{2K}(1-\cos 2\alpha)
\end{aligned}$$

Redefine $\frac{1}{2K} \to R$ and $2\alpha \to \theta$
We get:

$$\begin{aligned}
x &= R(\theta - \sin\theta + C_0) \\
y &= R(1-\cos\theta)
\end{aligned}$$

Initial Conditions: $A = (0,0)$, which means, at $\alpha = 0$, $x = y = 0$. By using this, we can find C_0:

$$\begin{aligned}
\implies 0 &= R(0 - 0 + C_0) \\
\implies C_0 &= 0
\end{aligned}$$

Backsubstituting C_0 value, we get,

$$\begin{aligned}
x &= R(\theta - \sin\theta) \\
y &= R(1-\cos\theta)
\end{aligned}$$

which is a cycloid equation.

Due to the explicit independence of the Lagrangian on x, we can solve this Lagrangian using Beltrami Identity

Beltrami Identity

In the calculus of variations, consider a functional

$$J[y] = \int L(y, y', x)\, dx$$

where the Lagrangian L does not depend explicitly on the independent variable x, i.e.

$$\frac{\partial L}{\partial x} = 0.$$

Then, the Euler–Lagrange equation admits a first integral known as the *Beltrami identity*:

$$L - y' \frac{\partial L}{\partial y'} = \text{constant.}$$

This identity reduces the order of the Euler–Lagrange equation by one and is especially useful in problems possessing translational symmetry in the independent variable.

Proof of Beltrami Identity

Start with a Lagrangian $L(y, y')$

If L does not depend on x, i.e., $\frac{\partial L}{\partial x} = 0$:

$$\frac{dL}{dx} = \frac{\partial L}{\partial y}\frac{dy}{dx} + \frac{\partial L}{\partial y'}\frac{dy'}{dx}$$

$$\frac{dL}{dx} = \frac{\partial L}{\partial y} y' + \frac{\partial L}{\partial y'} y'' \tag{2}$$

Using the Euler-Lagrange equation:

$$\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) - \frac{\partial L}{\partial y} = 0 \implies \frac{\partial L}{\partial y} = \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)$$

Multiply by y' on both sides, you get,

$$y' \frac{\partial L}{\partial y} = y' \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) \tag{3}$$

Now calculate the derivative of $y'\frac{\partial L}{\partial y}$:

$$\frac{d}{dx}\left(y'\frac{\partial L}{\partial y'}\right) = y''\frac{\partial L}{\partial y'} + y'\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)$$
$$y'\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = \frac{d}{dx}\left(y'\frac{\partial L}{\partial y'}\right) - y''\frac{\partial L}{\partial y'}$$

From equation no. (3), we get

$$y'\frac{\partial L}{\partial y} = \frac{d}{dx}\left(y'\frac{\partial L}{\partial y'}\right) - y''\frac{\partial L}{\partial y'}$$
$$y'\frac{\partial L}{\partial y} + y''\frac{\partial L}{\partial y'} = \frac{d}{dx}\left(y'\frac{\partial L}{\partial y'}\right)$$

From equation no. (2), we get,

$$\frac{dL}{dx} = \frac{d}{dx}\left(y'\frac{\partial L}{\partial y'}\right)$$

Integrating on both sides, we get,

$$L = y'\frac{\partial L}{\partial y'} + C_1$$

which is known as "Beltrami Identity"

Solving Lagrangian using Beltrami Identity

$$L - y'\frac{\partial L}{\partial y'} = C_1$$

$$L = \frac{\sqrt{1+(y')^2}}{\sqrt{y}} \implies \frac{\partial L}{\partial y'} = \frac{y'}{\sqrt{y}\sqrt{1+(y')^2}}$$

$$\implies \frac{\sqrt{1+(y')^2}}{\sqrt{y}} - \frac{(y')^2}{\sqrt{y}\sqrt{1+(y')^2}} = C_1$$

$$\frac{1}{\sqrt{y}\sqrt{1+(y')^2}} = C_1$$

$$1+(y')^2 = \frac{1}{C_1^2 y}$$

If you redefine $\frac{1}{C_1^2} \to c^2$, you get,

$$\implies 1+(y')^2 = \frac{c^2}{y}$$

which is equivalent to equation no. (1)

3 Sufficiency Conditions for the Brachistochrone Problem

Finding an extremal of a functional using the Euler–Lagrange equation only guarantees that the functional is *stationary* under small variations. This extremal could correspond to a minimum, a maximum, or a saddle point.

To determine whether it is truly a minimum, we check the following sufficiency conditions:

- **Legendre condition:** Ensures convexity of the Lagrangian in the slope y', preventing the extremal from being a maximum in that direction.
- **Jacobi condition:** Ensures there are no conjugate points in the interval, which could otherwise allow variations that lower the functional.
- **Weierstrass condition:** Excludes non-smooth or "corner-cutting" variations that could decrease the functional.

Together, these conditions guarantee that the extremal is a *strong local minimum* of the functional.

Euler-Lagrange Equation

The Euler–Lagrange equation is

$$\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) - \frac{\partial L}{\partial y} = 0.$$

Since the Lagrangian does not depend explicitly on x, the Beltrami identity applies:

$$L - y'\frac{\partial L}{\partial y'} = \text{constant}.$$

Substitution of L yields the cycloidal solution, hence the curve is an extremal.

The extremal is the cycloid

$$x(\theta) = a(\theta - \sin\theta),$$
$$y(\theta) = a(1 - \cos\theta).$$

$$\checkmark \quad \text{Euler–Lagrange condition satisfied}$$

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Legendre Condition

To examine the nature of the extremum, we evaluate the Legendre condition, which requires the second derivative of the Lagrangian with respect to the slope y' to be non-negative.

We compute

$$L_{y'y'} \equiv \frac{\partial^2 L}{\partial (y')^2}.$$

For the brachistochrone Lagrangian

$$L(y, y') = \frac{\sqrt{1+(y')^2}}{\sqrt{2gy}},$$

we obtain

$$L_{y'y'} = \frac{1}{\sqrt{2gy}} \frac{1}{\left(1+(y')^2\right)^{3/2}}.$$

Since

$$y > 0 \quad \text{and} \quad 1+(y')^2 > 0,$$

it follows that

$$L_{y'y'} > 0 \quad \text{everywhere along the curve.}$$

$$\checkmark \quad \text{Legendre condition satisfied.}$$

Jacobi Condition (No Conjugate Points)

The Jacobi equation is

$$\frac{d}{dx}\left(L_{y'y'}\,\eta'\right) - \left(L_{yy} - \frac{d}{dx} L_{yy'}\right)\eta = 0.$$

For autonomous problems, a Jacobi field is obtained by differentiating the extremal with respect to a parameter.

Let α be a parameter labeling nearby cycloids. Then

$$\eta(\theta) = \frac{\partial y(\theta, \alpha)}{\partial \alpha}.$$

For the cycloid,

$$y(\theta) = a(1 - \cos\theta) \quad \Rightarrow \quad \eta(\theta) \propto \sin\theta.$$

The first zero of η after $\theta = 0$ occurs at

$$\theta = \pi.$$

Physical endpoints satisfy $0 < \theta_2 < \pi$. Hence no conjugate points lie in (x_1, x_2).

$$\checkmark \quad \text{Jacobi condition satisfied}$$

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Weierstrass Condition

The Weierstrass condition provides a sufficient criterion for a *strong minimum.* It is formulated in terms of the Weierstrass excess function, defined as

$$E(x, y, y', v) = L(x, y, v) - L(x, y, y') - (v - y')\,\frac{\partial L}{\partial y'}.$$

Here,

$$L(x, y, v)$$

denotes the *time density associated with a competitor slope* v, while

$$L(x, y, y')$$

represents the *time density evaluated along the extremal slope* y'.

For the brachistochrone problem, the Lagrangian is

$$L(y, v) = \frac{\sqrt{1 + v^2}}{\sqrt{2gy}}.$$

A direct computation yields

$$E = \frac{1}{\sqrt{2gy}} \left[\sqrt{1 + v^2} - \sqrt{1 + (y')^2} - \frac{(v - y')\,y'}{\sqrt{1 + (y')^2}} \right].$$

Since $\frac{1}{\sqrt{2gy}} > 0$, the sign of the Weierstrass excess function E depends only on the quantity inside the brackets.

We therefore define

$$\Phi(v) = \sqrt{1+v^2} - \sqrt{1+(y')^2} - \frac{y'}{\sqrt{1+(y')^2}}(v-y').$$

Now observe that the function

$$f(v) = \sqrt{1+v^2}, \text{ which is convex}$$

So, $\Phi(v)$ becomes,

$$\Phi(v) = f(v) - f(y') - f'(y')(v-y')$$

For any convex function $f(v)$, the inequality

$$f(v) \geq f(v_0) + f'(v_0)(v-v_0)$$

holds for all v, with equality if and only if $v = v_0$.

Applying this inequality with $v_0 = y'$, we obtain

$$f(v) \geq f(y') + f'(y')(v-y')$$

$$f(v) - f(y') + f'(y')(v-y') \geq 0$$

Clearly $\Phi(v) \geq 0$, so,

$$E \geq 0,$$

with equality only when

$$v = y'.$$

✓ Weierstrass condition satisfied.

This excludes corner-cutting curves and establishes that the extremal corresponds to a *strong minimum.*

Conclusion

All four sufficiency conditions are satisfied:

$$\boxed{\text{The brachistochrone is a strict local minimum of time}}$$

4 Brachistochrone curve under linear gravity

We consider a an extension of the brachistochrone problem in which the gravitational acceleration is not constant but varies with vertical position according to

$$g(y) = g_0(1 - \alpha y),$$

where g_0 and α are positive constants. The motion takes place in the vertical xy-plane, with the y-axis directed vertically downward. The particle starts from rest and moves without friction.

Using conservation of mechanical energy, the speed of the particle at depth y is given by

$$\frac{1}{2}mv^2 = m\int_0^y g(y')\,dy'.$$

Substituting the form of $g(y)$,

$$\frac{1}{2}mv^2 = mg_0\int_0^y (1 - \alpha y')\,dy' = mg_0\left(y - \frac{\alpha y^2}{2}\right).$$

Hence, the speed as a function of y becomes

$$v(y) = \sqrt{2g_0\left(y - \frac{\alpha y^2}{2}\right)}.$$

For an infinitesimal arc length element

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx,$$

the corresponding time element is

$$dt = \frac{ds}{v} = \frac{\sqrt{1 + (y')^2}}{\sqrt{2g_0\left(y - \frac{\alpha y^2}{2}\right)}}\,dx.$$

Therefore, the total time of descent is expressed as the functional

$$T[y] = \int \frac{\sqrt{1 + (y')^2}}{\sqrt{2g_0\left(y - \frac{\alpha y^2}{2}\right)}}\,dx.$$

The brachistochrone curve in a non-uniform gravitational field is obtained by minimizing the functional $T[y]$.

Ignore constants, then we get,

$$L(y, y') = \frac{\sqrt{1 + (y')^2}}{\sqrt{y - \frac{\alpha y^2}{2}}}$$

Use Beltrami Identity:

$$L - y' \frac{\partial L}{\partial y'} = C$$

$$\frac{\sqrt{1 + y'^2}}{\sqrt{y - \frac{\alpha}{2} y^2}} - y' \left[\frac{y'}{\sqrt{y - \frac{\alpha}{2} y^2} \sqrt{1 + y'^2}} \right] = C$$

$$\frac{1}{\sqrt{y - \frac{\alpha}{2} y^2} \sqrt{1 + y'^2}} = C$$

$$y'^2 = \frac{1}{C^2 \left(y - \frac{\alpha}{2} y^2 \right)} - 1$$

We can only solve this differential equation using computational methods. For a fixed set of points, the deviation of the path from the cycloid depends upon $\frac{-\alpha y_f^2}{2}$ value.

5 Conclusion

In this work, we have revisited the classical brachistochrone problem and derived its solution using the calculus of variations. Under uniform gravity, the extremal curve is a cycloid, and we have verified that it satisfies the Euler–Lagrange equation as well as all sufficiency conditions for a strong minimum, including the Legendre, Jacobi, and Weierstrass criteria.

We have also extended the problem to a linearly varying gravitational field, deriving the corresponding time functional and differential equation. While this more general case generally requires numerical methods, it demonstrates how deviations from uniform gravity influence the path of quickest descent.

Overall, this study illustrates both the analytical elegance of classical variational methods and the practical considerations necessary when dealing with non-uniform gravitational fields, providing a comprehensive understanding of time-optimal paths in different gravity profiles.

References

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