Variational Analysis of Proper Time and Coordinate Time in Relativity

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Abstract

Variational principles play a central role in relativistic physics, where the motion of free particles is governed by the extremization of proper time. In this work, we derive the geodesic equation by extremizing the proper-time action in curved spacetime and then examine the extremization of coordinate time in a static, weak-field metric. By eliminating the proper-time parameter and taking the classical limit, we obtain an effective time functional whose extremization leads to a modified variational problem. A comparison with the classical brachistochrone reveals that relativistic time dilation prevents an exact correspondence, highlighting the distinction between proper and coordinate time extremization and clarifying the conditions under which classical results emerge.

1 Introduction

Variational principles provide a fundamental framework for describing physical motion. In general relativity, the trajectory of a free massive particle is obtained by extremizing the proper time along its worldline, leading to the geodesic equation and revealing the geometric nature of gravity.

Although proper time is the invariant quantity measured by a particle, coordinate time is often relevant in stationary spacetimes and practical applications. In such cases, it is meaningful to investigate whether particle trajectories can also be characterized by an extremization principle involving coordinate time, particularly in weak gravitational fields.

In this work, we derive the geodesic equation through the extremization of proper time and then examine the extremization of coordinate time in a static, weak-field spacetime. In the classical limit, the resulting time functional leads to a variational problem related to, but distinct from, the classical brachistochrone problem. This comparison highlights the differences between proper and coordinate time extremization and clarifies the role of relativistic corrections.

2 Extremization of Proper Time

In General Relativity, the trajectory of a free massive particle extremizes the **proper time** between two events:

$$\tau = \int d\tau, \qquad d\tau = \sqrt{-g_{\mu\nu}(x)\, dx^\mu dx^\nu}.$$

Parameterizing the worldline by an arbitrary parameter λ,

$$x^\mu = x^\mu(\lambda),$$

we have

$$d\tau = \sqrt{-g_{\mu\nu}(x)\, \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}\, d\lambda.$$

Thus the proper-time action becomes

$$S[\text{path}] = \int L\, d\lambda, \qquad L = \sqrt{-g_{\mu\nu}(x)\, \dot{x}^\mu \dot{x}^\nu},$$

where $\dot{x}^\mu \equiv dx^\mu/d\lambda$.

Remark: For mathematical convenience, we can equivalently use

$$L = \frac{1}{2} g_{\mu\nu}(x)\, \dot{x}^\mu \dot{x}^\nu,$$

since extremizing this gives the same geodesics.

Euler-Lagrange Equations

The Euler-Lagrange equations are

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) - \frac{\partial L}{\partial x^\mu} = 0.$$

Step 1: Compute $\partial L/\partial \dot{x}^\mu$

$$\frac{\partial L}{\partial \dot{x}^\mu} = \frac{\partial}{\partial \dot{x}^\mu}\left(\frac{1}{2}\, g_{\alpha\beta}(x)\, \dot{x}^\alpha \dot{x}^\beta\right) = \frac{1}{2}\, g_{\alpha\beta}(x)\, \frac{\partial}{\partial \dot{x}^\mu}\left(\dot{x}^\alpha \dot{x}^\beta\right)$$

Remark: (α, β) are dummy indices.

$$\frac{\partial}{\partial \dot{x}^\mu}\left(\dot{x}^\alpha \dot{x}^\beta\right) = \frac{\partial \dot{x}^\alpha}{\partial \dot{x}^\mu}\,\dot{x}^\beta + \dot{x}^\alpha\,\frac{\partial \dot{x}^\beta}{\partial \dot{x}^\mu}$$

$$\frac{\partial \dot{x}^\alpha}{\partial \dot{x}^\mu} = \delta^\alpha_\mu = \begin{cases} 1, & \alpha = \mu \\ 0, & \alpha \neq \mu \end{cases}$$

$$\Rightarrow \frac{\partial}{\partial \dot{x}^\mu}\left(\dot{x}^\alpha \dot{x}^\beta\right) = \delta^\alpha_\mu\,\dot{x}^\beta + \dot{x}^\alpha\,\delta^\beta_\mu$$

$$\frac{\partial L}{\partial \dot{x}^\mu} = \frac{1}{2}\,g_{\alpha\beta}\left(\delta^\alpha_\mu\,\dot{x}^\beta + \dot{x}^\alpha\,\delta^\beta_\mu\right)$$

$$g_{\alpha\beta}\,\delta^\alpha_\mu = g_{\mu\beta}, \qquad g_{\alpha\beta}\,\delta^\beta_\mu = g_{\alpha\mu}$$

$$\Rightarrow \frac{\partial L}{\partial \dot{x}^\mu} = \frac{1}{2}\left(g_{\mu\beta}\,\dot{x}^\beta + g_{\alpha\mu}\,\dot{x}^\alpha\right)$$

Since $g_{\mu\beta} = g_{\beta\mu}$,

$$= \frac{1}{2}\left(g_{\mu\beta}\,\dot{x}^\beta + g_{\mu\alpha}\,\dot{x}^\alpha\right)$$

α, β are dummy indices $\Rightarrow$ rename $\alpha, \beta \to \nu$

$$= \frac{1}{2}\left(g_{\mu\nu}\,\dot{x}^\nu + g_{\mu\nu}\,\dot{x}^\nu\right) = g_{\mu\nu}\,\dot{x}^\nu$$

$$= (\partial_\sigma g_{\mu\nu})\,\dot{x}^\sigma\,\dot{x}^\nu + g_{\mu\nu}\,\ddot{x}^\nu$$

$$\frac{\partial L}{\partial \dot{x}^\mu} = \frac{1}{2}\frac{\partial}{\partial \dot{x}^\mu}\left(g_{\alpha\beta}\,\dot{x}^\alpha \dot{x}^\beta\right) = g_{\mu\nu}\,\dot{x}^\nu.$$

Step 2: Take the Total Derivative

$$\frac{d}{d\lambda}\left(g_{\mu\nu}\,\dot{x}^\nu\right) = \frac{\partial g_{\mu\nu}}{\partial x^\sigma}\frac{dx^\sigma}{d\lambda}\dot{x}^\nu + g_{\mu\nu}\,\ddot{x}^\nu$$

$$= \frac{\partial g_{\mu\nu}}{\partial x^\sigma}\dot{x}^\sigma \dot{x}^\nu + g_{\mu\nu}\,\ddot{x}^\nu$$

$$= (\partial_\sigma g_{\mu\nu})\,\dot{x}^\sigma\,\dot{x}^\nu + g_{\mu\nu}\,\ddot{x}^\nu$$

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) = (\partial_\sigma g_{\mu\nu})\,\dot{x}^\sigma\,\dot{x}^\nu + g_{\mu\nu}\,\ddot{x}^\nu$$

Step 3: Compute $\partial L/\partial x^\mu$

$$\frac{\partial L}{\partial x^\mu} = \frac{1}{2}\frac{\partial}{\partial x^\mu}\left(g_{\alpha\beta}(x)\,\dot{x}^\alpha\dot{x}^\beta\right)$$

$$= \frac{1}{2}\left(\frac{\partial g_{\alpha\beta}(x)}{\partial x^\mu}\right)\dot{x}^\alpha\dot{x}^\beta = \frac{1}{2}\,\partial_\mu g_{\alpha\beta}\,\dot{x}^\alpha\dot{x}^\beta$$

$$\frac{\partial L}{\partial x^\mu} = \frac{1}{2}\,\partial_\mu g_{\alpha\beta}\,\dot{x}^\alpha\dot{x}^\beta$$

Assemble the Euler-Lagrange Equation

Combining the results,

$$g_{\nu\mu}\,\ddot{x}^\mu + (\partial_\sigma g_{\nu\mu})\,\dot{x}^\sigma\dot{x}^\mu - \frac{1}{2}\,\partial_\nu g_{\alpha\beta}\,\dot{x}^\alpha\dot{x}^\beta = 0$$

Raise the Index

Multiplying by the inverse metric $g^{\mu\rho}$,

$$g^{\rho\nu}g_{\nu\mu}\,\ddot{x}^\mu + g^{\rho\nu}(\partial_\sigma g_{\nu\mu})\,\dot{x}^\sigma\dot{x}^\mu - \frac{1}{2}\,g^{\rho\nu}\partial_\nu g_{\alpha\beta}\,\dot{x}^\alpha\dot{x}^\beta = 0$$

$$g^{\rho\nu}g_{\nu\mu} = \delta^\rho_\mu \rightarrow \text{forces } \rho = \nu$$

$$\ddot{x}^\rho + g^{\rho\nu}(\partial_\sigma g_{\nu\mu})\,\dot{x}^\sigma\dot{x}^\mu - \frac{1}{2}\,g^{\rho\nu}\partial_\nu g_{\alpha\beta}\,\dot{x}^\alpha\dot{x}^\beta = 0$$

Using the same summation logic used in the computation of $\partial L/\partial\dot{x}^\mu$, we get,

$$\partial_\sigma g_{\nu\mu} = \frac{1}{2}\left(\partial_\sigma g_{\nu\mu} + \partial_\mu g_{\nu\sigma}\right) \quad \text{(these two are the same under index symmetry)}$$

Rearranging indices symmetrically leads to the standard form:

$$\Rightarrow \quad \ddot{x}^\rho + \frac{1}{2}\,g^{\rho\nu}\left(\partial_\mu g_{\nu\sigma} + \partial_\sigma g_{\nu\mu} - \partial_\nu g_{\mu\sigma}\right)\dot{x}^\mu\dot{x}^\sigma = 0$$

Christoffel Symbols

Define the Christoffel symbols of the second kind:

$$\Gamma^\rho{}_{\nu\sigma} = \frac{1}{2}g^{\rho\mu}\left(\partial_\nu g_{\mu\sigma} + \partial_\sigma g_{\mu\nu} - \partial_\mu g_{\nu\sigma}\right).$$

Geodesic Equation

Substituting into the equation of motion yields the geodesic equation:

$$\boxed{\ddot{x}^\rho + \Gamma^\rho_{\ \nu\sigma}\,\dot{x}^\nu \dot{x}^\sigma = 0}$$

This equation describes the motion of a free particle in curved spacetime.

3 Extremization of Coordinate Time

$$ds^2 = dx^2 + dy^2$$

$$ds^2 = dx^2 + dy^2 - \left(1 + \frac{2\phi}{c^2}\right) c^2 dt^2$$

Consider

$$1 + \frac{2\phi}{c^2} \equiv \Phi(x)$$

$$ds^2 = dx^2 + dy^2 - \Phi(x)\, c^2 dt^2$$

We know that

$$ds^2 = -c^2 d\tau^2$$

Hence,

$$-c^2 d\tau^2 = dx^2 + dy^2 - \Phi(x)\, c^2 dt^2$$

Rearranging,

$$dt^2 = \frac{dx^2 + dy^2 + c^2 d\tau^2}{\Phi(x)\, c^2}$$

$$dt = \frac{1}{c}\sqrt{\frac{dx^2 + dy^2 + c^2 d\tau^2}{\Phi(x)}}$$

Therefore, the coordinate time functional is

$$T = \int \frac{1}{c}\sqrt{\frac{dx^2 + dy^2 + c^2 d\tau^2}{\Phi(x)}}$$

Write

$$dx^2 + dy^2 = \delta_{ij}\,\dot{x}^i \dot{x}^j$$

Thus,

$$T = \int \sqrt{\frac{\frac{\delta_{ij}\,\dot{x}^i\dot{x}^j}{c^2} + \dot{\tau}^2}{\Phi(x)}}$$

Eliminating $\dot{\tau}$

For now, consider this Lagrangian:

$$\mathcal{L} = \sqrt{\frac{\frac{\delta_{ij}\,\dot{x}^i\dot{x}^j}{c^2} + \dot{\tau}^2}{\Phi(x)}}$$

$$\tau \text{ is cyclic } \Rightarrow \frac{\partial \mathcal{L}}{\partial \dot{\tau}} = \text{constant}$$

So,

$$E = \frac{\partial \mathcal{L}}{\partial \dot{\tau}} = \frac{1}{\sqrt{\Phi(x)}} \frac{\dot{\tau}}{\sqrt{\frac{\delta_{ij}\dot{x}^i\dot{x}^j}{c^2} + \dot{\tau}^2}}$$

Squaring,

$$E^2 = \frac{1}{\Phi(x)} \frac{\dot{\tau}^2}{\frac{\delta_{ij}\dot{x}^i\dot{x}^j}{c^2} + \dot{\tau}^2}$$

Rearranging,

$$\frac{\frac{\delta_{ij}\dot{x}^i\dot{x}^j}{c^2} + \dot{\tau}^2}{\dot{\tau}^2} = \frac{1}{E^2\Phi(x)}$$

$$\frac{\delta_{ij}\dot{x}^i\dot{x}^j}{c^2\dot{\tau}^2} + 1 = \frac{1}{E^2\Phi(x)}$$

$$\frac{\delta_{ij}\dot{x}^i\dot{x}^j}{c^2\dot{\tau}^2} = \frac{1 - E^2\Phi(x)}{E^2\Phi(x)}$$

Hence,

$$\dot{\tau}^2 = \frac{E^2\Phi(x)}{1 - E^2\Phi(x)} \left(\frac{\delta_{ij}\dot{x}^i\dot{x}^j}{c^2} \right)$$

Rewriting the Time functional,

$$T = \int \frac{1}{\sqrt{\Phi(x)}} \sqrt{\frac{\delta_{ij}\dot{x}^i\dot{x}^j}{c^2} + \frac{E^2\Phi(x)}{1 - E^2\Phi(x)} \left(\frac{\delta_{ij}\dot{x}^i\dot{x}^j}{c^2} \right)}$$

$$T = \int \frac{1}{\sqrt{\Phi(x)}} \sqrt{\frac{\delta_{ij}\dot{x}^i\dot{x}^j}{c^2} \left[1 + \frac{E^2\Phi(x)}{1 - E^2\Phi(x)} \right]}$$

$$\mathcal{L} = \frac{1}{\sqrt{\Phi(x)}}\sqrt{\frac{\delta_{ij}\dot{x}^i\dot{x}^j}{c^2}\frac{1}{1-E^2\Phi(x)}}$$

Applying classical limit

Extremizing coordinate time is not a fundamental variational principle in GR. It becomes meaningful only in static, weak-field limits where a preferred time coordinate exists. When we go into the classical limit,

$$\sqrt{1-E^2\Phi(x)} \;\approx\; 1-\frac{E^2\Phi(x)}{2} \;\approx\; 1$$

So,

$$T = \int \frac{1}{\sqrt{\Phi(x)}}\sqrt{\frac{\delta_{ij}\dot{x}^i\dot{x}^j}{c^2}}$$

For weak-field approximation,

$$\Phi = \left(1+\frac{2\phi}{c^2}\right) \approx \left(1+\frac{2gy}{c^2}\right)$$

$$\frac{1}{\sqrt{\Phi}} \approx \left(1+\frac{2gy}{c^2}\right)^{-1/2} \approx \left(1-\frac{gy}{c^2}\right), \qquad (gy \ll c^2)$$

So the Time functional becomes,

$$T = \int \frac{1}{c}\left(1-\frac{gy}{c^2}\right)\sqrt{\delta_{ij}\dot{x}^i\dot{x}^j}$$

Executing δ_{ij},

$$T = \int \frac{1}{c}\left(1-\frac{gy}{c^2}\right)\sqrt{dx^2+dy^2}$$

$$T = \int \frac{1}{c}\left(1-\frac{gy}{c^2}\right)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$\Rightarrow \quad T[y(x)] = \int \frac{1}{c}\,(1-\frac{gy}{c^2})\sqrt{(1+y'^2)}\,dx$$

The Lagrangian is

$$L(y,y') = \sqrt{(1+y'^2)}\left(1-\frac{gy}{c^2}\right)$$

Since the Lagrangian has no explicit x-dependence,

$$\frac{\partial L}{\partial x} = 0,$$

The Beltrami identity applies.

Beltrami Identity

If $L = L(y, y')$ and $\partial L/\partial x = 0$, then

$$L - y'\frac{\partial L}{\partial y'} = \text{constant}.$$

Step 1: Compute $\partial L/\partial y'$

$$\frac{\partial L}{\partial y'} = \frac{1}{2}\left(1 - \frac{gy}{c^2}\right)(1 + y'^2)^{-1/2}(2y') = \left(1 - \frac{gy}{c^2}\right)\frac{y'}{\sqrt{1 + y'^2}}.$$

Step 2: Compute $L - y'\partial L/\partial y'$

$$\begin{aligned} L - y'\frac{\partial L}{\partial y'} &= \sqrt{1 + y'^2}\left(1 - \frac{gy}{c^2}\right) - y'\left(1 - \frac{gy}{c^2}\right)\frac{y'}{\sqrt{1 + y'^2}} \\ &= \left(1 - \frac{gy}{c^2}\right)\left[\sqrt{1 + y'^2} - \frac{y'^2}{\sqrt{1 + y'^2}}\right] \\ &= \left(1 - \frac{gy}{c^2}\right)\frac{1}{\sqrt{1 + y'^2}}. \end{aligned}$$

First Integral

Thus the Beltrami first integral is

$$\frac{1 - \dfrac{gy}{c^2}}{\sqrt{1 + y'^2}} = C,$$

where C is a constant.

Solving for the Slope

$$\begin{aligned} \sqrt{1 + y'^2} &= \frac{1 - \dfrac{gy}{c^2}}{C}, \\ 1 + y'^2 &= \frac{\left(1 - \dfrac{gy}{c^2}\right)^2}{C^2}, \\ y'^2 &= \frac{\left(1 - \dfrac{gy}{c^2}\right)^2}{C^2} - 1. \end{aligned}$$

This equation doesn't resemble the classical brachistochrone curve equation:

$$(y')^2 = \frac{c^2}{y} - 1$$

The classical brachistochrone extremizes Newtonian travel time, whereas here we extremize coordinate time derived from a relativistic metric. The difference reflects relativistic redshift effects and the non-equivalence of proper and coordinate time.

Therefore, we have to do a computational analysis for this differential equation.

4 Conclusion

We have analyzed variational principles based on both proper time and coordinate time in a relativistic setting. Extremization of proper time yields the geodesic equation, confirming that free particles follow paths determined by spacetime geometry.

A key outcome of this analysis is that the resulting differential equation does not reduce to the classical brachistochrone curve. This deviation highlights the influence of gravitational time dilation and demonstrates that extremizing coordinate time is not, in general, equivalent to extremizing Newtonian travel time. The classical brachistochrone emerges only under additional approximations that neglect relativistic corrections.

Overall, this study clarifies the conceptual and mathematical distinction between proper-time and coordinate-time extremization and illustrates how classical variational problems arise as limiting cases of relativistic principles. The approach presented here provides a useful bridge between classical mechanics and general relativity and offers a framework for further extensions, including stronger gravitational fields or fully relativistic numerical analyses.

References

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