Theoretical Calculation of Mercury's Perihelion Precession

Kalamata Chaitanya

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Abstract

We analyze Mercury's perihelion precession using both Newtonian mechanics and General Relativity. Starting from the classical Lagrangian, we derive Mercury's elliptical orbit and then include relativistic corrections via the Schwarzschild metric. Using a perturbative approach, we calculate the small relativistic shift in the orbit, leading to the famous precession. Numerical evaluation with Mercury's orbital parameters gives a precession of about 43 arcseconds per century, in excellent agreement with observations. This study highlights how relativistic effects refine classical predictions and provides a clear example of Einstein's theory in action.

1 Introduction

The motion of planets around the Sun has been a central problem in classical and modern physics. While Newton's law of universal gravitation successfully describes most planetary orbits, it fails to account for the observed precession of Mercury's perihelion fully. Mercury's orbit around the Sun reveals a small but measurable precession of its perihelion, which Newtonian mechanics cannot fully explain. This discrepancy was one of the early confirmations of Einstein's General Relativity, which accounts for the additional precession due to spacetime curvature.

In this work, we first derive Mercury's orbit using the Newtonian Lagrangian formalism, highlighting the elliptical nature of the orbit and the role of angular momentum. We then extend the analysis using the Schwarzschild metric to obtain the relativistic orbit equation. Applying a perturbative approach, we calculate the relativistic contribution to Mercury's perihelion precession and perform a numerical evaluation using the actual orbital parameters of Mercury. The results reproduce the observed precession of approximately 43 arcseconds per century, demonstrating the predictive power of General Relativity and the subtle differences between classical and relativistic orbital mechanics.

2 Mercury's Orbit: Newton's Approach

Lagrangian for Central Force

The Lagrangian L_{ag} in spherical coordinates is:

$$L_{ag} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\dot{\phi}^2) - V(r)$$

Assuming motion in a plane ($\theta = \pi/2$), we have:

$$L_{ag} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - V(r)$$

Radial Equation of Motion

Using the Euler-Lagrange equation for the radial coordinate r:

$$\frac{d}{dt}\left(\frac{\partial L_{ag}}{\partial \dot{r}}\right) - \frac{\partial L_{ag}}{\partial r} = 0$$

$$\frac{d}{dt}[\mu\dot{r}] - \mu r\dot{\phi}^2 + \frac{\partial V(r)}{\partial r} = 0$$

$$\mu\ddot{r} - \mu r\dot{\phi}^2 = -\frac{\partial V}{\partial r} \implies F(r) = \mu(\ddot{r} - r\dot{\phi}^2)$$

Conservation of Angular Momentum

Angular momentum L is given by:

$$L = \mu r^2\dot{\phi} \implies \dot{\phi} = \frac{L}{\mu r^2}$$

If you redefine $L \to L/\mu$, you get,

$$\dot{\phi} = \frac{L}{r^2}$$

Transformation of Variables

Introduce $u(\phi) = 1/r \implies r = 1/u$. The time derivative operator becomes

$$\frac{d}{dt} = \frac{d\phi}{dt}\frac{d}{d\phi} = \dot{\phi}\frac{d}{d\phi} = \frac{L}{r^2}\frac{d}{d\phi} = Lu^2\frac{d}{d\phi}$$

Calculating radial velocity $\dot{r}$:

$$\dot{r} = \frac{d}{dt}r = Lu^2\frac{d}{d\phi}\left(\frac{1}{u}\right) = Lu^2\left(-\frac{1}{u^2}\frac{du}{d\phi}\right) = L\frac{du}{d\phi}$$

Radial Acceleration $\ddot{r}$

$$\ddot{r} = \frac{d}{dt}(\dot{r}) = Lu^2\frac{d}{d\phi}\left(-L\frac{du}{d\phi}\right)$$

$$\boxed{\ddot{r} = -L^2u^2\frac{d^2u}{d\phi^2}}$$

Calculating the centrifugal term $r\dot{\phi}^2$:

$$r\dot{\phi}^2 = r\left(\frac{L}{r^2}\right)^2 = \frac{L^2}{r^3} = L^2u^3$$

$$\boxed{r\dot{\phi}^2 = L^2u^3}$$

Final Force Equation $F(r)$

Now substitute into $F(r) = \mu(\ddot{r} - r\dot{\phi}^2)$:

$$F(r) = \mu \left[-L^2 u^2 \frac{d^2u}{d\phi^2} - L^2 u^3 \right]$$

$$F(r) = -L^2 u^2 \mu \left[\frac{d^2u}{d\phi^2} + u \right]$$

Rearranging for the differential equation:

$$\boxed{\frac{d^2u}{d\phi^2} + u = -\frac{1}{\mu L^2 u^2} F(1/u)}$$

where $u = 1/r$.

Final form for gravitational force

$$F(r) = -\frac{GM\mu}{r^2} \quad \Rightarrow \quad F\left(\frac{1}{u}\right) = -GM\mu\, u^2$$

$$\frac{d^2u}{d\phi^2} + u = -\frac{1}{\mu L^2 u^2} \left(-GM\mu\, u^2 \right)$$

$$\frac{d^2u}{d\phi^2} + u = \frac{GM}{L^2}$$

Solution for equation no. (1) is

$$u = \frac{GM}{L^2} + A\cos\phi$$

$$u = \frac{GM}{L^2} \left[1 + \frac{AL^2}{GM} \cos\phi \right]$$

It looks like an ellipse's equation and resembles

$$u = \frac{1}{P}[1 + e\cos\phi]$$

Rewriting our equation, we get

$$\boxed{u = \frac{GM}{L^2}[1 + e\cos\phi]}$$

which is a perfectly closed ellipse.

3 Mercury's Orbit: Einstein's Approach

The spacetime metric outside a spherically symmetric mass M is given by the Schwarzschild solution:

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2 \right) \tag{1}$$

By spherical symmetry, the motion can be confined to the equatorial plane:

$$\theta = \frac{\pi}{2}, \quad \dot{\theta} = 0$$

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 d\phi^2 \tag{2}$$

Lagrangian for Timelike Geodesics

$$\begin{aligned} ds^2 &= g_{\mu\nu}\, dx^\mu dx^\nu \\ x^\mu &= x^\mu(\tau) \\ ds^2 &= g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}\, d\tau^2 \\ ds^2 &= g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu\, d\tau^2 \\ ds &= \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}\, d\tau \end{aligned} \tag{3}$$

$$s = \int ds \quad \Rightarrow \quad S = \int L\, d\tau, \quad \text{with} \quad L = \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}$$

$$\text{Alternatively, one often uses } L = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$$

By comparing (2) and (3), we get

$$g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = \left[-\left(1 - \frac{2GM}{rc^2}\right) c^2 \dot{t}^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} \dot{r}^2 + r^2 \dot{\phi}^2 \right] \tag{4}$$

Using the equation no. (4), the Lagrangian is

$$L = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = \frac{1}{2} \left[-\left(1 - \frac{2GM}{rc^2}\right) c^2 \dot{t}^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} \dot{r}^2 + r^2 \dot{\phi}^2 \right] \tag{5}$$

Conserved Quantities

Due to the explicit independence of L on t and ϕ, the corresponding conserved quantities are:

Energy Conservation (t cyclic)

$$\frac{\partial L}{\partial \dot{t}} = -\left(1 - \frac{2GM}{rc^2}\right) c^2 \dot{t} \equiv -E \quad \Rightarrow \quad \dot{t} = \frac{E}{c^2 \left(1 - \frac{2GM}{rc^2}\right)}, \text{where E is a constant} \tag{6}$$

Angular Momentum Conservation (ϕ cyclic)

$$\frac{\partial L}{\partial \dot{\phi}} = r^2 \dot{\phi} \equiv L \quad \Rightarrow \quad \dot{\phi} = \frac{L}{r^2}, \text{where L is a constant} \tag{7}$$

Radial Equation of Motion

For a massive particle:

$$ds^2 = -c^2 d\tau^2$$

We know

$$ds^2 = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu \, d\tau^2$$

By comparing, we get this

$$g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = -c^2 \tag{8}$$

Substitute the equation no. (4) in equation no. (8), we get,

$$-\left(1 - \frac{2GM}{rc^2}\right) c^2 \dot{t}^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} \dot{r}^2 + r^2 \dot{\phi}^2 = -c^2 \tag{9}$$

By substituting conserved quantities [equation no. (5) and (6)],

$$-\left(1 - \frac{2GM}{rc^2}\right) c^2 \frac{E^2}{c^4 \left[1 - \frac{2GM}{rc^2}\right]^2} + \left(1 - \frac{2GM}{rc^2}\right)^{-1} \dot{r}^2 + r^2 \frac{L^2}{r^4} = -c^2$$

$$-\left(1 - \frac{2GM}{rc^2}\right)^{-1} \frac{E^2}{c^2} + \left(1 - \frac{2GM}{rc^2}\right)^{-1} \dot{r}^2 + \frac{L^2}{r^2} = -c^2$$

$$-\frac{E^2}{c^2} + \dot{r}^2 + \left(\frac{L^2}{r^2} + c^2\right)\left(1 - \frac{2GM}{rc^2}\right) = 0$$

$$\dot{r}^2 = \frac{E^2}{c^2} - \left(1 - \frac{2GM}{rc^2}\right)\left(\frac{L^2}{r^2} + c^2\right)$$

If you redefine $E \to E/c$, you get

$$\dot{r}^2 = E^2 - \left(1 - \frac{2GM}{rc^2}\right)\left(c^2 + \frac{L^2}{r^2}\right) \tag{10}$$

Relativistic Orbit Equation

Introduce the reciprocal variable

$$u \equiv \frac{1}{r} \quad \Rightarrow \quad \dot{r} = -L\frac{du}{d\phi}$$

After algebra:

$$\left(\frac{du}{d\phi}\right)^2 = \frac{E^2}{L^2} - \left(1 - \frac{2GMu}{c^2}\right)\left(\frac{c^2}{L^2} + u^2\right)$$

Differentiating once with respect to ϕ,

$$2\left(\frac{du}{d\phi}\right)\frac{d^2u}{d\phi^2} = -\left[\left(1 - \frac{2GMu}{c^2}\right)\left(2u\frac{du}{d\phi}\right) + \left(\frac{c^2}{L^2} + u^2\right)\left(\frac{-2GM}{c^2}\frac{du}{d\phi}\right)\right]$$

$$-2\frac{d^2u}{d\phi^2} = 2u - 2u\left[\frac{2GMu}{c^2}\right] - \frac{2GM}{L^2} - \frac{2GM}{c^2}u^2$$

$$\frac{d^2u}{d\phi^2} + u = \frac{GM}{L^2} + \frac{3GM}{c^2}u^2 \tag{11}$$

which is the exact relativistic orbit equation.

We know, the Newtonian orbit equation, which is

$$\frac{d^2u_0}{d\phi^2} + u_0 = \frac{GM}{L^2} \tag{12}$$

and by solving this, we get

$$u_0(\phi) = \frac{GM}{L^2}\left(1 + e\cos\phi\right) \tag{13}$$

Perturbative Solution: Relativistic Correction

By observation, we can consider,

$$u = u_0 + u_1, \text{where } u_0 \text{ is Newtonian solution and } u_1 \text{ is small relativistic correction}$$

Substitute u in equation no. (11),

$$\frac{d^2(u_0+u_1)}{d\phi^2} + u_0 + u_1 = \frac{GM}{L^2} + \frac{3GM}{c^2}(u_0+u_1)^2$$

Drop u_1^2 and u_1u_0 terms,

$$\frac{d^2u_0}{d\phi^2} + \frac{d^2u_1}{d\phi^2} + u_0 + u_1 = \frac{GM}{L^2} + \frac{3GM}{c^2}u_0^2$$

By equation no. (12),

$$\frac{d^2u_1}{d\phi^2} + u_1 = \frac{3GM}{c^2}u_0^2$$

Square equation no. (13),

$$u_0^2 = \left(\frac{GM}{L^2}\right)^2 [1 + e\cos\phi]^2 \tag{14}$$

$$= \left(\frac{GM}{L^2}\right)^2 [1 + e^2\cos^2\phi + 2e\cos\phi] \tag{15}$$

We know, $\cos^2\phi = \frac{1+\cos 2\phi}{2}$, So,

$$u_0^2 = \left(\frac{GM}{L^2}\right)^2 \left[1 + 2e\cos\phi + \frac{e^2}{2} + \frac{e^2}{2}\cos 2\phi\right]$$

Substituting u_0^2 in u_2'' equation,

$$\frac{d^2u_1}{d\phi^2} + u_1 = \left(\frac{GM}{L^2}\right)^2\left(\frac{3GM}{c^2}\right)[2e\cos\phi] + C_0 + C_2\cos 2\phi \tag{16}$$

1. Constant Forcing Term C_0:

   We solve the differential equation

   $$\frac{d^2u_1}{d\phi^2} + u_1 = C_0.$$

   Try a constant solution:

   $$u_1 = A.$$

   Substituting,

   $$0 + A = C_0 \quad\Rightarrow\quad A = C_0.$$

   Hence, this term contributes

   $$u \longrightarrow u + \text{constant}.$$

Physical meaning:

- Shifts the average orbital radius.
- Slightly changes the semi-major axis.
- Independent of $\phi \Rightarrow$ cannot rotate the orbit.

2. $\cos 2\phi$ Forcing Term:
Now consider

$$\frac{d^2u_1}{d\phi^2} + u_1 = C_2 \cos 2\phi.$$

Try the ansatz

$$u_1 = A \cos 2\phi.$$

Then

$$u_1'' = -4A \cos 2\phi,$$

and

$$u_1'' + u_1 = (-4A + A) \cos 2\phi = -3A \cos 2\phi.$$

Matching coefficients gives

$$-3A = C_2 \quad \Rightarrow \quad A = -\frac{C_2}{3}.$$

Physical meaning:

- Produces a bounded oscillation.
- Slightly distorts the orbital shape (more "squashed" ellipse).
- Driving frequency $\neq 1 \Rightarrow$ no resonance.
- No secular (unbounded) growth.

So, the only meaningful term is $\cos \phi$. So, the equation no. (14) become,

$$\frac{d^2u_1}{d\phi^2} + u_1 = \left(\frac{GM}{L^2}\right)^2 \left(\frac{3GM}{c^2}\right) [2e \cos \phi]$$

$$\frac{d^2u_1}{d\phi^2} + u_1 = \alpha \cos \phi \text{ , where } \alpha = \frac{6G^3M^3e}{c^2L^4}$$

Solving this, we get,

$$u_1 = \frac{\alpha}{2} \phi \sin \phi$$

Perihelion Precession of Mercury

At first, we considered,

$$u = u_0 + u_1$$

By substituting u_0 and u_1,

$$u = \frac{GM}{L^2}[1 + e \cos \phi] + \frac{\alpha}{2} \phi \sin \phi$$

$$u = \frac{GM}{L^2} \left[1 + e \cos \phi + \frac{\alpha L^2}{2GM} \phi \sin \phi\right]$$

Let $\frac{\alpha L^2}{2GM} = \varepsilon \ll 1$

$$u = \frac{GM}{L^2}[1 + e\cos\phi + \varepsilon\phi\sin\phi]$$

$$u = \frac{GM}{L^2}\left[1 + e\left(\cos\phi + \frac{\varepsilon}{e}\phi\sin\phi\right)\right]$$

We know,

$$\cos\big((1-\delta)\phi\big) = \cos\phi + \delta\,\phi\sin\phi + O(\delta^2)$$

Neglecting δ^2 terms, we get,

$$\cos\big((1-\delta)\phi\big) = \cos\phi + \delta\,\phi\sin\phi + O(\delta^2)$$

Put $\delta = \varepsilon/e$. Then,

$$\cos\left[(1-\varepsilon/e)\phi\right] = \cos\phi + \frac{\varepsilon}{e}\phi\sin\phi \tag{17}$$

So, the equation becomes,

$$u = \frac{GM}{L^2}\left[1 + e\cos\left((1-\delta)\phi\right)\right] \tag{18}$$

where,"

$$\delta = \frac{\varepsilon}{e} = \frac{\alpha L^2}{2GMe} = \frac{3G^3M^3}{c^2L^4}$$

For a Newtonian Keplerian orbit, the (specific) angular momentum satisfies

$$L^2 = GMa(1-e^2).$$

Hence,

$$L^4 = G^2M^2a^2(1-e^2)^2.$$

Substituting into the expression for δ, we obtain

$$\delta = \frac{3G^3M^3}{c^2G^2M^2a^2(1-e^2)^2} = \frac{3GM}{a(1-e^2)c^2}.$$

$$\boxed{\delta = \frac{3GM}{a(1-e^2)c^2}}$$

The advance per orbit:

$$\Delta\phi = 2\pi\delta = \frac{6\pi GM}{a(1-e^2)c^2} \tag{19}$$

This is the standard general relativistic formula for the perihelion precession of Mercury.

4 Numerical Calculation of Mercury's Perihelion Precession

The dimensionless relativistic correction is given by

$$\delta = \frac{3GM}{a(1-e^2)c^2},$$

where

- G = gravitational constant

- M = mass of the Sun
- a = semi-major axis of Mercury
- e = eccentricity of Mercury's orbit
- c = speed of light

Constants (SI Units):

$$\begin{aligned} G &= 6.674 \times 10^{-11}\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2} \\ M &= 1.989 \times 10^{30}\,\mathrm{kg} \\ c &= 3.00 \times 10^{8}\,\mathrm{m/s} \\ a &= 5.79 \times 10^{10}\,\mathrm{m} \\ e &= 0.206 \end{aligned}$$

Step-by-step numerical evaluation

$$e^2 = (0.206)^2 = 0.0424, \quad 1 - e^2 = 1 - 0.0424 = 0.9576$$

$$GM = (6.674 \times 10^{-11})(1.989 \times 10^{30}) = 1.327 \times 10^{20}\,\mathrm{m}^3/\mathrm{s}^2$$

$$3GM = 3 \times 1.327 \times 10^{20} = 3.981 \times 10^{20}$$

$$c^2 = (3.00 \times 10^{8})^2 = 9.00 \times 10^{16}$$

$$a(1 - e^2) = (5.79 \times 10^{10})(0.9576) = 5.54 \times 10^{10}$$

$$a(1 - e^2)c^2 = (5.54 \times 10^{10})(9.00 \times 10^{16}) = 4.986 \times 10^{27}$$

Dimensionless relativistic parameter

$$\delta = \frac{3GM}{a(1-e^2)c^2} = \frac{3.981 \times 10^{20}}{4.986 \times 10^{27}} \approx 7.99 \times 10^{-8} \approx 8.0 \times 10^{-8}$$

Precession per orbit

$$\Delta\phi_{\mathrm{orbit}} = 2\pi\delta = 2\pi \times 7.99 \times 10^{-8} \approx 5.02 \times 10^{-7}\,\mathrm{rad}$$

Converting to arcseconds:

$$1\,\mathrm{rad} = 206265'' \quad \Rightarrow \quad \Delta\phi_{\mathrm{orbit}} = (5.02 \times 10^{-7})(206265) \approx 0.1036''$$

$$T = 0.2408\,\mathrm{yr} \quad \Rightarrow \quad \text{orbits per century} = \frac{100}{0.2408} \approx 415.2$$

Precession per Century

Total Precession per Century is

$$\Delta\phi_{\mathrm{century}} = 0.1036'' \times 415.2 \approx 43.0''$$

This agrees with the observed general relativistic precession of Mercury.

5 Conclusion

In this work, we studied Mercury's perihelion precession using both Newtonian mechanics and General Relativity. While the Newtonian analysis correctly predicts an elliptical orbit, it fails to account for the observed advance of the perihelion. By introducing relativistic corrections through the Schwarzschild metric and solving the resulting orbit equation perturbatively, we obtained an additional precession term. A numerical evaluation using Mercury's orbital parameters yields a precession of approximately 43 arcseconds per century, in excellent agreement with observations. This result highlights the necessity of General Relativity for accurately describing planetary motion in strong gravitational fields and stands as a classic confirmation of Einstein's theory.