

# Theoretical Calculation of Mercury's Perihelion Precession

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## **Abstract**

We analyze Mercury's perihelion precession using both Newtonian mechanics and General Relativity. Starting from the classical Lagrangian, we derive Mercury's elliptical orbit and then include relativistic corrections via the Schwarzschild metric. Using a perturbative approach, we calculate the small relativistic shift in the orbit, leading to the famous precession. Numerical evaluation with Mercury's orbital parameters gives a precession of about 43 arcseconds per century, in excellent agreement with observations. This study highlights how relativistic effects refine classical predictions and provides a clear example of Einstein's theory in action.

## 1 Introduction

About Mercury's orbit

## 2 Mercury's Orbit: Newton's Approach

### Lagrangian for Central Force

The Lagrangian  $L_{ag}$  in spherical coordinates is:

$$L_{ag} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2) - V(r)$$

Assuming motion in a plane ( $\theta = \pi/2$ ), we have:

$$L_{ag} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - V(r)$$

### Radial Equation of Motion

Using the Euler-Lagrange equation for the radial coordinate  $r$ :

$$\frac{d}{dt} \left( \frac{\partial L_{ag}}{\partial \dot{r}} \right) - \frac{\partial L_{ag}}{\partial r} = 0$$

$$\frac{d}{dt}[\mu\dot{r}] - \mu r\dot{\phi}^2 + \frac{\partial V(r)}{\partial r} = 0$$

$$\mu\ddot{r} - \mu r\dot{\phi}^2 = -\frac{\partial V}{\partial r} \implies F(r) = \mu(\ddot{r} - r\dot{\phi}^2)$$

### Conservation of Angular Momentum

Angular momentum  $L$  is given by:

$$L = \mu r^2 \dot{\phi} \implies \dot{\phi} = \frac{L}{\mu r^2}$$

If you redefine  $L \rightarrow L/\mu$ , you get,

$$\dot{\phi} = \frac{L}{r^2}$$

### Transformation of Variables

Introduce  $u(\phi) = 1/r \implies r = 1/u$ . The time derivative operator becomes:

$$\frac{d}{dt} = \frac{d\phi}{dt} \frac{d}{d\phi} = \dot{\phi} \frac{d}{d\phi} = \frac{L}{r^2} \frac{d}{d\phi} = Lu^2 \frac{d}{d\phi}$$

Calculating radial velocity  $\dot{r}$ :

$$\dot{r} = \frac{d}{dt}r = Lu^2 \frac{d}{d\phi} \left( \frac{1}{u} \right) = Lu^2 \left( -\frac{1}{u^2} \frac{du}{d\phi} \right) = L \frac{du}{d\phi}$$

## Radial Acceleration $\ddot{r}$

$$\ddot{r} = \frac{d}{dt}(\dot{r}) = Lu^2 \frac{d}{d\phi} \left( -L \frac{du}{d\phi} \right)$$

$$\boxed{\ddot{r} = -L^2 u^2 \frac{d^2 u}{d\phi^2}}$$

Calculating the centrifugal term  $r\dot{\phi}^2$ :

$$r\dot{\phi}^2 = r \left( \frac{L}{r^2} \right)^2 = \frac{L^2}{r^3} = L^2 u^3$$

$$\boxed{r\dot{\phi}^2 = L^2 u^3}$$

## Final Force Equation $F(r)$

Now substitute into  $F(r) = \mu(\ddot{r} - r\dot{\phi}^2)$ :

$$F(r) = \mu \left[ -L^2 u^2 \frac{d^2 u}{d\phi^2} - L^2 u^3 \right]$$

$$F(r) = -L^2 u^2 \mu \left[ \frac{d^2 u}{d\phi^2} + u \right]$$

Rearranging for the differential equation:

$$\boxed{\frac{d^2 u}{d\phi^2} + u = -\frac{1}{\mu L^2 u^2} F(1/u)}$$

where  $u = 1/r$ .

## Final form for gravitational force

$$F(r) = -\frac{GM\mu}{r^2} \Rightarrow F\left(\frac{1}{u}\right) = -GM\mu u^2$$

$$\frac{d^2 u}{d\phi^2} + u = -\frac{1}{\mu L^2 u^2} (-GM\mu u^2)$$

$$\frac{d^2 u}{d\phi^2} + u = \frac{GM}{L^2}$$

Solution for equation no. (1) is

$$u = \frac{GM}{L^2} + A \cos \phi$$

$$u = \frac{GM}{L^2} \left[ 1 + \frac{AL^2}{GM} \cos \phi \right]$$

It looks like an ellipse's equation and resembles

$$u = \frac{1}{P} [1 + e \cos \phi]$$

Rewriting our equation, we get

$$u = \frac{GM}{L^2} [1 + e \cos \phi]$$

### 3 Mercury's Orbit: Einstein's Approach

The spacetime metric outside a spherically symmetric mass  $M$  is given by the Schwarzschild solution:

$$ds^2 = - \left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (1)$$

By spherical symmetry, the motion can be confined to the equatorial plane:

$$\theta = \frac{\pi}{2}, \quad \dot{\theta} = 0$$

$$ds^2 = - \left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 d\phi^2 \quad (2)$$

### Lagrangian for Timelike Geodesics

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ x^\mu &= x^\mu(\tau) \\ ds^2 &= g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} d\tau^2 \\ ds^2 &= g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu d\tau^2 \\ ds &= \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} d\tau \end{aligned} \quad (3)$$

$$s = \int ds \quad \Rightarrow \quad S = \int L d\tau, \quad \text{with} \quad L = \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}$$

$$\text{Alternatively, one often uses } L = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$$

By comparing (2) and (3), we get

$$g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = \left[ - \left(1 - \frac{2GM}{rc^2}\right) c^2 \dot{t}^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} \dot{r}^2 + r^2 \dot{\phi}^2 \right] \quad (4)$$

Using the equation no. (4), the Lagrangian is

$$L = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = \frac{1}{2} \left[ - \left(1 - \frac{2GM}{rc^2}\right) c^2 \dot{t}^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} \dot{r}^2 + r^2 \dot{\phi}^2 \right] \quad (5)$$

### Conserved Quantities

Due to the explicit independence of  $L$  on  $t$  and  $\phi$ , the corresponding conserved quantities are:

### Energy Conservation ( $t$ cyclic)

$$\frac{\partial L}{\partial \dot{t}} = - \left( 1 - \frac{2GM}{rc^2} \right) c^2 \dot{t} \equiv -E \quad \Rightarrow \quad \dot{t} = \frac{E}{c^2 \left( 1 - \frac{2GM}{rc^2} \right)}, \text{ where } E \text{ is a constant} \quad (6)$$

### Angular Momentum Conservation ( $\phi$ cyclic)

$$\frac{\partial L}{\partial \dot{\phi}} = r^2 \dot{\phi} \equiv L \quad \Rightarrow \quad \dot{\phi} = \frac{L}{r^2}, \text{ where } L \text{ is a constant} \quad (7)$$

## Radial Equation of Motion

For a massive particle:

$$ds^2 = -c^2 d\tau^2$$

We know

$$ds^2 = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu d\tau^2$$

By comparing, we get this

$$g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = -c^2 \quad (8)$$

Substitute the equation no. (4) in equation no. (8), we get,

$$- \left( 1 - \frac{2GM}{rc^2} \right) c^2 \dot{t}^2 + \left( 1 - \frac{2GM}{rc^2} \right)^{-1} \dot{r}^2 + r^2 \dot{\phi}^2 = -c^2 \quad (9)$$

By substituting conserved quantities [equation no. (5) and (6)],

$$\begin{aligned} - \left( 1 - \frac{2GM}{rc^2} \right) c^2 \frac{E^2}{c^4 \left[ 1 - \frac{2GM}{rc^2} \right]^2} + \left( 1 - \frac{2GM}{rc^2} \right)^{-1} \dot{r}^2 + r^2 \frac{L^2}{r^4} &= -c^2 \\ - \left( 1 - \frac{2GM}{rc^2} \right)^{-1} \frac{E^2}{c^2} + \left( 1 - \frac{2GM}{rc^2} \right)^{-1} \dot{r}^2 + \frac{L^2}{r^2} &= -c^2 \\ - \frac{E^2}{c^2} + \dot{r}^2 + \left( \frac{L^2}{r^2} + c^2 \right) \left( 1 - \frac{2GM}{rc^2} \right) &= 0 \\ \dot{r}^2 = \frac{E^2}{c^2} - \left( 1 - \frac{2GM}{rc^2} \right) \left( \frac{L^2}{r^2} + c^2 \right) \end{aligned}$$

If you redefine  $E \rightarrow E/c$ , you get

$$\dot{r}^2 = E^2 - \left( 1 - \frac{2GM}{rc^2} \right) \left( c^2 + \frac{L^2}{r^2} \right) \quad (10)$$

## Relativistic Orbit Equation

Introduce the reciprocal variable

$$u \equiv \frac{1}{r} \quad \Rightarrow \quad \dot{r} = -L \frac{du}{d\phi}$$

After algebra:

$$\left( \frac{du}{d\phi} \right)^2 = \frac{E^2}{L^2} - \left( 1 - \frac{2GMu}{c^2} \right) \left( \frac{c^2}{L^2} + u^2 \right)$$

Differentiating once with respect to  $\phi$ ,

$$\begin{aligned}
2 \left( \frac{du}{d\phi} \right) \frac{d^2u}{d\phi^2} &= - \left[ \left( 1 - \frac{2GMu}{c^2} \right) \left( 2u \frac{du}{d\phi} \right) + \left( \frac{c^2}{L^2} + u^2 \right) \left( \frac{-2GM}{c^2} \frac{du}{d\phi} \right) \right] \\
-2 \frac{d^2u}{d\phi^2} &= 2u - 2u \left[ \frac{2GMu}{c^2} \right] - \frac{2GM}{L^2} - \frac{2GM}{c^2} u^2 \\
\frac{d^2u}{d\phi^2} + u &= \frac{GM}{L^2} + \frac{3GM}{c^2} u^2
\end{aligned} \tag{11}$$

which is the exact relativistic orbit equation.

We know, the Newtonian orbit equation, which is

$$\frac{d^2u_0}{d\phi^2} + u_0 = \frac{GM}{L^2} \tag{12}$$

and by solving this, we get

$$u_0(\phi) = \frac{GM}{L^2} (1 + e \cos \phi) \tag{13}$$

## Perturbative Solution: Relativistic Correction

By observation, we can consider,

$$u = u_0 + u_1, \text{ where } u_0 \text{ is Newtonian solution and } u_1 \text{ is small relativistic correction}$$

Substitute  $u$  in equation no. (11),

$$\frac{d^2(u_0 + u_1)}{d\phi^2} + u_0 + u_1 = \frac{GM}{L^2} + \frac{3GM}{c^2} (u_0 + u_1)^2$$

Drop  $u_1^2$  and  $u_1 u_0$  terms,

$$\frac{d^2u_0}{d\phi^2} + \frac{d^2u_1}{d\phi^2} + u_0 + u_1 = \frac{GM}{L^2} + \frac{3GM}{c^2} u_0^2$$

By equation no. (12),

$$\frac{d^2u_1}{d\phi^2} + u_1 = \frac{3GM}{c^2} u_0^2$$

Square equation no. (13),

$$u_0^2 = \left( \frac{GM}{L^2} \right)^2 [1 + e \cos \phi]^2 \tag{14}$$

$$= \left( \frac{GM}{L^2} \right)^2 [1 + e^2 \cos^2 \phi + 2e \cos \phi] \tag{15}$$

We know,  $\cos^2 \phi = \frac{1 + \cos 2\phi}{2}$ , So,

$$u_0^2 = \left( \frac{GM}{L^2} \right)^2 \left[ 1 + 2e \cos \phi + \frac{e^2}{2} + \frac{e^2}{2} \cos 2\phi \right]$$

Substituting  $u_0^2$  in equation no. ( ),

$$\frac{d^2 u_1}{d\phi^2} + u_1 = \left(\frac{GM}{L^2}\right)^2 \left(\frac{3GM}{c^2}\right) [2e \cos \phi] + C_0 + C_2 \cos 2\phi \quad (16)$$

1. Constant Forcing Term  $C_0$ :

We solve the differential equation

$$\frac{d^2 u_1}{d\phi^2} + u_1 = C_0.$$

Try a constant solution:

$$u_1 = A.$$

Substituting,

$$0 + A = C_0 \quad \Rightarrow \quad A = C_0.$$

Hence, this term contributes

$$u \longrightarrow u + \text{constant}.$$

**Physical meaning:**

- Shifts the average orbital radius.
- Slightly changes the semi-major axis.
- Independent of  $\phi \Rightarrow$  cannot rotate the orbit.

2.  $\cos 2\phi$  Forcing Term:

Now consider

$$\frac{d^2 u_1}{d\phi^2} + u_1 = C_2 \cos 2\phi.$$

Try the ansatz

$$u_1 = A \cos 2\phi.$$

Then

$$u_1'' = -4A \cos 2\phi,$$

and

$$u_1'' + u_1 = (-4A + A) \cos 2\phi = -3A \cos 2\phi.$$

Matching coefficients gives

$$-3A = C_2 \quad \Rightarrow \quad A = -\frac{C_2}{3}.$$

**Physical meaning:**

- Produces a bounded oscillation.
- Slightly distorts the orbital shape (more “squashed” ellipse).
- Driving frequency  $\neq 1 \Rightarrow$  no resonance.
- No secular (unbounded) growth.

So, the only meaningful term is  $\cos \phi$ . So, the equation no. (14) become,

$$\frac{d^2 u_1}{d\phi^2} + u_1 = \left(\frac{GM}{L^2}\right)^2 \left(\frac{3GM}{c^2}\right) [2e \cos \phi]$$

$$\frac{d^2 u_1}{d\phi^2} + u_1 = \alpha \cos \phi, \text{ where } \alpha = \frac{6G^3 M^3 e}{c^2 L^4}$$

Solving this, we get,

$$u_1 = \frac{\alpha}{2} \phi \sin \phi$$

## Perihelion Precession of Mercury

At first, we considered,

$$u = u_0 + u_1$$

By substituting  $u_0$  and  $u_1$ ,

$$\begin{aligned} u &= \frac{GM}{L^2} [1 + e \cos \phi] + \frac{\alpha}{2} \phi \sin \phi \\ u &= \frac{GM}{L^2} \left[ 1 + e \cos \phi + \frac{\alpha L^2}{2GM} \phi \sin \phi \right] \end{aligned}$$

Let  $\frac{\alpha L^2}{2GM} = \varepsilon \ll 1$

$$\begin{aligned} u &= \frac{GM}{L^2} [1 + e \cos \phi + \varepsilon \phi \sin \phi] \\ u &= \frac{GM}{L^2} \left[ 1 + e \left( \cos \phi + \frac{\varepsilon}{e} \phi \sin \phi \right) \right] \end{aligned}$$

We know,

$$\cos((1 - \delta)\phi) = \cos \phi + \delta \phi \sin \phi + O(\delta^2)$$

Neglecting  $\delta^2$  terms, we get,

$$\cos((1 - \delta)\phi) = \cos \phi + \delta \phi \sin \phi + O(\delta^2)$$

Put  $\delta = \varepsilon/e$ . Then,

$$\cos[(1 - \varepsilon/e)\phi] = \cos \phi + \frac{\varepsilon}{e} \phi \sin \phi \quad (17)$$

So, the equation becomes,

$$u = \frac{GM}{L^2} [1 + e \cos((1 - \delta)\phi)] \quad (18)$$

where,"

$$\delta = \frac{\varepsilon}{e} = \frac{\alpha L^2}{2GM e} = \frac{3G^3 M^3}{c^2 L^4}$$

For a Newtonian Keplerian orbit, the (specific) angular momentum satisfies

$$L^2 = GMa(1 - e^2).$$

Hence,

$$L^4 = G^2 M^2 a^2 (1 - e^2)^2.$$

Substituting into the expression for  $\delta$ , we obtain

$$\delta = \frac{3G^3 M^3}{c^2 G^2 M^2 a^2 (1 - e^2)^2} = \frac{3GM}{a(1 - e^2)c^2}.$$

$$\delta = \frac{3GM}{a(1 - e^2)c^2}$$



The advance per orbit:

$$\Delta\phi = 2\pi\delta = \frac{6\pi GM}{a(1-e^2)c^2} \quad (19)$$

This is the standard general relativistic formula for the perihelion precession of Mercury.

## 4 Numerical Calculation of Mercury's Perihelion Precession

The dimensionless relativistic correction is given by

$$\delta = \frac{3GM}{a(1-e^2)c^2},$$

where

- $G$  = gravitational constant
- $M$  = mass of the Sun
- $a$  = semi-major axis of Mercury
- $e$  = eccentricity of Mercury's orbit
- $c$  = speed of light

Constants (SI Units):

$$G = 6.674 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$$

$$M = 1.989 \times 10^{30} \text{ kg}$$

$$c = 3.00 \times 10^8 \text{ m/s}$$

$$a = 5.79 \times 10^{10} \text{ m}$$

$$e = 0.206$$

### Step-by-step numerical evaluation

$$e^2 = (0.206)^2 = 0.0424, \quad 1 - e^2 = 1 - 0.0424 = 0.9576$$

$$GM = (6.674 \times 10^{-11})(1.989 \times 10^{30}) = 1.327 \times 10^{20} \text{ m}^3/\text{s}^2$$

$$3GM = 3 \times 1.327 \times 10^{20} = 3.981 \times 10^{20}$$

$$c^2 = (3.00 \times 10^8)^2 = 9.00 \times 10^{16}$$

$$a(1 - e^2) = (5.79 \times 10^{10})(0.9576) = 5.54 \times 10^{10}$$

$$a(1 - e^2)c^2 = (5.54 \times 10^{10})(9.00 \times 10^{16}) = 4.986 \times 10^{27}$$

### Dimensionless relativistic parameter

$$\delta = \frac{3GM}{a(1 - e^2)c^2} = \frac{3.981 \times 10^{20}}{4.986 \times 10^{27}} \approx 7.99 \times 10^{-8} \approx 8.0 \times 10^{-8}$$

## Precession per orbit

$$\Delta\phi_{\text{orbit}} = 2\pi\delta = 2\pi \times 7.99 \times 10^{-8} \approx 5.02 \times 10^{-7} \text{ rad}$$

Converting to arcseconds:

$$1 \text{ rad} = 206265'' \quad \Rightarrow \quad \Delta\phi_{\text{orbit}} = (5.02 \times 10^{-7})(206265) \approx 0.1036''$$

$$T = 0.2408 \text{ yr} \quad \Rightarrow \quad \text{orbits per century} = \frac{100}{0.2408} \approx 415.2$$

## Precession per Century

Total Precession per Century is

$$\Delta\phi_{\text{century}} = 0.1036'' \times 415.2 \approx 43.0''$$

This agrees with the observed general relativistic precession of Mercury.