Strict Lyapunov Function for System with Nonsmooth PI Controller

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***Abstract*—PID controllers are the most popular controller in the industry. However, The great popularity of PID Controller goes hand in hand with their widespread misuse. This report investigates the problem of nonsmooth feedback stabilization against Lipschitz matched uncertainty. For achieving the specified goal the integral term of classical Proportional-Integral (PI) controller is replaced by an integral of discontinuous function. Replacing this integrator the overall control become absolutely continuous rather than discontinuous as in the first order sliding mode control. With this proposed scheme, the property of invariance with respect to matched Lipschitz uncertainty is still preserved. The main technical contribution of the paper is to give a sound and non-trival Lyapunov analysis of the closed loop system controlled by nonsmooth PI controller. The effectiveness of the proposed controller is illustrated by the simulation results.**

***Index Terms*—Nonsmooth PI, Stability and stabilization, Strict Lyapunov Function**

I. Introduction

THE idea of feedback to make corrective actions based on the difference between the desired and the actual values of a quantity can be implemented in many different ways. The benefits of feedback can be obtained by very simple feedback laws such as on-off control, proportional control and proportional-integral-derivative control.

Combining proportional, integral and derivative control, we obtain a controller that can be expressed mathematically as

$$u(t) = k_p e(t) + k_i \int_0^t e(t)\, d\tau + k_d \frac{de(t)}{dt} \tag{1}$$

The control action is thus a sum of three terms: the past as represented by the integral of the error, the present as represented by the proportional term and the future as represented by a linear extrapolation of the error (the derivative term). This form of feedback is called a proportional-integral-derivative (PID) controller and its action is illustrated in Figure 1.

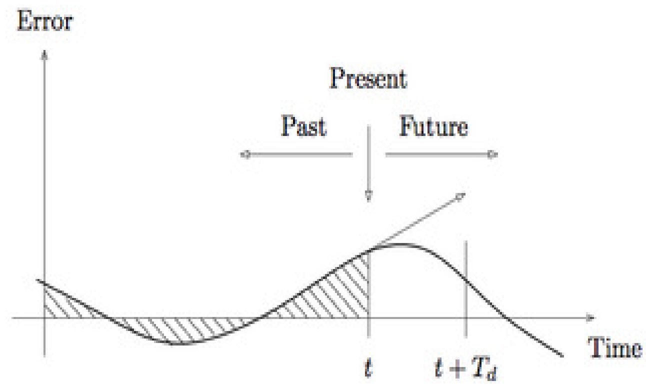

Fig. 1. Action of a PID controller. At time t, the proportional term depends on the instantaneous value of the error. The integral portion of the feedback is based on the integral of the error up to time t (shaded portion). The derivative term provides an estimate of the growth or decay of the error over time by looking at the rate of change of the error. T d represents the approximate amount of time in which the error is projected forward

Stability of perturbed system is one of the classical problem in the control literature [1]. There are several ways to address this problem. For example, consider the system

$$\dot{x} = f(t,x) + g(t,x) + u$$

where $u \in \mathbb{R}^m$, $f : [0,\infty) \times D \rightarrow D \subseteq \mathbb{R}^n$ and $g : [0,\infty) \times D \rightarrow D \subseteq \mathbb{R}^n$ are piecewise continuous functions in t and locally Lipschitz in x on $[0,\infty) \times D$ and $D \subset \mathbb{R}^n$ is a domain that contains the origin $x = 0$. In the above representation one can assume that $f(t,x)$ is nominal part and $g(t,x)$ contains modeling errors, aging, or uncertainties and disturbances which exist in any realistic problem. In reality $g(t,x)$ is not known to us but we are knowing the upper bound of $\|g(t,x)\|$ in the deterministic assumption. Suppose that if $f(t,x)$ also contains some uncertainties then one can always write $\dot{x} = \tilde{f}(t,x) + f(t,x) - \tilde{f}(t,x) + g(t,x) + u$.

The main intention of control researchers is to investigate the problem that, if the above system with $g(t,x) = 0$ is uniformly asymptotically stable at origin, then what can be said about the stability and behavior of the perturbed system for $g(t,x) \neq 0$? There are several approaches available in literature to solve this problem. In [1], it is reported that if $g(t,x)$ is vanishes at the equilibrium point then classical state feedback is able to guarantee the asymptotic stability. However, for nonvanishing disturbances at the origin the state feedback control (proportional control) doesn't ensure the uniformly asymptotically stability of the system [1]. It is found that some class of nonvanishing perturbations are taken care by the simple PI (proportional-integral feedback control) but it fails to handle the time varying perturbations [2].

Several delicate controllers have been investigated to enhance control performance and robustness. Sliding mode control (SMC) [3], [4] is one of them. It is one of the

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most promising control technique for controlling plants under uncertainty conditions. However, this control is not desirable from the implementation point of view due to (i) the oscillations caused by the high-frequency switching discontinuous controller excite the unmodelled dynamics in the closed loop, and (ii) the undesirable effects on the actuators caused by the chattering [3], [4], [6].

In the last two decades, a number of methods have been proposed to alleviate the chattering effect (see [3], [6] and references therein). "Higher order sliding mode" [7]-[11] is more promising among them. Its main idea is to reduce not only the sliding function, but also its high order derivatives zero.

The main idea is to reduce the chattering by artificially introducing one or more integrators in the system such that the control signal become a continuous function [11]. For instance, to obtain the absolute continuous control signal for the system $\dot{x} = f(t,x) + g(t,x) + u, \; x \in \mathbb{R}$, an integrator is introduced to increase the order of the system by one and then discontinuous higher order sliding mode algorithm can be used. However, the implementation of these controllers require the knowledge of $\dot{x}$. In this case, we can reconstruct perturbation, by computing $g(t,x) = \dot{x} - f(t,x) - u$, and it would be possible to compensate it without a discontinuous control [12].

To avoid the above mentioned drawbacks, the Super-twisting algorithm (STA) [8], [14] is proposed. However, recently it is found in literature that the chattering is still there because of nonlipschitz term in STA which generates infinite force at the origin [13], [14].

Motivating from the above fact, it seems that some more work is required in this area for following goals,

- modify control such that it is able to handle all kind of Lipschitz disturbances either vanishing or nonvanishing at origin. (One can further note that if disturbance is discontinuous no continuous control can handle it).
- design control such that overall control is absolutely continuous, so that chattering effects can be eliminated.
- propose a controller such that, it does not require any extra information other than the state variables.

For achieving the specified goal integral part of PI controller is replaced by an discontinuous integrator. Adding this extra integrator overall control is still absolutely continuous rather than first order sliding mode control, but the property of invariance with respect to Lipschitz matched uncertainties is still preserved. Finally we prove the stability of the closed loop system via a homogeneous, continuously differentiable and strict Lyapunov function.

The rest of the paper is organized as follows. In Section II the motivation is formulated. The main results of the paper are presented in Section III-A. Finally, some concluding remarks are included in Section V.

II. Motivation

For illustration of the proposed control strategy consider the following first order system

$$\dot{x}_1 = f(t,x) + u + d, \quad x_1 \in \mathbb{R} \tag{2}$$

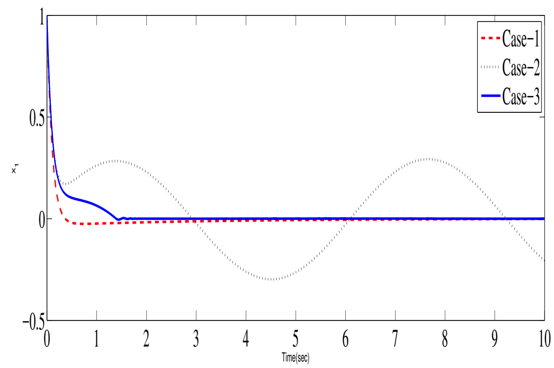

Fig. 2. Evolution of state

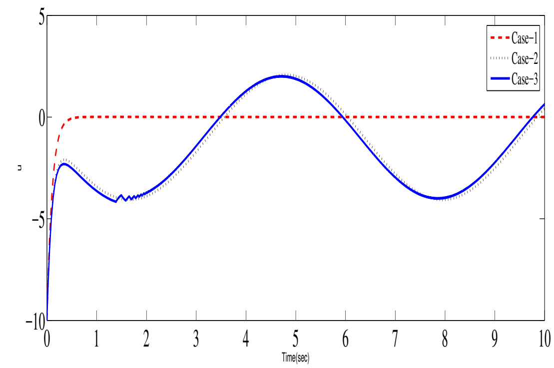

Fig. 3. Control

where $f(t,x)$ is known function and d is Lipschitz perturbations/disturbances. The proposed control u is taken as

$$u = -f(t,x) - k_1 x_1 - k_2 \int_0^t \text{sign}(x_1(\tau)) d\tau \tag{3}$$

where $k_i > 0$, for $i = 1,2$. On substitution of the proposed controller (3) into (2), the closed loop system is given by

$$\dot{x}_1 = -k_1 x_1 + z, \;\; \dot{z} = -k_2 \text{sign}(x_1) + \dot{d} \tag{4}$$

where $z(t) := -k_2 \int_0^t \text{sign}(x_1(\tau)) d\tau + d$. Solution of (4) are understood in the sense of Fillipov [15]. To get better insight and for further generalization of result closed loop system (4) has been simulated in Matlab environment. For the simulation the controller gains are selected as $k_1 = 10$, $k_2 = 3$ and the initial condition of state is chosen as $x_1(0) = 1$. Case-1 of Fig. 2 shows state evolution, when only state feedback control is applied for the disturbance free system, Case-2 of Fig. 2 demonstrate the state evolution of the system in the presence of disturbance $d = 1 + 3\sin(t)$ when classical state feedback is applied and finally Case-3 of Fig. 2 shows the state evolution in the case of proposed controller. One can observed that in the presence of disturbance also the proposed controller provide the same response as in the disturbance free case. Simulation results of Fig. 3 in Case-3 also confirm that the the control effort is continuous. Due to the continuous action of the control, system will be free from wear and tear effect of the actuators.

Remark 1. *The main benefits of the proposed controller over the first order sliding mode is that overall control is continuous and property of insensitive with respect to Lipschtiz disturbance is retained. Another advantage of proposed controller over first order sliding mode is it also able to reject ramp time varying disturbances.*

III. Main Results

The following Theorem gives the asymptotic stability of the system (4).

Theorem 1. *Consider the system* (4) *and let* $|\dot{d}| < k_2$. *Then the system of differential inclusion* (4) *is asymptotically stable in spite of disturbance* $\dot{d}$ *if* $k_1 > 0$ *and* $|\dot{d}| \leq k_2 \leq L(t)\left(\pi_1 + \frac{2^{\frac{3}{2}}}{3}\pi_2\right)$ *with* $\frac{2^2 2^{\frac{5}{6}}}{3^2}\pi_2 \leq \pi_1 \leq \frac{2^{\frac{3}{2}}}{3}\pi_2$ *where* $\pi_i > 0; i = 1,\ 2$ *and* $L(t), \dot{L}(t) > 0$.

Although system (4) is not homogenous or weighted homogeneous but one can use the weighted homogeneous Lyapunov function to prove stability. Now we are going to recall a result about continuous real-valued homogeneous functions ([[5], Lemma 4.2]), which will be used in the proof of Theorem 1.

Lemma 1. *Suppose* V_1 *and* V_2 *are continuous real-valued functions* $V_i : \mathbb{R}^n \to \mathbb{R}$, *homogeneous with the same weights and degrees* $l_1 > 0$ *and* $l_2 > 0$, *respectively, and* V_1 *is positive-definite. Then for every* $x \in \mathbb{R}^n$,

$$\left[\min_{\{z:V_1(z)=1\}}\right][V_1(x)]^{\frac{l_2}{l_1}} \leq V_2(x) \leq \left[\max_{\{z:V_1(z)=1\}}\right][V_1(x)]^{\frac{l_2}{l_1}}.$$

Next result states the detailed proof of Theorem 1.

PROOF. *By introducing time-varying change of variables*

$$z_1(t) = \frac{x(t)}{L(t)},\ z_2(t) = \frac{z(t)}{L(t)},\ L(t) > 0, \quad \forall t \geq 0 \tag{5}$$

In the new co-ordinates, system (4) *is given by*

$$\begin{aligned} \dot{z_1} &= -\left(k_1 + \frac{\dot{L}}{L}\right)z_1 + z_2 \\ \dot{z_2} &= -\frac{k_2}{L}sign(z_1) + \frac{\dot{d}}{L} - z_2\frac{\dot{L}}{L} \end{aligned} \tag{6}$$

Consider the following Lyapunov function in the new co-ordinates

$$V(z) = \left(\pi_1|z_1| + \frac{1}{2}z_2^2\right)^{\frac{3}{2}} + \pi_2 z_1 z_2.$$

Now using Young's inequality we are going to show that proposed Lyapunov function is upper bounded above by zero.

$$\begin{aligned} V(z) &\geq (\pi_1|z_1|)^{\frac{3}{2}} + \left(\frac{1}{2}z_2^2\right)^{\frac{3}{2}} \\ &- \pi_2\left(\frac{2}{3}g^{\frac{3}{2}}|z_1|^{\frac{3}{2}} + \frac{1}{3}g^{-3}|z_2|^3\right), \quad g \geq 0 \\ &= \left(\pi_1^{\frac{3}{2}} - \frac{2}{3}\pi_2 g^{\frac{3}{2}}\right)|z_1|^{\frac{3}{2}} + \left(\left(\frac{1}{2}\right)^{\frac{3}{2}} - \frac{1}{3}\pi_2 g^{-3}\right)|z_2|^3. \end{aligned}$$

For $V \geq 0; \forall z$, *each of* $\left(\pi_1^{\frac{3}{2}} - \frac{2}{3}\pi_2 g^{\frac{3}{2}}\right)$ *and* $\left(\frac{1}{2}\right)^{\frac{3}{2}} - \frac{1}{3}\pi_2 g^{-3}$ *should be greater than 0. Suppose* $2^{\frac{1}{2}}\left(\frac{\pi_2}{3}\right)^{\frac{1}{3}} < g < \left(\frac{3}{2\pi_2}\right)^{\frac{2}{3}}\pi_1$, *which implies* $\left(\frac{3}{2\pi_2}\right)^{\frac{2}{3}}\pi_1 \geq 2^{\frac{1}{2}}\left(\frac{\pi_2}{3}\right)^{\frac{1}{3}}$. *Thus,* $\pi_1 \geq \frac{2^{\frac{1}{2}}2^{\frac{2}{3}}}{3}\pi_2$. *Selecting* g *to be the linear combination of* $2^{\frac{1}{2}}\left(\frac{\pi_2}{3}\right)^{\frac{1}{3}}$ *and* $\left(\frac{3}{2\pi_2}\right)^{\frac{2}{3}}\pi_1$ *will lead to* $V \geq 0$. *Thus* $g = \alpha 2^{\frac{1}{2}}\left(\frac{\pi_2}{3}\right)^{\frac{1}{3}} + (1-\alpha)\left(\frac{3}{2\pi_2}\right)^{\frac{2}{3}}\pi_1, \quad 0 \leq \alpha \leq 1$. *Now our next aim is to show* $\dot{V} < 0$,

$$\begin{aligned} \dot{V} &= \left\{\frac{3}{2}\left(\pi_1|z_1| + \frac{1}{2}z_2^2\right)^{\frac{1}{2}}\pi_1 sign(z_1) + \pi_2 z_2\right\}\dot{z_1} \\ &+ \left\{\frac{3}{2}\left(\pi_1|z_1| + \frac{1}{2}z_2^2\right)^{\frac{1}{2}}z_2 + \pi_2 z_1\right\}\dot{z_2} \\ &= -\frac{3}{2}\left(\pi_1|z_1| + \frac{1}{2}z_2^2\right)\chi + \pi_2 z_2^2 - \pi_2\left(k_1 + \frac{\dot{L}}{L}\right)z_1 z_2 \\ &- \pi_2\frac{k_2}{L}sign(z_1)z_1 + \pi_2 z_1\frac{\dot{d}}{L} - \pi_2 z_1 z_2\frac{\dot{L}}{L} \end{aligned}$$

where

$$\begin{aligned} \chi :&= \pi_1 sign(z_1)\left(\left(k_1 + \frac{\dot{L}}{L}\right)z_1 - z_2\right) \\ &+ z_2\left(\frac{k_2}{L}sign(z_1) + z_2\frac{\dot{L}}{L} - \frac{\dot{d}}{L}\right). \end{aligned}$$

One can also rewrite $\dot{V}$ *as,*

$$\dot{V} = -W_1(z)\left(\frac{\dot{L}}{L}\right) + W_2(z)\left(\frac{\dot{d}}{L}\right) - W_3^*(z) \tag{7}$$

where

$$\begin{aligned} W_1(z) &= \frac{3}{2}\left(\pi_1|z_1| + \frac{1}{2}z_2^2\right)^{\frac{1}{2}}\left(\pi_1|z_1| + z_2^2\right) + 2\pi_2 z_1 z_2 \\ W_2(z) &= \frac{3}{2}\left(\pi_1|z_1| + \frac{1}{2}z_2^2\right)^{\frac{1}{2}}z_2 + \pi_2 z_1 \\ W_3^*(z) &= \left(\frac{3}{2}\left(\pi_1|z_1| + \frac{1}{2}z_2^2\right)^{\frac{1}{2}}k_1\pi_1 + \pi_2\frac{k_2}{L}\right)|z_1| \\ &+ \frac{3}{2}\left(\pi_1|z_1| + \frac{1}{2}z_2^2\right)^{\frac{1}{2}}\left(\frac{k_2}{L} - \pi_1\right)sign(z_1 z_2)|z_2| \\ &- \pi_2 z_2^2 + \pi_2 k_1 z_1 z_2 \end{aligned} \tag{8}$$

We are going to show that $W_3^*(z)$ *would dominate over* $W_2(z)$, *given that* $\left|\dot{d}\right| < k_2$ *Since,* $\frac{3}{2}\left(\pi_1|z_1| + \frac{1}{2}z_2^2\right)^{\frac{1}{2}}|z_2| \geq \frac{3}{2}\left(\frac{1}{2}\right)^{\frac{1}{2}}z_2^2$, *therefore,* $\pi_2(2)^{\frac{1}{2}}\left(\pi_1|z_1| + \frac{1}{2}z_2^2\right)^{\frac{1}{2}}|z_2| \geq -\pi_2 z_2^2$ *and* $W_3^* \leq W_3^{'}$, *where,*

$$\begin{aligned} W_3^{'} &= \left\{\frac{3}{2}\left(\pi_1|z_1| + \frac{1}{2}z_2^2\right)^{\frac{1}{2}}k_1\pi_1 + \pi_2\frac{k_2}{L}\right\}|z_1| \\ &+ \left\{\frac{3}{2}\left(\pi_1 + \frac{1}{2}z_2^2\right)^{\frac{1}{2}}\chi_1\right\}|z_2| + \pi_2 k_1 z_1 z_2, \end{aligned} \tag{9}$$

where $\chi_1 := \left(\frac{k_2}{L} sign\left(z_1 z_2\right) - \pi_1 sign\left(z_1 z_2\right) + \frac{2^{\frac{3}{2}}}{3}\pi_2\right)$. *Again, as* $z_1 z_2 \leq \frac{2}{3}c^{\frac{3}{2}}\left|z_1\right|^{\frac{3}{2}} + \frac{1}{3}c^{-3}\left|z_2\right|^3$, *so* $W_3' \leq W_3$, *where* W_3 *can be written as*

$$W_3 = \left\{\frac{3}{2}\left(\pi_1\left|z_1\right| + \frac{1}{2}z_2^2\right)^{\frac{1}{2}} k_1\pi_1 + \pi_2\frac{k_2}{L}\right\}\left|z_1\right| + \left\{\frac{3}{2}\left(\pi_1\left|z_1\right| + \frac{1}{2}z_2^2\right)^{\frac{1}{2}}\chi_1\right\}\left|z_2\right| + + \pi_2 k_1\left(\frac{2}{3}c^{\frac{3}{2}}\left|z_1\right|^{\frac{3}{2}} + \frac{1}{3}c^{-3}\left|z_2\right|^3\right). \tag{10}$$

Since $\left(\pi_1\left|z_1\right| + \frac{1}{2}z_2^2\right)^{\frac{1}{2}}\left|z_1\right| \geq \pi_1^{\frac{1}{2}}\left|z_1\right|^{\frac{3}{2}}$, $2\left(\pi_1\left|z_1\right| + \frac{1}{2}z_2^2\right)\left|z_2\right| \geq \left|z_2\right|^3$, $W_3^f \geq W_3$, *where* W_3^f *can be written as,*

$$W_3^f = \left\{\chi_2\left(k_1\pi_1 + \frac{4}{9}\pi_2 k_1\pi_1^{-\frac{1}{2}}c^{\frac{3}{2}}\right) + \pi_2\frac{k_2}{L}\right\}\left|z_1\right| + \chi_2\left(\chi_1 + \frac{4}{9}\pi_2 k_1 c^{-3}\left(\pi_1\left|z_1\right| + \frac{1}{2}z_2^2\right)^{\frac{1}{2}}\right)\left|z_2\right|, \tag{11}$$

where $\chi_2 := \frac{3}{2}\left(\pi_1\left|z_1\right| + \frac{1}{2}z_2^2\right)^{\frac{1}{2}}$. *For* W_3^f *to be greater than zero* $\forall z$, *both the coefficients of equation* (11) *should be independently greater than zero, that is, if*

$$\chi_2\left(k_1\pi_1 + \frac{4}{9}\pi_2 k_1\pi_1^{-\frac{1}{2}}c^{\frac{3}{2}}\right) + \pi_2\frac{k_2}{L} \geq 0, \tag{12}$$

$$\left(\frac{k_2}{L} - \pi_1\right) sign\left(z_1 z_2\right) + \frac{2^{\frac{3}{2}}}{3}\pi_2 + \frac{8}{27}\pi_2 k_1 c^{-3}\chi_2 \geq 0. \tag{13}$$

These two inequalities are satisfied if,

$$\begin{aligned} \frac{k_2}{L} - \pi_1 + \frac{2^{\frac{3}{2}}}{3}\pi_2 + \frac{8}{27}\pi_2 k_1 c^{-3}\chi_2 &\geq 0 \\ -\frac{k_2}{L} + \pi_1 + \frac{2^{\frac{3}{2}}}{3}\pi_2 + \frac{8}{27}\pi_2 k_1 c^{-3}\chi_2 &\geq 0, \end{aligned} \tag{14}$$

which can be rewritten as

$$0 \leq k_2 \leq L\left(\pi_1 + \frac{2^{\frac{3}{2}}}{3}\pi_2\right). \tag{15}$$

$$W_3^f(z) \geq W_3^{f^*}(z) \triangleq \alpha\left|z_1\right| + \beta\left(\pi_1\left|z_1\right| + \frac{1}{2}z_2^2\right)^{\frac{1}{2}}\left|z_2\right|, \tag{16}$$

with

$$\begin{aligned} \alpha &= \min_z\left[\chi_2\left(k_1\pi_1 + \frac{4}{9}\pi_2 k_1\pi_1^{-\frac{1}{2}}c^{\frac{3}{2}}\right) + \pi_2\frac{k_2}{L}\right] \geq 0 \\ \beta &= \min_z\left[\chi_1 + \frac{8}{27}\pi_2 k_1 c^{-3}\chi_2\right] \geq 0, \end{aligned} \tag{17}$$

$W_3^{f^*}(z)$ *is a continuous and homogeneous positive definite function. According to Lemma 1, it follows that* $\forall z \in \mathbb{R}^2$, $W_2(z) \leq \gamma W_3^{f^*}(z)$ *is satisfied, with* $\gamma = \max_{\left\{z: W_3^{f^*}(z)=1\right\}} > 0$, *because both* $W_2(z)$ *and* $W_3^{f^*}(z)$ *are continuous and homogeneous with same weights and degree. Finally,*

$$\begin{aligned} W_1(z) &= \frac{3}{2}\left(\pi_1\left|z_1\right| + \frac{1}{2}z_2^2\right)^{\frac{1}{2}}\left(\pi_1\left|z_1\right| + z_2^2\right) + 2\pi_2 z_1 z_2 \\ &\geq \frac{3}{2}\left(\pi_1\left|z_1\right|\right)^{\frac{1}{2}}\pi_1\left|z_1\right| + \frac{3}{2}\left(\frac{1}{2}z_2^2\right)^{\frac{1}{2}} z_2^2 \\ &\quad - 2\pi_2\left(\frac{2}{3}g^{\frac{3}{2}}\left|z_1\right|^{\frac{3}{2}} + \frac{1}{3}g^{-3}\left|z_2\right|^3\right) \\ &= \left(\frac{3}{2}\pi_1^{\frac{3}{2}} - \frac{4}{3}\pi_2 g^{\frac{3}{2}}\right)\left|z_1\right|^{\frac{3}{2}} + \left(\frac{3}{2^{\frac{3}{2}}} - \frac{2\pi_2}{3}g^{-3}\right)\left|z_2\right|^3 \end{aligned} \tag{18}$$

$W_1(z)$ *is positive-definite if* $\frac{3}{2}\pi_1^{\frac{3}{2}} - \frac{4}{3}\pi_2 g^{\frac{3}{2}} > 0$ *and* $\frac{3}{2^{\frac{3}{2}}} - \frac{2\pi_2}{3}g^{-3} > 0$ *or equivalently*

$$\frac{2^{\frac{5}{6}}}{3^{\frac{2}{3}}}\pi_2^{\frac{1}{3}} < g < \frac{3^{\frac{4}{3}}}{2^2}\frac{\pi_1}{\pi_2^{\frac{2}{3}}} \tag{19}$$

and such a g exists if $\pi_1 > \frac{2^{\frac{5}{6}}2^2}{3^2}\pi_2$. *Thus* π_1 *should be selected such that* $\pi_1 > \frac{2^{\frac{5}{6}}2^2}{3^2}\pi_2$. *It can be noted that it also fulfills* $\pi_1 \geq \frac{2^{\frac{1}{2}}2^{\frac{2}{3}}}{3}\pi_2$ *required for* $V \geq 0$. *This completes the proof.* □

A. Nonsmooth PI for the Higher Order Uncertain Chain of Integrators

For n^{th} order uncertain chain of integrators, the proposed nonsmooth PI controller is given as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_n &= f(t,x) + u + d, \end{aligned} \tag{20}$$

where $\mathbf{X}^\top = \begin{bmatrix} x_1 & x_2 & \ldots & x_n \end{bmatrix} \in \mathbb{R}^{1\times n}$ and u is the proposed control and is taken as

$$u = -f(t,x) - \mathbf{K_p X} - \int_0^t K_I \text{sign}\left(\mathbf{K_P X}\right) d\tau \tag{21}$$

where $\mathbf{K_p} = \begin{bmatrix} k_1 & k_2 & \ldots & k_n \end{bmatrix}$ with $k_i > 0$ for $i = 1,\ldots,n$ and $K_I > 0$. After applying proposed controller (21) into (20), the closed loop system is given by

$$\begin{aligned} \dot{\mathbf{X}} &= \mathbf{AX} - \mathbf{BK_P X} + \mathbf{B}Z \\ \dot{Z} &= -K_I \text{sign}\left(\mathbf{K_P X}\right) + \dot{d} \end{aligned} \tag{22}$$

where, $Z = -\int_0^t K_I \text{sign}\left(\mathbf{K_P X}\right) d\tau + d$, and

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

The following Theorem gives the asymptotic stability of the system (20).

Theorem 2. *Consider the system* (20) *and let* $|\dot{d}| < K_I$. *Then the system of differential inclusion* (20) *is asymptotically stable in spite of disturbance* d *if* $\mathbf{K_P}$ *is selected such that* $Q := (\mathbf{BK_P} - \mathbf{A})$ *has positive eigenvalues and*

$\left|\dot{d}\right| \leq K_I \leq L(t)\left(-\pi_1\|\mathbf{B}\| + \frac{2^{\frac{3}{2}}}{3}\pi_\mathbf{2}\mathbf{B}\right)$ *with* $\frac{2^2 2^{\frac{5}{6}}}{3^2}\|\pi_\mathbf{2}\| \leq \pi_1 \leq \frac{2^{\frac{3}{2}}}{3}\|\pi_\mathbf{2}\|$ *where* $\pi_1 > 0$, $L(t), \dot{L}(t) > 0$ *and* $\pi_\mathbf{2} = \begin{bmatrix}\pi_{21} & \pi_{22} & \dots & \pi_{2n}\end{bmatrix}$ *with* $\pi_{2i} > 0$ *for* $i = 1, \dots, n$.

PROOF. *On applying the following time-varying change of variables,* $\mathbf{Z_1}(t) := \frac{\mathbf{X(t)}}{L(t)}$, $Z_2(t) := \frac{\bar{Z}(t)}{L(t)}$, *one can rewrite the* (22) *as*

$$\begin{aligned}\dot{\mathbf{Z_1}} &= -\left(\frac{\dot{L}}{L}\mathbf{I} + \mathbf{BK_P} - \mathbf{A}\right)\mathbf{Z_1} + \mathbf{B}Z_2\\ \dot{Z}_2 &= -\frac{\dot{L}}{L}Z_2 - \frac{K_I}{L}\frac{\mathbf{K_P Z_1}}{\|\mathbf{K_P Z_1}\|} + \frac{\dot{d}}{L},\end{aligned} \tag{23}$$

where $\mathbf{I}$ *is an identity matrix and* $L(t)$ *is some continuously differentiable time varying positive function* $\mathbb{C}^1$ *i.e.,* $L(t) > 0 \quad \forall t \geq 0$ *and* $\dot{L} > 0$ *exists. Consider the* $V(Z)$ *be a Lyapunov function in the new co-ordinates*

$$V(Z) = \left(\pi_1\|\mathbf{Z_1}\| + \frac{1}{2}Z_2^2\right)^{\frac{3}{2}} + \pi_\mathbf{2}\mathbf{Z_1}Z_2, \tag{24}$$

where $Z := [\mathbf{Z_1}^\top \; Z_2]^\top$, $\pi_1 > 0$ *and* $\pi_\mathbf{2} = \begin{bmatrix}\pi_{21} & \pi_{22} & \dots & \pi_{2n}\end{bmatrix}$ *with* $\pi_{2i} > 0$ *for* $i = 1, \dots, n$. *Next using Young's and norm inequalities, we are going to show that proposed Lyapunov function* (24) *is lower bounded from zero*

$$\begin{aligned}V(Z) &\geq (\pi_1\|\mathbf{Z_1}\|)^{\frac{3}{2}} + \left(\frac{1}{2}Z_2^2\right)^{\frac{3}{2}}\\ &- \|\pi_\mathbf{2}\|\left(\frac{2}{3}g^{\frac{3}{2}}\|\mathbf{Z_1}\|^{\frac{3}{2}} + \frac{1}{3}g^{-3}|Z_2|^3\right), \quad g \geq 0\\ &= \left(\pi_1^{\frac{3}{2}} - \frac{2}{3}\|\pi_\mathbf{2}\|g^{\frac{3}{2}}\right)\|\mathbf{Z_1}\|^{\frac{3}{2}} + \left(\left(\frac{1}{2}\right)^{\frac{3}{2}} - \frac{1}{3}\|\pi_\mathbf{2}\|g^{-3}\right)|Z_2|^3.\end{aligned} \tag{25}$$

For $V \geq 0; \; \forall Z$, *each of* $\left(\pi_1^{\frac{3}{2}} - \frac{2}{3}\|\pi_\mathbf{2}\|g^{\frac{3}{2}}\right)$ *and* $\left(\left(\frac{1}{2}\right)^{\frac{3}{2}} - \frac{1}{3}\|\pi_\mathbf{2}\|g^{-3}\right)$ *should be greater than 0. Suppose* $2^{\frac{1}{2}}\left(\frac{\|\pi_\mathbf{2}\|}{3}\right)^{\frac{1}{3}} < g < \left(\frac{3}{2\|\pi_\mathbf{2}\|}\right)^{\frac{2}{3}}\pi_1$, *which further implies* $\left(\frac{3}{2\|\pi_\mathbf{2}\|}\right)^{\frac{2}{3}}\pi_1 \geq 2^{\frac{1}{2}}\left(\frac{\|\pi_\mathbf{2}\|}{3}\right)^{\frac{1}{3}}$. *Thus,* $\pi_1 \geq \frac{2^{\frac{1}{2}}2^{\frac{2}{3}}}{3}\|\pi_\mathbf{2}\|$. *Selecting* g *to be the linear combination of* $2^{\frac{1}{2}}\left(\frac{\|\pi_\mathbf{2}\|}{3}\right)^{\frac{1}{3}}$ *and* $\left(\frac{3}{2\|\pi_\mathbf{2}\|}\right)^{\frac{2}{3}}\pi_1$ *will lead to* $V \geq 0$. *Thus* $g = \alpha 2^{\frac{1}{2}}\left(\frac{\pi_\mathbf{2}}{3}\right)^{\frac{1}{3}} + (1-\alpha)\left(\frac{3}{2\|\pi_\mathbf{2}\|}\right)^{\frac{2}{3}}\pi_1, \quad 0 \leq \alpha \leq 1$.

Time derivative of Lyapunov function (24) *along the system trajectory* (23)

$$\begin{aligned}\dot{V}(Z) &= \left(\Theta\pi_1 \mathit{SIGN}\left(\mathbf{Z_1}^\top\right) + \pi_\mathbf{2}\mathbf{Z_2}\right)\dot{\mathbf{Z_1}} + (\Theta Z_2 + \pi_\mathbf{2}\mathbf{Z_1})\dot{Z}_2\\ &= \left(\Theta\pi_1 \mathit{SIGN}\left(\mathbf{Z_1}^\top\right) + \pi_\mathbf{2}\mathbf{Z_2}\right)\\ &\quad \left(-\left(\frac{\dot{L}}{L}\mathbf{I} + \mathbf{BK_P} - \mathbf{A}\right)\mathbf{Z_1} + \mathbf{B}Z_2\right)\\ &\quad + (\Theta Z_2 + \pi_\mathbf{2}\mathbf{Z_1})\left(-\frac{\dot{L}}{L}Z_2 - \frac{K_I}{L}\frac{\mathbf{K_P Z_1}}{\|\mathbf{K_P Z_1}\|} + \frac{\dot{d}}{L}\right),\end{aligned} \tag{26}$$

where $\Theta := \frac{3}{2}\left(\pi_1\|\mathbf{Z_1}\| + \frac{1}{2}Z_2^2\right)^{\frac{1}{2}}$ *or*

$$\dot{V}(Z) = -W_1\left(\frac{\dot{L}}{L}\right) + W_2\left(\frac{\dot{d}}{L}\right) - W_3^*, \tag{27}$$

where,

$$W_1 = \frac{3}{2}\left(\pi_1\|\mathbf{Z_1}\| + \frac{1}{2}Z_2^2\right)^{\frac{1}{2}}\left(\pi_1\|\mathbf{Z_1}\| + Z_2^2\right) + 2Z_2\pi_\mathbf{2}\mathbf{Z_1} \tag{28a}$$

$$W_2 = \frac{3}{2}\left(\pi_1\|\mathbf{Z_1}\| + \frac{1}{2}Z_2^2\right)^{\frac{1}{2}}Z_2 + \pi_\mathbf{2}\mathbf{Z_1} \tag{28b}$$

$$\begin{aligned}W_3^* &= \frac{3}{2}\left(\pi_1\|\mathbf{Z_1}\| + \frac{1}{2}Z_2^2\right)^{\frac{1}{2}}\Xi + \pi_2 Z_2(\mathbf{BK_P} - \mathbf{A})\mathbf{Z_1}\\ &- Z_2^2\pi_\mathbf{2}\mathbf{B} + \frac{K_I}{L}\pi_\mathbf{2}\mathbf{Z_1}\mathit{sign}(\mathbf{K_P Z_1})\end{aligned} \tag{28c}$$

where

$$\Xi := \pi_1\frac{\mathbf{Z_1}^\top(\mathbf{BK_P} - \mathbf{A})\mathbf{Z_1}}{\|\mathbf{Z_1}\|} - \pi_1 Z_2\frac{\mathbf{Z_1}^\top\mathbf{B}}{\|\mathbf{Z_1}\|} + Z_2\frac{K_I}{L}\mathit{sign}(\mathbf{K_P Z_1}).$$

We are going to show that W_3^* *would dominate over* W_2, *given that* $\left|\dot{d}\right| < K_I$.
Since, we have selected $\mathbf{K_P}$ *such that* $\mathbf{Q} := \mathbf{BK_P} - \mathbf{A}$ *has positive eigenvalues. Then using Rayleigh inequality one can write* $\lambda_{\min}\{\mathbf{Q}\}\|\mathbf{Z_1}\|^2 \leq \mathbf{Z_1}^\top\mathbf{Q}\mathbf{Z_1} \leq \lambda_{\max}\{Q\}\|\mathbf{Z_1}\|^2$, *where* $\lambda_{\min}\{\mathbf{Q}\}$ *and* $\lambda_{\max}\{\mathbf{Q}\}$ *are the minimum and maximum eigenvalues of the matrix* $\mathbf{Q}$. *One can further write* $\pi_1\frac{\mathbf{Z_1}^\top(\mathbf{BK_P} - \mathbf{A})\mathbf{Z_1}}{\|\mathbf{Z_1}\|} \leq \pi_1\lambda_{\max}\{Q\}\|\mathbf{Z_1}\|$. *Furthermore,* $\frac{K_I}{L}\pi_\mathbf{2}\mathbf{Z_1}\mathit{sign}(\mathbf{K_P Z_1}) \leq \frac{K_I}{L}\|\pi_\mathbf{2}\|\|\mathbf{Z_1}\|$, *provided* $K_I \geq 0$ *and* $\pi_1 Z_2\frac{\mathbf{Z_1^T B}}{\|\mathbf{Z_1}\|} \leq \pi_1\|\mathbf{B}\||Z_2|$. *Therefore,* $W_3^* < W_3^{''}$, *where*

$$\begin{aligned}W_3^{''} &:= \frac{3}{2}\left(\pi_1\|\mathbf{Z_1}\| + \frac{1}{2}Z_2^2\right)^{\frac{1}{2}}\Theta_1 + \pi_\mathbf{2}Z_2(\mathbf{BK_P} - \mathbf{A})\mathbf{Z_1}\\ &- Z_2^2\pi_\mathbf{2}\mathbf{B} + \frac{K_I}{L}\alpha_1\|\mathbf{K_P}\|\|\mathbf{Z_1}\|.\end{aligned} \tag{29}$$

where $\alpha_1 := \pi_\mathbf{2}\mathbf{K_P}^\top\left(\mathbf{K_P}\mathbf{K_P}^\top\right)^{-1} > 0$ *and* $\Theta_1 := \pi_1\lambda_{\max}\{Q\}\|\mathbf{Z_1}\| + \pi_1\|\mathbf{B}\|Z_2 + \frac{K_I}{L}\mathit{sign}(Z_2\mathbf{K_P Z_1})|Z_2|$. *Since,* $\frac{3}{2}\left(\pi_1\|\mathbf{Z_1}\| + \frac{1}{2}Z_2^2\right)^{\frac{1}{2}}|Z_2| \geq \frac{3}{2}\left(\frac{1}{2}\right)^{\frac{1}{2}}z_2^2$, *therefore,*

$$\begin{aligned}W_3^{iv} &= \frac{3}{2}\left(\pi_1\|\mathbf{Z_1}\| + \frac{1}{2}Z_2^2\right)^{\frac{1}{2}}\Theta_2 + \pi_\mathbf{2}Z_2(\mathbf{BK_P} - \mathbf{A})\mathbf{Z_1}\\ &+ \frac{K_I}{L}\alpha_1\|\mathbf{K_P}\|\|\mathbf{Z_1}\|,\end{aligned} \tag{30}$$

where

$$\begin{aligned}\Theta_2 :&= \pi_1\lambda_{\max}\{Q\}\|\mathbf{Z_1}\| + \pi_1\|\mathbf{B}\|Z_2\\ &+ \frac{K_I}{L}\mathit{sign}(Z_2\mathbf{K_P Z_1})|Z_2| + \frac{2^{\frac{3}{2}}}{3}\pi_\mathbf{2}\mathbf{B}|Z_2|.\end{aligned} \tag{31}$$

Again, as

$$\|\mathbf{Z_1}\||Z_2| \leq \frac{2}{3}c^{\frac{3}{2}}\|\mathbf{Z_1}\|^{\frac{3}{2}} + \frac{1}{3}c^{-3}|Z_2|^3; c > 0, \tag{32a}$$

$$\left(\pi_1\|\mathbf{Z_1}\| + \frac{1}{2}Z_2^2\right)^{\frac{1}{2}}\|\mathbf{Z_1}\| \geq \pi_1^{\frac{1}{2}}\|\mathbf{Z_1}\|^{\frac{3}{2}}, \tag{32b}$$

$$2\left(\pi_1\|\mathbf{Z_1}\| + \frac{1}{2}Z_2^2\right)|Z_2| \geq |Z_2|^3, \tag{32c}$$

$$\frac{3}{2}\left(\pi_1\|\mathbf{Z_1}\| + \frac{1}{2}Z_2^2\right)^{\frac{1}{2}} \geq \frac{3}{2}\left(\frac{1}{2}\right)^{\frac{1}{2}}Z_2^2. \tag{32d}$$

So $W_3^{iv} \leq W_3^{vi}$, where W_3^{vi} can be written as

$$W_3^{vi} = \left(\Theta\Theta_3 + \frac{K_I}{L}\alpha_1 \|\mathbf{K_P}\|\right)\|\mathbf{Z_1}\| + \Theta\left(\Theta_4 + \frac{8}{27}c^{-3}\Theta\left(\|\pi_\mathbf{2}\|\|\mathbf{BK_P}-\mathbf{A}\|\right)\right)|Z_2|, \tag{33}$$

where,

$$\begin{aligned}\Theta_3 &:= \pi_1\lambda_{\max}\{Q\} + \frac{4}{9}c^{\frac{3}{2}}\pi_1^{\frac{-1}{2}}\left(\|\pi_\mathbf{2}\|\|\mathbf{BK_P}-\mathbf{A}\|\right),\\ \Theta_4 &:= \pi_1\|\mathbf{B}\|\, sign\,(Z_2) + \frac{K_I}{L}sign\,(Z_2\mathbf{K_P}\mathbf{Z_1}) + \frac{2^{\frac{3}{2}}}{3}\pi_\mathbf{2}\mathbf{B}\end{aligned} \tag{34}$$

For W_3^{vi} to be greater than zero $\forall Z$, both the coefficients Θ_3 and Θ_4 of equation (33) *should be independently greater than zero , that is, if*

$$\pi_1\|\mathbf{B}\| + \frac{K_I}{L} + \frac{2^{\frac{3}{2}}}{3}\pi_\mathbf{2}\mathbf{B} \geq 0 \tag{35a}$$

$$\pi_1\|\mathbf{B}\| - \frac{K_I}{L} + \frac{2^{\frac{3}{2}}}{3}\pi_\mathbf{2}\mathbf{B} \geq 0 \tag{35b}$$

$$-\pi_1\|\mathbf{B}\| - \frac{K_I}{L} + \frac{2^{\frac{3}{2}}}{3}\pi_\mathbf{2}\mathbf{B} \geq 0 \tag{35c}$$

$$-\pi_1\|\mathbf{B}\| + \frac{K_I}{L} + \frac{2^{\frac{3}{2}}}{3}\pi_\mathbf{2}\mathbf{B} \geq 0 \tag{35d}$$

For inequality $\Theta_4 \geq 0$ to be satisfied, all of inequalities (35) *have to be satisfied, which can be re-written as*

$$0 \leq K_I \leq L\left(-\pi_1\|\mathbf{B}\| + \frac{2^{\frac{3}{2}}}{3}\pi_\mathbf{2}\mathbf{B}\right). \tag{36}$$

$$W_3^{vi}(Z) \geq \overset{*}{W_3^{vi}}(Z) \triangleq \alpha\|\mathbf{Z_1}\| + \beta\left(\pi_1\|\mathbf{Z_1}\| + \frac{1}{2}Z_2^2\right)^{\frac{1}{2}}|Z_2|, \tag{37}$$

with

$$\begin{aligned}\alpha &= \min_Z\left[\Theta\Theta_3 + \frac{K_I}{L}\alpha\|\mathbf{K_P}\|\right] \geq 0\\ \beta &= \min_Z\left[\Theta_4 + \frac{8}{27}c^{-3}\Theta\left(\|\pi_\mathbf{2}\|\|\mathbf{BK_P}-\mathbf{A}\|\right)\right] \geq 0,\end{aligned} \tag{38}$$

$\overset{*}{W_3^{vi}}(Z)$ is a continuous and homogeneous positive definite function. According to Lemma 1, it follows that $\forall Z \in \mathbb{R}^{n+1}$, $W_2(Z) \leq \gamma \overset{*}{W_3^{vi}}(Z)$ is satisfied, with $\gamma = \max_{\left\{Z:\overset{*}{W_3^{vi}}(Z)=1\right\}} > 0$, because both $W_2(Z)$ and $\overset{*}{W_3^{vi}}(Z)$ are continuous and homogeneous with same weights and degree.

$$\begin{aligned}W_1(Z) &= \frac{3}{2}\left(\pi_1\|\mathbf{Z_1}\| + \frac{1}{2}Z_2^2\right)^{\frac{1}{2}}\left(\pi_1\|\mathbf{Z_1}\| + Z_2^2\right) + 2Z_2\pi_\mathbf{2}\mathbf{Z_1}\\ &\geq \frac{3}{2}\left(\pi_1||\mathbf{Z_1}||\right)^{\frac{1}{2}}\pi_1||\mathbf{Z_1}|| + \frac{3}{2}\left(\frac{1}{2}Z_2^2\right)^{\frac{1}{2}}Z_2^2\\ &- 2||\pi_\mathbf{2}||\left(\frac{2}{3}g^{\frac{3}{2}}||\mathbf{Z_1}||^{\frac{3}{2}} + \frac{1}{3}g^{-3}|Z_2|^3\right)\\ &= \left(\frac{3}{2}\pi_1^{\frac{3}{2}} - \frac{4}{3}\|\pi_\mathbf{2}\|g^{\frac{3}{2}}\right)\|\mathbf{Z_1}\|^{\frac{3}{2}}\\ &+ \left(\frac{3}{2^{\frac{3}{2}}} - \frac{2\|\pi_\mathbf{2}\|}{3}g^{-3}\right)|Z_2|^3.\end{aligned} \tag{39}$$

$W_1(z)$ is positive-definite if $\frac{3}{2}\pi_1^{\frac{3}{2}} - \frac{4}{3}\|\pi_\mathbf{2}\|g^{\frac{3}{2}} > 0$ and $\frac{3}{2^{\frac{3}{2}}} - \frac{2\|\pi_\mathbf{2}\|}{3}g^{-3} > 0$ or equivalently

$$\frac{2^{\frac{5}{6}}}{3^{\frac{2}{3}}}\|\pi_\mathbf{2}\|^{\frac{1}{3}} < g < \frac{3^{\frac{4}{3}}}{2^2}\frac{\pi_1}{\|\pi_\mathbf{2}\|^{\frac{2}{3}}} \tag{40}$$

and such a g exists if $\pi_1 > \frac{2^{\frac{5}{6}}2^2}{3^2}\|\pi_\mathbf{2}\|$. Thus π_1 should be selected such that $\pi_1 > \frac{2^{\frac{5}{6}}2^2}{3^2}\|\pi_\mathbf{2}\|$. It can be noted that it also fulfills $\pi_1 \geq \frac{2^{\frac{1}{2}}2^{\frac{2}{3}}}{3}\pi_2$ required for $V \geq 0$. This completes the proof. □

IV. Simulation

we demonstrate the robustness of nonsmooth PI control for the second order uncertain chain of integrators containing constant or time-varying matched disturbances. Consider the following second order system

$$\dot{x}_1 = x_2; \dot{x}_2 = f(t,x) + u + d, \quad x^\top = [x_1, x_2] \in \mathbb{R}^2 \tag{41}$$

where the proposed control u is taken as

$$u = -f(t,x) - k_1x_1 - k_2x_2 + \int_0^t -k_3\text{sign}(k_1x_1 + k_2x_2)d\tau \tag{42}$$

where $k_i > 0$ for $i = 1, \ldots, 3$. Evolution of states

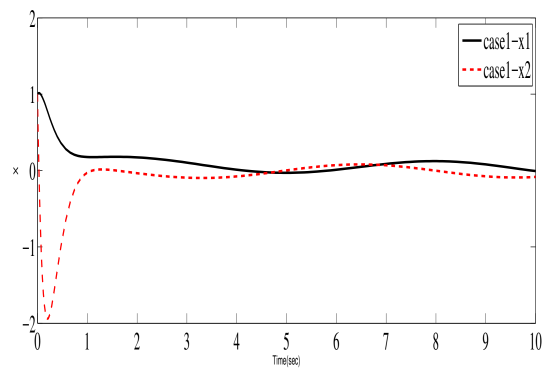

Fig. 4. Evolution of states with disturbance

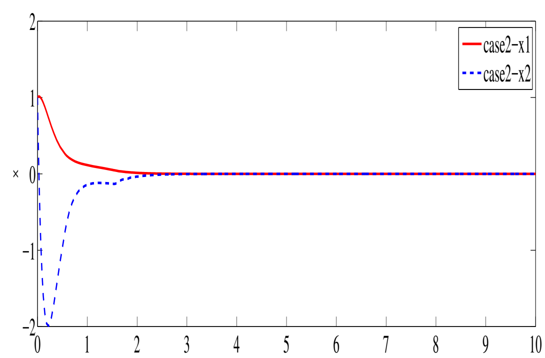

Fig. 5. Evolution of states with disturbance using proposed controller

are shown in the Fig. 4 using linear state feedback in the presence of disturbance $d = 1 + 3\sin(t)$. Above mathematical

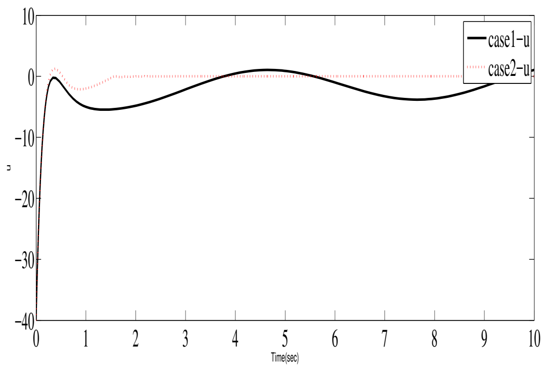

Fig. 6. Control

formulation is shown by simulations in Fig. 5, which shows the convergence of the states in the presence of disturbance $d = 1+3\sin(t)$. Case-1 and Case -2 of Fig. 6 shows the control effort using linear state feedback and above mathematical formulation respectively. In the simulation, controller gains and the initial conditions are chosen as $k_1 = 30, k_2 = 11, k_3 = 5$ and $x_0 = [1\ 1]^\top$ respectively. It is also confirmed from the simulation shown in Fig. 6, that the control is continuous.

V. Conclusion

The stabilization of systems with state feedback controller under uncertainty is studied in this paper. Here a modified a state feedback controller is proposed to achieve the asymptotic stability in the presence of disturbances. The proposed method completely rejects the Lipschitz matched disturbances. The insensitivity to the disturbances are obtained by incorporating the discontinuous part in the controller. The stability of this controller for uncertain chain of integrator is established for first order system, then it is extended to uncertain chain of integrator. The performances of the controller is evaluated.

References

[1] H. K. Khalil, *Nonlinear systems*, Upper Saddle River: Prentice hall, 2002.

[2] J. Alvarez-Ramirez, F. Guillermo, and S. Rodolfo, “A PI controller configuration for robust control of a class of nonlinear systems”, *J. of the Franklin Institute*, vol. 339, no. 1, pp. 29-41,2002.

[3] V. Utkin, *Sliding Mode Control in Electro-Mechanical Systems*, CRC Press, Second edition, Automation and Control Engineering, 2009.

[4] C. Edwards and S. Spurgeon, *Sliding Mode Control*, Taylor and Francis, Eds.London:, 1998.

[5] S. P. Bhat, D. S. Bernstein, “Geometric homogeneity with applications to finite-time stability ,” *Mathematics of Control Signals and Systems*, vol. 17, pp. 101–127, 2005.

[6] I. Boiko, *Discontinuous control systems: frequency-domain analysis and design.* Springer, 2009.

[7] A. Levant, “Homogeinity approach to high-order sliding mode design”, *Automatica*, no. 41, pp. 823–830, 2005.

[8] A. Levant, “Sliding order and sliding accuracy in sliding mode control,” *Int. J. Control*, vol. 58, no. 6, pp. 1247–1263, 1993.

[9] A. Levant, “Principles of 2-sliding mode design”, *Automatica*, vol. 43, no. 4 pp. 576–586, 2007.

[10] A. Levant, “Quasi-Continuous High-Order Sliding-Mode Controllers”. *IEEE Trans. Autom. Control*, vol. 50, no. 11, pp. 1812–1816, 2005.

[11] A. Levant, “Higher-order sliding modes, differentiation and output-feedback control,” *Int. J. Control*, vol. L. 76, no. 9/10, pp. 924–941, 2003.

[12] L. Fridman, “Sliding mode enforcement after 1990: main results and some open problems.” *Sliding Modes after the First Decade of the 21st Century*. Springer Berlin Heidelberg, pp. 3-57, 2012.

[13] V. Utkin and S. P. Alexander, “Adaptive Sliding Mode Control,” *Advances in Sliding Mode Control*, Springer Berlin Heidelberg, pp. 21-53, 2013.

[14] V. Utkin, “On convergence time and disturbance rejection of Super-twisting control”, *IEEE Trans. Autom. Control*, FP-12-348, 2013.

[15] A. F. Filippov, *Diffrential equations and discontinuous right hand side*, Kluwer, Dordrecht, The Netherlands, 1988.