Loopy belief propagation

CS B553
Spring 2013

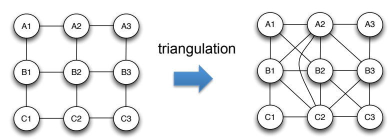
Announcements

- A2 due tonight
  - Reminder of late policy: 10% off for up to 48 hours late
- A3 coming soon!

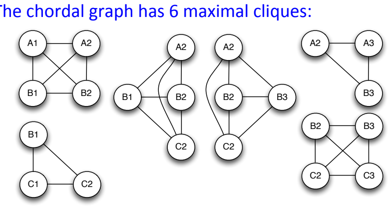
Constructing a clique tree

- One way to find a clique tree is to choose a variable elimination ordering and “run” VE
- Another approach is to use a graph construction
  - If necessary, **moralize** to produce an undirected graph G.
  - **Triangulate** G to produce a chordal graph H. (Would like one with minimum clique size, but this is NP hard.)
  - Find **maximal cliques** in H. (Not NP hard for chordal graphs.)
  - **Construct graph** with nodes corresponding to max cliques in H, edges weighted according to degree of overlap. I.e. edge between C1 and C2 has weight |C1 ⋂ C2 |
  - **Find a max spanning tree** on this graph to yield a clique tree.

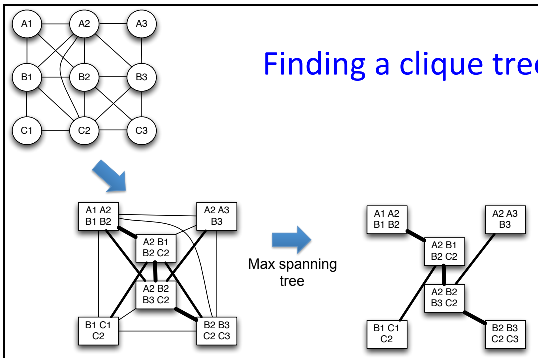
Clique trees for grid graphs



- The chordal graph has 6 maximal cliques:



Finding a clique tree



(Edge thickness indicates edge weight, i.e. size of intersection of variables)

Tree width

- The *tree width* of a graph G is equal to m-1,
  - Where m is the size of the largest clique in the triangulated (chordal) version of G
- **The worst-case running time of exact inference on a Markov or Bayes network is exponential in the tree width of the moralized graph.**
  - E.g., the tree width of an n x n grid graph is n, so inference takes $O(s^n)$ time, where s is # of possible values of each random variable

Making inference tractable

- Three strategies for efficient inference in practice:
  1. **Use a graphical model with low tree width.**
  2. **Make some additional assumptions about the problem.** (Efficient algorithms exist for some special cases, e.g. when the clique potentials have a specific form.)
  3. **Settle for an approximate inference algorithm.**

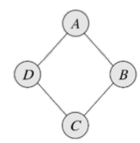
Making inference tractable

- In practice, making inference tractable is a key challenge in applying graphical models to applications
- Typically, the options are:
  - **Exact inference** with **arbitrary potentials** on a graphical model, but with a **simplified structure**
  - **Exact inference** on a graphical model with **arbitrary structure**, but **restricted potentials**
  - Graphical model with **arbitrary structure** and **arbitrary potentials,** but settle for **approximate inference**

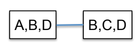
Approximate inference

- So far, we've covered *exact* inference algorithms
  - I.e. algorithms that can compute marginal probability distributions exactly, with no error
  - They are fast for some graphical models, like trees, but are prohibitively expensive for most applications
- But is exact inference worth it?
  - Our models aren't perfect anyway…
- Much more efficient *approximate* inference algorithms exist
  - If we're willing to settle for inexact answers

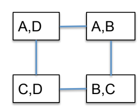
Consider a simple cycle:



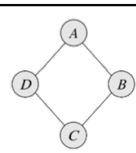
- A clique tree for this cycle is:



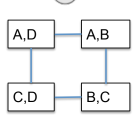
  - Recall that a clique tree is a special *cluster graph* (each node is a clique of the original graph, edges connect subset of nodes sharing a common variable)
- Here's another cluster graph for the above cycle:
  - It's not a clique tree
  - It's a *loopy* graph



Back to sum-product…



- Recall that sum-product involves sending messages to neighbors,



$$\delta_{i\to j}(\mathbf{S_{i,j}}) = \sum_{\mathbf{C_i}-\mathbf{C_j}} \psi_i(\mathbf{C_i}) \prod_{k\in\mathcal{N}(i)-\{j\}} \delta_{k\to i}(\mathbf{S_{k,i}})$$

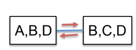
- We derived this for use on a clique tree, but the definition of messages (above) works on any graph!
  - I.e., there's nothing that prevents us from running Sum-Product Belief Propagation on a loopy cluster graph
  - There's just no theoretical justification for doing this…

Sum product

$$\delta_{i\to j}(\mathbf{S_{i,j}}) = \sum_{\mathbf{C_i}-\mathbf{C_j}} \psi_i(\mathbf{C_i}) \prod_{k\in\mathcal{N}(i)-\{j\}} \delta_{k\to i}(\mathbf{S_{k,i}})$$

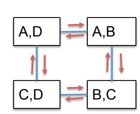
BP on a clique tree

- Messages start at leaves, propagate up to root, then propagate down to leaves



BP on loopy cluster graph

- Nodes send messages to their neighbors, based on their clique potential and messages from neighbors



Loopy Belief Propagation

- Construct a cluster graph from the Markov network
- Each node sends messages to its neighbor,

$$\delta_{i \to j}(\mathbf{S_{i,j}}) = \sum_{\mathbf{C_i} - \mathbf{C_j}} \psi_i(\mathbf{C_i}) \prod_{k \in \mathcal{N}(i) - \{j\}} \delta_{k \to i}(\mathbf{S_{k,i}})$$

  – Problems with this?
- How to begin?
  – Initialize by having every node send a "fake" initial message, consisting of just a uniform distribution
- How to end?
  – Keep running iterations of BP until convergence
  – Unfortunately, convergence might not happen…

Historical note: Turbo codes

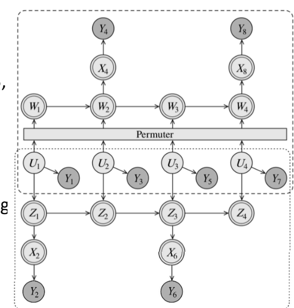
- Loopy Belief Propagation (LBP) has been known for ~25 years, but people assumed it was not useful
- In the 1990s, a seemingly unrelated discovery sparked renewed interest in LBP
  – Paper on a new error-correcting transmission algorithm, for noisy links (e.g. wireless networks)
  – Breakthrough in communications technology; modern wireless phones still use this (or a related) technique

Noisy link model

- Suppose we want to send bits U1, U2, U3, U4 across a noisy communications link
  – Need some redundancy to detect and correct errors
  – So, we encode these bits as some new bits X1 – X8
  – The observer receives some bits Y1—Y8, which may be a corrupted copy of X1 – X8
- From a probabilistic inference standpoint, receiver wants to compute distribution over **X** given **Y**

Turbo codes

- Berrou (1993)
- Solve for U1—U4, given Y1 – Y8
- Hack: First solve for Y1, Y3, Y5, Y7.
- Hold these constant, then solve for Y2, Y4, Y6, Y8.
- Then repeat, iteratively; i.e. "Turbocharge" estimates using iterative feedback loop
- Unbeknownst to them, they had (re-)discovered Loopy Belief Propagation!



Loopy BP properties

- Recall that clique trees exhibited the *running intersection property*:
  – If variable X is part of node C and node D, then every node along the path from C to D also contains X
- Loopy cluster graphs won't satisfy this condition
- Instead, we'll require the following weaker property:
  – If variable X is part of node C and node D, then there exists exactly one path between C and D such that the scope of the messages along the path include X
  – Intuitively, helps to prevent endless propagation cycles

Loopy BP properties

- Loopy BP can be **much** faster than exact inference
  – Running time?
- Loopy BP is not guaranteed to converge to a solution
- Even if it does converge, it's not guaranteed to converge to the correct solution
  – However, in practice it seems to converge to a reasonably good solution for a variety of Markov net problems
- Rather little is known about its theoretical properties
  – Including when it will work well and when it won't
  – We'll see (later) one explanation for why it works

Loopy BP on grid graphs

- In the special case of grid graphs, we can run BP on the Markov net directly, instead of the cluster graph
- This is possible because we can create a cluster graph with a very simple structure
- The computation is exactly the same, but it's more convenient to understand and implement this way

Belief propagation

- Messages are passed between the variables
  - Message from node *i* to *j* at iteration *t* is,

$$m^{(t)}_{i\to j}(X_j) = \sum_{X_i} \phi(X_i, X_j) \prod_{k\in\mathcal{N}(X_i)-\{X_j\}} m^{(t-1)}_{k\to i}(X_i)$$

  - Intuitively (and anthropomorphically) a message from me to you says, for each of your states, "If you choose state Xj, here's how happy my neighbors and I would be."