Graph cuts

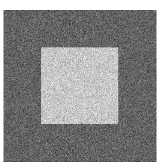
CS B553
Spring 2013

Faster MAP inference?

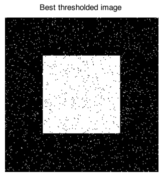
- We've now seen two algorithms for MAP inference:
  - Variable elimination: Exact, but potentially very slow
  - Loopy Max-product BP: Fast, but approximate
- It turns out that in some cases, MAP problems are easier than Marginal inference problems
  - One interesting case: With binary random variables, and potential functions that satisfy (relatively weak) restrictions, exact inference on a pairwise Markov network is efficient

Consider a simpler problem…

- Suppose we want to find a bright object against a dark background
  - But some of the pixel values are slightly wrong
  - So want to estimate a 0 or 1 label for each pixel

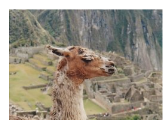
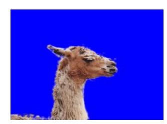


Best thresholded image



A slightly more interesting problem…

- Foreground vs background segmentation
  - We want to label every pixel of an image with a 0 or a 1, indicating whether it's a background or foreground pixel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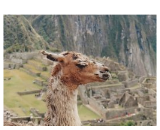
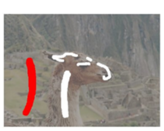
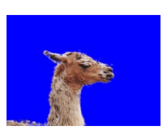
A slightly more interesting problem…

- One approach: Use some human input
  - Use sketches out a few strokes to indicate foreground and background, then we try to classify rest of pixels.

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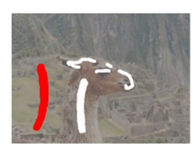
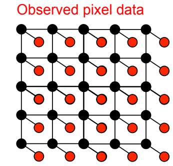
Solving with an MRF

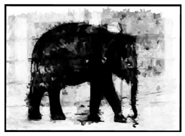


Observed pixel data

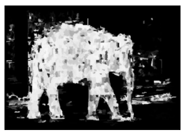


Unobservable binary labels

Data cost



D(fg, Y)



D(bg, Y)

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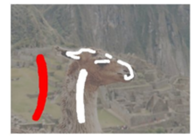
Solving with an MRF



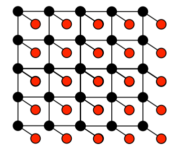
- So, we want to solve a problem of the form:

$$X^* = \arg\min_X \sum_i D_i(X_i, Y_i) + \sum_{(i,j)\in\mathcal{E}} V(X_i, X_j)$$

  - Y variables are given
  - X variables are binary-valued
  - D cost functions have any form
  - V cost functions have the form:

$$V(X_i, X_j) = \begin{cases} 0 & \text{if } X_i = X_j \\ k & \text{otherwise} \end{cases}$$

Observed pixel data



Unobservable binary labels

Network flow can help

- The minimization problem with 2 labels can be solved exactly using network flow
  - Construction probably due to [Hammer et al. 65]
  - First applied to images by [Greig et al. 86]
- Classical Computer Science problem reduction
  - Turn a new problem into a problem we can solve!

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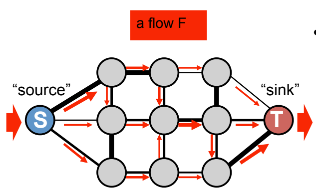
Maximum flow problem



A graph with two terminals

- Max flow problem:
  - Each edge is a "pipe"
  - Find the largest flow F of "water" that can be sent from the "source" to the "sink" along the pipes
  - Source output = sink input = flow value
  - Edge weights give the pipe's capacity

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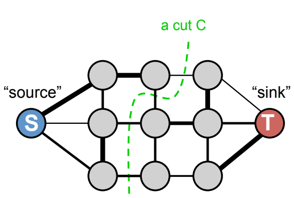
Minimum cut problem



A graph with two terminals

- Min cut problem:
  - Find the cheapest way to cut the edges so that the "source" is separated from the "sink"
  - Cut edges going from source side to sink side
  - Edge weights now represent cutting "costs"

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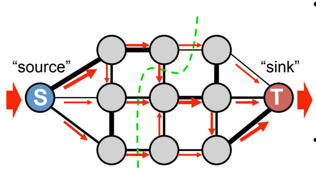
Max flow/Min cut theorem



A graph with two terminals

- Max Flow = Min Cut:
  - Proof sketch: value of a flow is value over any cut
  - Maximum flow saturates the edges along the minimum cut
    - Ford and Fulkerson, 1962
    - Problem reduction!
- Ford and Fulkerson gave first polynomial time algorithm for globally optimal solution

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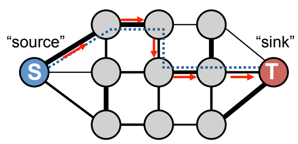
"Augmenting Path" algorithms



A graph with two terminals

- Find a path from S to T along non-saturated edges
- Increase flow along this path until some edge saturates

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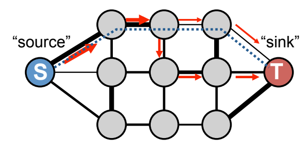
"Augmenting Path" algorithms



A graph with two terminals

- Find a path from S to T along non-saturated edges
- Increase flow along this path until some edge saturates
- Find next path…
- Increase flow…

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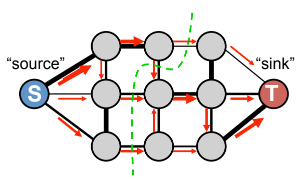
"Augmenting Path" algorithms



A graph with two terminals

- Find a path from S to T along non-saturated edges
- Increase flow along this path until some edge saturates

Iterate until all paths from S to T have at least one saturated edge

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Min flow algorithms

- Ford-Fulkerson (1962) is the classic algorithm
  - Takes time O(|E| f), where f is the maximum flow
  - May not converge in some cases
- Edmonds-Karp (1972) gave an improved version
  - Same as F-F, but the augmented path is always the shortest with available capacity. Can be found using breadth-first search.
  - Takes time O(|V| $|E|^2$)

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Back to binary MRFs…

- We want to solve a problem of the form:

$$X^* = \arg\min_X \sum_i D_i(X_i, Y_i) + \sum_{(i,j)\in\mathcal{E}} V(X_i, X_j)$$

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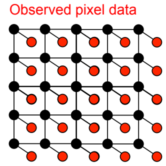
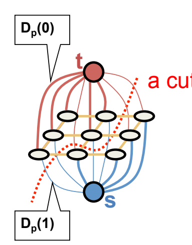


Unobservable binary labels

Basic graph cut construction

- One non-terminal vertex per pixel
  - Each pixel connects directly to s,t, and to its neighbors
  - Edge to t has weight Dp(0), edge to s has weight Dp(1)
  - Edge (p,q) has weight $V_{pq}(0,1)$
- Cost of cut is the cost of the entire labeling
  - Pixel p labeled 1 if connected to t, labeled 0 if connected to s



$$E(x_1,\ldots,x_n) = \sum_p D_p(x_p) + \sum_{p,q} V_{p,q}(x_p, x_q)$$

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