Assignment 1 - B553

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Question 1

Total number of ways of arranging 4 parakeets = 8!
Total number of arrangements in which no two adjacent parakeets have the same color = 2!(4!*4!)

$$\begin{aligned} \Pr(\text{No two adjacent parakeets have the same color}) &= \frac{2!(4! * 4!)}{8!} \\ &= \frac{1}{35} \\ &= 0.02857142857143 \end{aligned}$$

Question 2

(a)

For a CPU to have all functioning cores, means that all cores should not be defective. Since the defects in the cores are all independent of one another,

$$\begin{aligned} \Pr(\text{no core has defect}) &= (0.7)^8 \\ &= 0.05764801 \end{aligned}$$

(b)

First we calculate the probability of each type of computer being made. We can consider this as a binomial distribution with 8 trials and the probability of success p as 0.7. The binomial probability can be expressed as:

$$B(8, 0.7, x) = \binom{8}{x} * (0.7)^x * (0.3)^{(8-x)}$$

$$\begin{aligned}
\Pr(\text{Great model}) &= \Pr(\text{one,two or three cores working}) \\
&= \sum_{x=1}^{3} \left(\binom{8}{x} * (0.7)^x * (0.3)^{8-x} \right) \\
&= 0.05790
\end{aligned}$$

$$\begin{aligned}
\Pr(\text{Advanced model}) &= \Pr(\text{at least 4 cores functioning}) \\
&= \sum_{x=4}^{7} \left(\binom{8}{x} * (0.7)^x * (0.3)^{8-x} \right) \\
&= 0.88438
\end{aligned}$$

$$\begin{aligned}
\Pr(\text{Extreme model}) &= \Pr(\text{all 8 cores functioning}) \\
&= (0.7)^8 \\
&= 0.05764801
\end{aligned}$$

Expectation of number of each of these models is then given by the equation:

$$E(X) = \sum_{x \in X} (x * pr(x))$$

Using this, we get,

$$\begin{aligned}
E(\text{Great model}) &= 1000 * 0.05790 \\
&= 57.9 \approx 58
\end{aligned}$$

$$\begin{aligned}
E(\text{Advanced model}) &= 1000 * 0.88438 \\
&= 884.38 \approx 884
\end{aligned}$$

$$\begin{aligned}
E(\text{Extreme model}) &= 1000 * 0.05764801 \\
&= 57.64801 \approx 58
\end{aligned}$$

(c)

Expected revenue can be calculated as:

$$\begin{aligned}
E(revenue) &= \$(58 * 50 * 0.057) + \$(100 * 884 * 0.884) + \$(58 * 1000 * .057) \\
&= \$81,616
\end{aligned}$$

Question 3

Let us define the following events:

$$\begin{aligned}
A_1 &= \text{Judge 1 votes guilty} \\
A_2 &= \text{Judge 2 votes guilty} \\
A_3 &= \text{Judge 3 votes guilty} \\
G &= \text{The person is guilty}
\end{aligned}$$

We are given the following things:

$$\begin{aligned}
Pr(G) &= 0.7 \\
Pr(A_1|G) = Pr(A_2|G) &= Pr(A_3|G) = 0.7 \\
Pr(A_1|\bar{G}) = Pr(A_2|\bar{G}) &= Pr(A_3|\bar{G}) = 0.2
\end{aligned}$$

(a)

Using Bayes rule,

$$\begin{aligned}
Pr(G|A_1) &= \frac{Pr(A_1|G) * Pr(G)}{Pr(A_1)} \\
&= \frac{Pr(A_1|G) * Pr(G)}{(Pr(A_1|G) * Pr(G)) + \left(Pr(A_1|\bar{G}) * Pr(\bar{G})\right)} \\
&= \frac{0.7 * 0.7}{(0.7 * 0.7) + (0.2 * 0.3)} \\
&= 0.8909
\end{aligned}$$

(b)

$$\begin{aligned}
Pr(G|A_1, A_2, A_3) &= \left(\frac{Pr(G, A_1, A_2, A_3)}{Pr(A_1, A_2, A_3)}\right) \\
&= \left(\frac{Pr(A_1|G) * Pr(A_2|G) * Pr(A_3|G) * Pr(G)}{(\sum_{g \in G} Pr(A_1|G = g) * Pr(A_2|G = g) * Pr(A_3|G = g) * Pr(G = g))}\right) \\
&= \left(\frac{0.7^4}{(0.7^4) + (0.2^3 * 0.3)}\right) \\
&= 0.9901
\end{aligned}$$

(c)

$$
\begin{aligned}
Pr(A_3|\bar{A}_1, \bar{A}_2) &= \frac{Pr(\bar{A}_1, \bar{A}_2, A_3)}{Pr(\bar{A}_1, \bar{A}_2)} \\
&= \frac{\Sigma_{g \in G}\left(Pr(\bar{A}_1|g) * Pr(\bar{A}_2|g) * Pr(A_3|g) * Pr(g)\right)}{\Sigma_{g \in G}\left(Pr(\bar{A}_1|g) * Pr(\bar{A}_2|g) * Pr(g)\right)} \\
&= \frac{0.0825}{0.255} \\
&= 0.3235
\end{aligned}
$$

Question 4

Consider a set of $k+1$ events of which k are undesirable. Assume that the probability of the desired event is p. Then the expected number of trials to get the desired event is $\frac{1}{p}$ [1].

(a)

Since we have 3 of the 4 figurines, getting them in the next trial is an undesired event. Here the probability of getting the figurine that we want is $\frac{1}{4}$. Hence,

$$E[\text{Number of trials to get one figurine}] = \frac{1}{p} = \frac{1}{\frac{1}{4}} = 4$$

(b)

We can break down this question into four parts.

$$
\begin{aligned}
Pr(\text{Getting 1st new figurine}) &= 1 \\
Pr(\text{Getting 2nd new figurine}) &= \frac{3}{4} \\
Pr(\text{Getting 3rd new figurine}) &= \frac{1}{2} \\
Pr(\text{Getting 4th new figurine}) &= \frac{1}{4}
\end{aligned}
$$

Hence,

$$\begin{aligned}
E[\text{Number of trials to get all four figurines}] &= 1 + \frac{1}{\frac{3}{4}} + \frac{1}{\frac{1}{2}} + \frac{1}{\frac{1}{4}} \\
&= 1 + \frac{4}{3} + 2 + 4 \\
&= 8.333
\end{aligned}$$

Question 5

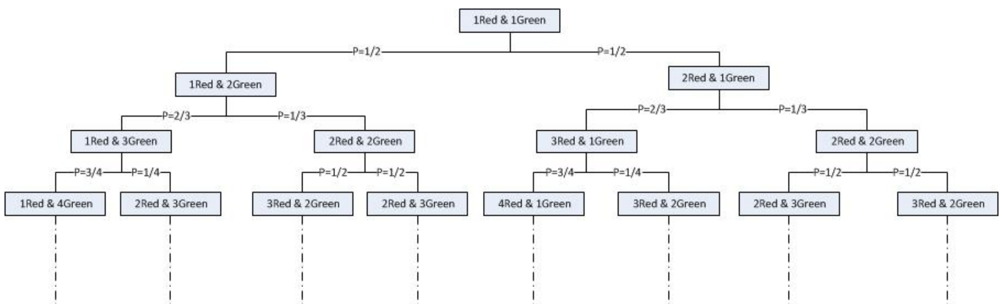

Figure 5a

Figure 1: Graph for 5a

Figure 1 demonstrates the graph can be generated for the given experiment. From it we get,

$$Pr[R = 1, G = (n+1)] = \left(\frac{1}{2} * \frac{2}{3} * \frac{3}{4} * \cdots * \frac{n}{(n+1)} \right)$$

If we move down each path in the given graph we find the following:

$$\begin{aligned}
Pr[R = 1, G = (n+1)] &= \frac{1}{n+1} \\
Pr[R = 2, G = n] &= \frac{1}{n+1} \\
Pr[R = n, G = 2] &= \frac{1}{n+1} \\
Pr[R = n+1, G = 1] &= \frac{1}{n+1}
\end{aligned}$$

where R,G are the number of red and green bills

This shows that the probability of getting red bills and green bills is the same.

(a)

By linearity of expectation,

$$E[G-R] = E[G] - E[R]$$

From the observation above, we get,

$$E[G-R] = 0$$

For the variance, we have the expression,

$$V[G-R] = \sum_{-n}^{n} (x-\mu)^2 * Pr(x)$$

Values of $G-R$ range from $-n$ to n
This can be split as:

$$V[G-R] = \sum_{-n}^{0} \big((x-\mu)^2 * Pr(x)\big) + \sum_{1}^{n} \big((x-\mu)^2 * Pr(x)\big)$$

Since our μ is 0, we get

$$V[G-R] = 2 * \sum_{1}^{n} (x)^2 * Pr(x)$$

Substituting the values from above, we get,

$$= 2 * 44469.1666$$
$$= 88938.33$$

(b)

If we need to owe the government at least \$300, then,

$$R - G \geq 300$$
$$\text{and}$$
$$R + G = 367$$
$$\Rightarrow G \geq 334$$
$$= \sum_{x=334}^{365} \big(x^2 * Pr(x)\big)$$
$$= 31.55 \quad \textit{(by calculation)}$$

Parts (a) and (b) have been discussed with [2]

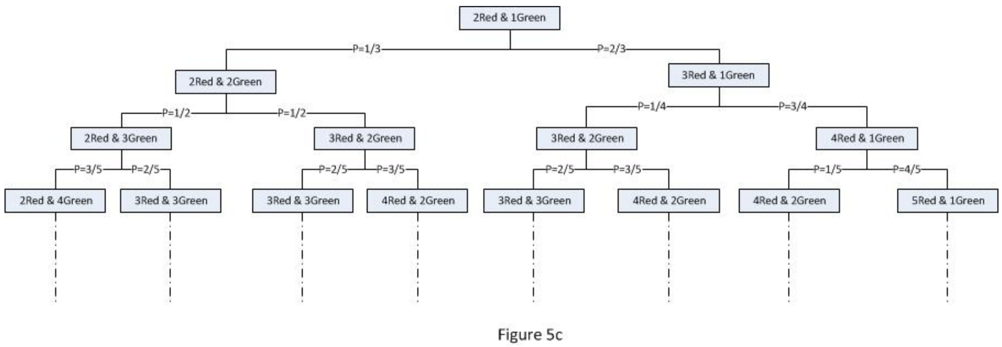

Figure 2: Graph for 5c

(c)

In this case *Figure* 2 shows the graph. As in part (a) of this question we get,

$$
\begin{aligned}
Pr[R = 2, G = (n+1)] &= \left(\frac{1}{3} * \frac{2}{4} * \frac{3}{5} * \cdots * \frac{(n-1)}{(n+1)}\right) \\
&= \frac{2}{n * (n+1)} \\
Pr[R = n+2, G = 1] &= \left(\frac{2}{3} * \frac{3}{4} * \frac{4}{5} * \cdots * \frac{n}{(n+1)}\right) \\
&= \frac{2}{(n+1)}
\end{aligned}
$$

To get the expectation, we again use linearity of expectation, after which we get,

$$
\begin{aligned}
E[G - R] &= E[G] - E[R] \\
&= \sum_{1}^{n+1} (x * Pr(x)) - \sum_{1}^{n+2} (x * Pr(x)) \\
&= \sum_{1}^{n+1} \left(x * \frac{2}{(n-x)(n+1)}\right) - \sum_{1}^{n+2} \left(x * \frac{2}{(n-x)(n+1)}\right) \\
&= -\frac{(n+2)}{(n+1)} \\
&= -\frac{367}{366} \\
&= -1.027
\end{aligned}
$$

Another way to solve this would be to consider the use of graph theory. Notice that the graph is a binary tree of depth 365. Hence the number of nodes is 2^{365}. Also note that at depth k, there are $k-1$ nodes that are repeated k times (that is, there are $k-1$ nodes that have the same combination of green bills and red bills as each other, and there are k such nodes).So, at level 365, we will have 364 nodes that have the same combination of green and red bills and there are 365 such nodes. So, the probability of r red bills and g green bills can be computed by using a modification of Djikstra's algorithm (where the path length is a multiplication of the individual edges, each edge representing the probability of picking a new bill) to these nodes.

References

[1] Mathematical expectation from CodeChef *http://www.codechef.com/wiki/tutorial-expectation*

[2] Nasheed Moiz, Shrutika Poyrekar