Mathematical Appendix: Estimation and Derivations

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Notation

Let $y_t = r_t^{\text{spread}}$. Define the inflation forecast error

$$s_t = \pi_t^{\text{yoy}} - \mathbb{E}[\pi_t^{\text{yoy}} \mid \mathcal{F}_{t-12}] \equiv \pi_t^{\text{yoy}} - \mathbb{E}_{t-12}[\pi_t^{\text{yoy}}].$$

For horizon h, define $y_t^{(h)} = y_{t+h}$. Let L denote the number of lags. The regressor vector is

$$x_{t,h} = (1,\ s_t,\ y_{t-1}, \ldots, y_{t-L},\ s_{t-1}, \ldots, s_{t-L})' \in \mathbb{R}^k, \qquad k = 2 + 2L.$$

Stacking observations gives $Y_h \in \mathbb{R}^{n_h}$ and $X_h \in \mathbb{R}^{n_h \times k}$.

Frequentist Local Projections

$$\hat{\theta}_h = (X_h' X_h)^{-1} X_h' Y_h, \qquad \hat{\beta}_h = e_2' \hat{\theta}_h.$$

Identification.

$$\mathbb{E}[u_t^{(h)} \mid s_t, y_{t-1:t-L}, s_{t-1:t-L}] = 0 \quad \Rightarrow \quad \operatorname{plim}_{n_h \to \infty} \hat{\theta}_h = \theta_h.$$

FWL. Partition $X_h = [\mathbf{s}\ W]$. Let $M_W = I - W(W'W)^{-1}W'$. Then

$$\hat{\beta}_h = \frac{\mathbf{s}' M_W Y_h}{\mathbf{s}' M_W \mathbf{s}} = \frac{\tilde{s}' \tilde{Y}_h}{\tilde{s}' \tilde{s}}.$$

HAC Inference

Let $\hat{u}_t^{(h)} = y_t^{(h)} - x_{t,h}' \hat{\theta}_h$, $g_t = x_{t,h} \hat{u}_t^{(h)}$.

$$\widehat{\Omega}_h = \widehat{\Gamma}_0 + \sum_{\ell=1}^{\bar{\ell}_h} w_\ell (\widehat{\Gamma}_\ell + \widehat{\Gamma}_\ell'), \quad \widehat{\Gamma}_\ell = \frac{1}{n_h} \sum_{t=\ell+1}^{n_h} g_t g_{t-\ell}', \quad w_\ell = 1 - \frac{\ell}{\bar{\ell}_h + 1}.$$

$$\widehat{\operatorname{Var}}(\hat{\theta}_h) = (X_h' X_h)^{-1} \widehat{\Omega}_h (X_h' X_h)^{-1}.$$

Cumulative LPs

Define $y_t^{\text{cum}(h)} = \sum_{j=1}^h y_{t+j}$ and estimate

$$y_t^{\text{cum}(h)} = x_{t,h}' \theta_h^{\text{cum}} + \varepsilon_t^{(h)}.$$

Because cumulative outcomes use overlapping forward windows, errors are serially correlated; HAC inference is therefore required.

Bayesian Ridge LP

$$Y_h \mid \theta_h, \sigma^2 \sim N(X_h\theta_h, \sigma^2 I), \quad \theta_h \mid \sigma^2 \sim N(0, \sigma^2\lambda^{-1}I), \quad p(\sigma^2) \propto 1/\sigma^2.$$

$$V_h = (X_h' X_h + \lambda I)^{-1}, \quad \mu_h = V_h X_h' Y_h, \quad \theta_h \mid \sigma^2, Y_h \sim N(\mu_h, \sigma^2 V_h).$$

$$\hat{\theta}_h^{\text{Bayes}} = (X_h' X_h + \lambda I)^{-1} X_h' Y_h.$$

Marginally integrating out σ_h^2, any linear combination $a'\theta_h \mid Y_h$ follows a Student-t distribution with location $a'\mu_h$ and scale proportional to $a'V_h a$, as implied by the conjugate Normal–Inverse-Gamma posterior.

Rate-Controlled Extension

Augmenting with yield changes Δy_t^{10} gives

$$\hat{\beta}_h^{(r)} = \frac{\mathbf{s}' M_Z Y_h}{\mathbf{s}' M_Z \mathbf{s}}, \qquad M_Z = I - Z_h (Z_h' Z_h)^{-1} Z_h'.$$