End Semester Examination | PH17B011 Chaithanya

PART-A

1. (a) i) Assuming pi = M_PI
= 3.14159.

Answer: 3.14159.

ii). $\epsilon_{total} = \sqrt{N}\epsilon_m + \frac{\beta}{N^4}$; β is a constant

For minimizing total error,

$$\frac{d\epsilon}{dN} = 0 \Rightarrow \frac{\sqrt{\epsilon_m}}{2\sqrt{N}} - \frac{4\beta}{N^5} = 0$$

$$\Rightarrow N^{9/2} = \frac{8\beta}{\sqrt{\epsilon_m}}$$

$$\Rightarrow N \simeq \left(\frac{8\beta}{\sqrt{\epsilon_m}}\right)^{2/9}$$, β is a constant of proportionality

$$\therefore N \sim \epsilon_m^{-9}$$

2. b) $f(x \pm h) = f(x) \pm h f'(x) + \frac{h^2}{2!} f'' \pm \frac{h^3}{3!} f''' + \frac{h^4}{4!} f^{(4)} \pm \frac{h^5}{5!} f^5 + \dots$

$$f(x \pm 2h) = f(x) \pm 2h f'(x) + \frac{2^2 h^2}{2!} f'' \pm \frac{2^3 h^3}{3!} f''' + \frac{2^4 h^4}{4!} f^4 \pm \frac{h^5 2^5}{5!} f^5 + \dots$$

~~$f(x+2h) - 2f(x+h) + 2f(x-h) - f(x-2h)$:~~

$$f(x+2h) - f(x-2h) = 2\left[2h f'(x) + \frac{2^3 h^3}{3!} f''' + \frac{2^5 h^5}{5!} f^{(5)} + O(h^7)\right]$$

$$f(x-h) - f(x+h) = -2\left[h f' + \frac{h^3}{3!} f''' + \frac{h^5}{5!} f^{(5)} + O(h^7)\right]$$

$$\Rightarrow f(x+2h) - 2f(x+h) + 2f(x-h) - f(x-2h)$$

$$= 2\left[2h f' + \frac{2^3 h^3}{3!} f''' + \frac{2^5 h^5}{5!} f^{(5)} + O(h^7)\right] - 2 \times 2\left[h f' + \frac{h^3}{3!} f''' + \frac{h^5}{5!} f^{(5)} + O(h^7)\right]$$

$$= h^3 \left(\frac{16-4}{3!}\right) f''' + \left(\frac{64-4}{5!}\right) h^5 f^{(5)} + O(h^7)$$

$$\Rightarrow f''' = \frac{f(x+2h) - 2f(x+h) + 2f(x-h) - f(x-2h)}{2h^3} - \frac{1}{4} h^2 f^{(5)} + O(h^4)$$

$\therefore$ Truncation error $= \boxed{\frac{h^2 f^{(5)}}{4}}$

with $f(x) = x^3$,

$$\frac{f(x+2h) - 2f(x+h) + 2f(x-h) - f(x-2h)}{2h^3}$$

$$= \frac{(x+2h)^3 - 2(x+h)^3 + 2(x-h)^3 - (x-2h)^3}{2h^3}$$

$$= \left[2\times(3x^2(2h) + (2h)^3) - 4(3x^2h + h^3)\right] / 2h^3$$

$$= \frac{12x^2h + 16h^3 - 12x^2h - 4h^3}{2h^3}$$

$$= \frac{6h^3}{h^3} = \underline{\underline{6}}$$

Truncation error $= f'''''\frac{h^2}{4}$

since $f'''''(x^3) = 0$
There is no truncation error and the answer obtained above is exact.

e) $I = \int_a^b dx\, f(x) \simeq \int_a^b P_N(x)$

$P_N(x) = \sum_i^N l_i(x) f(x_i)$, $l_i = \prod_{j=0, j\neq i}^{N} \frac{x - x_j}{x_i - x_j}$

$0 \leq i \leq N$

For $N=1$, taking the points $x_0 = a$, $x_1 = b$,

$$W_i = \int_a^b l_i(x)\,dx$$

$$\Rightarrow W_0 = \int_a^b l_0(x)\,dx = \int_a^b \frac{x-b}{a-b}\,dx$$

$$= \frac{1}{(a-b)} \left[\frac{x^2}{2} - bx\right]\Bigg|_a^b$$

$$= \frac{1}{(a-b)} \left(\frac{b^2}{2} - b^2 - \frac{a^2}{2} + ab\right)$$

$$= +\frac{(b-a)}{2}$$

$$W_1 = \int_a^b l_1(x)\,dx = \int_a^b \frac{x-a}{b-a}\,dx$$

$$= \frac{1}{(b-a)} \left(\frac{b^2}{2} - ab + a^2 - a^2/2\right) = \frac{b-a}{2}$$

$$\Rightarrow \cancel{\not{}} I = \sum_i f(x_i)\,w_i$$

$$= f(x_0)W_0 + f(x_i)W_1$$

$$= \frac{(a-b)}{2}\left(f(x_1) - f(x_0)\right)$$

This is the <u>trapezoidal method</u> with only one interval for integration approximation.

d) options (i) and (ii) have subtractions of numbers that are close to each other. Subtraction of two close numbers result in a very small number which can amplify the total error in the resulting number as:

$a = b - c$

$\Rightarrow a_c = b_c - c_c$; a_c b_c & c_c are roundoffs

$\Rightarrow \frac{a_c}{a} \simeq 1 + \frac{b}{a}(\epsilon_b - \epsilon_c)$

if b and c are close ~~together~~ since a is small, subtraction error is high.

Hence option (iii) is preferable.

PART-B

2. Critical discussion of the three methods for finding roots.

1) Bisection Method:

This algorithm is applicable to a continuous f on an interval [a,b] where f has atleast one root ~~and~~ ~~tot~~ around which sign of f changes.

1. The algorithm begins with the bounds [a,b] given $f(a) \times f(b) < 0$ and ~~$f(a) \neq 0$ & $f(b) \neq 0$~~

2. Mid point of $[a,b] = \frac{a+b}{2}$ is calculated.

3. If $f(\frac{a+b}{2} = c) = 0$, c is the root.

   Else if $f(a).f(c) < 0$, $b = c$

   else $a = c$ and the process is repeated from step 2 till tolerance is reached.

   $|a-b| <=$ tolerance.

   In the code, tolerance $= 10^{-5}$

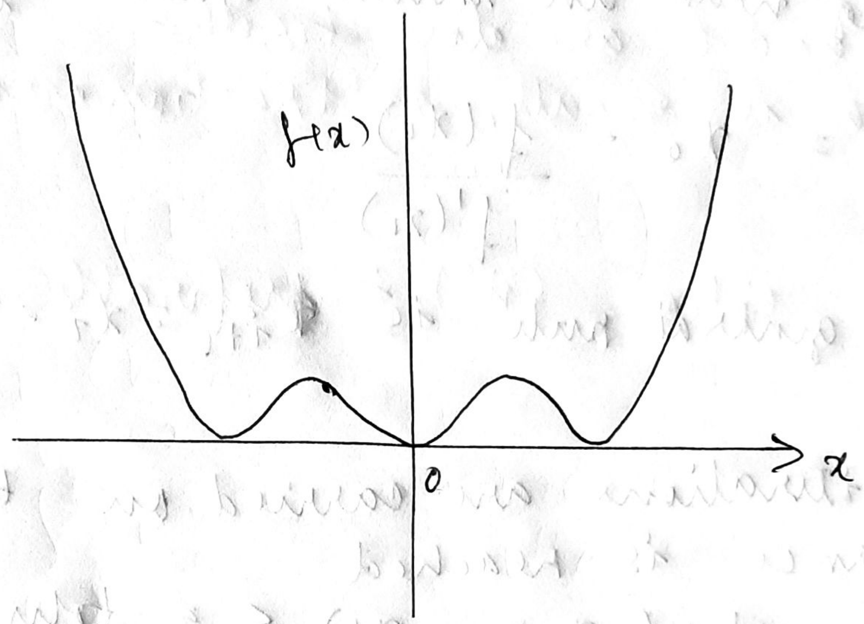
Advantages:

- Simple to implement.
- Solution is guaranteed as if f changes signs in the interval, $f = 0$ must exist at some point.
- Increasing number of iterations gives better results.

For the given function

$f(x) = x^2 - 4x\sin x + (2\sin x)^2$,

the approximate graph is shown below.



There is no change in sign of $f(x)$ in any interval.

Hence bisection algorithm cannot be used.

However, $f(x) = (x - 2\sin x)^2$

$\therefore$ Finding root of $x - 2\sin x$ can give a solution to $f(x)$.

The positive solution obtained is 1.89549 after 17 iterations.

The initial range was taken to be [1.5, 2] for positive root solution.

Disadvantages:

Slow convergence.

Can't apply to functions that do not change signs around roots (as in this case).

11) Newton - Raphson Method

For a differentiable function f that does not have $f' = 0$ at any point; starting with an initial guess x_0,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

The general rule is $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$

The iterations are carried out till tolerance is reached.

e.g. $abs(x_{i+1} - x_i) <$ tolerance.

Tolerance $= 10^{-5}$ was used in the codes.

Advantages:

- Faster convergence rate
- Only one initial guess is required.
- Non-sign changing functions can be used.

Disadvantages:

- Convergence is not guaranteed.
- Difficult to apply on functions that have a complex derivative.
- Derivative of function must not be zero at any point.

In the implementation, the algorithm terminated in 44 steps for initial $x_0 = 1.5$.

The solution on termination was 1.89532

iii) Secant Method

Applicable to a continuous function on an initial interval $[a,b]$.
Taking $a = x_0$ & $b = x_1$; next steps in the iterations are calculated as

$$x_{i+1} = \frac{(x_{i-1}) f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}$$

given that $f(x_i) - f(x_{i-1}) \neq 0$.
The iterations are carried out till tolerance $> |x_{i+1} - x_i|$ is reached.
In this implementation, tolerance $= 10^{-5}$
~~&~~ the intial guesses were $x_0 = 1.5$ and $x_1 = 2$.
This was chosen to avoid NaN errors.
The algorithm gave a solution of 1.89567 after 18 iterations.

Advantages:

- Since it is not linear, it converges faster than bisection for equal implementations. (Not this case, since bisection was carried out on $x - a\sin x$ & not the original function).

Disadvantages:

- Convergence may not occur (NaN values)
- if $f(x_{i+1}) - f(x_i) = 0$, secant method ~~canot~~ cannot be used

For the given question, secant method provided the fastest convergence.