In[ ]:=
```
Remove["Global`*"]
```

Bounded Trajectories

Numerically solving the equations parametrically, with parameter μ (mass ratio) for bounded trajectories and generating plots for various μ.

$\frac{d}{dt}v(t) = (r(t)\ \omega(t)^2\ +\ g\cos(\theta(t) - \mu))/(1+\mu)$

$\frac{d}{dt}\omega(t)\ = (-2v(t)\omega(t) - g\sin(\theta(t)))/r(t)$

with initial conditions $r(0) = 1$, $\theta(0) = 90^0$, $v(0) = 0$, $\omega(0) = 0$

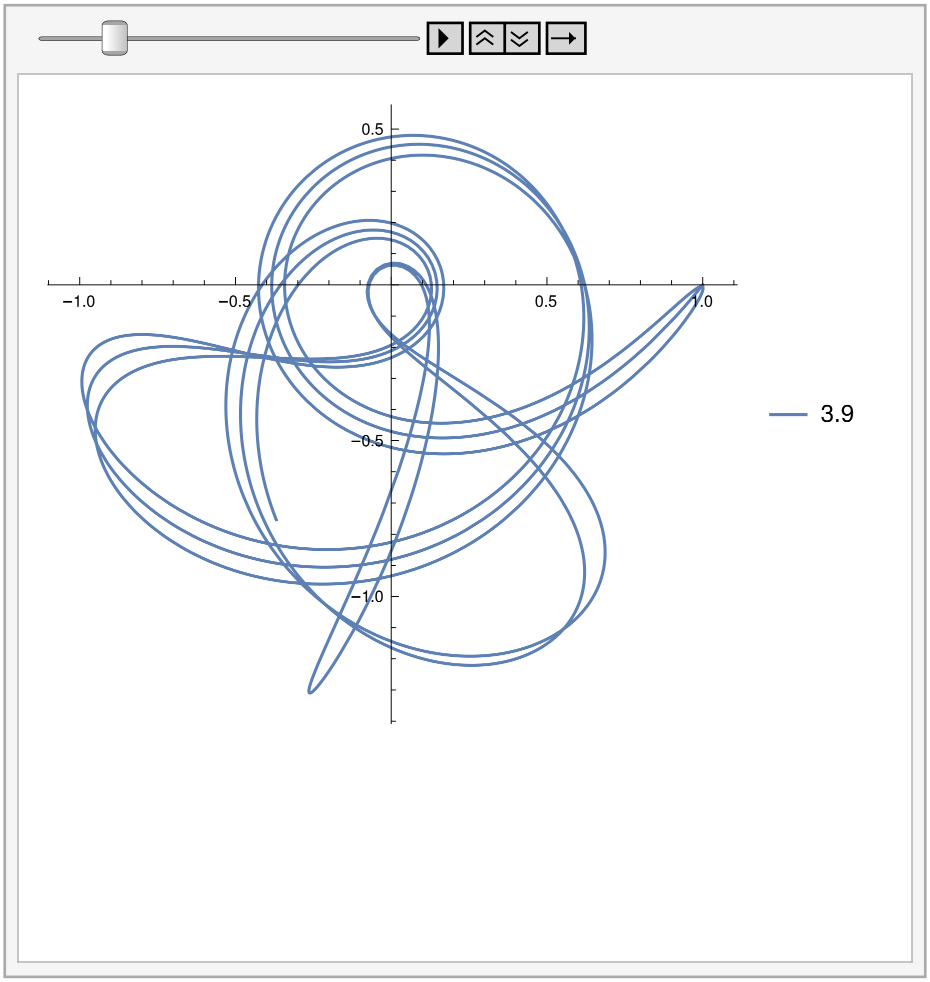
In[ ]:=
```
sol = ParametricNDSolve[{ v '[t] == ( r[t] × ω[t]^2 + 9.8 (Cos[θ[t]] - μ)) / (1 + μ),
   ω '[t] == (- 2 v[t] × ω[t] - 9.8 Sin[θ[t]]) / r[t], r '[t] == v[t], θ '[t] == ω[t], r[0] == 1,
   θ[0] == Pi / 2, v[0] == 0, ω[0] == 0}, {r, v, θ, ω}, {t, 0, 100}, {μ}, MaxSteps → 200 000]
```

Out[ ]=
```
{r → ParametricFunction[ Expression: r  Parameters: {μ} ],
 v → ParametricFunction[ Expression: v  Parameters: {μ} ],
 θ → ParametricFunction[ Expression: θ  Parameters: {μ} ],
 ω → ParametricFunction[ Expression: ω  Parameters: {μ} ]}
```

In[ ]:=
```
fr = Table[ParametricPlot[Evaluate[{r[μ][t] Sin[θ[μ][t]], -r[μ][t] Cos[θ[μ][t]]} /. sol],
    {t, 0, 10}, PlotLegends → {μ}], {μ, 1, 20, 0.1}];
```

In[ ]:=
```
ListAnimate[fr]
```

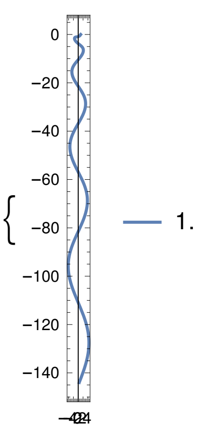
Out[ ]=



Plots for varying μ, with μ indicated on right edge

In[ ]:=
```
Table[ParametricPlot[Evaluate[{r[μ][t] Sin[θ[μ][t]], -r[μ][t] Cos[θ[μ][t]]} /. sol],
  {t, 0, 50}, PlotLegends → {μ}, Frame → True], {μ, 1, 10, 0.1}]
```

Out[ ]= {



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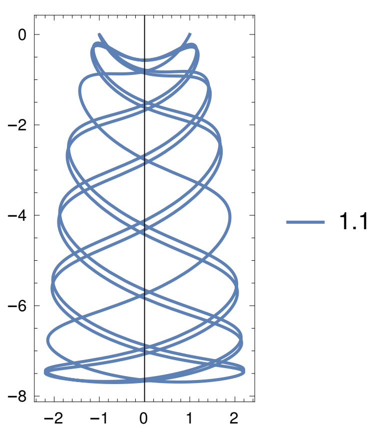

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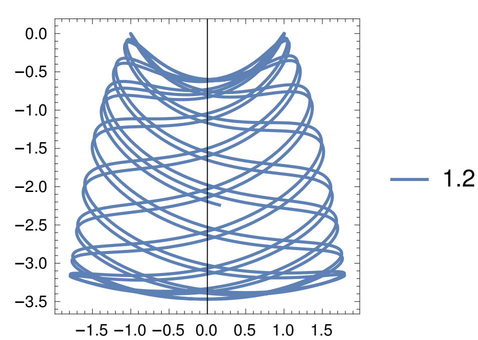

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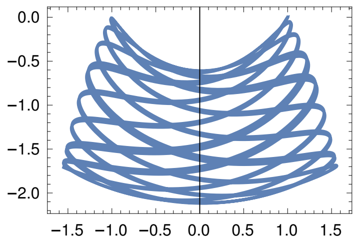

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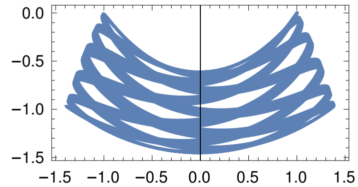

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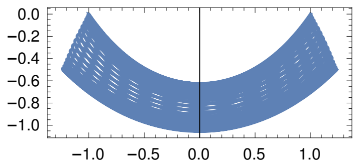

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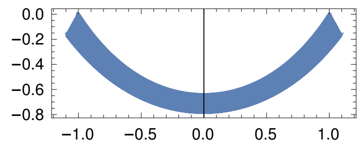

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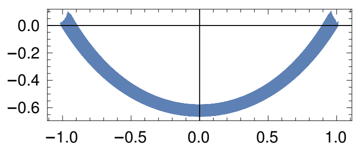

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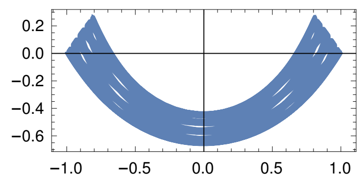

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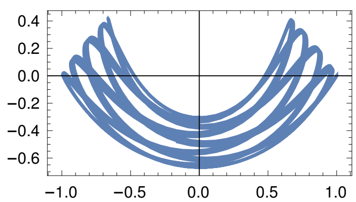

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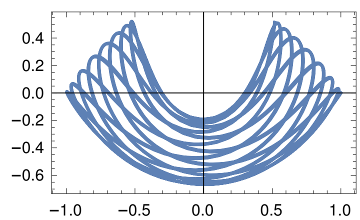

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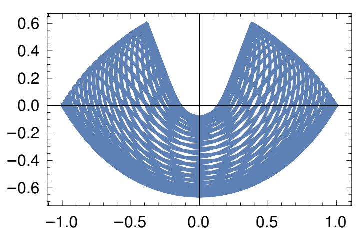

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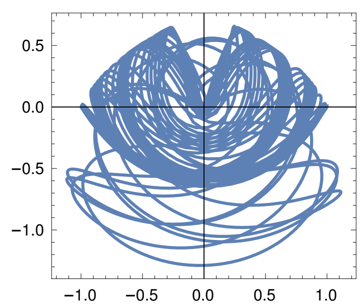

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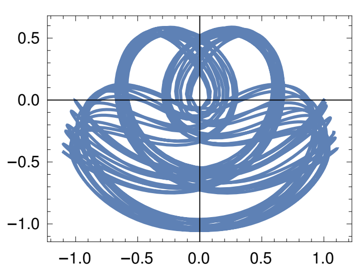

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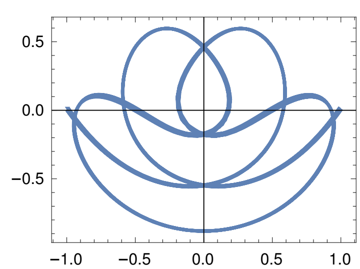

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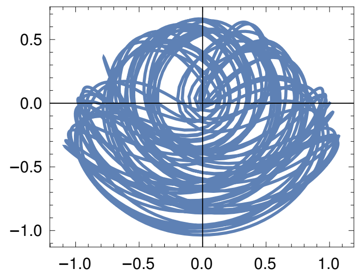

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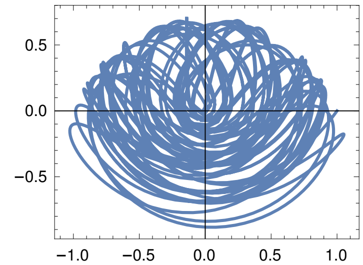

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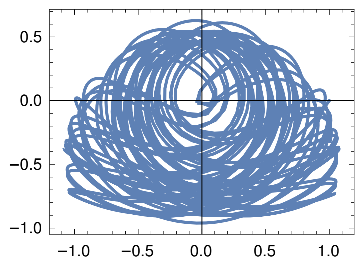

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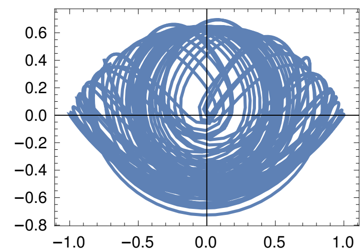

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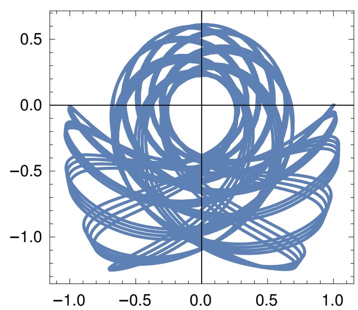

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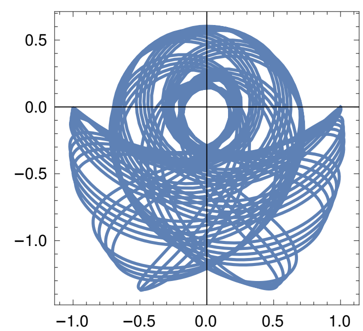

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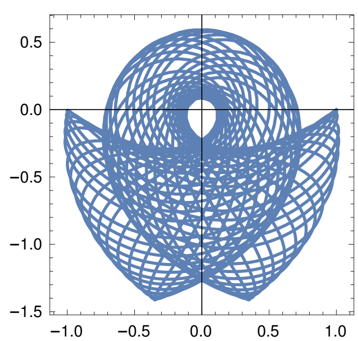

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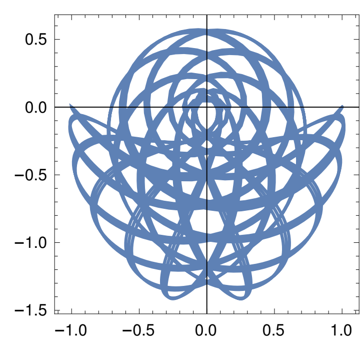

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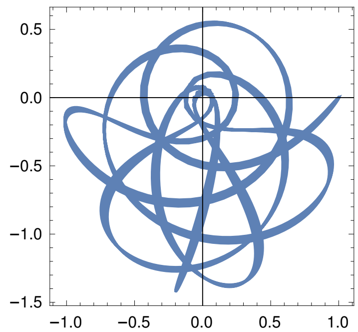

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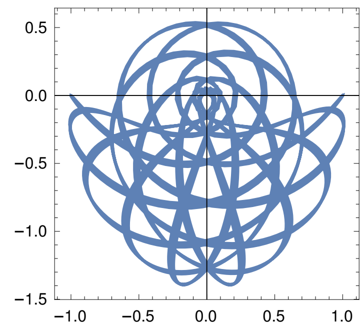

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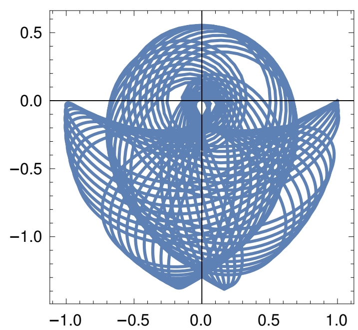

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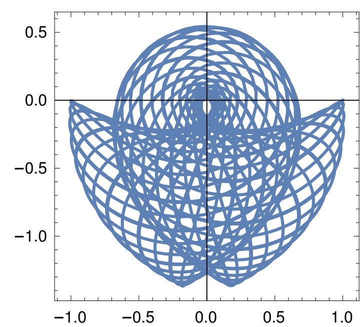

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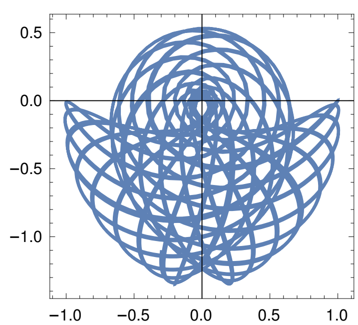

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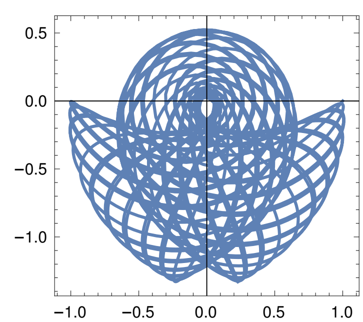

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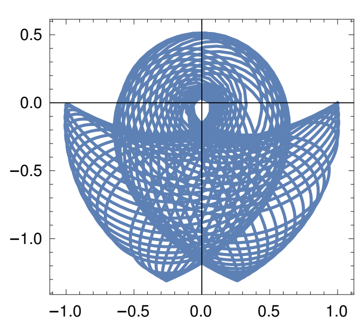

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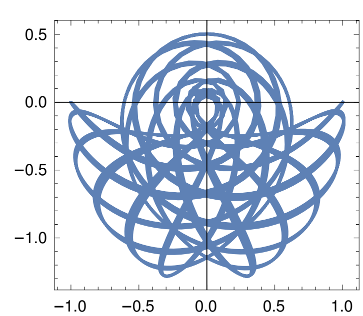

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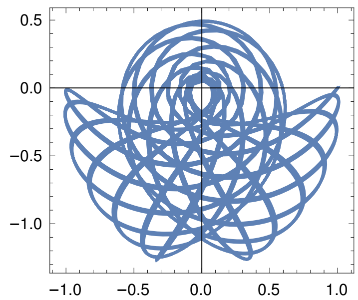

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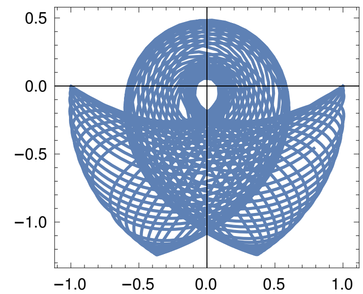

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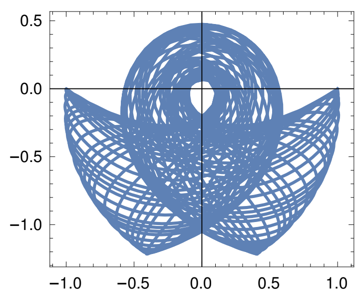

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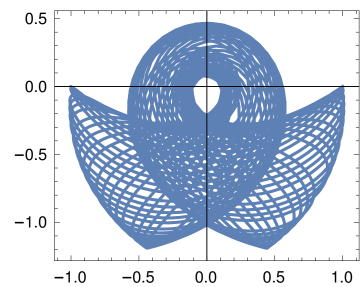

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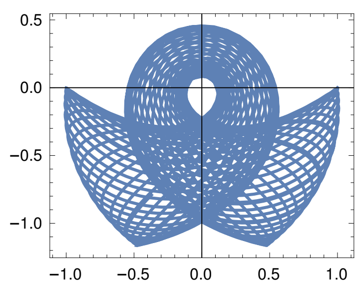

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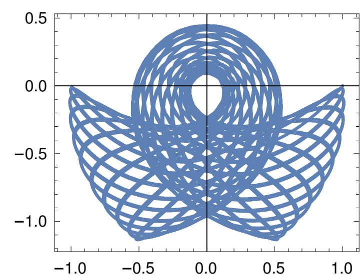

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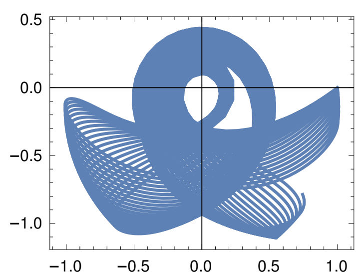

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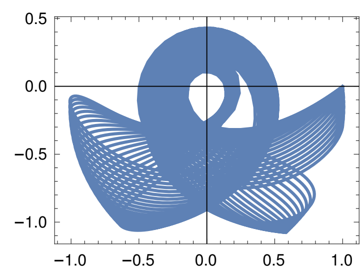

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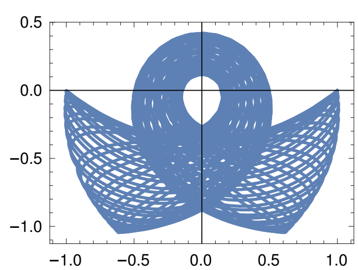

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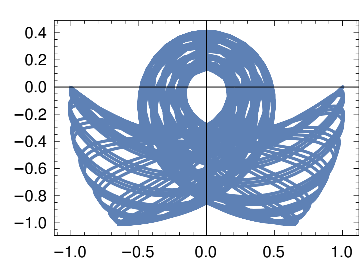

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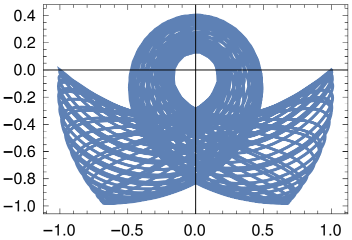

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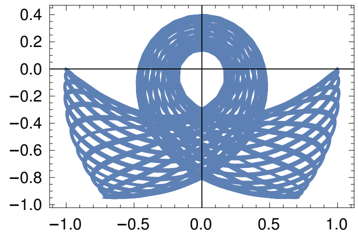

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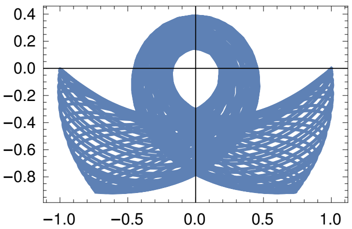

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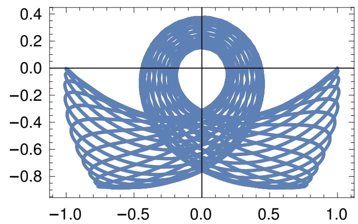

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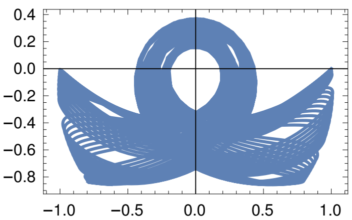

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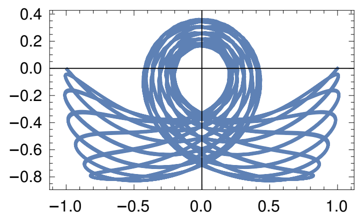

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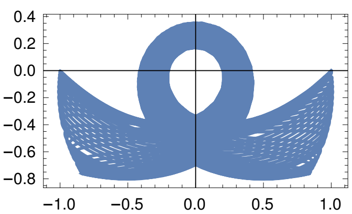

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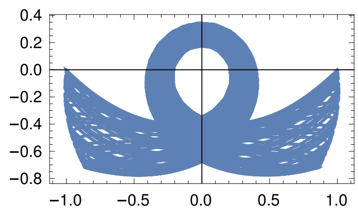

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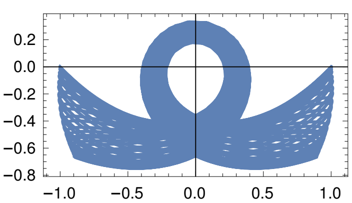

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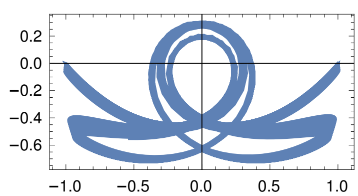

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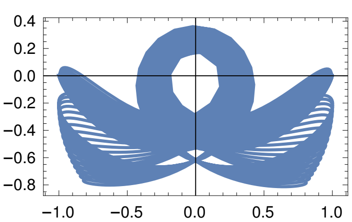

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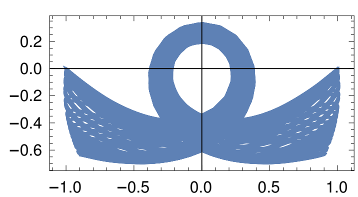

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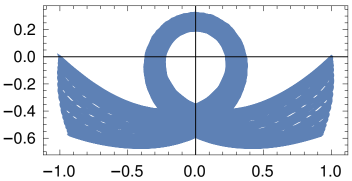

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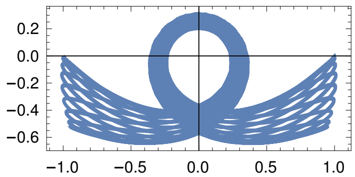

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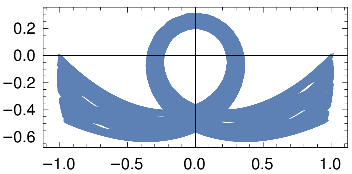

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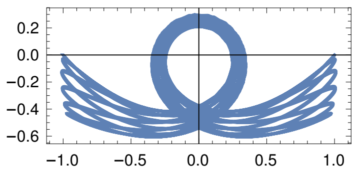

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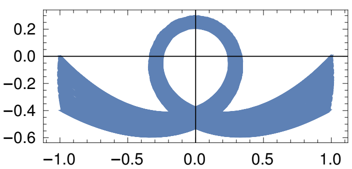

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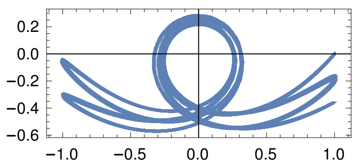

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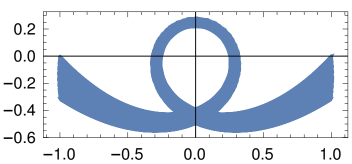

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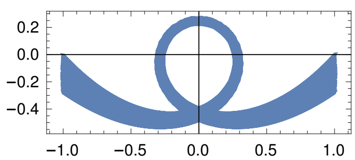

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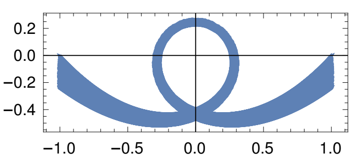

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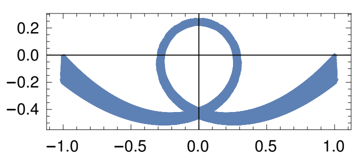

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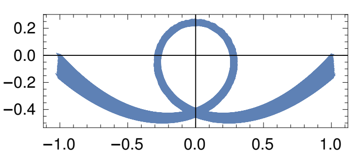

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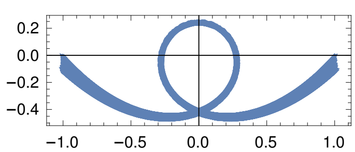

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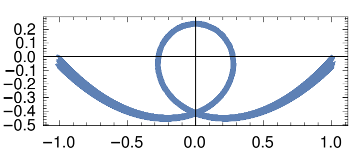

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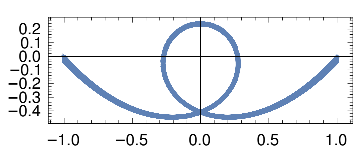

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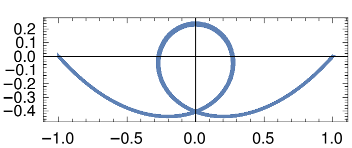

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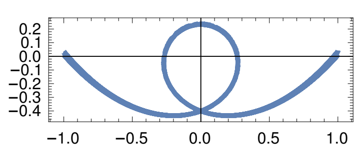

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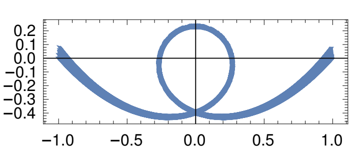

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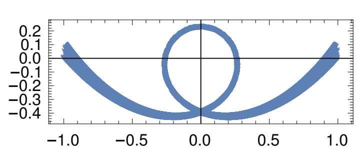

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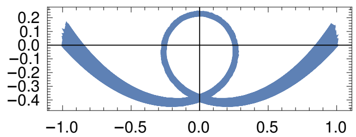

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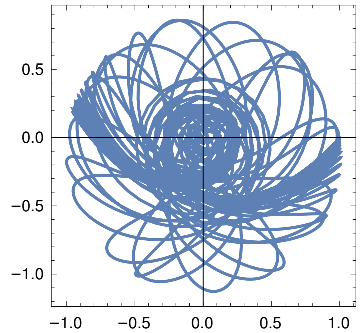

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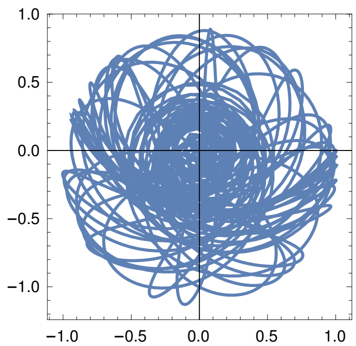

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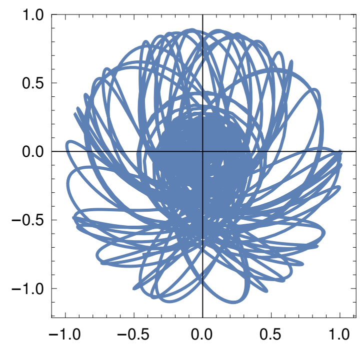

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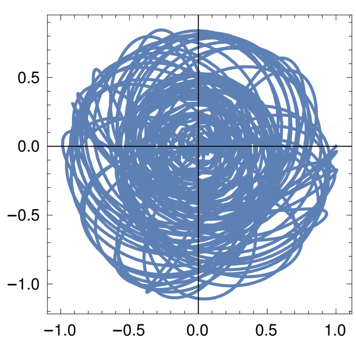

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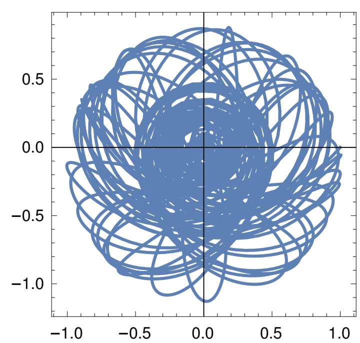

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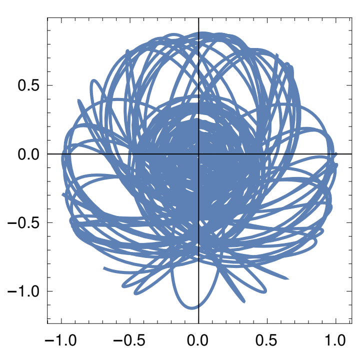

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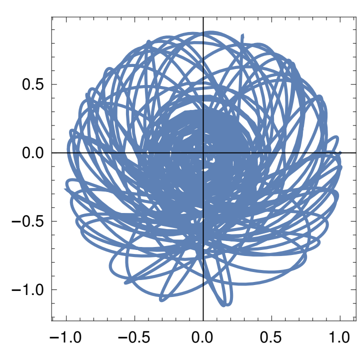

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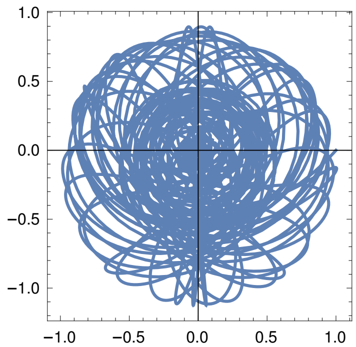

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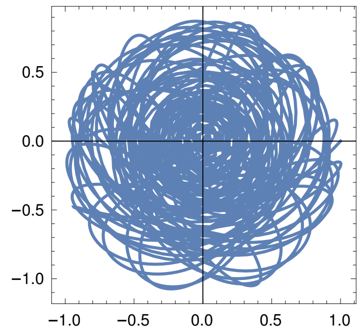

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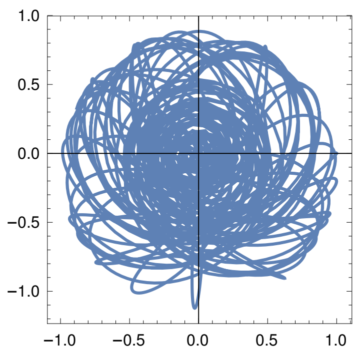

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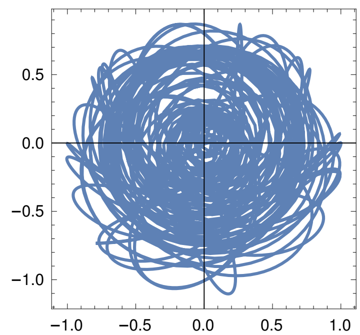

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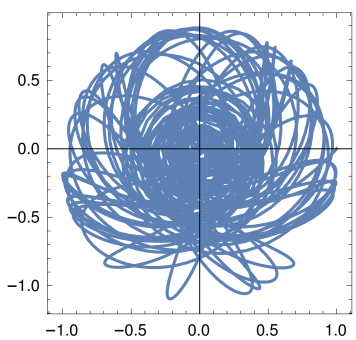

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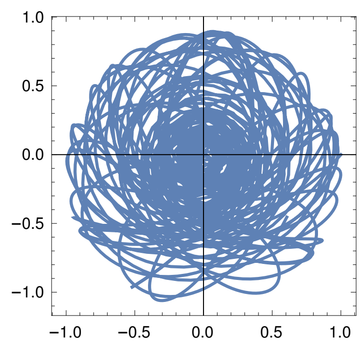

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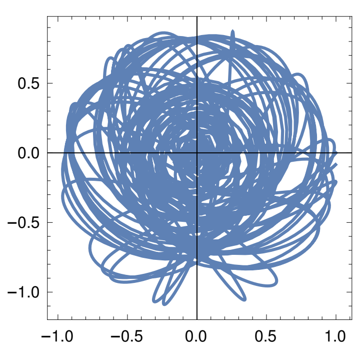

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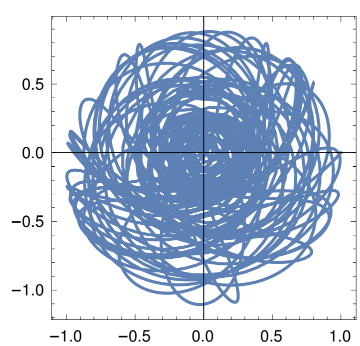

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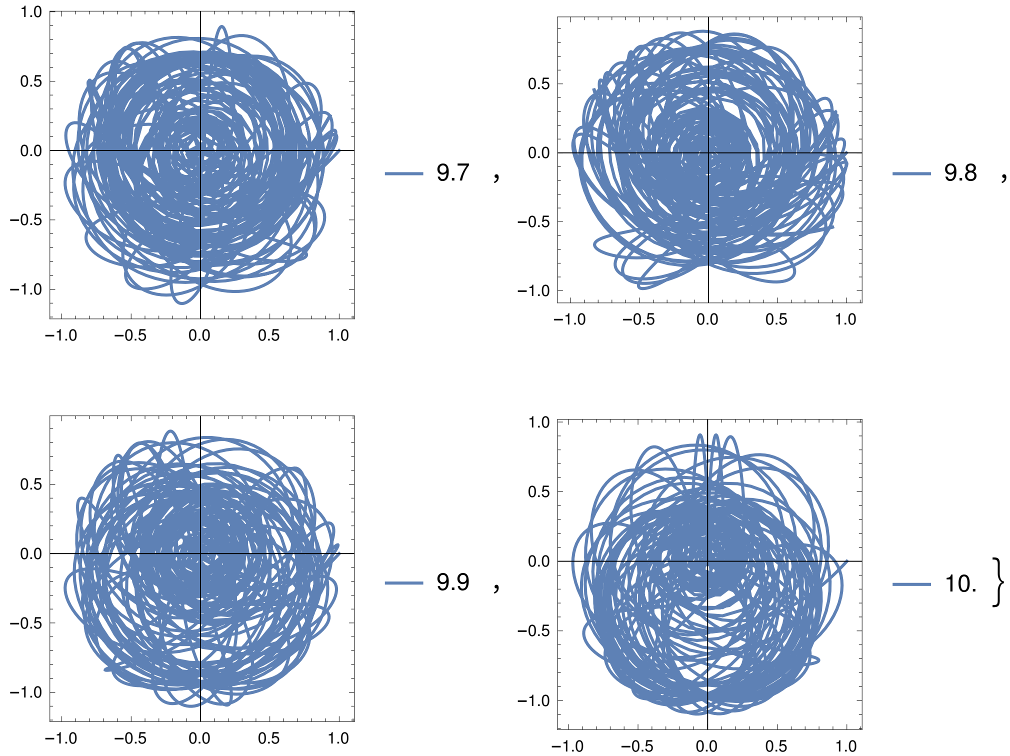


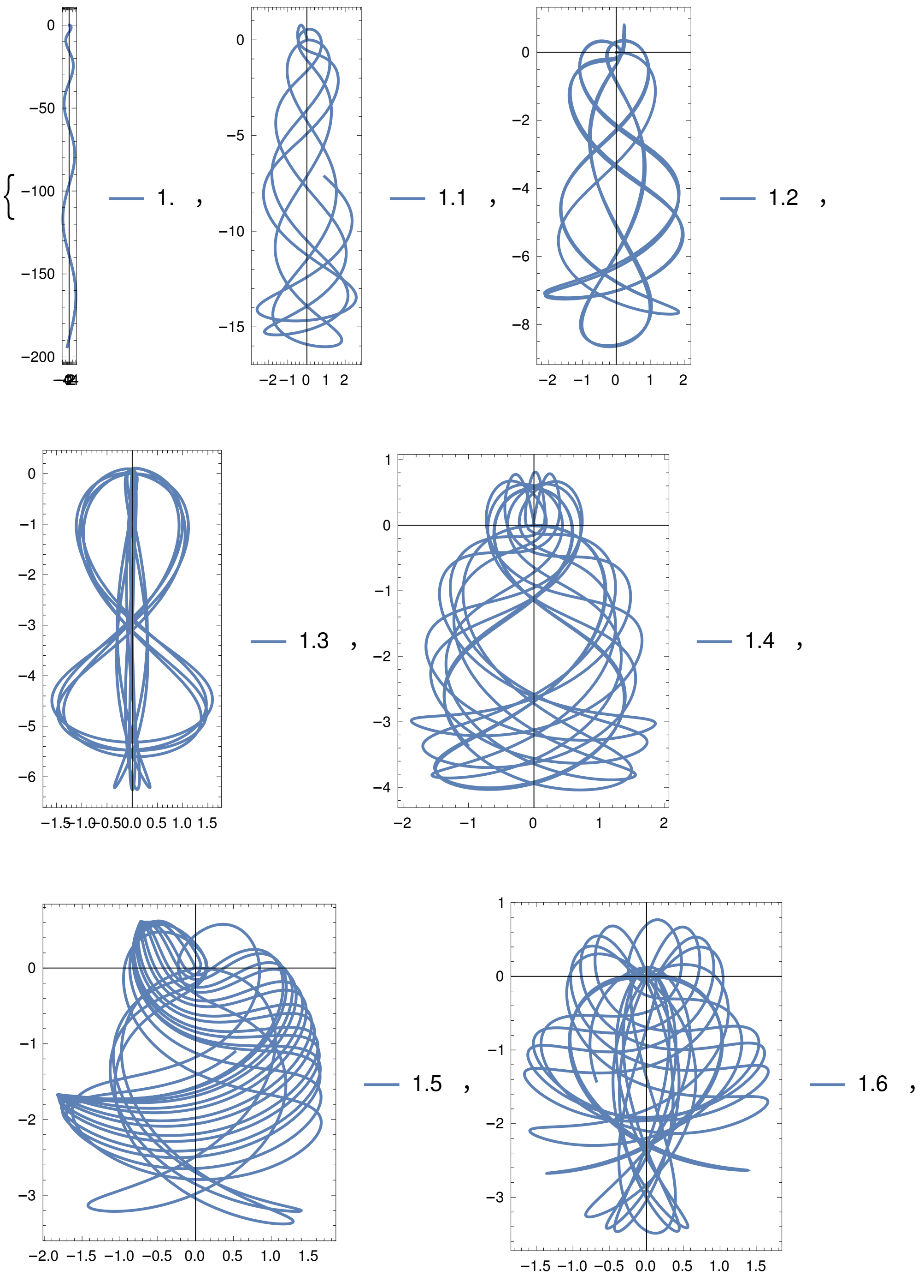
Singular Trajectories

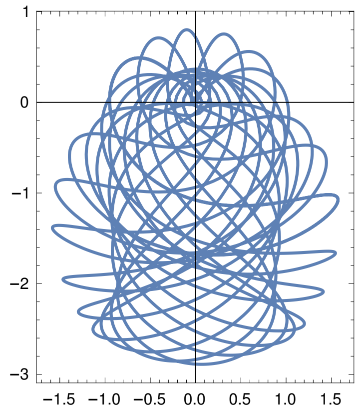
Numerically solving for for singular trajectories with initial conditions $r(0) = 0$, $\theta(0) = 90^0$, $v(0) = 4$ and $\omega(0) = 0$ and generating plots :

In[ ]:=
```
sol1 = ParametricNDSolve[{ v '[t] == ( r[t] × ω[t] ^ 2 + 9.8 (Cos[θ[t]] − μ)) / (1 + μ),
   ω '[t] == (− 2 v[t] × ω[t] − 9.8 Sin[θ[t]]) / r[t], r '[t] == v[t], θ '[t] == ω[t], r[0] == 10 ^ (−10),
   θ[0] == Pi / 2, v[0] == 4, ω[0] == 0}, {r, v, θ, ω}, {t, 0, 100}, {μ}, MaxSteps → 200 000]
```

Out[ ]=
```
{r → ParametricFunction[ Expression: r Parameters: {μ} ],
 v → ParametricFunction[ Expression: v Parameters: {μ} ],
 θ → ParametricFunction[ Expression: θ Parameters: {μ} ],
 ω → ParametricFunction[ Expression: ω Parameters: {μ} ]}
```

In[ ]:=
```
fr1 = Table[ParametricPlot[Evaluate[{r[μ][t] Sin[θ[μ][t]], -r[μ][t] Cos[θ[μ][t]]} /. sol1],
    {t, 0, 50}, PlotLegends → {μ}, Frame -> True], {μ, 1, 10, 0.1}]
```

Out[ ]=




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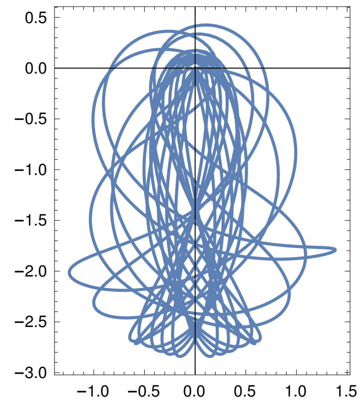

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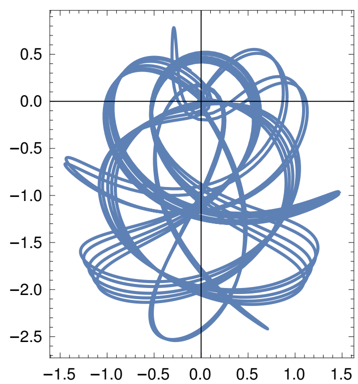

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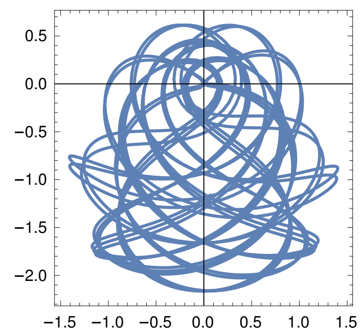

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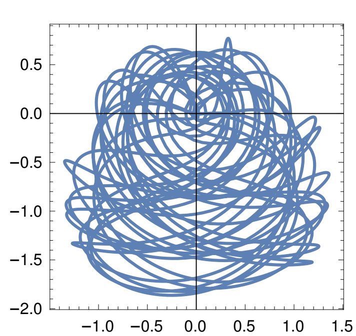

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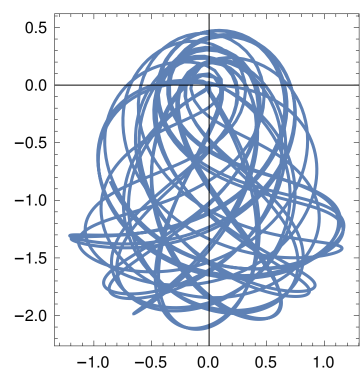

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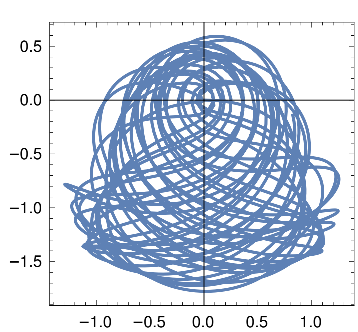

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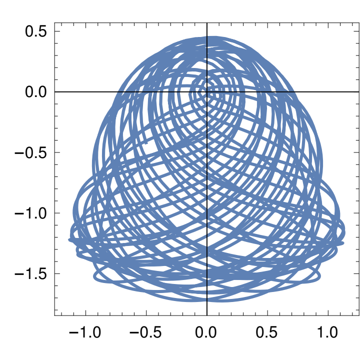

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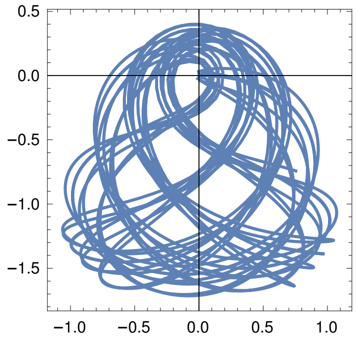

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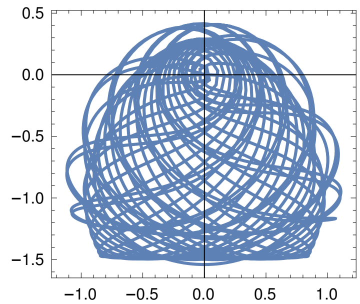

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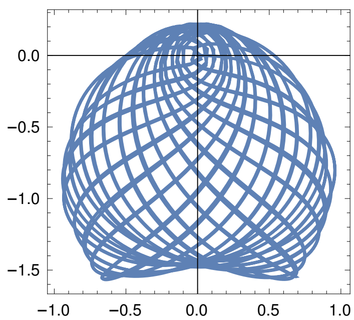

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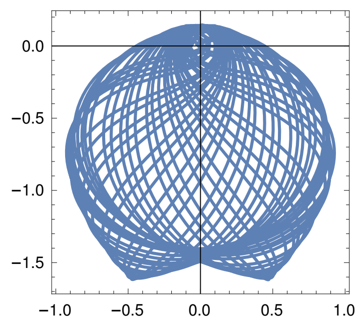

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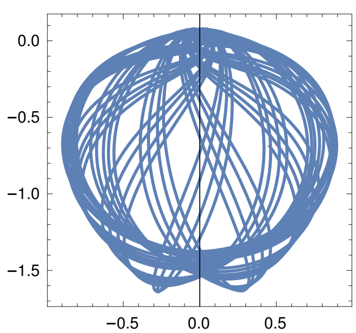

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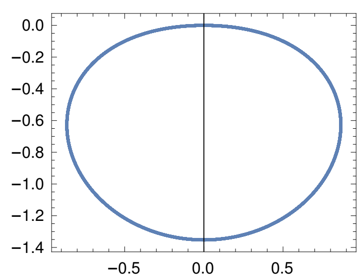

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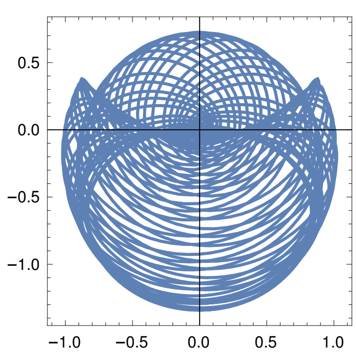

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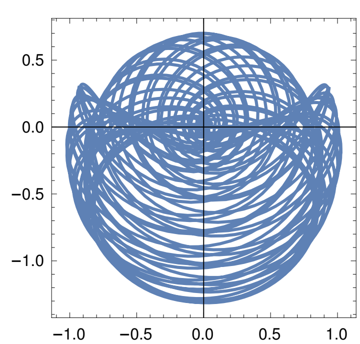

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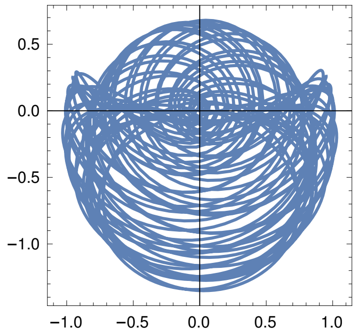

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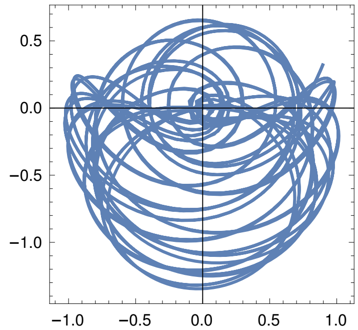

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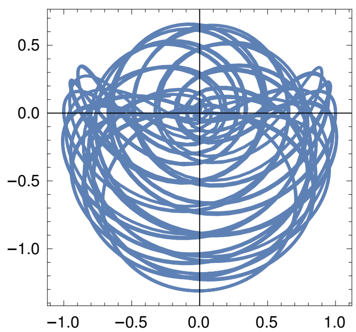

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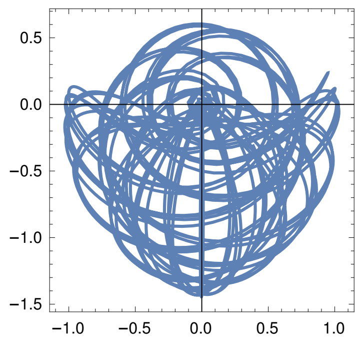

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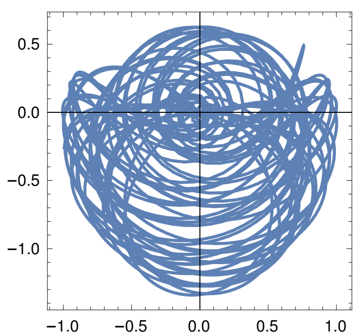

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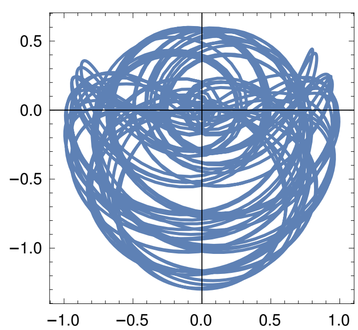

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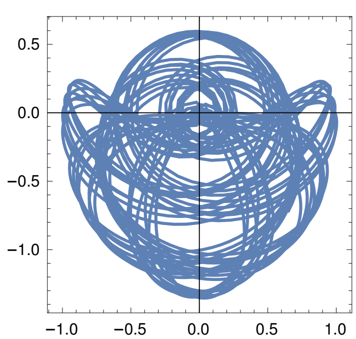

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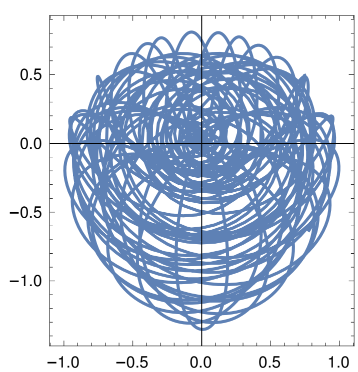

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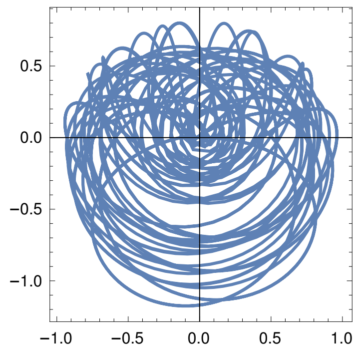

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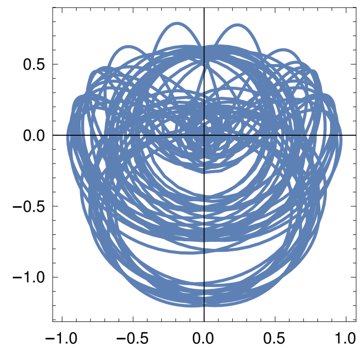

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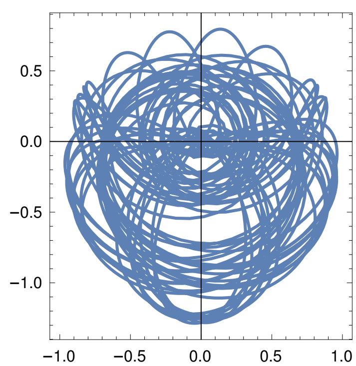

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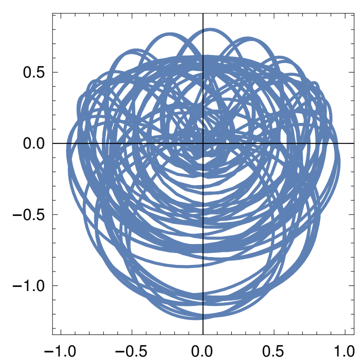

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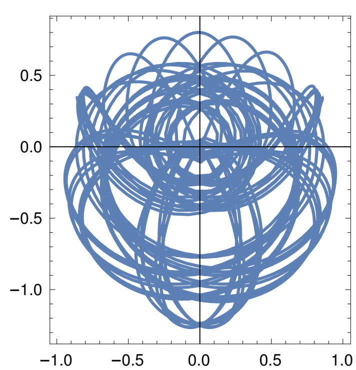

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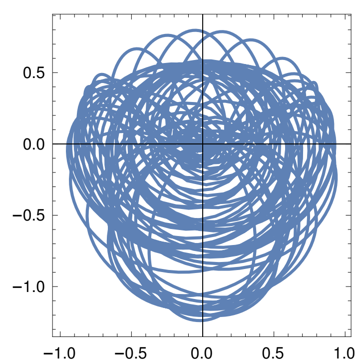

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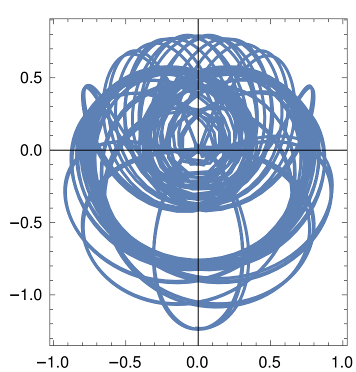

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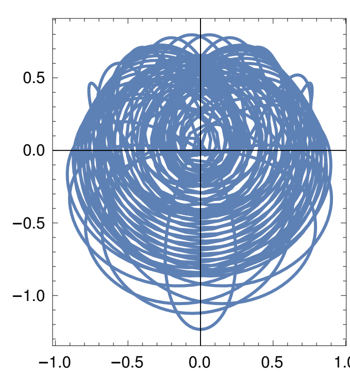

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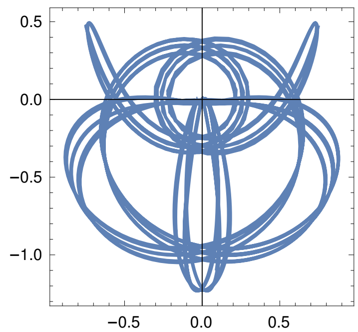

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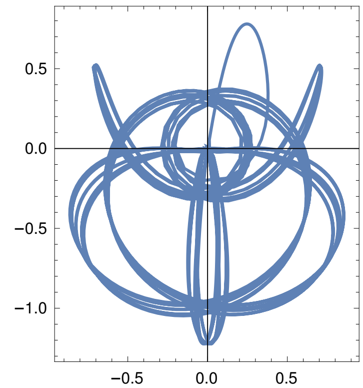

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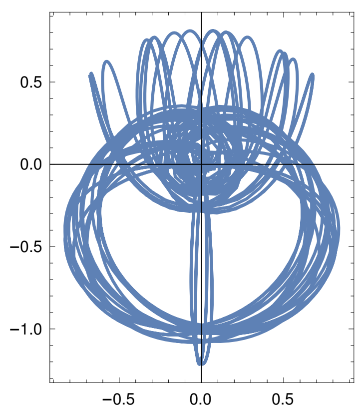

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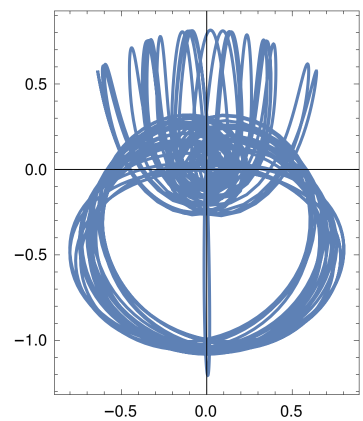

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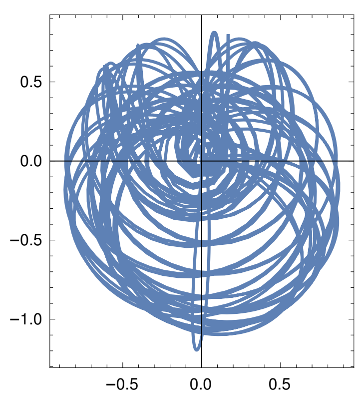

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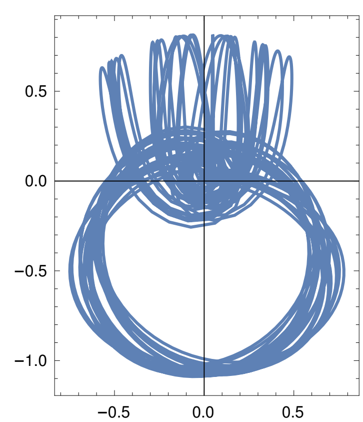

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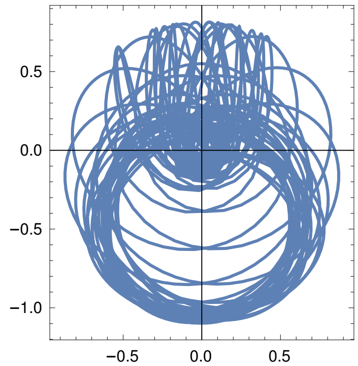

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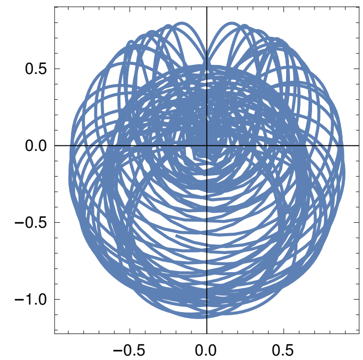

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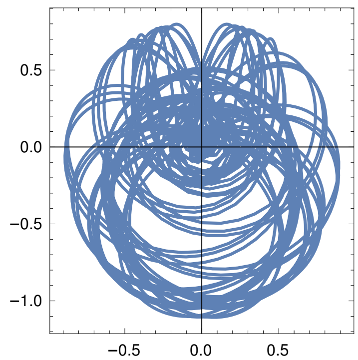

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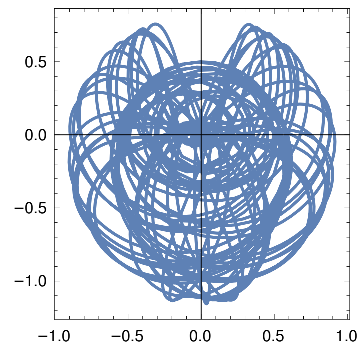

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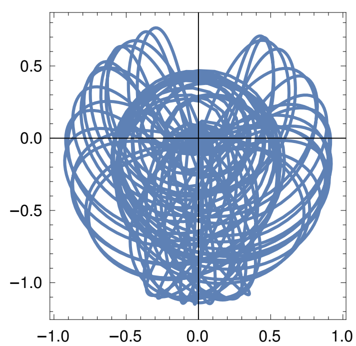

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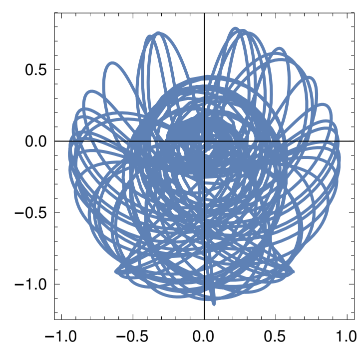

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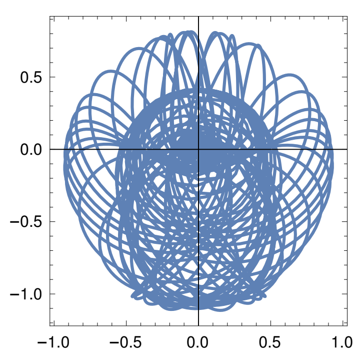

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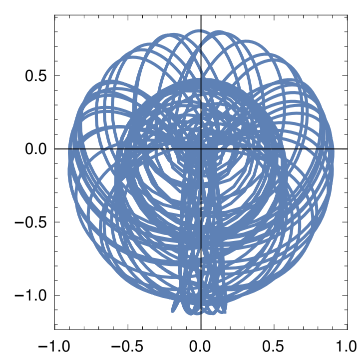

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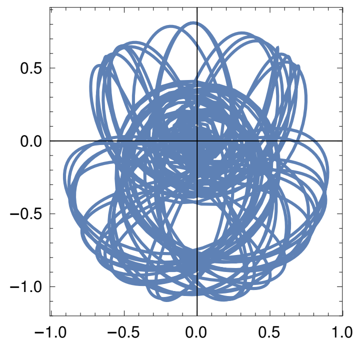

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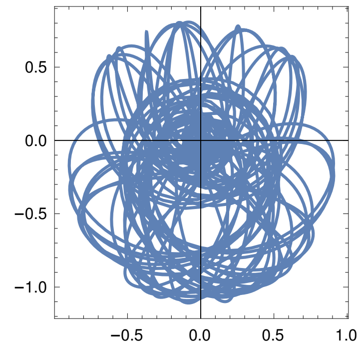

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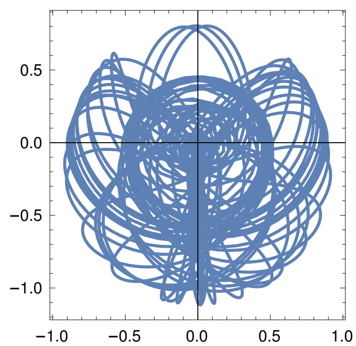

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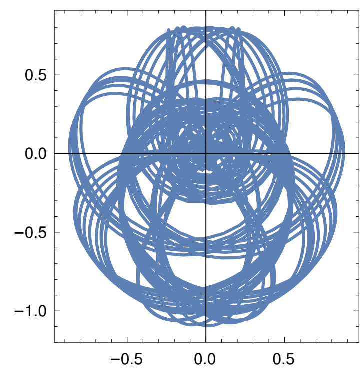

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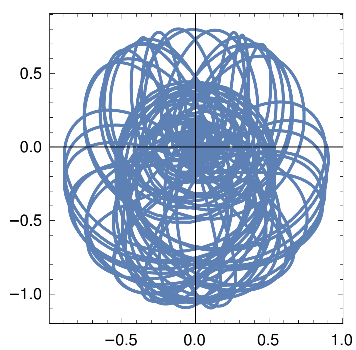

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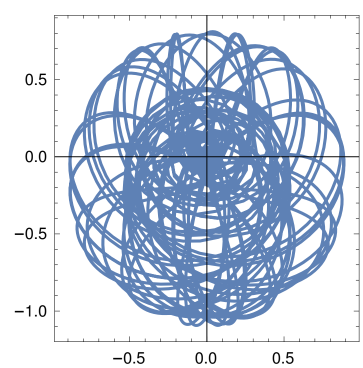

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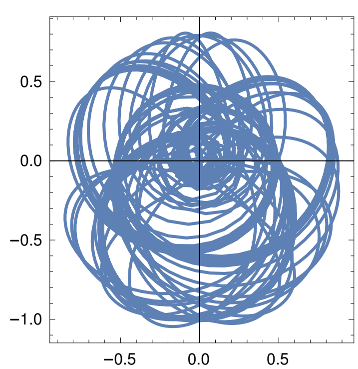

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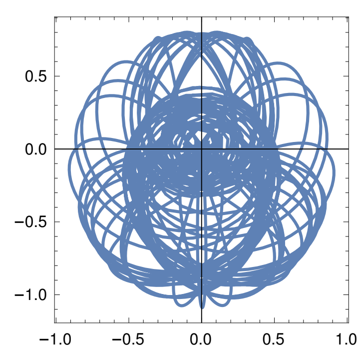

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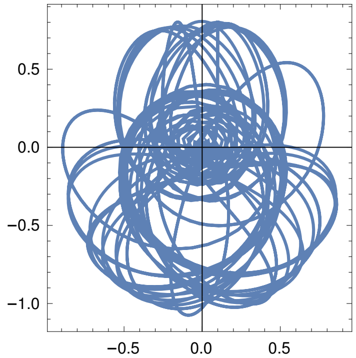

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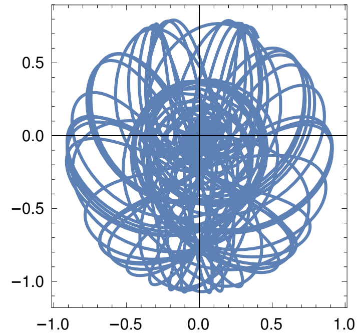

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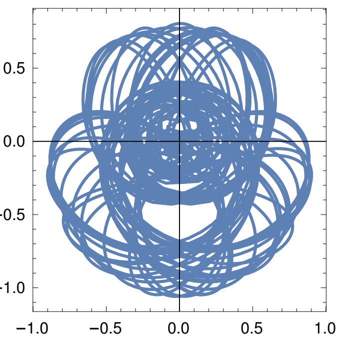

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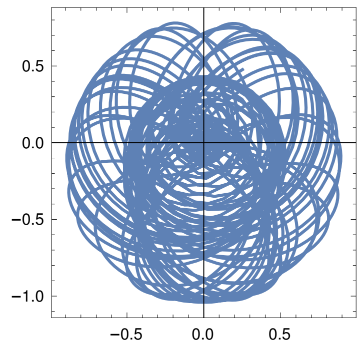

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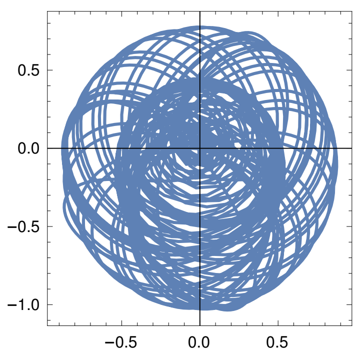

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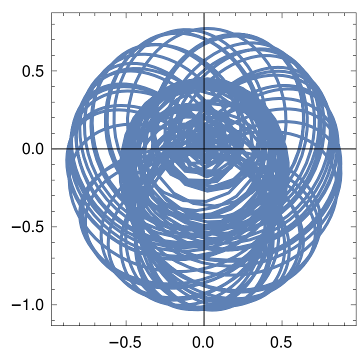

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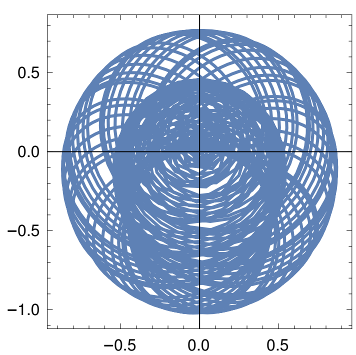

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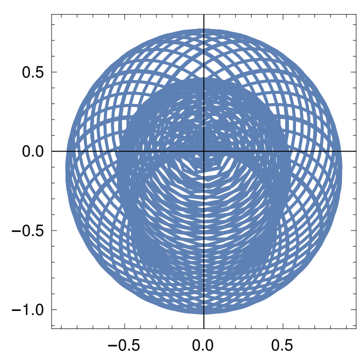

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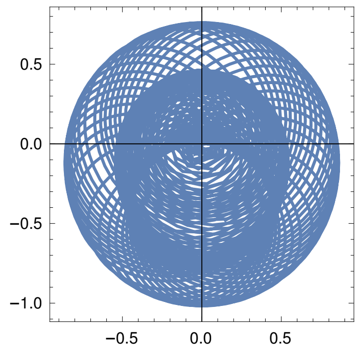

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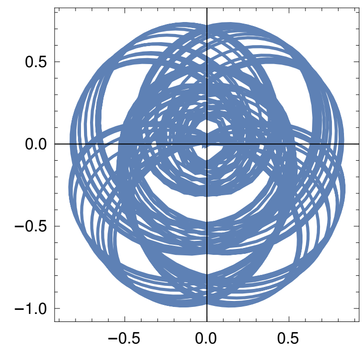

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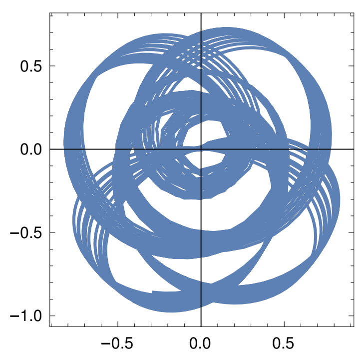

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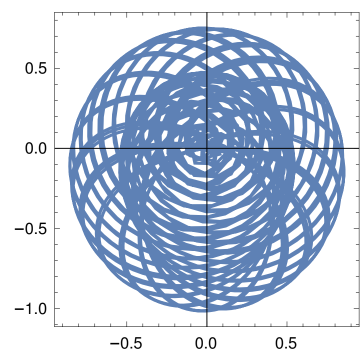

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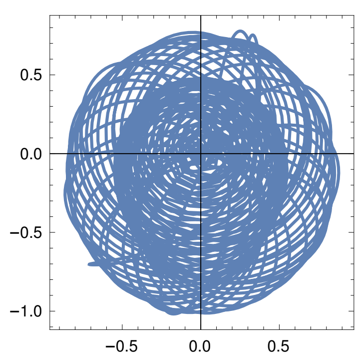

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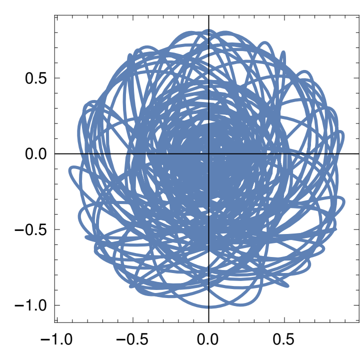

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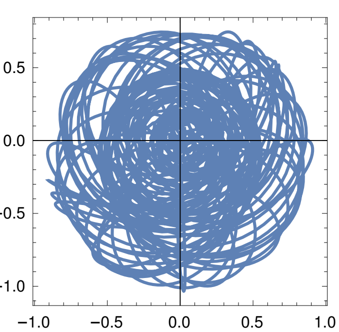

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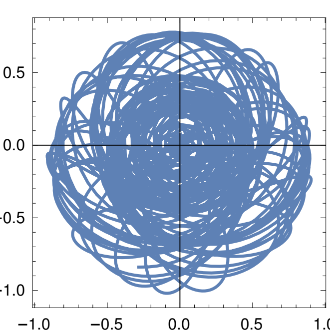

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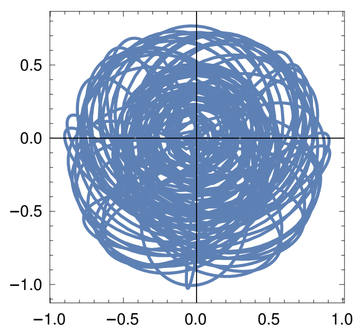

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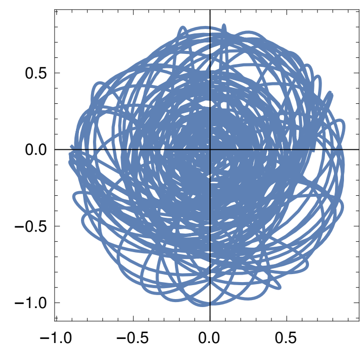

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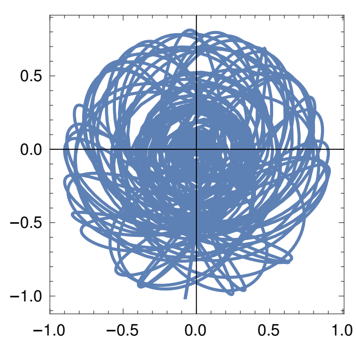

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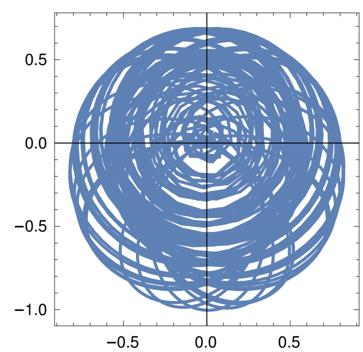

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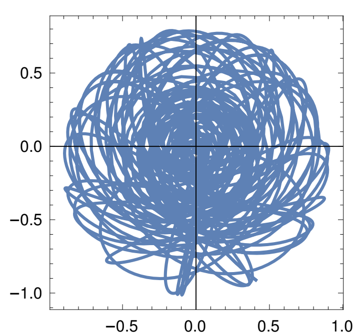

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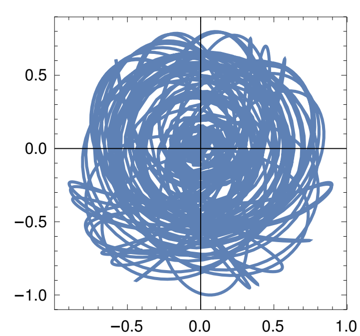

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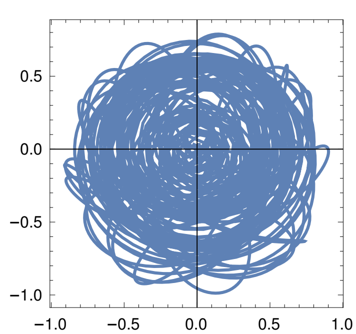

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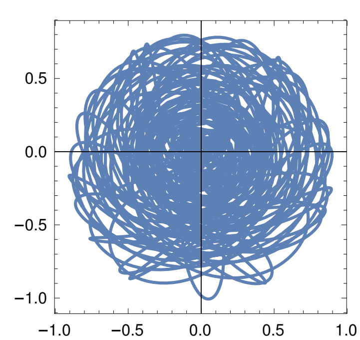

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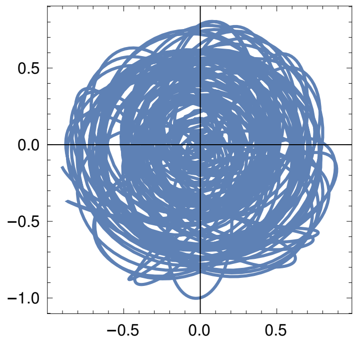

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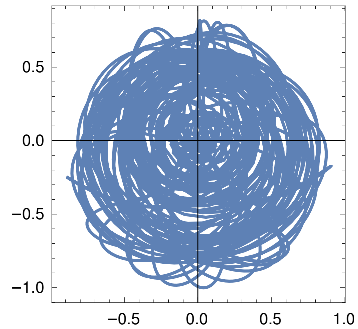

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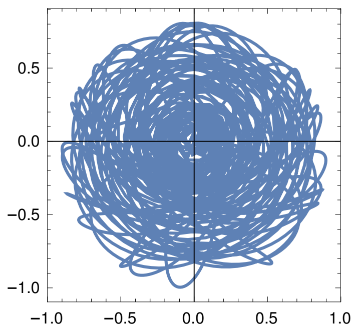

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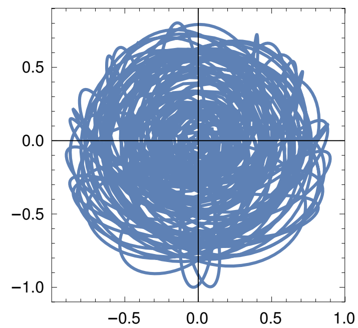

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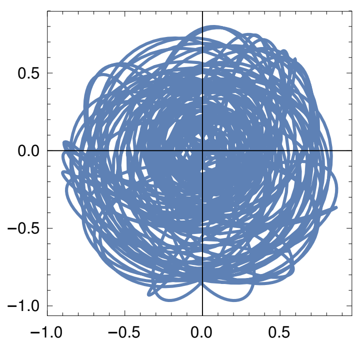

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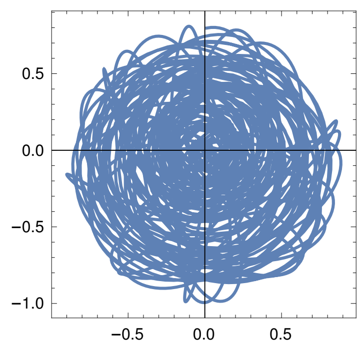


— 10. }

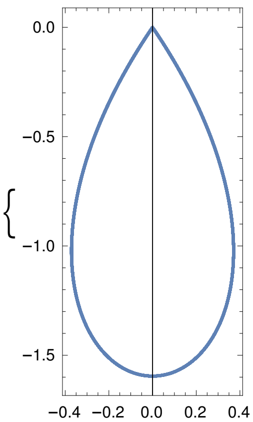
Terminating Trajectories

Varying the initial angle $\theta(0)$ for mass ratio $\mu = 3$, we obtain terminating trajectories that always return to the origin

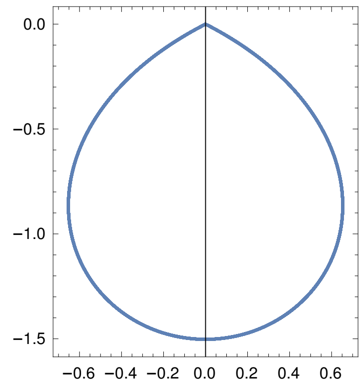
```
In[ ]:= sol2 = ParametricNDSolve[{ v '[t] == ( r[t] × ω[t] ^ 2 + 9.8 (Cos[θ[t]] - 3)) / (4),
        ω '[t] == (- 2 v[t] × ω[t] - 9.8 Sin[θ[t]]) / r[t], r '[t] == v[t], θ '[t] == ω[t], r[0] == 10 ^ (-10),
        θ[0] == θ0, v[0] == 4, ω[0] == 0}, {r, v, θ, ω}, {t, 0, 50}, {θ0}, MaxSteps → 200 000]

Out[ ]= {r → ParametricFunction[ Expression: r Parameters: {θ0} ],
         v → ParametricFunction[ Expression: v Parameters: {θ0} ],
         θ → ParametricFunction[ Expression: θ Parameters: {θ0} ],
         ω → ParametricFunction[ Expression: ω Parameters: {θ0} ]}
```

In[ ]:=
```
Table[ParametricPlot[Evaluate[{r[θ0][t] Sin[θ[θ0][t]], -r[θ0][t] Cos[θ[θ0][t]]} /. sol2],
  {t, 0, 50}, PlotLegends → {θ0}, Frame -> True], {θ0, 0.6, Pi, 0.5}]
```

Out[ ]= {



0.6 ,



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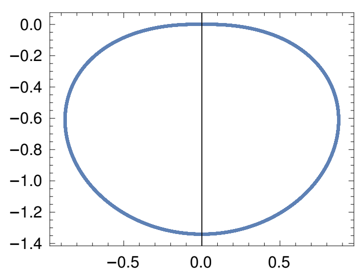

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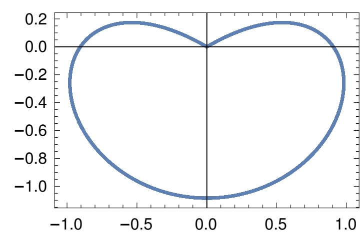

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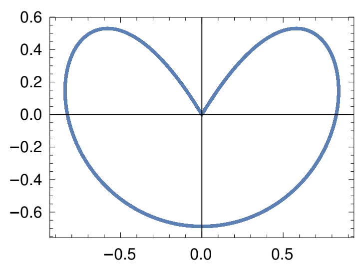

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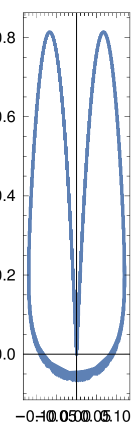

3.1 }