Swinging Atwood's Machine

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PH17B011

November 8, 2019

Abstract

The trajectory of the Swinging Atwood's Machine (SAM) was simulated by computational methods. The SAM resembles a simple Atwood's machine where one of the masses is allowed to swing in a 2-dimensional plane, producing a variety of trajectories - bounded, periodic, singular and terminating. Representative trajectories have been presented. The critical mass ratio for the integrable solution was verified, where the trajectory is terminating irrespective of the initial conditions.

1 Introduction

The Swinging Atwood's Machine is similar to a simple Atwood's machine, with two masses connected by an inextensible rope for zero mass, and two frictionless pulleys with zero radii. The difference from the regular Atwood's system is that one of the masses is allowed to move in the two dimensional place of the system around its pulley without hitting the pulley or the other mass. This extra degree of freedom gives rise to a plethora of trajectories, both integrable and chaotic, depending on the mass ratio of the two masses μ and initial conditions of position and velocity, with the bound that M $\geq$ m (referring to Figure 1 below) to avoid runaway solutions. The centrifugal force on the swinging mass balances the extra weight of the hanging mass and imparts the system dynamic equillibrium.

Figure 1: Swinging Atwood's Machine

This work was primarily regarding the results by Nicholas B. Tufillaro.[1] In the Section II the equations for the trajectory of the smaller mass will be derived, followed by classification of orbits in Section III, and analytical derivation of critical mass in section IV. This will be followed by Conclusion and References and the code used for generating the plots.

2 Equations of Motion

The equations of motion for the SAM can be derived from the Euler-Lagrangian equations for the system. The system has two degrees of freedom - r, θ as seen in Figure 1. The Kinetic Energy of the system is

$$T = \frac{1}{2}M\dot{r}^2 + \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \tag{1}$$

, and the Potential Energy term is

$$V = gr(M - m\cos\theta) \tag{2}$$

The Lagrangian (L = T-V) for the system is, therefore

$$L = \frac{1}{2}M\dot{r}^2 + \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - gr(M - m\cos\theta) \tag{3}$$

Using the Euler-Lagrange relations, the equations of motion of the system is

$$(m + M)\ddot{r} = mr\dot{\theta}^2 + g(m\cos\theta - M)$$

$$\frac{d}{dt}(mr^2\dot{\theta}) = -mgr\sin\theta$$

Simplifying using μ = M/m, we get

$$(1 + \mu)\ddot{r} = r\dot{\theta}^2 + g(\cos\theta - \mu) \tag{4}$$

called the radial equation, and,

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} + gr\sin\theta = 0 \tag{5}$$

the angular equation. The path followed by the system is entirely dependent upon the initial conditions of position and velocity - r(0) = r_0, $\theta(0) = \theta_0$, v(0) $\omega(0) = \omega_0$ These equations were solved numerically for varying mass ratios and plotted.

3 Classification of Trajectories

We consider only *Bounded Orbits*, i.e. r$\leq r_{max}$, with mass ratio $\mu \geq 1$.

- *Periodic orbits*: r(t+τ) = r(t) and θ(t+τ) = θ(t); the orbit repeats itself with time τ
- *Singular orbits*: r(0) = 0; orbits that return to the origin at some time, can also be such that the motion begins with an initial velocity from the origin (the pulley of the mass m)
- *Terminating Singular orbits*: r(τ) = r(0) = 0; the mass m returns to the origin after time τ

Table 1 shows some representative plots for bounded orbits, with $\theta(0) = 90°$.
The plots were studied for mass ratio μ from 1 to 10 with increments of 0.1 for several time periods. Most plots are ergodic in nature and show chaotic behaviour, however for some mass ratios, periodic plots were obtained as shown in Table 2.

Singular Trajectories are when the mass m is launched from the origin with horizontal velocity v(0) = 4 and $\theta(0) = 90°$. A representation of the plots is shown in Table 3.
The mass ratio μ was varied from 1 to 10 as before.
Plots in which the trajectory returned close to the origin are Terminating. Table 4 shows some of the terminating trajectories. The mass was launched from the origin horizontally with an initial velocity v(0) = 4.

The trajectories that are periodic are **Type A**, the most basic example being that of μ = 1.665 of Table 2, similar to a simple pendulum. **Type B** trajectories are that which are terminating, the simplest one being that of μ = 3 in Table 4.

4 Critical Mass Ratio

We can see for mass ratio μ = 3 in Table 4 where $\theta(0) = 90°$ and v(0) = 4 was used, the trajectory returns to the origin. Table 5 shows the plots obtained by varying the initial angle $\theta(0)$, with initial velocity v(0) = 4. It can be seen that all the trajectories execute one symmetrical loop and return to the origin, irrespective of initial conditions of angle and velocity.

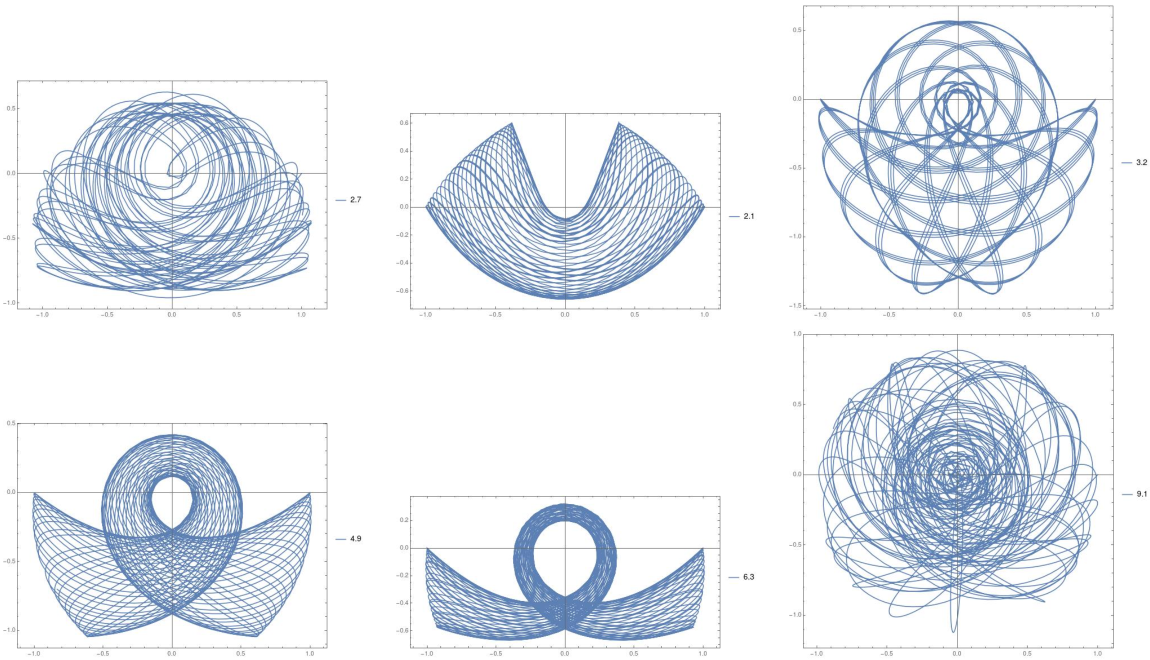

Table 1: Bounded trajectories with $\theta(0) = 90°$, r(0) = 1, v(0) = 0, $\omega(0) = 0$.The mass ratio is indicated on the right edge of each plot.

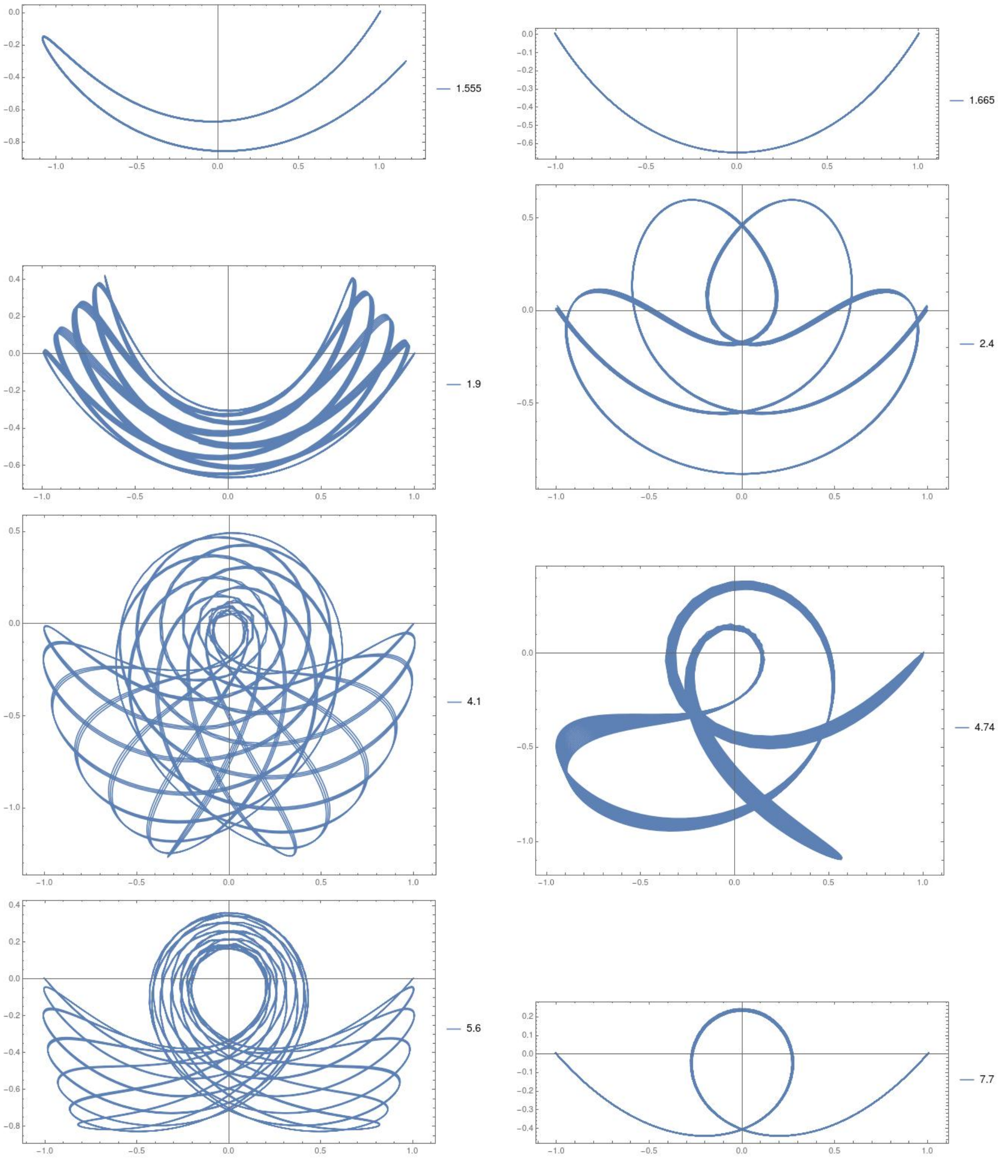

Table 2: Periodic trajectories with $\theta(0) = 90°$, r(0) = 1, v(0) = 0, $\omega(0) = 0$.The mass ratio is indicated on the right edge of each plot.

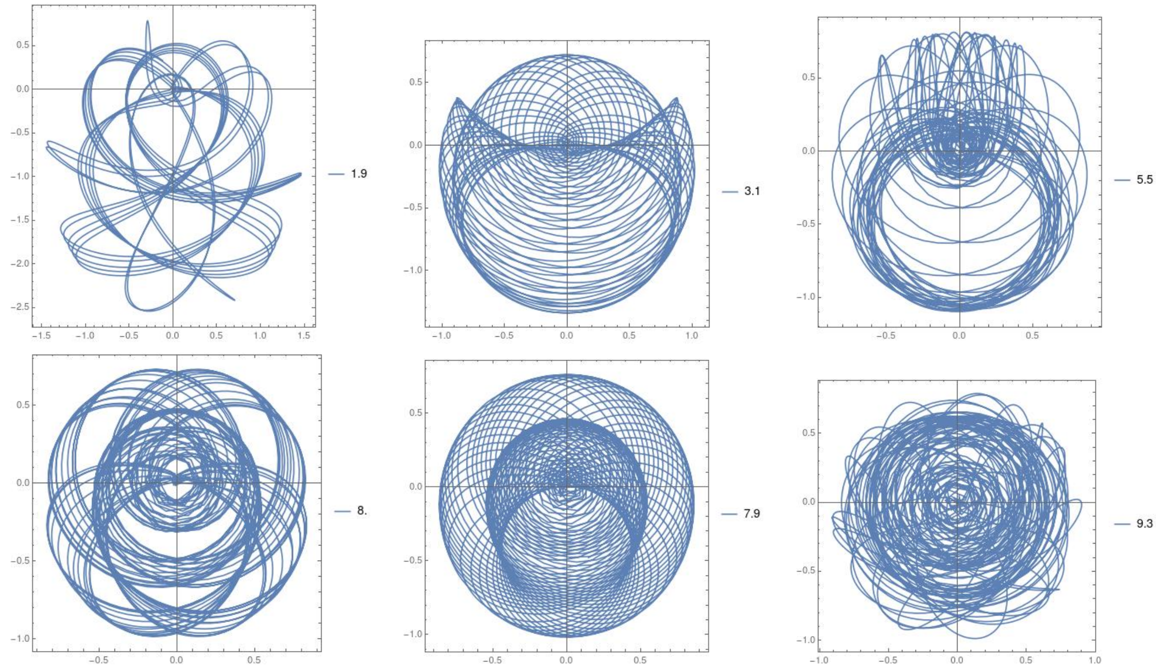

Table 3: Singular trajectories with $\theta(0) = 90°$, r(0)=0, $\omega(0) = 0$ and v(0) $= 4$. The mass ratio is indicated on the right edge of each plot.

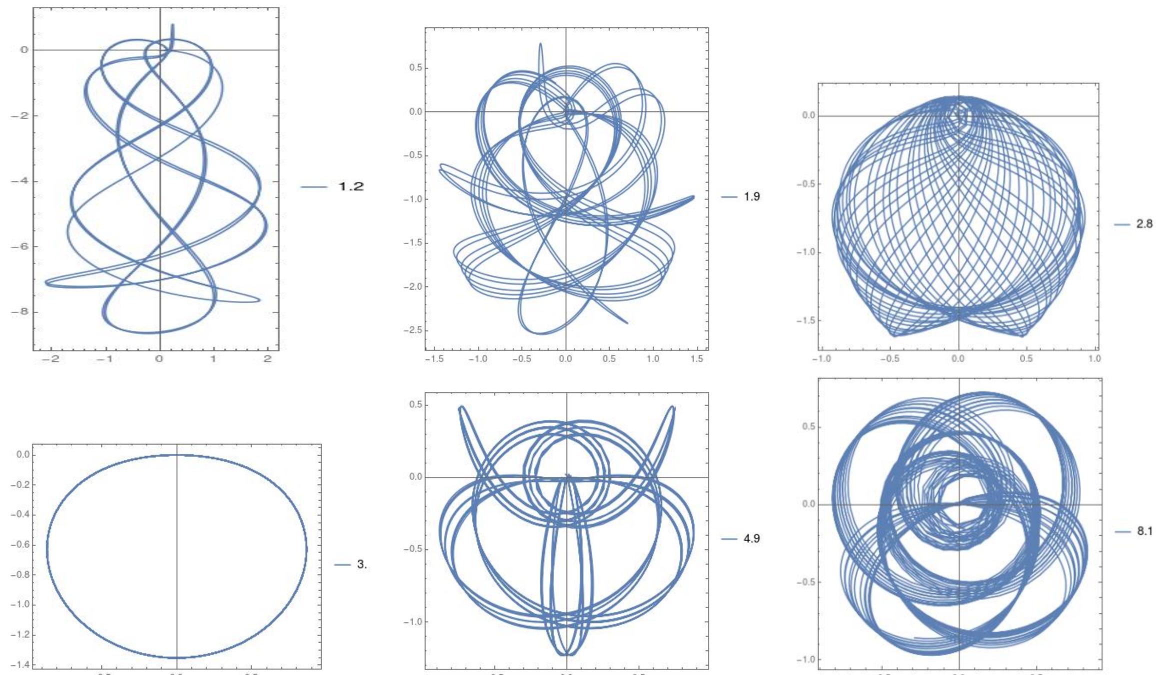

Table 4: Terminating trajectories with $\theta(0) = 90°$, r(0)=0, $\omega(0) = 0$ and v(0) $= 4$.The mass ratio is indicated on the right edge of each plot.

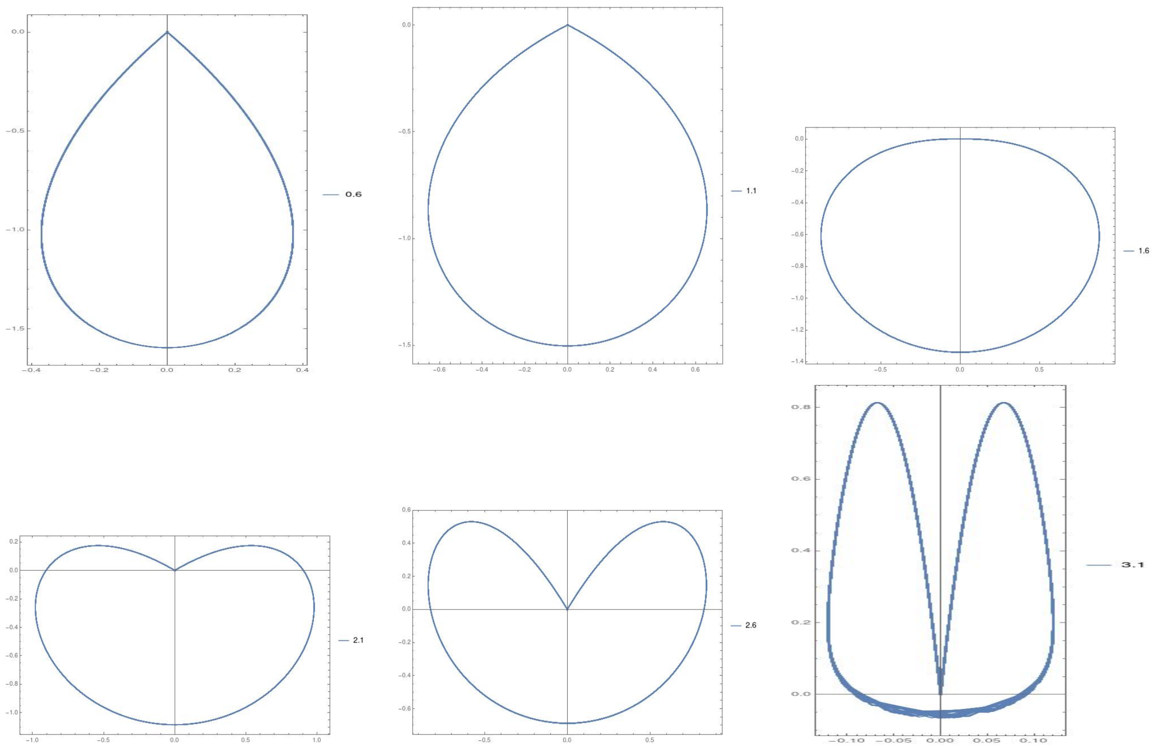

Table 5: Terminating trajectories with $\mu = 3$, r(0)=0, $\omega(0) = 0$ and v(0) = 4.The initial angle $\theta(0)$ is indicated on the right edge of each plot.

5 Conclusion

In this project, the bounded motion of the Simple Atwood's Machine has been explored. It is seen that most of the trajectories are chaotic, with exceptions of a few periodic and terminating trajectories for the special mass ratio $\mu = 3$, where the trajectory always terminated irrespective of initial conditions. The theoretical derivation for appropriate mass ratios corresponding to initial angles can be derived by approximation methods, which has not been done here. Trajectories for $\mu < 1$ also has not been explored. Nevertheless, it can be seen that the SAM provides a good and simple system to explore chaotic motion. The full set of plots can be seen in the Mathematica notebook attached.

References

[1] Nicholas B. Tufillaro, Tyler A. Abbott, and David J. Griffiths. Swinging atwood's machine. *American Journal of Physics*, 52(10):895–903, 1984.